ON THE INVARIANCE GROUPS OF THE BERWALD-MOÓR METRIC OF ORDER TWO AND THREE

M. Neagu[1], **H. Raeisi-Dehkordi**[2]

[1]*University Transilvania of Braşov, Braşov, Romania*

[2]*Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran*

mircea.neagu@unitbv.ro, hengameh_62@aut.ac.ir

In this paper we describe the groups of local transformations of coordinates which preserve unchanged on tangent bundles the two and three dimensional Berwald-Moór metrics. Some algebraic properties of these groups are studied. Finally, we suggest the possible structure of these transformations in the general n-dimensional case.

Key Words: tangent bundles, Berwald-Moór metrics of order two and three, local transformation groups of invariance.

1 Introduction

The geometrical Berwald-Moór structure [4, 10] was intensively investigated by P.K. Rashevski [14] and further physically substantiated and developed by G.S. Asanov [1], D.G. Pavlov and G.I. Garas'ko [5,12,13]. These physical studies emphasize the importance of the Finsler geometry characterized by the total equality in rights of all non-isotropic directions in the theory of space-time structure, gravitation and electromagnetism. For this a reason, one underlines the important role played by the Berwald-Moór metric

$$F_n : TM^n \to \mathbb{R}, \qquad F_n(y) = \sqrt[n]{y^1 y^2 ... y^n}, \qquad n \geq 2,$$

whose Finslerian geometry is studied on tangent bundles by M. Matsumoto and H. Shimada [8], and, in a jet geometrical approach, by V. Balan and M. Neagu [3]. From a physical point of view, an Einstein relativistic law says that the form of all physical laws must be the same in any inertial reference frame (local chart of coordinates). For such a reason, we study in this paper the geometrical transformations which keep unchanged the Berwald-Moór metrics of order two and three. Notice that the geometrical translation of the previous Einstein's physical law is that any geometrical object used in theory must have the same local form in any local chart of coordinates.

In this perspective, let us remind that $(x, y) = (x^i, y^i)$ are the coordinates of the tangent bundle TM^n (associated to an n-dimensional real manifold M^n), which transform by the rules (the Einstein convention of summation is assumed everywhere):

$$\widetilde{x}^i = \widetilde{x}^i(x^j), \qquad \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} y^j, \tag{1.1}$$

where $i, j = \overline{1, n}$ and rank $(\partial \widetilde{x}^i / \partial x^j) = n$.

2 The invariance group of the Berwald-Moór metric of order two

Let us consider the Berwald-Moór metric of order two on the tangent vector bundle TM^2, which is expressed by

$$F_2(y) = \sqrt{y^1 y^2}, \qquad y^1 y^2 \geq 0. \tag{2.1}$$

We recall that the general transformations of coordinates on the tangent bundle TM^2 are given by

$$\begin{gathered} \widetilde{x}^1 = \widetilde{x}^1(x^1, x^2), \qquad \widetilde{x}^2 = \widetilde{x}^2(x^1, x^2), \\ \widetilde{y}^1 = \partial^1_1 y^1 + \partial^1_2 y^2, \quad \widetilde{y}^2 = \partial^2_1 y^1 + \partial^2_2 y^2, \end{gathered} \tag{2.2}$$

where $\partial_j^i := \partial\widetilde{x}^i/\partial x^j$, $\forall\ i,j=\overline{1,2}$. Consequently, the Berwald-Moór metric of order two (2.1) has a global geometrical character if the equality $F_2(\widetilde{y}) = F_2(y)$ is true. This means that for any y^1 and y^2 we must have

$$\widetilde{y}^1\widetilde{y}^2 = y^1y^2 = \partial_1^1\partial_1^2\left(y^1\right)^2 + \partial_2^1\partial_2^2\left(y^2\right)^2 + \left(\partial_1^1\partial_2^2 + \partial_2^1\partial_1^2\right)y^1y^2.$$

Proposition 2.1. *The local transformations of coordinates that preserve invariant the Berwald-Moór metric of order two (2.1) are given by the affine transformations*

$$\mathcal{T}_{(a,b,c)} : \begin{pmatrix} \widetilde{x}^1 \\ \widetilde{x}^2 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 1/a & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} b \\ c \end{pmatrix} \tag{2.3}$$

and

$$\mathcal{S}_{(A,B,C)} : \begin{pmatrix} \widetilde{x}^1 \\ \widetilde{x}^2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1/A \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} B \\ C \end{pmatrix}, \tag{2.4}$$

where $a,b,c,A,B,C \in \mathbb{R}$, with $a \neq 0$ and $A \neq 0$.

Proof. The local transformations of coordinates which invariate the Berwald-Moór metric of order two (2.1) verify the following first order PDE system:

$$\begin{cases} \partial_1^1\partial_1^2 = 0 \\ \partial_2^1\partial_2^2 = 0 \\ \partial_1^1\partial_2^2 + \partial_2^1\partial_1^2 = 1. \end{cases}$$

Case 1. If we consider that $\partial_1^1 = 0$, then we deduce that $\widetilde{x}^1 = \widetilde{x}^1(x^2)$ and $\widetilde{x}^2 = \widetilde{x}^2(x^1)$. Moreover, the following equality is true:

$$\partial_2^1 = \frac{1}{\partial_1^2} = a \in \mathbb{R}^*.$$

It follows that

$$\mathcal{T}_{(a,b,c)} : \begin{cases} \widetilde{x}^1 = ax^2 + b \\ \widetilde{x}^2 = \dfrac{1}{a}x^1 + c, \end{cases}$$

where $b,c \in \mathbb{R}$.

Case 2. By analogy, if we consider that we have $\partial_1^1 \neq 0$, then we find

$$\mathcal{S}_{A,B,C} : \begin{cases} \widetilde{x}^1 = Ax^1 + B \\ \widetilde{x}^2 = \dfrac{1}{A}x^2 + C, \end{cases}$$

where $A,B,C \in \mathbb{R}$ and $A \neq 0$. ■

Remark 2.2. *The affine transformations $\mathcal{S}_{A,B,C}$ include all translations of the plane $\mathbb{R}^2$, which are defined by $\mathcal{S}_{1,x_0,y_0}$. The plane rotations from the set of the affine transformations (2.3) and (2.4) are only the following rotations:*

$$\mathcal{R}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{R}_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Corollary 2.3. *The set of the local transformations of coordinates (which are given by $\mathcal{T}_{a,b,c}$ and $\mathcal{S}_{A,B,C}$), that invariates the Berwald-Moór metric of order two (2.1), has an algebraic structure of non-commutative group with respect to the operation of composition of functions.*

Proof. By direct computations, we observe that we have the following relations:

$$\begin{aligned}
\mathcal{T}_{(a,b,c)} \circ \mathcal{T}_{(a',b',c')} &= \mathcal{S}_{(a/a',b+ac',c+b'/a)},\\
\mathcal{T}_{(a,b,c)} \circ \mathcal{S}_{(A,B,C)} &= \mathcal{T}_{(a/A,b+aC,c+B/a)},\\
\mathcal{S}_{(A,B,C)} \circ \mathcal{T}_{(a,b,c)} &= \mathcal{T}_{(Aa,Ab+B,c/A+C)},\\
\mathcal{S}_{(A,B,C)} \circ \mathcal{S}_{(A',B',C')} &= \mathcal{S}_{(AA',B+AB',C+C'/A)}.
\end{aligned}$$

The neutral element of this group is $\mathcal{S}_{(1,0,0)}$. Moreover, we have

$$\mathcal{T}^{-1}_{(a,b,c)} = \mathcal{T}_{(a,-ac,-b/a)}, \qquad \mathcal{S}^{-1}_{(A,B,C)} = \mathcal{S}_{(1/A,-B/A,-AC)}.$$

■

Using the form of the affine transformations (2.3) and (2.4), let us introduce the following sets of matrices:

$$\mathcal{H}_1 = \left\{ \mathfrak{T} = \left(\begin{array}{cc} 0 & a \\ 1/a & 0 \end{array} \right) \middle| \, a \in \mathbb{R}^* \right\}$$

and

$$\mathcal{H}_2 = \left\{ \mathfrak{S} = \left(\begin{array}{cc} A & 0 \\ 0 & 1/A \end{array} \right) \middle| \, A \in \mathbb{R}^* \right\},$$

where we have $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$. In this context, we can prove the following algebraic results of characterization:

Proposition 2.4. *A matrix $\mathfrak{M} \in M_2(\mathbb{R})$ belongs to the set $\mathcal{H}_1$ if and only if it verifies the following conditions:*

(1) $\mathfrak{M}^{-1} = \mathfrak{M}$;

(2) $\det \mathfrak{M} = -1$;

(3) $\mathfrak{M}^T = E \cdot \mathfrak{M} \cdot E$, *where* $E = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$.

Proof. It is easy to see that any matrix $\mathfrak{T} \in \mathcal{H}_1$ verifies the relations (1), (2) and (3). Conversely, let us consider a matrix

$$\mathfrak{M} = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(\mathbb{R})$$

which verifies the relations (1), (2) and (3). It follows that we obtain

$$\begin{cases} ad - bc = -1, & a = -d \\ a = d, & bc = 1. \end{cases}$$

Solving this system, we find what we were looking for. ■

Proposition 2.5. *A matrix $\mathfrak{X} \in M_2(\mathbb{R})$ belongs to the set $\mathcal{H}_2$ if and only if it verifies the following conditions:*

(1) $\mathfrak{X}^T = \mathfrak{X}$;

(2) $\det \mathfrak{X} = 1$;

(3) $\mathfrak{X}^{-1} = E \cdot \mathfrak{X} \cdot E$.

Proof. It is easy to see that any matrix $\mathfrak{S} \in \mathcal{H}_2$ verifies the relations (1), (2) and (3). Conversely, let us consider a matrix

$$\mathfrak{X} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathbb{R})$$

which verifies the relations (1), (2) and (3). It follows that we obtain

$$\begin{cases} B = C, & AD - BC = 1 \\ B = -C, & AD = 1. \end{cases}$$

Solving this system, we get $B = C = 0$ and $AD = 1$. ■

3 The invariance group of the Berwald-Moór metric of order three

Let us consider the Berwald-Moór metric of order three on the tangent vector bundle TM^3, which is expressed by

$$F_3(y) = \sqrt[3]{y^1 y^2 y^3}. \tag{3.1}$$

The general transformations of coordinates on TM^3 are given by

$$\begin{gathered} \widetilde{x}^1 = \widetilde{x}^1(x^1, x^2, x^3), \quad \widetilde{x}^2 = \widetilde{x}^2(x^1, x^2, x^3), \quad \widetilde{x}^3 = \widetilde{x}^3(x^1, x^2, x^3), \\ \widetilde{y}^1 = \partial_1^1 y^1 + \partial_2^1 y^2 + \partial_3^1 y^3, \\ \widetilde{y}^2 = \partial_1^2 y^1 + \partial_2^2 y^2 + \partial_3^2 y^3, \\ \widetilde{y}^3 = \partial_1^3 y^1 + \partial_2^3 y^2 + \partial_3^3 y^3, \end{gathered} \tag{3.2}$$

where $\partial_j^i := \partial \widetilde{x}^i / \partial x^j$, $\forall\ i, j = \overline{1, 3}$.

It follows that the Berwald-Moór metric of order three (3.1) has a global geometrical character if the equality $F_3(\widetilde{y}) = F_3(y)$ is true. This means that for any y^1, y^2 and y^3 we must have

$$\begin{gathered} \widetilde{y}^1 \widetilde{y}^2 \widetilde{y}^3 = y^1 y^2 y^3 = \\ = \left(\partial_1^1 \partial_2^2 \partial_3^3 + \partial_1^1 \partial_3^2 \partial_2^3 + \partial_2^1 \partial_1^2 \partial_3^3 + \partial_2^1 \partial_3^2 \partial_1^3 + \partial_3^1 \partial_1^2 \partial_2^3 + \partial_3^1 \partial_2^2 \partial_1^3\right) y^1 y^2 y^3 + \\ + \partial_1^1 \partial_1^2 \partial_1^3 \left(y^1\right)^3 + \partial_2^1 \partial_2^2 \partial_2^3 \left(y^2\right)^3 + \partial_3^1 \partial_3^2 \partial_3^3 \left(y^3\right)^3 + \\ + \left(\partial_1^1 \partial_1^2 \partial_2^3 + \partial_1^1 \partial_2^2 \partial_1^3 + \partial_2^1 \partial_1^2 \partial_1^3\right) \left(y^1\right)^2 y^2 + \\ + \left(\partial_1^1 \partial_1^2 \partial_3^3 + \partial_1^1 \partial_3^2 \partial_1^3 + \partial_3^1 \partial_1^2 \partial_1^3\right) \left(y^1\right)^2 y^3 + \\ + \left(\partial_2^1 \partial_2^2 \partial_1^3 + \partial_2^1 \partial_1^2 \partial_2^3 + \partial_1^1 \partial_2^2 \partial_2^3\right) \left(y^2\right)^2 y^1 + \\ + \left(\partial_2^1 \partial_2^2 \partial_3^3 + \partial_2^1 \partial_3^2 \partial_2^3 + \partial_3^1 \partial_2^2 \partial_2^3\right) \left(y^2\right)^2 y^3 + \\ + \left(\partial_3^1 \partial_3^2 \partial_1^3 + \partial_3^1 \partial_1^2 \partial_3^3 + \partial_1^1 \partial_3^2 \partial_3^3\right) \left(y^3\right)^2 y^1 + \\ + \left(\partial_3^1 \partial_3^2 \partial_2^3 + \partial_3^1 \partial_2^2 \partial_3^3 + \partial_2^1 \partial_3^2 \partial_3^3\right) \left(y^3\right)^2 y^2. \end{gathered}$$

Inspired by the form of the affine transformations (2.3) and (2.4), we will use the following matrix notations for three dimensional affine maps:

$$\widetilde{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}, \quad \mathcal{A} \in M_3(\mathbb{R}), \quad \mathcal{B} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \mathcal{X} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}, \quad \widetilde{\mathcal{X}} = \begin{pmatrix} \widetilde{x}^1 \\ \widetilde{x}^2 \\ \widetilde{x}^3 \end{pmatrix}.$$

In this context, we can give the following geometrical result:

Proposition 3.1. *The local transformations of coordinates that preserve invariant the Berwald-Moór metric of order three (3.1) are given by the following affine transformations:*

$$\mathcal{T}_{(a_1,a_2,b_1,b_2,b_3)} : \widetilde{\mathcal{X}} = \mathcal{A}_1\mathcal{X} + \mathcal{B}_1, \tag{3.3}$$

$$\mathcal{S}_{(a_3,a_4,b_4,b_5,b_6)} : \widetilde{\mathcal{X}} = \mathcal{A}_2\mathcal{X} + \mathcal{B}_2, \tag{3.4}$$

$$\mathcal{R}_{(a_5,a_6,b_7,b_8,b_9)} : \widetilde{\mathcal{X}} = \mathcal{A}_3\mathcal{X} + \mathcal{B}_3, \tag{3.5}$$

$$\mathcal{M}_{(a_7,a_8,b_{10},b_{11},b_{12})} : \widetilde{\mathcal{X}} = \mathcal{A}_4\mathcal{X} + \mathcal{B}_4, \tag{3.6}$$

$$\mathcal{N}_{(a_9,a_{10},b_{13},b_{14},b_{15})} : \widetilde{\mathcal{X}} == \mathcal{A}_5\mathcal{X} + \mathcal{B}_5, \tag{3.7}$$

$$\mathcal{O}_{(a_{11},a_{12},b_{16},b_{17},b_{18})} : \widetilde{\mathcal{X}} = \mathcal{A}_6\mathcal{X} + \mathcal{B}_6, \tag{3.8}$$

where $0 \neq a_i \in \mathbb{R},\ \forall\ i = \overline{1,12},\ b_j \in \mathbb{R},\ \forall\ j = \overline{1,18},$ *and we have*

$$\mathcal{A}_1 = \begin{pmatrix} 0 & a_1 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & \dfrac{1}{a_1a_2} \end{pmatrix}, \qquad \mathcal{A}_2 = \begin{pmatrix} 0 & a_3 & 0 \\ 0 & 0 & a_4 \\ \dfrac{1}{a_3a_4} & 0 & 0 \end{pmatrix},$$

$$\mathcal{A}_3 = \begin{pmatrix} 0 & 0 & a_5 \\ a_6 & 0 & 0 \\ 0 & \dfrac{1}{a_5a_6} & 0 \end{pmatrix}, \qquad \mathcal{A}_4 = \begin{pmatrix} 0 & 0 & a_7 \\ 0 & a_8 & 0 \\ \dfrac{1}{a_7a_8} & 0 & 0 \end{pmatrix},$$

$$\mathcal{A}_5 = \begin{pmatrix} a_9 & 0 & 0 \\ 0 & a_{10} & 0 \\ 0 & 0 & \dfrac{1}{a_9a_{10}} \end{pmatrix}, \qquad \mathcal{A}_6 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{12} \\ 0 & \dfrac{1}{a_{11}a_{12}} & 0 \end{pmatrix},$$

$$\mathcal{B}_r = \begin{pmatrix} b_{3r-2} \\ b_{3r-1} \\ b_{3r} \end{pmatrix}, \ \forall\ r = \overline{1,6}.$$

Proof. The local transformations of coordinates which invariate the Berwald-Moór metric of order three (3.1) verify the following first order PDE system:

$$\begin{cases} \partial_1^1\partial_1^2\partial_1^3 = 0 & (I) \\ \partial_2^1\partial_2^2\partial_2^3 = 0 & (II) \\ \partial_3^1\partial_3^2\partial_3^3 = 0 & (III) \\ \partial_1^1\partial_1^2\partial_2^3 + \partial_1^1\partial_2^2\partial_1^3 + \partial_2^1\partial_1^2\partial_1^3 = 0 & (IV) \\ \partial_1^1\partial_1^2\partial_3^3 + \partial_1^1\partial_3^2\partial_1^3 + \partial_3^1\partial_1^2\partial_1^3 = 0 & (V) \\ \partial_2^1\partial_2^2\partial_1^3 + \partial_2^1\partial_1^2\partial_2^3 + \partial_1^1\partial_2^2\partial_2^3 = 0 & (VI) \\ \partial_2^1\partial_2^2\partial_3^3 + \partial_2^1\partial_3^2\partial_2^3 + \partial_3^1\partial_2^2\partial_2^3 = 0 & (VII) \\ \partial_3^1\partial_3^2\partial_1^3 + \partial_3^1\partial_1^2\partial_3^3 + \partial_1^1\partial_3^2\partial_3^3 = 0 & (VIII) \\ \partial_3^1\partial_3^2\partial_2^3 + \partial_3^1\partial_2^2\partial_3^3 + \partial_2^1\partial_3^2\partial_3^3 = 0 & (IX) \\ \partial_1^1\partial_2^2\partial_3^3 + \partial_1^1\partial_3^2\partial_2^3 + \partial_2^1\partial_1^2\partial_3^3 + \partial_2^1\partial_3^2\partial_1^3 + \partial_3^1\partial_1^2\partial_2^3 + \partial_3^1\partial_2^2\partial_1^3 = 1. & (X) \end{cases}$$

It follows that we have the situations:

$$I:\begin{cases}\partial_1^1=0,\quad II:\begin{cases}\partial_2^1=0\\ \partial_2^2=0\\ \partial_2^3=0\end{cases}\\ \partial_1^2=0,\quad II:\begin{cases}\partial_2^1=0\\ \partial_2^2=0\\ \partial_2^3=0\end{cases}\\ \partial_1^3=0,\quad II:\begin{cases}\partial_2^1=0\\ \partial_2^2=0\\ \partial_2^3=0.\end{cases}\end{cases}$$

For more convenience, let us rewrite the above relations as follows:

$$\begin{aligned}
&\partial_1^1=0,\quad \partial_2^1=0 &&(1)\\
&\partial_1^1=0,\quad \partial_2^2=0 &&(2)\\
&\partial_1^1=0,\quad \partial_2^3=0 &&(3)\\
&\partial_1^2=0,\quad \partial_2^1=0 &&(4)\\
&\partial_1^2=0,\quad \partial_2^2=0 &&(5)\\
&\partial_1^2=0,\quad \partial_2^3=0 &&(6)\\
&\partial_1^3=0,\quad \partial_2^1=0 &&(7)\\
&\partial_1^3=0,\quad \partial_2^2=0 &&(8)\\
&\partial_1^3=0,\quad \partial_2^3=0. &&(9)
\end{aligned}$$

Consequently, we have the following nine cases[1]:

Case (i):

$$(1),(III):\begin{cases}\partial_3^2=0,\ (V):\begin{cases}\partial_1^2=0\to\partial_2^3=0\to\partial_3^3=0 & \surd\\ \partial_1^3=0,\ (VII):\begin{cases}\partial_2^2=0\to\partial_3^3=0 & \surd\\ \partial_2^3=0\to\partial_1^2=0 & \sharp\end{cases}\end{cases}\\ \partial_3^3=0,\ (V):\begin{cases}\partial_1^2=0,\ (VII):\begin{cases}\partial_2^2=0\to\partial_1^3=0 & \sharp\\ \partial_2^3=0\to\partial_3^2=0 & \surd\end{cases}\\ \partial_1^3=0\to\partial_2^2=0\to\partial_3^2=0 & \surd.\end{cases}\end{cases}$$

The sets of the corresponding solutions are given by

$$\begin{cases}\partial_1^1=\partial_2^1=\partial_2^2=\partial_3^2=\partial_1^3=\partial_3^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^3),\widetilde{x}^2=\widetilde{x}^2(x^1),\widetilde{x}^3=\widetilde{x}^3(x^2)\\ \partial_1^1=\partial_2^1=\partial_1^2=\partial_3^2=\partial_2^3=\partial_3^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^3),\widetilde{x}^2=\widetilde{x}^2(x^2),\widetilde{x}^3=\widetilde{x}^3(x^1).\end{cases}$$

Case (ii):

$$(2),(III):\begin{cases}\partial_3^1=0,\ (IV):\begin{cases}\partial_1^2=0\to\partial_2^3=0\to\partial_3^3=0 & \surd\\ \partial_1^3=0,\ (VI):\begin{cases}\partial_1^2=0\to\partial_2^3=0 & \sharp\\ \partial_2^3=0\to\partial_3^2=0 & \surd\end{cases}\end{cases}\\ \partial_3^2=0,\ (IV):\begin{cases}\partial_2^1=0\to\partial_1^3=0\to\partial_3^3=0 & \surd\\ \partial_1^3=0,\ (VI):\begin{cases}\partial_2^1=0\to\partial_3^3=0 & \surd\\ \partial_2^3=0\to\partial_3^1=0 & \surd\end{cases}\end{cases}\\ \partial_3^3=0,\ (IV):\begin{cases}\partial_2^1=0,\ (V):\begin{cases}\partial_1^2=0\to\partial_1^3=0\sharp\\ \partial_1^3=0\to{}_{23}=0 & \surd\end{cases}\\ \partial_1^2=0,\ (VII):\begin{cases}\partial_2^1=0\to\partial_1^3=0 & \sharp\\ \partial_2^3=0\to\partial_3^1=0 & \surd\end{cases}\\ \partial_1^3=0,\ (VI):\begin{cases}\partial_2^1=0\to\partial_3^2=0 & \surd\\ \partial_1^2=0\to\partial_2^1=0 & \sharp.\end{cases}\end{cases}\end{cases}$$

[1] In this proof, the symbols $\surd$ and $\sharp$ show the acceptable and non-acceptable solutions of our PDE system.

$$\begin{cases} \partial_1^1 = \partial_3^1 = \partial_2^2 = \partial_3^2 = \partial_1^3 = \partial_2^3 = 0 \\ \widetilde{x}^1 = \widetilde{x}^1(x^2), \widetilde{x}^2 = \widetilde{x}^2(x^1), \widetilde{x}^3 = \widetilde{x}^3(x^3) \\ \partial_1^1 = \partial_3^1 = \partial_1^2 = \partial_2^2 = \partial_2^3 = \partial_3^3 = 0 \\ \widetilde{x}^1 = \widetilde{x}^1(x^2), \widetilde{x}^2 = \widetilde{x}^2(x^3), \widetilde{x}^3 = \widetilde{x}^3(x^1) \\ \partial_1^1 = \partial_2^1 = \partial_2^2 = \partial_3^2 = \partial_1^3 = \partial_3^3 = 0 \\ \widetilde{x}^1 = \widetilde{x}^1(x^3), \widetilde{x}^2 = \widetilde{x}^2(x^1), \widetilde{x}^3 = \widetilde{x}^3(x^2). \end{cases}$$

Case (iii):

$$(3),\ (III):\begin{cases} \partial_3^1 = 0,\ (IV): \begin{cases} \partial_1^2{=}0,\ (VI){:}\begin{cases} \partial_2^2{=}0{\to}\partial_3^3{=}0 & \surd \\ \partial_1^3{=}0{\to}\partial_2^2{=}0 & \sharp \end{cases} \\ \partial_1^3{=}0{\to}\partial_2^2{=}0{\to}\partial_3^2{=}0 \quad \surd \end{cases} \\ \partial_3^2 = 0,\ (IV): \begin{cases} \partial_2^1{=}0,\ (V){:}\begin{cases} \partial_1^2{=}0{\to}\partial_3^3{=}0 & \surd \\ \partial_1^3{=}0{\to}\partial_1^2{=}0 & \sharp \end{cases} \\ \partial_1^2{=}0,\ (VI){:}\begin{cases} \partial_2^1{=}0{\to}\partial_3^3{=}0 & \surd \\ \partial_1^3{=}0{\to}\partial_2^1{=}0 & \sharp \end{cases} \\ \partial_1^3{=}0,\ (VII){:}\begin{cases} \partial_2^1{=}0{\to}\partial_1^2{=}0 & \sharp \\ \partial_2^2{=}0{\to}\partial_3^1{=}0 & \surd \end{cases} \end{cases} \\ \partial_3^3 = 0,\ (IV): \begin{cases} \partial_2^1{=}0{\to}\partial_1^2{=}0{\to}\partial_3^2{=}0 \quad \surd \\ \partial_1^2{=}0,\ (VI){:}\begin{cases} \partial_2^1{=}0{\to}\partial_3^2{=}0 & \surd \\ \partial_2^2{=}0{\to}\partial_3^1{=}0 & \surd. \end{cases} \end{cases} \end{cases}$$

$$\begin{cases} \partial_1^1 = \partial_3^1 = \partial_2^2 = \partial_3^2 = \partial_1^3 = \partial_2^3 = 0 \\ \widetilde{x}^1 = \widetilde{x}^1(x^2), \widetilde{x}^2 = \widetilde{x}^2(x^1), \widetilde{x}^3 = \widetilde{x}^3(x^3) \\ \partial_1^1 = \partial_2^1 = \partial_1^2 = \partial_3^2 = \partial_2^3 = \partial_3^3 = 0 \\ \widetilde{x}^1 = \widetilde{x}^1(x^3), \widetilde{x}^2 = \widetilde{x}^2(x^2), \widetilde{x}^3 = \widetilde{x}^3(x^1) \\ \partial_1^1 = \partial_3^1 = \partial_1^2 = \partial_2^2 = \partial_2^3 = \partial_3^3 = 0 \\ \widetilde{x}^1 = \widetilde{x}^1(x^2), \widetilde{x}^2 = \widetilde{x}^2(x^3), \widetilde{x}^3 = \widetilde{x}^3(x^1). \end{cases}$$

Case (iv):

$$(4),\ (III):\begin{cases} \partial_3^1 = 0,\ (IV): \begin{cases} \partial_2^2{=}0{\to}\partial_1^3{=}0{\to}{}_{33}{=}0 \quad \surd \\ \partial_1^3{=}0,\ (VI){:}\begin{cases} \partial_2^2{=}0{\to}\partial_3^3{=}0 & \surd \\ \partial_2^3{=}0{\to}\partial_3^2{=}0 & \surd \end{cases} \end{cases} \\ \partial_3^2 = 0, (IV): \begin{cases} \partial_1^1{=}0{\to}\partial_2^3{=}0{\to}\partial_3^3{=}0 \quad \surd \\ \partial_1^3{=}0,\ (VI){:}\begin{cases} \partial_1^1{=}0{\to}\partial_2^3{=}0 & \sharp \\ \partial_2^3{=}0{\to}\partial_3^1{=}0 & \surd \end{cases} \end{cases} \\ \partial_3^3 = 0, (IV): \begin{cases} \partial_1^1{=}0,\ (VII){:}\begin{cases} \partial_2^2{=}0{\to}\partial_1^3{=}0 & \sharp \\ \partial_2^3{=}0{\to}\partial_3^2{=}0 & \surd \end{cases} \\ \partial_2^2{=}0,\ (V){:}\begin{cases} \partial_1^1{=}0{\to}\partial_1^3{=}0 & \sharp \\ \partial_1^3{=}0{\to}\partial_3^1{=}0 & \surd \end{cases} \\ \partial_1^3{=}0,\ (VI){:}\begin{cases} \partial_1^1{=}0{\to}\partial_2^2{=}0 & \sharp \\ \partial_2^2{=}0{\to}\partial_3^1{=}0 & \surd. \end{cases} \end{cases} \end{cases}$$

$$\begin{cases} \partial_2^1 = \partial_3^1 = \partial_1^2 = \partial_3^2 = \partial_1^3 = \partial_2^3 = 0 \\ \widetilde{x}^1 = \widetilde{x}^1(x^1), \widetilde{x}^2 = \widetilde{x}^2(x^2), \widetilde{x}^3 = \widetilde{x}^3(x^3) \\ \partial_2^1 = \partial_3^1 = \partial_1^2 = \partial_2^2 = \partial_1^3 = \partial_3^3 = 0 \\ \widetilde{x}^1 = \widetilde{x}^1(x^1), \widetilde{x}^2 = \widetilde{x}^2(x^3), \widetilde{x}^3 = \widetilde{x}^3(x^2) \\ \partial_1^1 = \partial_2^1 = \partial_1^2 = \partial_3^2 = \partial_2^3 = \partial_3^3 = 0 \\ \widetilde{x}^1 = \widetilde{x}^1(x^3), \widetilde{x}^2 = \widetilde{x}^2(x^2), \widetilde{x}^3 = \widetilde{x}^3(x^1). \end{cases}$$

Case (v):

$$(5),\ (III):\begin{cases}\partial_3^1=0,\ (V):\begin{cases}\partial_1^1{=}0\to\partial_2^3{=}0\to\partial_3^3{=}0 & \surd\\ \partial_1^3{=}0,\ (VII){:}\begin{cases}\partial_2^1{=}0\to\partial_3^3{=}0 & \surd\\ \partial_2^3{=}0\to\partial_1^1{=}0 & \sharp\end{cases}\end{cases}\\ \partial_3^3=0,\ (V):\begin{cases}\partial_1^1{=}0,\ (VII){:}\begin{cases}\partial_2^1{=}0\to\partial_1^3{=}0 & \sharp\\ \partial 32{=}0\to\partial_3^1{=}0 & \surd\end{cases}\\ \partial_1^3{=}0\to\partial_2^1{=}0\to\partial_3^1{=}0 & \surd.\end{cases}\end{cases}$$

$$\begin{cases}\partial_1^1=\partial_3^1=\partial_1^2=\partial_2^2=\partial_2^3=\partial_3^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^2),\widetilde{x}^2=\widetilde{x}^2(x^3),\widetilde{x}^3=\widetilde{x}^3(x^1)\\ \partial_2^1=\partial_3^1=\partial_1^2=\partial_2^2=\partial_1^3=\partial_3^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^1),\widetilde{x}^2=\widetilde{x}^2(x^3),\widetilde{x}^3=\widetilde{x}^3(x^2).\end{cases}$$

Case (vi):

$$(6),\ (III):\begin{cases}\partial_3^1=0,\ (IV):\begin{cases}\partial_1^1{=}0,\ (VI){:}\begin{cases}\partial_2^2{=}0\to\partial_3^3{=}0 & \surd\\ \partial_1^3{=}0\to\partial_2^2{=}0 & \sharp\end{cases}\\ \partial_2^2{=}0,\ (V){:}\begin{cases}\partial_1^1{=}0\to\partial_3^3{=}0 & \surd\\ \partial_1^3{=}0\to\partial_1^1{=}0 & \sharp\end{cases}\\ \partial_1^3{=}0,\ (VII){:}\begin{cases}\partial_2^1{=}0\to\partial_3^2{=}0 & \surd\\ \partial_2^2{=}0\to\partial_1^1{=}0 & \sharp\end{cases}\end{cases}\\ \partial_3^2=0,\ (IV):\begin{cases}\partial_1^1{=}0,\ (VI){:}\begin{cases}\partial_2^1{=}0\to\partial_3^3{=}0\surd\\ \partial_1^3{=}0\to\partial_2^1{=}0 & \sharp\end{cases}\\ \partial_1^3{=}0\to\partial_2^1{=}0\to\partial_3^1{=}0 & \surd\end{cases}\\ \partial_3^3=0,\ (IV):\begin{cases}\partial_1^1{=}0,\ (VI){:}\begin{cases}\partial_2^1{=}0\to\partial_3^2{=}0 & \surd\\ \partial_2^2{=}0\to\partial_3^1{=}0 & \surd\end{cases}\\ \partial_2^2{=}0\to\partial_1^1{=}0\to\partial_3^1{=}0 & \surd.\end{cases}\end{cases}$$

$$\begin{cases}\partial_2^1=\partial_3^1=\partial_1^2=\partial_3^2=\partial_1^3=\partial_2^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^1),\widetilde{x}^2=\widetilde{x}^2(x^2),\widetilde{x}^3=\widetilde{x}^3(x^3)\\ \partial_1^1=\partial_2^1=\partial_1^2=\partial_3^2=\partial_2^3=\partial_3^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^3),\widetilde{x}^2=\widetilde{x}^2(x^2),\widetilde{x}^3=\widetilde{x}^3(x^1)\\ \partial_1^1=\partial_3^1=\partial_1^2=\partial_2^2=\partial_2^3=\partial_3^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^2),\widetilde{x}^2=\widetilde{x}^2(x^3),\widetilde{x}^3=\widetilde{x}^3(x^1).\end{cases}$$

Case (vii):

$$(7),\ (III):\begin{cases}\partial_3^1=0,\ (IV):\begin{cases}\partial_1^2{=}0,\ (VI){:}\begin{cases}\partial_2^2{=}0\to\partial_3^3{=}0 & \surd\\ \partial_2^3{=}0\to\partial_3^2{=}0 & \surd\end{cases}\\ \partial_2^3{=}0\to\partial_1^2\to\partial_3^2{=}0 & \surd\end{cases}\\ \partial_3^2=0,\ (IV):\begin{cases}\partial_1^1{=}0,\ (VII){:}\begin{cases}\partial_2^2{=}0\to\partial_3^3{=}0 & \surd\\ \partial_2^3{=}0\to\partial_1^2{=}0 & \sharp\end{cases}\\ \partial_1^2{=}0,\ (VI){:}\begin{cases}\partial_1^1{=}0\to\partial_2^3{=}0 & \sharp\\ \partial_2^3{=}0\to\partial_3^1{=}0 & \surd\end{cases}\\ \partial_2^3{=}0,\ (V){:}\begin{cases}\partial_1^1{=}0\to\partial_1^2{=}0 & \sharp\\ \partial_1^2\to\partial_3^1{=}0 & \surd\end{cases}\end{cases}\\ \partial_3^3=0,\ (IV):\begin{cases}\partial_1^1{=}0\to\partial_2^2{=}0\to\partial_3^2{=}0 & \surd\\ \partial_1^2{=}0,\ (VI){:}\begin{cases}\partial_1^1{=}0\to\partial_2^2{=}0 & \sharp\\ \partial_2^2{=}0\to\partial_1^1{=}0 & \sharp\end{cases}\end{cases}\end{cases}$$

$$\begin{cases}\partial_2^1=\partial_3^1=\partial_1^2=\partial_2^2=\partial_1^3=\partial_3^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^1),\widetilde{x}^2=\widetilde{x}^2(x^3),\widetilde{x}^3=\widetilde{x}^3(x^2)\\ \partial_2^1=\partial_3^1=\partial_1^2=\partial_3^2=\partial_1^3=\partial_2^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^1),\widetilde{x}^2=\widetilde{x}^2(x^2),\widetilde{x}^3=\widetilde{x}^3(x^3)\\ \partial_1^1=\partial_2^1=\partial_2^2=\partial_3^2=\partial_1^3=\partial_3^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^3),\widetilde{x}^2=\widetilde{x}^2(x^1),\widetilde{x}^3=\widetilde{x}^3(x^2).\end{cases}$$

Case (viii):

$$(8),\ (III):\begin{cases}\partial_3^1=0,\ (IV):\begin{cases}\partial_1^1{=}0,\ (VI){:}\begin{cases}\partial_1^2{=}0{\to}\partial_2^3{=}0 & \sharp\\ \partial_2^3{=}0{\to}\partial_3^2{=}0 & \surd\end{cases}\\ \partial_1^2{=}0,\ (VII){:}\begin{cases}\partial_2^1{=}0{\to}\partial_3^3{=}0 & \surd\\ \partial_2^3{=}0{\to}\partial_1^1{=}0 & \sharp\end{cases}\\ \partial_2^3{=}0,\ (V){:}\begin{cases}\partial_1^1{=}0{\to}\partial_3^2{=}0 & \surd\\ \partial_1^2{=}0{\to}\partial_1^1{=}0 & \sharp\end{cases}\end{cases}\\ \partial_3^2=0,\ (IV):\begin{cases}\partial_1^1{=}0,\ (VI){:}\begin{cases}\partial_2^1{=}0{\to}\partial_3^3{=}0 & \surd\\ \partial_2^3{=}0{\to}\partial_3^1{=}0 & \surd\end{cases}\\ \partial_2^3{=}0{\to}\partial_1^1{=}0{\to}\partial_3^1{=}0 \quad \surd\end{cases}\\ \partial_3^3=0,\ (IV):\begin{cases}\partial_1^1{=}0,\ (VI){:}\begin{cases}\partial_2^1{=}0{\to}\partial_3^2{=}0 & \surd\\ \partial_1^2{=}0{\to}\partial_2^1{=}0 & \sharp\end{cases}\\ \partial_1^2{=}0{\to}\partial_2^1{=}0{\to}\partial_3^1{=}0 \quad \surd.\end{cases}\end{cases}$$

$$\begin{cases}\partial_1^1=\partial_3^1=\partial_2^2=\partial_3^2=\partial_1^3=\partial_2^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^2),\widetilde{x}^2=\widetilde{x}^2(x^1),\widetilde{x}^3=\widetilde{x}^3(x^3)\\ \partial_1^1=\partial_2^1=\partial_2^2=\partial_3^2=\partial_1^3=\partial_3^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^3),\widetilde{x}^2=\widetilde{x}^2(x^1),\widetilde{x}^3=\widetilde{x}^3(x^2)\\ \partial_2^1=\partial_3^1=\partial_1^2=\partial_2^2=\partial_1^3=\partial_3^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^1),\widetilde{x}^2=\widetilde{x}^2(x^3),\widetilde{x}^3=\widetilde{x}^3(x^2)\end{cases}$$

Case (ix):

$$(9),\ (III):\begin{cases}\partial_3^1=0,\ (V):\begin{cases}\partial_1^1{=}0{\to}\partial_2^2{=}0{\to}\partial_3^2{=}0 \quad \surd\\ \partial_1^2{=}0,\ (VII){:}\begin{cases}\partial_2^1{=}0{\to}\partial_3^2{=}0 & \surd\\ \partial_2^2{=}0{\to}\partial_1^1{=}0 & \sharp\end{cases}\end{cases}\\ \partial_3^2=0,\ (V):\begin{cases}\partial_1^1{=}0,\ (VII){:}\begin{cases}\partial_2^1{=}0{\to}\partial_1^2{=}0 & \sharp\\ \partial_2^2{=}0{\to}\partial_3^1{=}0 & \surd\end{cases}\\ \partial_1^2{=}0{\to}\partial_2^1{=}0{\to}\partial_3^1{=}0 \quad \surd.\end{cases}\end{cases}$$

$$\begin{cases}\partial_1^1=\partial_3^1=\partial_2^2=\partial_3^2=\partial_1^3=\partial_2^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^2),\widetilde{x}^2=\widetilde{x}^2(x^1),\widetilde{x}^3=\widetilde{x}^3(x^3)\\ \partial_2^1=\partial_3^1=\partial_1^2=\partial_3^2=\partial_1^3=\partial_2^3=0\\ \widetilde{x}^1=\widetilde{x}^1(x^1),\widetilde{x}^2=\widetilde{x}^2(x^2),\widetilde{x}^3=\widetilde{x}^3(x^3).\end{cases}$$

Finally, by eliminating the repetitive situations and replacing them into relation (X), we obtain the following sets of solutions:

$$\mathcal{T}_{(a_1,a_2,b_1,b_2,b_3)}:\begin{cases}\widetilde{x}^1=a_1x^2+b_1\\ \widetilde{x}^2=a_2x^1+b_2\\ \widetilde{x}^3=\dfrac{1}{a_1a_2}x^3+b_3\end{cases}\Longleftrightarrow \widetilde{\mathcal{X}}=\mathcal{A}_1\mathcal{X}+\mathcal{B}_1,$$

$$\mathcal{S}_{(a_3,a_4,b_4,b_5,b_6)}:\begin{cases}\widetilde{x}^1=a_3x^2+b_4\\ \widetilde{x}^2=a_4x^3+b_5\\ \widetilde{x}^3=\dfrac{1}{a_3a_4}x^1+b_6\end{cases}\Longleftrightarrow \widetilde{\mathcal{X}}=\mathcal{A}_2\mathcal{X}+\mathcal{B}_2,$$

$$\mathcal{R}_{(a_5,a_6,b_7,b_8,b_9)}:\begin{cases}\widetilde{x}^1=a_5x^3+b_7\\ \widetilde{x}^2=a_6x^1+b_8\\ \widetilde{x}^3=\dfrac{1}{a_5a_6}x^2+b_9\end{cases}\Longleftrightarrow \widetilde{\mathcal{X}}=\mathcal{A}_3\mathcal{X}+\mathcal{B}_3,$$

$$\mathcal{M}_{(a_7,a_8,b_{10},b_{11},b_{12})}:\begin{cases}\widetilde{x}^1=a_7x^3+b_{10}\\ \widetilde{x}^2=a_8x^2+b_{11}\\ \widetilde{x}^3=\dfrac{1}{a_7a_8}x^1+b_{12}\end{cases}\Longleftrightarrow \widetilde{\mathcal{X}}=\mathcal{A}_4\mathcal{X}+\mathcal{B}_4,$$

$$\mathcal{N}_{(a_9,a_{10},b_{13},b_{14},b_{15})}:\begin{cases}\widetilde{x}^1=a_9x^1+b_{13}\\ \widetilde{x}^2=a_{10}x^2+b_{14}\\ \widetilde{x}^3=\dfrac{1}{a_9a_{10}}x^3+b_{15}\end{cases}\Longleftrightarrow \widetilde{\mathcal{X}}=\mathcal{A}_5\mathcal{X}+\mathcal{B}_5,$$

$$\mathcal{O}_{(a_{11},a_{12},b_{16},b_{17},b_{18})}:\begin{cases}\widetilde{x}^1=a_{11}x^1+b_{16}\\ \widetilde{x}^2=a_{12}x^3+b_{17}\\ \widetilde{x}^3=\dfrac{1}{a_{11}a_{12}}x^2+b_{18}\end{cases}\Longleftrightarrow \widetilde{\mathcal{X}}=\mathcal{A}_6\mathcal{X}+\mathcal{B}_6.$$

■

Remark 3.2. *The affine transformations $\mathcal{N}_{(a_9,a_{10},b_{13},b_{14},b_{15})}$ include all translations of the space $\mathbb{R}^3$, by putting $\mathcal{N}_{(1,1,x_0,y_0,z_0)}$. The spatial rotations from the set of the above affine transformations are only the following rotations:*

$$\mathcal{R}_0:=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \mathcal{R}_1:=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\quad \mathcal{R}_2:=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix},$$

$$\mathcal{R}_3:=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix},\quad \mathcal{R}_4:=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\quad \mathcal{R}_5:=\begin{pmatrix}0&1&0\\0&0&-1\\-1&0&0\end{pmatrix},$$

$$\mathcal{R}_6:=\begin{pmatrix}0&-1&0\\0&0&1\\-1&0&0\end{pmatrix},\ \mathcal{R}_7:=\begin{pmatrix}0&-1&0\\0&0&-1\\1&0&0\end{pmatrix},\ \mathcal{R}_8:=\begin{pmatrix}0&0&1\\-1&0&0\\0&-1&0\end{pmatrix},$$

$$\mathcal{R}_9:=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\ \mathcal{R}_{10}:=\begin{pmatrix}0&0&-1\\1&0&0\\0&-1&0\end{pmatrix},\ \mathcal{R}_{11}:=\begin{pmatrix}0&0&-1\\-1&0&0\\0&1&0\end{pmatrix}.$$

There are some results for the Berwald-Moór metric of order three, which are similar with those from the case of second order:

Corollary 3.3. *The set of the local transformations of coordinates that invariates the Berwald-Moór metric of order three (3.1) has an algebraic structure of non-commutative group with respect to the operation of composition of functions.*

Proof. The axioms of algebraic group are proved by direct computations. For instance, we give one of them (the other ones can be similarly obtained):

$$\mathcal{T}_{(a_1,a_2,b_1,b_2,b_3)}\circ\mathcal{T}_{\left(a_1',a_2',b_1',b_2',b_3'\right)}=\mathcal{O}_{\left(a_2'a_1,a_2a_1',a_1b_2'+b_1,a_2b_1'+b_2,\frac{b_3'}{a_1a_2}+b_3\right)}.$$

In this group, the neutral element is $\mathcal{O}_{(1,1,0,0,0)}$. At the same time, we have

$$\mathcal{T}^{-1}_{(a_1,a_2,b_1,b_2,b_3)} = \mathcal{T}_{\left(\frac{1}{a_2},\frac{1}{a_1},\frac{-b_2}{a_2},\frac{-b_1}{a_1},-b_3a_1a_2\right)},$$
$$\mathcal{S}^{-1}_{(a_3,a_4,b_4,b_5,b_6)} = \mathcal{R}_{\left(a_3a_4,\frac{1}{a_3},-b_6a_3a_4,\frac{-b_4}{a_3},\frac{-b_5}{a_4}\right)},$$
$$\mathcal{R}^{-1}_{(a_5,a_6,b_7,b_8,b_9)} = \mathcal{S}_{\left(\frac{1}{a_6},a_5a_6,\frac{-b_8}{a_6},-b_9a_5a_6,\frac{-b_7}{a_5}\right)},$$
$$\mathcal{M}^{-1}_{(a_7,a_8,b_{10},b_{11},b_{12})} = \mathcal{M}_{\left(a_7a_8,\frac{1}{a_8},-b_{12}a_7a_8,\frac{-b_{11}}{a_8},\frac{-b_{10}}{a_7}\right)},$$
$$\mathcal{N}^{-1}_{(a_9,a_{10},b_{13},b_{14},b_{15})} = \mathcal{N}_{\left(\frac{1}{a_9},\frac{1}{a_{10}},\frac{-b_{13}}{a_9},\frac{-b_{14}}{a_{10}},-b_{15}a_9a_{10}\right)},$$
$$\mathcal{O}^{-1}_{(a_{11},a_{12},b_{16},b_{17},b_{18})} = \mathcal{O}_{\left(\frac{1}{a_{11}},a_{11}a_{12},\frac{-b_{16}}{a_{11}},-b_{18}a_{11}a_{12},\frac{-b_{17}}{a_{12}}\right)}.$$

■

We put the matrices mentioned in (3.3) – (3.8) into the following sets:

$$\mathcal{V}_1 = \left\{ \left(\begin{array}{ccc} a_9 & 0 & 0 \\ 0 & a_{10} & 0 \\ 0 & 0 & \frac{1}{a_9a_{10}} \end{array} \right) \middle| \, a_9, a_{10} \in \mathbb{R}^* \right\},$$

$$\mathcal{V}_2 = \left\{ \left(\begin{array}{ccc} 0 & a_3 & 0 \\ 0 & 0 & a_4 \\ \frac{1}{a_3a_4} & 0 & 0 \end{array} \right) \middle| \, a_3, a_4 \in \mathbb{R}^* \right\},$$

$$\mathcal{V}_3 = \left\{ \left(\begin{array}{ccc} 0 & 0 & a_5 \\ a_6 & 0 & 0 \\ 0 & \frac{1}{a_5a_6} & 0 \end{array} \right) \middle| \, a_5, a_6 \in \mathbb{R}^* \right\},$$

$$\mathcal{W}_1 = \left\{ \left(\begin{array}{ccc} 0 & 0 & a_7 \\ 0 & a_8 & 0 \\ \frac{1}{a_7a_8} & 0 & 0 \end{array} \right) \middle| \, a_7, a_8 \in \mathbb{R}^* \right\},$$

$$\mathcal{W}_2 = \left\{ \left(\begin{array}{ccc} 0 & a_1 & 0 \\ a_2 & 0 & 0 \\ 0 & 0 & \frac{1}{a_1a_2} \end{array} \right) \middle| \, a_1, a_2 \in \mathbb{R}^* \right\},$$

$$\mathcal{W}_3 = \left\{ \left(\begin{array}{ccc} a_{11} & 0 & 0 \\ 0 & 0 & a_{12} \\ 0 & \frac{1}{a_{11}a_{12}} & 0 \end{array} \right) \middle| \, a_{11}, a_{12} \in \mathbb{R}^* \right\}.$$

Notice that the intersection of each two sets from the above sets of matrices is the empty set. In this context, we can show that the following result of characterization holds good:

Proposition 3.4. *For a matrix $\mathfrak{X} \in M_3(\mathbb{R})$ the following statements are true:*

(1) $\mathfrak{X} \in \mathcal{V}_1$ *if and only if*

(a) $\det \mathfrak{X} = 1$;

(b) *The vectors* $\mathbf{e_1} = (1,0,0)$, $\mathbf{e_2} = (0,1,0)$, $\mathbf{e_3} = (0,0,1)$ *are eigenvectors of the matrix* $\mathfrak{X}$.

(2) $\mathfrak{X} \in \mathcal{V}_2$ *if and only if* $\mathfrak{X} \cdot E_1 \in \mathcal{V}_1$, *where* $E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

(3) $\mathfrak{X} \in \mathcal{V}_3$ *if and only if* $\mathfrak{X} \cdot E_2 \in \mathcal{V}_1$, *where* $E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

(4) $\mathfrak{X} \in \mathcal{W}_1$ *if and only if* $\mathfrak{X} \cdot E_3 \in \mathcal{V}_1$, *where* $E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

(5) $\mathfrak{X} \in \mathcal{W}_2$ *if and only if* $\mathfrak{X} \cdot E_4 \in \mathcal{V}_1$, *where* $E_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(6) $\mathfrak{X} \in \mathcal{W}_3$ *if and only if* $\mathfrak{X} \cdot E_5 \in \mathcal{V}_1$, *where* $E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Proof. In order to prove the statement (1), note that any matrix $\mathfrak{X} \in \mathcal{V}_1$ verifies the relations (a) and (b). Conversely, let us consider an arbitrary matrix

$$\mathfrak{X} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

which verifies relations (a) and (b). Let us suppose that λ_1, λ_2 and λ_3 are the corresponding eigenvalues of the eigenvectors $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$. In this case, we have $\mathfrak{X} \cdot \mathbf{e_1} = \lambda_1 \mathbf{e_1}$. It follows that $a = \lambda_1$, $d = 0$, $g = 0$. By analogy, we have $b = 0$, $e = \lambda_2$, $h = 0$ and $c = 0$, $f = 0$, $k = \lambda_3$. Now, the condition $\det \mathfrak{X} = 1$ implies $\lambda_1 \lambda_2 \lambda_3 = 1$. This completes the proof of the statement (1).

The proof of the other statements can be similarly done. ∎

4 Conclusions and work in progress

The affine transformations obtained in the study of invariance of the Berwald-Moór metric in two and three dimensions suggest that there exist some similar results for an arbitrary Berwald-Moór metric of order n. It is our work in progress to study and prove that the matrices of order n (corresponding to the affine maps which invariate the n-dimensional Berwald-Moór metric) are exactly the matrices $\mathcal{A}_\sigma$, where σ is a permutation of the set $\{1, 2, ..., n\}$, having as non-zero entries only the entries $a_{1\sigma(1)}$, $a_{2\sigma(2)}$, ..., $a_{n\sigma(n)}$, verifying the equality

$$a_{1\sigma(1)} \cdot a_{2\sigma(2)} \cdot ... \cdot a_{n\sigma(n)} = 1.$$

It follows that we have $\det \mathcal{A}_\sigma = \varepsilon(\sigma) = \pm 1$. Obviously, the number of such kind of matrices is equal to $n!$, that is exactly the number of all permutations of order n. For example, in the case of the Berwald-Moór metric of order four, we must have $4! = 24$ matrices corresponding to the affine transformations of invariance of the Berwald-Moór metric of order four. These must be

$$A_1 = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & \dfrac{1}{a_1 a_2 a_3} \end{pmatrix}, \quad ... \; A_{23} = \begin{pmatrix} 0 & 0 & a_{67} & 0 \\ 0 & 0 & 0 & a_{68} \\ 0 & a_{69} & 0 & 0 \\ \dfrac{1}{a_{67} a_{68} a_{69}} & 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} a_4 & 0 & 0 & 0 \\ 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & a_6 \\ 0 & 0 & \dfrac{1}{a_4 a_5 a_6} & 0 \end{pmatrix}, \; A_{24} = \begin{pmatrix} 0 & 0 & 0 & a_{70} \\ 0 & 0 & a_{71} & 0 \\ 0 & a_{72} & 0 & 0 \\ \dfrac{1}{a_{70} a_{71} a_{72}} & 0 & 0 & 0 \end{pmatrix}.$$

Finally, we would like to underline that a rigourously mathematical study of the above possible algebraic-geometrical results related to the invariance group of the Berwald-Moór metric of order $n \geq 4$ represents the aim of a subsequent research paper which is already in our attention.

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ИНВАРИАНТНЫЕ ГРУППЫ МЕТРИКИ БЕРВАЛЬДА-МООРА ВТОРОГО И ТРЕТЬЕГО ПОРЯДКА

М. Неагу[1], Х. Рейши-Дехкорди[2]

[1]*Трансильванский университет, Брашов, Румыния*

[2]*Тегеранский технологический университет Амиркабир (Тегеранский Политехникум), Тегеран, Иран*

mircea.neagu@unitbv.ro, hengameh_62@aut.ac.ir

В этой работе мы описываем группы локальных преобразований координат, которые сохраняют неизменной на касательных расслоениях двух- и трехмерные метрики Бервальда-Моора. Изучены некоторые алгебраические свойства этих групп. Также предложена возможная структура этих преобразований в общем n-мерном случае..

Ключевые слова: касательное расслоение, метрика Бервальда-Моора порядка два и три, инвариантные группы локальных преобразований.