DIFFERENTIAL GEOMETRY OF FINSLER-SPACETIME TANGENT BUNDLE

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I draw on my earlier work to review various aspects of the differential geometry of a Finsler-spacetime tangent bundle, all based on the possible existence of a physical upper bound on proper acceleration. In particular, the bundle connection and associated differential geometric fields are calculated for a Finsler-spacetime tangent bundle particularized for the case of a statioinary measuring device.

Key Words: Finsler geometry, spacetime tangent bundle, maximal proper acceleration, physical Finsler coordinates, spacetime structure, Kaehler spacetime, complex spacetime, torsion.

1 Introduction

In the present work, I draw on my earlier work to review various aspects of the differential geometry of a Finsler-spacetime tangent bundle, all based on the possible existence of a physical upper bound on proper acceleration. The appropriate bundle coordinates are the spacetime coordinates of the measuring device in the base manifold and the four-velocity coordinates of the measuring device in the tangent space manifold [1]. Of particular interest is the case of a stationary measuring device in which it is implicit that the gravitational force on the device is balanced by a non-gravitational force, as in the case of an ordinary scale measuring weight. For a Finsler-spacetime tangent bundle, the Levi-Civita connection coefficients reduce to the form given by Yano and Davies for a generic tangent bundle of a Finsler manifold [2, 3]. The components of the connection in the spacetime-spacetime-fiber sector have a form consistent with Cartan's theory of Finsler space, provided that the gauge curvature field vanishes. A vanishing gauge curvature field is equivalent to the condition that the four-velocity of the measuring device be a parallel vector field. The latter is equivalent to Cartan's condition that there be absolute parallelism of the line elements, and that the tangent space coordinates form a parallel vector field [3]. This is consistent with Deicke, who proved that a Finsler space cannot be represented as a nonholonomic subspace of a Riemannian space unless the latter condition is imposed [3, 4]. Deicke subsequently proved that a Finsler space can always be represented as a nonholonomic subspace of a space with torsion [3, 5, 6]. If bundle torsion is included in the Finsler-spacetime tangent bundle, then the bundle connection becomes compatible with Cartan's connection for Finsler space if a component of the contorsion is made to cancel the contribution of the gauge curvature field to the connection in the spacetime-spacetime-fiber sector [7]. The spacetime tangent bundle of a Finsler spacetime is almost complex [8]. Also provided that the gauge curvature field is vanishing, then the Finsler-spacetime tangent bundle is Kaehlerian with vanishing covariant derivative of the almost complex structure [8]. The vanishing of the gauge curvature field is also the condition that the Finsler-spacetime tangent bundle have a vanishing Nijenhuis tensor (torsion of the almost complex structure) in the anholonomic frame adapted to the spacetime connection, and that it be complex [9]. If bundle torsion satisfying prescribed conditions is introduced, the Finsler-spacetime tangent bundle can be made to remain almost complex, and the covariant derivative of the almost complex structure can be made to remain vanishing, without the need to impose the relatively restrictive condition of vanishing gauge curvature field [7, 10]. However, the Finsler-spacetime tangent bundle cannot be complex unless the gauge curvature field is vanishing. In the present work, drawing on this earlier work, the bundle connection and associated differential geometric fields are calculated for a Finsler-spacetime tangent bundle particularized for the case of a stationary measuring device.

2 Bundle connection

The gauge curvature field for the Finsler-spacetime tangent bundle is given by [2]

$$F^{\alpha}_{\ \mu\nu} = \rho_0 v^{\lambda} \overline{R}^{\alpha}_{\ \lambda\mu\nu}, \tag{1}$$

where Greek indices range from 0 to 3, corresponding to the time component and the three space components, respectively; ρ_0 is a constant of the order of the Planck length; $v^{\lambda}$is the four-velocity of the measuring device; and $\overline{R}^{\alpha}_{\lambda\mu\nu}$ is the spacetime Riemann curvature tensor

$$\overline{R}^{\alpha}_{\ \lambda\mu\nu} = \Gamma^{\alpha}_{\ \beta\delta,\gamma} - \Gamma^{\alpha}_{\ \beta\gamma,\delta} + \Gamma^{\alpha}_{\ \mu\gamma}\Gamma^{\mu}_{\ \beta\delta} - \Gamma^{\alpha}_{\ \mu\delta}\Gamma^{\mu}_{\ \beta\gamma}, \tag{2}$$

written in an anholonomic basis adapted to the spacetime affine connection $\Gamma^{\alpha}_{\beta\delta}$. In the anholonomic basis, a comma followed by a lower-case Greek index implicitly denotes

$$_{,\mu} \equiv \frac{\partial}{\partial x^{\mu}} - \rho_0^{-1} A^{\beta}_{\ \mu} \frac{\partial}{\partial v^{\beta}}, \tag{3}$$

expressed in terms of the gauge potential

$$A^{\beta}_{\ \mu} = \rho_0 v^{\lambda} \Gamma^{\beta}_{\ \lambda\mu}. \tag{4}$$

The condition of vanishing gauge curvature field, Eq. (1), for the stationary measuring device is thus given by

$$F^{\alpha}_{\ \mu\nu} = \rho_0 v^{0} \overline{R}^{\alpha}_{\ 0\mu\nu} = 0. \tag{5}$$

The four-velocity v^{μ} of the measuring device satisfies

$$g_{\mu\nu} v^{\mu} v^{\nu} = 1, \tag{6}$$

and the Finsler metric function L then becomes [2]

$$L^2 = g_{\mu\nu} v^{\mu} v^{\nu} = 1, \tag{7}$$

which is satisfied by the indicatrix. From Eq. (6), it follows that for the stationary measuring device, v^0 is a solution to

$$g_{00}(x, v^0, 0, 0, 0) v^{02} = 1. \tag{8}$$

It is to be noted that for a Finsler metric $g_{\mu\nu}(x, v)$, there may be multiple stationary frames for the measuring device satisfying Eq. (8).

Thus for a stationary measuring device, the four-velocity is

$$\left\{v^0, v^1, v^2, v^3\right\} = \left\{v^0, 0, 0, 0\right\}, \tag{9}$$

and the gauge curvature field, Eq. (1), becomes

$$F^{\alpha}_{\ \mu\nu} = \rho_0 v^{0} \overline{R}^{\alpha}_{\ 0\mu\nu}. \tag{10}$$

Also, the gauge potential, Eq. (4), becomes

$$A^{\beta}_{\ \mu} = \rho_0 v^{0} \Gamma^{\beta}_{\ 0\mu}. \tag{11}$$

It follows from the homogeneity of the Finsler metric function that

$$v^\alpha \frac{\partial}{\partial v^\alpha} g_{\mu\nu}(x,v) = 0, \tag{12}$$

$$\frac{\partial}{\partial v^\alpha} g_{\mu\nu} = \frac{\partial}{\partial v^\mu} g_{\alpha\nu}, \tag{13}$$

and

$$v^\alpha \frac{\partial}{\partial v^\alpha} g_{\mu\nu} = v^\alpha \frac{\partial}{\partial v^\mu} g_{\alpha\nu} = 0. \tag{14}$$

Thus, for a stationary measuring device, it follows from Eqs. (12) - (14) that

$$v^0 \frac{\partial}{\partial v^0} g_{\mu\nu} = v^0 \frac{\partial}{\partial v^\mu} g_{0\nu} = 0. \tag{15}$$

Also from Eq. (6), it follows that

$$\frac{\partial}{\partial v^\mu} L^2(x,v) = v^{02} \frac{\partial}{\partial v^\mu} g_{00} + 2v^0 g_{0\mu} = 0, \tag{16}$$

and

$$\frac{\partial^2}{\partial v^\mu \partial v^\nu} L^2(x,v) = v^{02} \frac{\partial^2}{\partial v^\mu \partial v^\nu} g_{00} + 4v^0 \frac{\partial}{\partial v^\nu} g_{0\mu} + 2g_{\mu\nu} = 0. \tag{17}$$

For the Finsler spacetime, the Christoffel symbols for the four-velocity tangent space are given by [2]

$$\Pi^\mu{}_{\alpha\beta} = \frac{1}{2} \rho_0^{-1} g^{\mu\lambda} \frac{\partial}{\partial v^\lambda} g_{\alpha\beta}, \tag{18}$$

which for the stationary measuring device, according to Eq. (15), for non-vanishing v^0, becomes

$$\Pi^\mu{}_{\alpha\beta} = \frac{1}{2} \rho_0^{-1} g^{\mu k} \frac{\partial}{\partial v^k} g_{\alpha\beta}, \tag{19}$$

in which the lower-case Latin index k ranges from 1 to 3.

If the spacetime connection $\Gamma^\mu_{\alpha\beta}$ is of the Levi-Civita form, then in the adapted anholonomic basis, the gauge potential is given by [2]

$$A^\beta{}_\mu = \rho_0 v^\lambda \overline{\left\{{}^\beta{}_{\lambda\mu}\right\}}, \tag{20}$$

in which the Christoffel symbols are given by

$$\overline{\left\{{}^\beta{}_{\lambda\mu}\right\}} = \frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}), \tag{21}$$

and for the stationary measuring device, the gauge potential becomes

$$A^\beta{}_\mu = \rho_0 v^0 \overline{\left\{{}^\beta{}_{0\mu}\right\}}. \tag{22}$$

The bundle connection in the spacetime sector of the Finsler-spacetime tangent bundle is given by [2]

$$^{(8)}\Gamma^\mu{}_{\alpha\beta} = \overline{\left\{{}^\mu{}_{\alpha\beta}\right\}} = \left\{{}^\mu{}_{\alpha\beta}\right\} - g^{\mu\nu} \left(A^\lambda{}_\alpha \Pi_{\lambda\beta\nu} + A^\lambda{}_\beta \Pi_{\lambda\alpha\nu} - A^\lambda{}_\nu \Pi_{\lambda\alpha\beta} \right), \tag{23}$$

in which $\left\{{}^\mu{}_{\alpha\beta}\right\}$ is the canonical Levi-Civita symbol written in a coordinate basis. Next substituting Eq. (23) in Eq. (20), one has

$$A^\lambda{}_\alpha = \rho_0 v^\delta \left\{{}^\lambda{}_{\delta\alpha}\right\} - \rho_0 v^\delta g^{\lambda\phi} \left(A^\psi{}_\delta \Pi_{\psi\alpha\phi} + A^\psi{}_\alpha \Pi_{\psi\delta\phi} - A^\psi{}_\phi \Pi_{\psi\delta\alpha} \right). \tag{24}$$

From Eq. (18), one has

$$\rho_0^{-1}\frac{\partial}{\partial v^\lambda}g_{\alpha\beta} = 2\Pi_{\lambda\alpha\beta}, \tag{25}$$

and also, from the symmetry of the spacetime metric,

$$\Pi_{\mu\alpha\beta} = \Pi_{\alpha\mu\beta} = \Pi_{\beta\alpha\mu} = \Pi_{\mu\beta\alpha}. \tag{26}$$

Next substituting Eq. (25) in Eq. (14) and using Eq. (26), one obtains for a Finsler spacetime:

$$v^\alpha\Pi_{\alpha\mu\nu} = v^\alpha\Pi_{\mu\alpha\nu} = v^\alpha\Pi_{\mu\nu\alpha} = 0. \tag{27}$$

Also, substituting Eqs. (27) and (20) in Eq. (24), and solving for the gauge potential, one obtains

$$A^\lambda{}_\alpha = \rho_0 v^\delta \left\{{}^\lambda{}_{\delta\alpha}\right\} - \rho_0^2 v^\gamma v^\delta \overline{\left\{{}^\psi{}_{\gamma\delta}\right\}}\Pi_{\psi\alpha}{}^\lambda. \tag{28}$$

Furthermore, from Eq. (23) it follows that

$$v^\gamma v^\delta\overline{\left\{{}^\psi{}_{\gamma\delta}\right\}} = v^\gamma v^\delta\left\{{}^\psi{}_{\gamma\delta}\right\} - v^\gamma v^\delta g^{\psi\alpha}\left(A^\beta{}_\gamma\Pi_{\beta\delta\alpha} + A^\beta{}_\delta\Pi_{\beta\gamma\alpha} - A^\beta{}_\alpha\Pi_{\beta\gamma\delta}\right). \tag{29}$$

Next using Eq. (27) in Eq. (29), one obtains

$$v^\gamma v^\delta\overline{\left\{{}^\psi{}_{\gamma\delta}\right\}} = v^\gamma v^\delta\left\{{}^\psi{}_{\gamma\delta}\right\}, \tag{30}$$

and then substituting Eq. (30) in Eq. (28), and using Eq. (26), one gets

$$A^\lambda{}_\alpha = \rho_0 v^\delta\left\{{}^\lambda{}_{\delta\alpha}\right\} - \rho_0^2 v^\gamma v^\delta\left\{{}^\psi{}_{\gamma\delta}\right\}\Pi^\lambda{}_{\alpha\psi}. \tag{31}$$

For a stationary measuring device, this becomes

$$A^\lambda{}_\alpha = \rho_0 v^0\left\{{}^\lambda{}_{0\alpha}\right\} - \rho_0^2 v^{02}\left\{{}^\psi{}_{00}\right\}\Pi^\lambda{}_{\alpha\psi}\ . \tag{32}$$

The bundle connection in the spacetime-spacetime-fiber and spacetime-fiber-spacetime sectors is given by [2]

$${}^{(8)}\Gamma^\mu{}_{\alpha b} ={}^{(8)}\Gamma^\mu{}_{b\alpha} = \Pi^\mu{}_{\alpha b} + \frac{1}{2}\overline{R}_{b\lambda\alpha}{}^\mu\rho_0 v^\lambda, \tag{33}$$

and for a stationary measuring device, this becomes

$${}^{(8)}\Gamma^\mu{}_{\alpha b} ={}^{(8)}\Gamma^\mu{}_{b\alpha} = \Pi^\mu{}_{\alpha b} + \frac{1}{2}\rho_0 v^0\overline{R}_{b0\alpha}{}^\mu. \tag{34}$$

The bundle connection in the base-fiber-fiber sector is given by [2]

$${}^{(8)}\Gamma^\mu{}_{ab} = \rho_0 v^\phi\frac{\overline{D}}{Dx^\phi}\Pi_{ab}{}^\mu, \tag{35}$$

in which

$$\frac{\overline{D}}{Dx^\phi}\Pi_{ab}{}^\mu = \Pi_{ab}{}^\mu{}_{,\phi} + \Gamma^\mu{}_{\delta\phi}\Pi_{ab}{}^\delta - \Gamma^\delta{}_{a\phi}\Pi_{\delta b}{}^\mu - \Gamma^\delta{}_{b\phi}\Pi_{a\delta}{}^\mu \tag{36}$$

in the anholonomic basis. Alternatively, one has [2]

$${}^{(8)}\Gamma^\mu{}_{ab} = \nabla_o A_{ab}{}^\mu, \tag{37}$$

where

$$\nabla_0 \equiv \frac{1}{L}\rho_0 v^\phi \frac{\overline{D}}{Dx^\phi}, \tag{38}$$

and

$$A_{a\delta}{}^{\mu} = L\Pi_{a\phi}{}^{\mu}. \tag{39}$$

For a stationary measuring device, Eqs. (35) and (37), using Eq. (6), become

$$^{(8)}\Gamma^{\mu}{}_{ab} = \rho_0 v^0 \frac{\overline{D}}{Dx^0}\Pi_{ab}{}^{\mu}. \tag{40}$$

The bundle connection in the fiber-base-base sector is [2]

$$^{(8)}\Gamma^{m}{}_{\alpha\beta} = -\Pi_{\alpha\beta}{}^{m} + \frac{1}{2}\overline{R}^{m}{}_{\lambda\alpha\beta}\rho_0 v^\lambda, \tag{41}$$

which for a stationary measuring device becomes

$$^{(8)}\Gamma^{m}{}_{\alpha\beta} = -\Pi_{\alpha\beta}{}^{m} + \frac{1}{2}\rho_0 v^0 \overline{R}^{m}{}_{0\alpha\beta}. \tag{42}$$

In the fiber-base-fiber sector, the bundle connection is [2]

$$^{(8)}\Gamma^{m}{}_{\alpha b} = -\rho_0 v^\phi \frac{\overline{D}}{Dx^\phi}\Pi_{b}{}^{m}{}_{\alpha,}, \tag{43}$$

or alternatively

$$^{(8)}\Gamma^{m}{}_{ab} = -\nabla_o A_{b\alpha}{}^{m}. \tag{44}$$

For a stationary measuring device, Eqs. (43) and (44) become, using Eq. (6),

$$^{(8)}\Gamma^{m}{}_{\alpha b} = -\rho_0 v^0 \frac{\overline{D}}{Dx^0}\Pi_{\phi}{}^{m}{}_{\alpha,}. \tag{45}$$

The bundle connection in the fiber-fiber-base sector is [2]

$$^{(8)}\Gamma^{m}{}_{b\alpha} = \overline{\left\{{}^{m}{}_{b\alpha}\right\}}, \tag{46}$$

and this will clearly also hold for a stationary measuring device.

In the fiber sector, the bundle connection is [2]

$$^{(8)}\Gamma^{m}{}_{ab} = \Pi^{m}{}_{ab}. \tag{47}$$

Summarizing the expressions for the connection in the various sectors of the Finsler-spacetime bundle in the anholonomic frame and for a stationary measuring device, one has

$$^{(8)}\Gamma^{\mu}{}_{\alpha\beta} = \overline{\left\{{}^{\mu}{}_{\alpha\beta}\right\}}, \tag{48}$$

$$^{(8)}\Gamma^{\mu}{}_{\alpha b} =^{(8)} \Gamma^{\mu}{}_{b\alpha} = \Pi^{\mu}{}_{\alpha b} + \frac{1}{2}\rho_0 v^0 \overline{R}_{b0\alpha}{}^{\mu}, \tag{49}$$

$$^{(8)}\Gamma^{\mu}{}_{ab} = \rho_0 v^0 \frac{\overline{D}}{Dx^0}\Pi_{ab}{}^{\mu}, \tag{50}$$

$$^{(8)}\Gamma^{m}{}_{\alpha\beta} = -\Pi_{\alpha\beta}{}^{m} + \frac{1}{2}\rho_0 v^0 \overline{R}^{m}{}_{0\alpha\beta}, \tag{51}$$

$$^{(8)}\Gamma^{m}{}_{\alpha b} = -\rho_0 v^0 \frac{\overline{D}}{Dx^0}\Pi_{b}{}^{m}{}_{\alpha,}, \tag{52}$$

$$^{(8)}\Gamma^{m}{}_{b\alpha} = \overline{\{^{m}{}_{b\alpha}\}}. \tag{53}$$

$$^{(8)}\Gamma^{m}{}_{ab} = \Pi^{m}{}_{ab}. \tag{54}$$

in which the Christoffel symbols of four-velocity space are given by

$$\Pi^{\mu}{}_{\alpha\beta} = \frac{1}{2}\rho_0^{-1} g^{\mu\lambda} \frac{\partial}{\partial v^{\lambda}} g_{\alpha\beta}, \tag{55}$$

and one also has

$$\overline{\{^{\mu}{}_{\alpha\beta}\}} = \{^{\mu}{}_{\alpha\beta}\} - A^{\lambda}{}_{\alpha}\Pi_{\lambda\beta}{}^{\mu} - A^{\lambda}{}_{\beta}\Pi_{\lambda\alpha}{}^{\mu} - A^{\lambda\mu}\Pi_{\lambda\alpha\beta}, \tag{56}$$

in which the gauge potential is given by

$$A^{\lambda}{}_{\alpha} = \rho_0 v^0 \{^{\lambda}{}_{0\alpha}\} - \rho_0^2 v^{02} \left\{^{\psi}{}_{00}\right\} \Pi^{\lambda}{}_{\alpha\psi} \ . \tag{57}$$

3 Curvature scalar

The Riemann curvature scalar of a Finsler-spacetime tangent bundle is [2]

$$\begin{aligned} ^{(8)}R \ &= \ \overline{R} - \frac{1}{4}F^{\alpha\beta\gamma}F_{\alpha\beta\gamma} - 2\Pi^{\alpha\gamma}{}_{\alpha}\Pi^{\beta}{}_{\gamma\beta} - 2\frac{D}{D\rho_0 v_\beta}\Pi^{\alpha}{}_{\beta\alpha} \\ &\quad -\nabla\Pi^{\alpha\beta\gamma}\nabla\Pi_{\alpha\beta\gamma} - \nabla\Pi^{\alpha\gamma}{}_{\alpha}\nabla\Pi^{\beta}{}_{\gamma\beta} + 2\frac{\overline{D}}{Dx_\beta}\nabla\Pi^{\alpha}{}_{\beta\alpha}. \end{aligned} \tag{58}$$

A special case of the Finsler-spacetime tangent bundle is the Riemannian-spacetime tangent bundle, for which the four-velocity space Christoffel symbols $\Pi^{\mu}{}_{\gamma\beta}$ are vanishing. In this case Eq. (58) reduces to

$$^{(8)}R = \overline{R} - \frac{1}{4}F^{\alpha\beta\gamma}F_{\alpha\beta\gamma}. \tag{59}$$

For a stationary measuring device, using Eqs. (1), (7), and (9), this becomes

$$^{(8)}R = \overline{R} - \frac{1}{4}\rho_0^2 \overline{R}^{\alpha 0\beta\gamma}\overline{R}_{\alpha 0\beta\gamma}. \tag{60}$$

Equation (60) was used earlier in a perturbative calculation of possible corrections to the gravitational red shift for a static emitter on a Schwarzschild star and for a stationary measuring device at a large distance from the star [11]. A stationary measuring device was also invoked in reducing the action for the spacetime tangent bundle to the ordinary vacuum Einstein's field equations in a Riemannian spacetime in the mathematical limit of infinite maximal proper acceleration [12].

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ДИФФЕРЕНЦИАЛЬНАЯ ГЕОМЕТРИЯ КАСАТЕЛЬНЫХ РАССЛОЕНИЙ ФИНСЛЕРОВОГО ПРОСТРАНСТВА-ВРЕМЕНИ

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В своих более ранних работах автор рассмотрел различные аспекты дифференциальной геометрии касательных расслоений финслерова пространства-времени, которые основываются на возможном существовании верхней границы релятивистски равноускоренного движения. В частности, вычислены связность расслоения и ассоциированные дифференциально-геометрические поля для касательного расслоения финслерова пространства-времени для случая стационарного измерительного прибора.

Ключевые слова: финслерова геометрия, касательное расслоение пространства-времени, максимальное собственное ускорение, физические финслеровы координаты, структура пространства-времени, пространство-время Кэлера, комплексное пространство-время, вращение.