LINE INTEGRATION AND SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS OVER CAYLEY-DICKSON ALGEBRAS

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Line integration of generalized functions is studied. Second order partial differential equations with piecewise continuous and generalized variable coefficients over Cayley-Dickson algebras are investigated. Formulas for integrations of such equations are deduced. For this purpose a non-commutative line integration is used. Examples of solutions of partial differential equations are given.

Key Words: Cayley-Dickson algebra, partial differential equation, line integration, generalized function.

1 Introduction

It is well-known, that differential equations have many-sided applications in different sciences including physics, mechanics, other natural sciences, techniques, economics, etc. The differential equations also are very important for mathematics [7, 13, 34, 15, 29, 37]. Predominantly differential equations are considered over fields such as real, complex, or with non-archimedean norms. Recently they are also begun to be studied over Clifford algebras [9, 10, 11].

Such algebras have a long history, because quaternions were first introduced by W.R. Hamilton in 1843. He had planned to use them for problems of mechanics and mathematics [12, 33]. Their generalization known as the octonion algebra was introduced by J.T. Graves and A. Cayley in 1843-45. Then Dickson had investigated more general algebras known now as the Cayley-Dickson algebras [1, 2, 14].

The Cayley-Dickson algebras, particularly, octonions and quaternions are widely used in physics, but mainly algebraically. Already Maxwell had utilized quaternions to derive his equations of electrodynamics, but then he had rewritten them in real coordinates.

In the 50-th of the 20-th century Yang and Mills had used them in quantum field theory, but theory of functions over octonions and quaternions in their times was not sufficiently developed to satisfy their needs. Discussing that situation they have formulated the problem of developing analysis over octonions and quaternions [8]. This is natural, because quantum fields are frequently non-abelian [35]. Dirac had used complexified quaternions to solve the Klein-Gordon hyperbolic differential equation with constant coefficients.

This work continues previous articles of the author. In those articles (super)-differentiable functions of Cayley-Dickson variables and their non-commutative line integrals were investigated [22, 23, 21, 25, 26]. In the papers [24, 20] differential equations and their systems over octonions and quaternions were studied.

The Cayley-Dickson algebras $\mathcal{A}_r$ have the even generator $i_0 = 1$ and the purely imaginary odd generators $i_1, ..., i_{2^r-1}$, $2 \le r$, $i_k^2 = -1$ and $i_0 i_k = i_k$ and $i_k i_l = -i_l i_k$ for each $1 \le k \ne l$. For $3 \le r$ the multiplication of these generators is generally non-associative, so they form not a group, but a non-commutative quasi-group with the property of alternativity $i_k(i_k i_l) = (i_k^2) i_l$ and $(i_l i_k) i_k = i_l (i_k^2)$ instead of associativity. Ordinary super-analysis operates with graded algebras over Abelian groups. Therefore, super-analysis over the Cayley-Dickson algebras is in some respect more complicated than usual super-analysis, for example, over the Grassman algebras.

The aim of this paper is in developing of Dirac's approach on partial differential equations with variable piecewise continuous or generalized coefficients.

The technique presented there is developed here below for solutions of partial differential equations of the second order of arbitrary signatures and with variable coefficients which may also be piecewise continuous or generalized functions. Moreover, signatures may change piecewise in a domain. Formulas for integrations of such equations are deduced. For this purpose a non-commutative line integration of generalized functions is developed. Examples of partial differential equations are given. Moreover the approach of §§2-25 over the Cayley-Dickson algebras $\mathcal{A}_v$ gives the fundamental solution of any first and second order linear partial differential equation with variable z-differentiable $\mathcal{A}_v$-valued coefficients, $z \in U \subset \mathcal{A}_v$, where U is a domain in $\mathcal{A}_v$ satisfying some mild convexity conditions described below. These results can be used for solutions of concrete partial differential equations or their systems of different orders with piecewise continuous or generalized coefficients, for example, of Helmholtz' or Klein-Gordon's types, which are important in optics of composite materials or quantum field theory. Finally solutions of some types of non-linear partial differential equations over Cayley-Dickson algebras are studied.

Main results of this paper are obtained for the first time.

2 Partial differential equations of the second order.

1. Remarks and notations. For a subset U in either the quaternion skew field $\mathbf{H} = \mathcal{A}_2$ or in the octonion algebra $\mathbf{O} = \mathcal{A}_3$ or the Cayley-Dickson algebra $\mathcal{A}_r$, $r \geq 4$, we put $\pi_{\mathsf{s},\mathsf{p},\mathsf{t}}(U) := \{\mathsf{u} : z \in U, z = \sum_{\mathsf{v} \in \mathbf{b}} w_{\mathsf{v}} \mathsf{v},\ \mathsf{u} = w_{\mathsf{s}} \mathsf{s} + w_{\mathsf{p}} \mathsf{p}\}$ for each $\mathsf{s} \neq \mathsf{p} \in \mathbf{b}$, where $\mathsf{t} := \sum_{\mathsf{v} \in \mathbf{b} \setminus \{\mathsf{s},\mathsf{p}\}} w_{\mathsf{v}} \mathsf{v} \in \mathcal{A}_{r,\mathsf{s},\mathsf{p}} := \{z \in \mathcal{A}_r : z = \sum_{\mathsf{v} \in \mathbf{b}} w_{\mathsf{v}} \mathsf{v},\ w_{\mathsf{s}} = w_{\mathsf{p}} = 0,\ w_{\mathsf{v}} \in \mathbf{R}\ \forall \mathsf{v} \in \mathbf{b}\}$, where $\mathbf{b} := \{i_0, i_1, ..., i_{2^r-1}\}$ is the family of standard generators of the algebra $\mathcal{A}_r$ so that $i_j^2 = -1$, for each $j \geq 1$, $i_j i_k = -i_k i_j$ for each $j \neq k \geq 1$, $i_0 = 1$. Geometrically the domain $\pi_{\mathsf{s},\mathsf{p},\mathsf{t}}(U)$ means the projection on the complex plane $\mathbf{C}_{\mathsf{s},\mathsf{p}}$ of the intersection U with the plane $\tilde{\pi}_{\mathsf{s},\mathsf{p},\mathsf{t}} \ni \mathsf{t}$, $\mathbf{C}_{\mathsf{s},\mathsf{p}} := \{a\mathsf{s} + b\mathsf{p} : a, b \in \mathbf{R}\}$, since $\mathsf{s}\mathsf{p}^* \in \hat{b} := \mathbf{b} \setminus \{1\}$. Recall that in §§2.5-7 [22] for each continuous function $f : U \to \mathcal{A}_r$ it was defined the operator $\hat{f}$ by each variable $z \in \mathcal{A}_r$. If a function f is z-differentiable by the Cayley-Dickson variable $z \in U \subset \mathcal{A}_r$, $2 \leq r$, then $\hat{f}(z) = dg(z)/dz$, where $(dg(z)/dz).1 = f(z)$.

A Hausdorff topological space X is said to be n-connected for $n \geq 0$ if each continuous map $f : S^k \to X$ from the k-dimensional real unit sphere into X has a continuous extension over $\mathbf{R}^{\mathbf{k+1}}$ for each $k \leq n$ (see also [36]). A 1-connected space is also said to be simply connected.

It is supposed further, that a domain U in $\mathcal{A}_r^m$ has the property that

$(D1)$ each projection $\mathbf{p}_j(U) =: U_j$ is $(2^r - 1)$-connected;

$(D2)$ $\pi_{\mathsf{s},\mathsf{p},\mathsf{t}}(U_j)$ is simply connected in $\mathbf{C}$ for each $k = 0, 1, ..., 2^{r-1}$, $\mathsf{s} = i_{2k}$, $\mathsf{p} = i_{2k+1}$, $\mathsf{t} \in \mathcal{A}_{r,\mathsf{s},\mathsf{p}}$ and $u \in \mathbf{C}_{\mathsf{s},\mathsf{p}}$, for which there exists $z = \mathsf{u} + \mathsf{t} \in U_j$,
where $e_j = (0, ..., 0, 1, 0, ..., 0) \in \mathcal{A}_r^m$ is the vector with 1 on the j-th place, $\mathbf{p}_j(z) = {}^j z$ for each $z \in \mathcal{A}_r^m$, $z = \sum_{j=1}^m {}^j z e_j$, ${}^j z \in \mathcal{A}_r$ for each $j = 1, ..., m$, $m \in \mathbf{N} := \{1, 2, 3, ...\}$. Frequently we take $m = 1$. Henceforward, we consider a domain U satisfying Conditions $(D1, D2)$ if any other is not outlined.

The family of all $\mathcal{A}_r$ locally analytic functions $f(x)$ on U with values in $\mathcal{A}_r$ is denoted by $\mathcal{H}(U, \mathcal{A}_r)$. It is supposed that a locally analytic function $f(x)$ is written in the x-representation $\nu(x)$, also denoted by $\nu = \nu^f$. The latter is equivalent to the super-differentiability of f (see [23, 22, 24]). Each such f is supposed to be specified by its phrase ν.

For each super-differentiable function $f(x)$ its non-commutative line integral $\int_\gamma f(x)dx$ in U is defined along a rectifiable path γ in U. It is the integral of a differential form $\hat{f}(x).dx$, where

$(I1)$ $\hat{f}(x) = dg(x)/dx$,

$(I2)$ $[dg(x)/dx)].1 = f(x)$ for each $x \in U$.

A branch of the non-commutative line integral can be specified with the help of either the left or right algorithm (see [23, 22, 24]). We take further for definiteness the left algorithm if something another will not described. For $f \in \mathcal{H}(U, \mathcal{A}_r)$ and a rectifiable path $\gamma : [a, b] \to \mathcal{A}_r$ the integral $\int_\gamma f(x)dx$ depends only on an initial $\alpha = \gamma(a)$ and final $\beta = \gamma(b)$ points due to the non-commutative analog of the homotopy theorem in U, where $a < b \in \mathbf{R}$. When initial and final points or a path are not marked we denote the operation of the non-commutative line integration in the domain U simply by $\int f(x)dx$ analogously to the indefinite integral.

To rewrite a function from real variables z_j in the z-representation the following identities are used:

(1) $z_j = (-zi_j + i_j(2^r - 2)^{-1}\{-z + \sum_{k=1}^{2^r-1} i_k(zi_k^*)\})/2$
for each $j = 1, 2, ..., 2^r - 1$,

$$(2) \quad z_0 = (z + (2^r - 2)^{-1}\{-z + \sum_{k=1}^{2^r-1} i_k(zi_k^*)\})/2,$$

where $2 \le r \in \mathbf{N}$, z is a Cayley-Dickson number decomposed as

(3) $z = z_0 i_0 + ... + z_{2^r-1} i_{2^r-1} \in \mathcal{A}_r$, $z_j \in \mathbf{R}$ for each j, $i_k^* = \tilde{i}_k = -i_k$ for each $k > 0$, $i_0 = 1$, since $i_k(i_0 i_k^*) = i_0 = 1$, $i_k(i_j i_k^*) = -i_k(i_k^* i_j) = -(i_k i_k^*)i_j = -i_j$ for each $k \ge 1$ and $j \ge 1$ with $k \ne j$ (shortly $k \ne j \ge 1$), $i_k(i_k i_k^*) = i_k$ for each $k \ge 0$.

As usually $C^0(U, \mathcal{A}_v)$ denotes the $\mathbf{R}$-linear space of all continuous $\mathcal{A}_v$-valued functions $f : U \to \mathcal{A}_v$. More generally $C^n(U, \mathcal{A}_v)$ denotes the $\mathbf{R}$-linear space of all n times continuously differentiable by real variables $z_0, ..., z_{2^v-1}$ functions $f : U \to \mathcal{A}_v$, where $n \in \mathbf{N}$. Certainly, $C^n(U, \mathcal{A}_v)$ can be supplied with the structure of left- and right-module over the Cayley-Dickson algebra $\mathcal{A}_v$ using point-wise multiplication of functions $f(z)$ on Cayley-Dickson numbers from the left and the right.

2. Factorization and integration of equations.

We consider the second order partial differential equation:

(1) $Af = g$, where

$$A = \sum_{l,m=1}^{k} \mathbf{a}_{l,m}\partial^2/\partial\tau_l\partial\tau_m + \sum_{l=1}^{k} \alpha_l \partial/\partial\tau_l$$

is a partial differential operator of the second order. Let us suppose that the quadratic form

$$a(\tau) := \sum_{l,m} \mathbf{a}_{l,m}\tau_l\tau_m$$

is non-degenerate and is not always negative, because otherwise we can consider $-A$. Moreover, let a matrix of coefficients be real and symmetric $\mathbf{a}_{l,m}(\tau) = \mathbf{a}_{m,l}(\tau) \in \mathbf{R}$, $\alpha_l, \tau_l \in \mathbf{R}$ for each $l, m = 1, ..., k$. Then we reduce this form $a(\tau)$ by an invertible $\mathbf{R}$ linear operator $C = C(\tau)$ to the sum of squares. This means, that

$$(2) \quad A = \sum_{l=1}^{k} \mathbf{b}_l \partial^2/\partial s_l^2 + \sum_{l=1}^{k} \beta_l \partial/\partial s_l,$$

where $\partial s_j/\partial\tau_l = C_{l,j}(\tau)$, $C = (C_{l,j})$, with real-valued functions $\mathbf{b}_l$ and β_l for each l. Here

$$\mathbf{b}_l\delta_{j,l} = \sum_{p,m} \mathbf{a}_{p,m}C_{p,j}C_{m,l} \text{ and}$$

$$\beta_j = \sum \mathbf{a}_{p,m}(\partial C_{p,j}/\partial\tau_m) + \sum_{v=1}^{k} \alpha_v C_{v,j}$$

for all $j, l = 1, ..., k$. In the case when coefficients of A are constant, using a multiplier of the type $\exp(\sum_l \epsilon_l s_l)$ it is possible to reduce this equation to the case so that if $\mathbf{b}_l \ne 0$, then $\beta_l = 0$ (see §3, Chapter 4 in [34]). Therefore, one can as usually simplify the operator with the help of such change of coordinates and consider that only β_1 may be non-zero if $\mathbf{b}_1 = 0$.

Thus one can choose an invertible real matrix $(c_{h,m})_{h,m=1,...,k}$ corresponding to $C = C(\tau)$ so that $\mathbf{b}_l \le 0$ for $p+1 \le l \le k$ and $\mathbf{b}_l \ge 0$ for $0 < l \le p$, where $0 < p \le k$, $q := k - p$. When $q = 0$ and $\beta_l = 0$ for each l the operator is elliptic, for $q = 0$ and $\beta_1 \ne 0$ the operator is parabolic, for $0 < p < k$ and $\beta_l = 0$ for each l the operator is hyperbolic. Sometimes the matrix C can be chosen constant on a domain, where the signature (p, q) of the quadratic form $a(\tau)$ is constant. We suppose that the sums $\sum_{l=1}^p \mathbf{b}_l^2(x) > 0$ and $\sum_{l=p+1}^k \mathbf{b}_l^2(x) > 0$ are positive λ-almost everywhere on a domain U, where λ is the measure induced by the Lebesgue measure on the real shadow of the Cayley-Dickson algebra. Generally the natural number $k - p = q(x)$ may either be constant or change while crossing the surface $\{x \in U : \sum_{l=1}^k \mathbf{b}_l^2(x) = 0\}$, when the domain U satisfies Conditions $1(D1, D2)$.

We consider elliptic and hyperbolic partial differential operators reduced to the sum of squares

$$(3) \quad A = [\sum_{l=0}^{k} \mathbf{b}_l(x)\partial^2/\partial x_l^2],$$

where $\mathbf{b}_l(x) \in \mathbf{R}$ for all $x = x_0 i_0 + ... + x_{2^r-1} i_{2^r-1}$ in the open domain $U \subset \mathcal{A}_r$ satisfying Conditions $1(D1, D2)$ in the Cayley-Dickson algebra $\mathcal{A}_r$, $1 \le k \le 2^r - 1$, $2 \le r \le 3$. Practically the coefficient $\mathbf{b}_l$ can depend only on $x_0, ..., x_k$ remaining z-differentiable in definite z-representations due to Formulas $1(1-3)$ for each l.

More generally we can consider partial differential operators of the form

(4) $A = c_1 B_1 + ... + c_m B_m$, where $c_j B_j f = c_j(B_j f)$, while each

$$(4') \quad B_j = \sum_{k=m_1+...+m_{j-1}+1}^{m_1+...+m_j} \mathbf{b}_k(x)\partial^2/\partial x_k^2$$

is an elliptic partial differential operator of the second order by variables $x_{m_1+...+m_{j-1}+1},...,x_{m_1+...+m_j}$; $c_j \in \mathcal{A}_r$ with $Re(c_j) \ge 0$ for each $1 \le j \le l$, $Re(c_j) < 0$ for every $j > l$, with $|c_j| = 1$ for each $j = 1, ...m$, where $1 \le r$, $1 \le l < m$, $m_0 = 0$.

We remind, that Dirac had used complexified bi-quaternions to solve Klein-Gordon's hyperbolic partial differential equation with constant coefficients appearing in spin problems. That is, he had decomposed d'Alembert's operator $\partial^2/\partial t^2 - \nabla^2$ as the product $\mathbf{i}^*\sigma\mathbf{i}\sigma$ over the complexified bi-quaternion algebra $\mathbf{H}_\mathbf{C}$ with the first order differential operator σ.

If follow this approach one takes the complexified Cayley-Dickson algebra

(5) $(\mathcal{A}_r)_\mathbf{C} = \mathcal{A}_r \oplus \mathcal{A}_r\mathbf{i}$,

where $\mathbf{i}$ is taken to be commuting with i_j for each $j = 0, ..., 2^r - 1$. Now the algebra $(\mathcal{A}_r)_\mathbf{C}$ is already not the division algebra even for $2 \le r \le 3$, that is two non zero elements with zero product occur in it. Then each element $z = (z_1, 0)$ in $(\mathcal{A}_r)_\mathbf{C}$ can be written in the 2×2 matrix form $\begin{pmatrix} z_1 & 0 \\ 0 & z_1 \end{pmatrix}$ and $z = (0, z_2)$ can be written in the form $\begin{pmatrix} 0 & z_2 \\ -z_2 & 0 \end{pmatrix}$, where entries $z_1, z_2 \in \mathcal{A}_r$ are Cayley-Dickson numbers, $\mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let each coefficient c_j be written in the polar form

(6) $c_j = \exp(i_{\kappa(j)}\gamma_j)$

with $0 \le |\gamma_j| \le \pi$, $j = 1, ..., m$, $1 \le r$, $1 \le \kappa(j) \le \kappa(j+1)$ for each j. Put $p = p_1 + ... + p_m$, where $p_j = 0$ for either $\gamma_j = 0$ or $\kappa(j) = \kappa(j-1)$, while $p_j = 1$ for $\gamma_j \ne 0$ and $\kappa(j) \ne \kappa(j-1)$. Up to an isomorphism we take the Cayley-Dickson algebra $\mathcal{A}_v$ with $v \ge r$ satisfying inequalities $2^{v-1} < 2^p(m+1) \le 2^v$. Further we make the complexification $(\mathcal{A}_v)_\mathbf{C}$ of the Cayley-Dickson algebra $\mathcal{A}_v$.

Take two non-negative integer numbers $0 \le r$ and v with $r \le v \in \mathbf{Z}$. We consider the quotient algebra over the real field $\mathcal{A}_v/\mathcal{A}_r =: \mathcal{A}_{r,v}$. For $r = v$ this algebra is isomorphic with the real field $\mathbf{R}$. For $r < v$ the algebra $\mathcal{A}_{r,v}$ is isomorphic with $\bigoplus_{k=0}^{2^{v-r}-1} \mathbf{R} i_{2^r k}$. The latter algebra is produced by generators $\{i_{2^r\beta} : \ \beta = 2^\gamma - 1; \ \gamma = 0, 1, ..., v-r\}$ and their finite ordered products, that gives the generators set $\{i_{2^r k} : \ k = 0, ..., 2^{v-r} - 1\}$, where generators satisfying the numbering rule $i_j i_{2^s} = i_{j+2^s}$ for each $1 \le s$, $j = 0, 1, ..., 2^s - 1$ can be taken up to an isomorphism of the Cayley-Dickson algebra $\mathcal{A}_{s+1}$. Therefore, the algebra $\mathcal{A}_{r,v}$ is isomorphic with the Cayley-Dickson algebra $\mathcal{A}_{v-r}$, since the doubling procedure can be started from another suitable purely imaginary Cayley-Dickson numbers such as generators [1, 14]. But we consider in $\mathcal{A}_{r,v}$ its specific generators basis $\{i_{2^r k} : \ k = 0, ..., 2^{v-r} - 1\}$.

For each Cayley-Dickson numbers $x, y \in \mathcal{A}_r$ we define the real-valued scalar product

(RS) $(x, y) = (x, y)_r := Re(x\tilde{y})$,

where $\tilde{z} = z^*$ denotes the conjugated number, while $Re(y) := (y + y^*)/2$ denotes the real part of y.

The real scalar product $(.,.)_r$ in $\mathcal{A}_r$ we extend on the algebra $\mathcal{A}_{r,v}$ as

$$(SP) \quad < x, y >_{r,v} = x\tilde{y} = \sum_{j,k=0}^{2^{v-r}-1} x_{2^r j} y_{2^r k} i_{2^r j} i^*_{2^r k}$$

for each $x, y \in \mathcal{A}_{r,v}$, $x = \sum_{j=0}^{2^{v-r}-1} x_{2^r j} i_{2^r j}$, $x_{2^r j} \in \mathbf{R}$ for each $j = 0, ..., 2^{v-r} - 1$. Particularly, one gets $< x, y >_{0,v} = < x, y >_v$. In the case of the complexified algebra $(\mathcal{A}_{r,v})_{\mathbf{C}}$ the scalar product is:

$$(SPC) \quad < (a, b), (c, d) >_{r,v} = < (a, b), (c, d) > = (< a, c > - < b, d >, < a, d > + < b, c >),$$

for all (a, b) and $(c, d) \in (\mathcal{A}_{r,v})_{\mathbf{C}}$.

We recall the doubling procedure for the Cayley-Dickson algebra $\mathcal{A}_{r+1}$ from $\mathcal{A}_r$. Each Cayley-Dickson number $z \in \mathcal{A}_{r+1}$ is written in the form $z = \xi + \eta \mathbf{l}$, where $\mathbf{l}^2 = -1$, $\mathbf{l} \notin \mathcal{A}_r$, $\xi, \eta \in \mathcal{A}_r$. The addition of such numbers is componentwise. The conjugate of any Cayley-Dickson number z is given by the formula:

$(M1)$ $z^* := \xi^* - \eta \mathbf{l}$.

The multiplication in $\mathcal{A}_{r+1}$ is defined by the following equation:

$(M2)$ $(\xi + \eta \mathbf{l})(\gamma + \delta \mathbf{l}) = (\xi\gamma - \tilde{\delta}\eta) + (\delta\xi + \eta\tilde{\gamma})\mathbf{l}$

for each ξ, η, γ, $\delta \in \mathcal{A}_r$, $z := \xi + \eta \mathbf{l} \in \mathcal{A}_{r+1}$, $\zeta := \gamma + \delta \mathbf{l} \in \mathcal{A}_{r+1}$.

Using Formula $(M2)$ we get: $(b i_{2^r k})(i_{2^r k} b)^* = (b i_{2^r k})(b^* i^*_{2^r k}) = b^2 = (b^2 i_{2^r k}) i^*_{2^r k} = i_{2^r k}(i_{2^r k} b^2)^*$ for each $k \ge 1$ and $b \in \mathcal{A}_r$, since $i^*_j = -i_j$ for each $j \ge 1$. Another useful identity is the following: $(i_s i_{2^r j}) i^*_{2^r k} = -(i_s i_{2^r k}) i^*_{2^r j}$ for each $0 \le s \le 2^r - 1$ and $k \ne j$ with $k \ge 1$ and $j \ge 1$, since $(i_s i_{2^r j}) i_{2^r k} = (i_s i^*_{2^r k}) i_{2^r j}$. Certainly also the equality $(i_s i_0) i^*_j + (i_s i_j) i^*_0 = 0$ holds for each $j \ge 1$ and $1 \le s \le 2^r - 1$, since $i_0 = 1$. Therefore, Formulas (SP) and $(4')$ together with the latter identities imply:

$$(6) \quad < cBy, y >_v = \sum_{j=0}^{2^{v-r}-1} c < B y_{2^r j}, y_{2^r j} >_{r,v}$$

for each $c \in \mathcal{A}_r$ and a twice differentiable function y with values in $\mathcal{A}_{r,v}$.

Relative to the complex scalar product given by Equality (SPC) we decompose the operator A (see $(4, 4')$ above) in the form

(7) $A = (\mathbf{i}\sigma)(\mathbf{i}\sigma_1) + Q = -\sigma\sigma_1 + Q$,

where σ, σ_1 and Q are partial differential operators of the first order, each complex number

$\alpha \in \mathbf{C}$ is presented as a real 2×2 matrix. Particularly, $\mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{i}^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Each subalgebra $\mathbf{g}_{j,k}$ constructed from two generators $i_j \ne i_k$ is associative, consequently, $(wi_k)(w^*i_k^*) = w^2$ and $w((wi_k)i_k) = -w^2$ for each $w = w_0 + w_j i_j$ with $w_0, w_j \in \mathbf{R}$. Therefore, we can take

$$(8) \quad \sigma f(z) = \sum_{j=1}^{m} \sum_{k=m_1+...+m_{j-1}+1}^{m_1+...+m_j} \mathbf{a}_k(z)(\partial f/\partial z_{2^r k})[w_j^* i_{2^r k}^*] \text{ and}$$

$$(9) \quad \sigma_1 f(z) = \sum_{j=1}^{m} \sum_{k=m_1+...+m_{j-1}+1}^{m_1+...+m_j} \mathbf{a}_k(z)(\partial f/\partial z_{2^r k})[w_j i_{2^r k}^*]$$

on the space of $\mathcal{A}_r$-valued (super-)differentiable functions f for $r \le 2$ or real-valued functions f for $3 \le r$ of the $\mathcal{A}_{r,v}$ variable, since $(\mathbf{i}i_k)^2 = \mathbf{i}^2 i_k^2 = (-1)^2 = 1$ for each $k \ge 1$, where $w_j^2 = c_j$ for all j and $\mathbf{a}_k^2(x) = \mathbf{b}_k(x)$ for each k and x, $w_j \in \mathcal{A}_r$, $\mathbf{a}_k(x) \in \mathbf{R}$ for all k and x, $i_{2^r k} \in \mathcal{A}_{r,v}$, $z_{2^r k} = x_k$, $z = \sum_k z_{2^r k} i_{2^r k} \in \mathcal{A}_{r,v}$, $\partial f(z)/\partial z_{2^r k} = (df(z)/dz).i_{2^r k}$. For $b = \partial^2 f/\partial z_{2^r k}^2$ and $\mathbf{l} = i_{2^r k}$ and $w \in \mathcal{A}_r$ one has the identities: $(b(w\mathbf{l}))(w^*\mathbf{l}) = ((wb)\mathbf{l})(w^*\mathbf{l}) = -w(wb) = -w^2 b$ and $(((b\mathbf{l})w^*)\mathbf{l})w = (((bw)\mathbf{l})\mathbf{l})w = -(bw)w = -bw^2$ in the considered here cases. The operator Q is given by the equality:

$$(10) \quad Qf(z) = \sum_{j_1,j_2=1}^{m} \sum_{k_1=m_1+...+m_{j_1-1}+1}^{m_1+...+m_{j_1}} \sum_{k_2=m_1+...+m_{j_2-1}+1}^{m_1+...+m_{j_2}}$$

$$\mathbf{a}_{k_1}(z)\{(\partial \mathbf{a}_{k_2}(z)/\partial z_{2^r k_1})(\partial f/\partial z_{2^r k_2})[w_{j_2} i_{2^r k_2}^*]\}[w_{j_1}^* i_{2^r k_1}^*],$$

since $\mathbf{i}i_k = i_k\mathbf{i}$ in the complexified Cayley-Dickson algebra $(\mathcal{A}_v)_{\mathbf{C}}$ for each k. The latter equality (10) shows, that the differential operator Q is non-zero, when $\mathbf{a}_k(z)$ are non-constant coefficients.

If use $i_0 = 1$ and $\partial/\partial z_0$ also one can write out d'Alembert's operator in our notation

(11) $\partial^2/\partial z_0^2 - \sum_{j=1}^3 \partial^2/\partial z_j^2 = (\mathbf{i}^*\partial/\partial z_0 + i_1\partial/\partial z_1 + i_2\partial/\partial z_2 + i_3\partial/\partial z_3)(\mathbf{i}\partial/\partial z_0 + i_1\partial/\partial z_1 + i_2\partial/\partial z_2 + i_3\partial/\partial z_3)$.

We recall, that the Cayley-Dickson algebra $\mathcal{A}_r$ is power associative, that is $z^k z^l = z^{k+l}$ for all natural numbers k and l. But the complexified Cayley-Dickson algebra $(\mathcal{A}_r)_{\mathbf{C}}$ is not power associative for $r \ge 3$, since the Cayley-Dickson algebra $\mathcal{A}_r$ is not associative for $r \ge 3$. Therefore, we do not widely use the complexified Cayley-Dickson algebras, but we utilize the Cayley-Dickson algebras $\mathcal{A}_v$ over the real field $\mathbf{R}$, when something other will not be specified.

With these decomposition of operators given by Equations $(7-9, 11)$ the differential equation (1) can be integrated with the help of the non-commutative line integration. We consider at first the partial differential equation

(12) $\Upsilon f = g$

on an open domain U in $\mathcal{A}_v$, where

(13) $\Upsilon f = \sum_{j=0}^{2^v-1} (\partial f/\partial z_j)[i_j^*\psi_j(z)]$,

f and g and $\psi_j(z)$ are $\mathcal{A}_v$-valued functions on the domain U satisfying Conditions $1(D1, D2)$, where g, $\psi_j \in C^0(U, \mathcal{A}_v)$ for each j, particularly they may be $\mathcal{A}_v$ (super-)differentiable functions.

3. Line integration over Cayley-Dickson algebras. Take any phrase

(1) $\mu(z) = \sum_m \{c_m, z^m\}_{q(m)}$

corresponding to the function f, where

$\{c_m, z^m\}_{q(m)} = \{c_{1,m_1} z^{m_1} ... c_{k,m_k} z^{m_k}\}_{q(m)}$,

$q(m)$ is a vector indicating on an order of multiplications in the curled brackets, $c_{j,m_j} \in \mathcal{A}_v$ for each j, $m = (m_1, ..., m_k)$, $k \in \mathbf{N}$, $0 \le m_j \in \mathbf{Z}$ for each j, $z^k = (...((zz)z)...)z$. We put for convenience $z^0 = 1$ in the considered phrases. Though the symbol z^0 can be retained when necessary to specify a branch of the line integral over the Cayley-Dickson algebra $\mathcal{A}_r$ (see [22, 23, 24]). Using the shift $z \mapsto (z - {}_0z)$ we can consider such series with the center at a point ${}_0z$ instead of zero. Then the derivative of the phrase is:

(2) $d\mu(z)/dz =$
$\sum_{m,l,j}\{c_{1,m_1}z^{m_1}...c_{j-1,m_{j-1}}z^{m_{j-1}}c_{j,m_j}((z^{m_j-l-1}I)z^l)c_{j+1,m_{j+1}}z^{m_{j+1}}...c_{k,m_k}z^{m_k}\}_{q(m)}$,
where I denotes the unit operator, so that $d\mu/dz$ is the operator valued derivative function, $0 \le l \le m_j - 1,\ \ j = 1, ..., k$. From Equality (2) it follows that

(3) $(d\mu(i_px)/dx).1 = (d\mu(z)/dz).i_p = \partial\mu(z)/\partial z_p$ for $z = i_px$.

If $\gamma : [a, b] \to \mathcal{A}_r$ is a function, then

$V_a^b\gamma := \sup_P |\gamma(t_{j+1}) - \gamma(t_j)|$

is called the variation of γ on the segment $[a, b] \subset \mathbf{R}$, where the supremum is taken by all finite partitions P of the segment $[a, b]$, $P = \{t_0 = a < t_1 < ... < t_n = b\}$, $n \in \mathbf{N}$. A continuous function $\gamma : [a, b] \to \mathcal{A}_r$ with the finite variation $V_a^b\gamma < \infty$ is called a rectifiable path. It is convenient to take the unit segment $[a, b] = [0, 1]$ using a suitable reparametrization.

We say that a function ν on U is absolutely continuous on U if for each rectifiable path $\gamma : [0, 1] \to \mathcal{A}_v$ for each $\epsilon > 0$ and each $\tau \in [0, 1]$ a positive number $\delta > 0$ exists so that $V_\tau^{\min(1,\tau+\delta)}\nu(\gamma) < \epsilon$ and $V_{\max(0,\tau-\delta)}^\tau\nu(\gamma) < \epsilon$.

We call a function ν of bounded variation on U if for each rectifiable path $\gamma : [a, b] \to U$ the variation $V_a^b\nu(\gamma) < \infty$ is finite. The family of all functions $\nu : U \to \mathcal{A}_v$ of bounded variation will be denoted by $\mathcal{V}(U, \mathcal{A}_v)$.

The non-commutative line integral $\int_\gamma f(z)d\nu(z)$ along a rectifiable path $\gamma : [0, 1] \to U \subset \mathcal{A}_v$ for a phrase μ and a given function ν of bounded variation is the limit by partitions $P = \{0 = \tau_0 < \tau_1 < ... < \tau_n = 1\}$ with their diameter $\delta(P) = \sup_j |\tau_{j+1} - \tau_j|$ tending to zero of integral sums

$\int_\gamma f(z)d\nu(z) := \lim_{\delta(P)\to 0} \sum_j (d\kappa(z)/dz)|_{z=\gamma(\tau_j)}.[\nu(\gamma(\tau_{j+1})) - \nu(\gamma(\tau_j))]$,
where $(d\kappa(z)/dz).1 = \mu(z)$ for all $z \in U$. The notation

$\hat{f}(z) = dg(z)/dz$ and $\hat{\mu}(z) = d\kappa(z)/dz$

is also used, where $g(z)$ is a super-differentiable function to which the phrase κ corresponds.

If f is a continuous function we fix for it a sequence $f^n(z)$ of super-differentiable functions and their phrases $\mu^n(z)$ such that $f^n(z)$ converges to $f(z)$ on each compact subset of the domain U, where $n \in \mathbf{N}$. The non-commutative line integral has a continuous extension on the $\mathbf{R}$-linear space, left and right $\mathcal{A}_v$ module, of continuous functions $C^0(U, \mathcal{A}_v)$ for a marked function $\nu(z)$ of bounded variation and a given rectifiable path γ:

$\int_\gamma f(z)d\nu(z) := \lim_{n\to\infty} \int_\gamma f^n(z)d\nu(z)$.

This means that the $\mathbf{R}$ homogeneous $\mathcal{A}_v$ additive operator $\hat{f}(z)$ is defined for the continuous function f in the sense of distributions:

$(\hat{f}; \nu, \gamma) := \int_\gamma f(z)d\nu(z)$

for each rectifiable path γ in U and every function $\nu(z)$ of bounded variation. Particularly, $\nu(z) = id(z) = z$ on U can also be taken.

If ν and f are super-differentiable functions such that the derivative $d\nu(z)/dz$ of ν is the invertible $\mathbf{R}$ homogeneous $\mathcal{A}_v$ additive operator for each $z \in U$, then a super-differentiable solution of the differential equation

$(dg(z)/dz).(d\nu(z)/dz) = dq((\nu(z))/dz$

on U exists, since $dz/d\nu = (d\nu/dz)^{-1}$. That is, $(dg(z)/dz).d\nu(z) = (dq(\nu(z))/dz).dz$. Therefore,

$\int_\gamma f(z)d\nu(z) = \lim_{\delta(P)\to 0} \sum_j (dq(\nu)/d\nu)|_{\nu=\nu(\gamma(\tau_j))}.[\nu(\gamma(\tau_{j+1})) - \nu(\gamma(\tau_j))] = \int_{\nu(\gamma)} p(y)dy$,
where $p(\nu) = (dq(\nu)/d\nu).1$ (see also Theorems 2.11 and 2.13 in [27]).

A function $\nu : U \to \mathcal{A}_v$ is called piecewise continuous or differentiable or super-differentiable on a domain U in the Cayley-Dickson algebra U, if a family of open or canonical closed subsets U_j of U exists so that each restriction $\nu|_{U_j}$ is continuous or differentiable or super-differentiable respectively, where $U = \bigcup_j U_j$ and $U_j \cap U_k = \partial U_j \cap \partial U_k$ for each $j \ne k$, $\partial U_j := cl(U_j) \setminus Int(U_j)$, $cl(U_j)$ denotes the closure of U_j in $\mathcal{A}_v$ and $Int(U_j)$ denotes the interior of U_j in $\mathcal{A}_v$.

If f is a continuous function and ν is a function of bounded variation for which the limits $\lim_n f^n = f$ and $\nu = \lim_n \nu^n$ uniformly converge on each compact subset of U and phrases μ^n of f^n and ϵ^n of ν^n are specified, where f^n are super-differentiable functions and ν^n are piecewise super-differentiable functions on U so that $\lim_n V_0^1(\nu^n(\gamma) - \nu(\gamma)) = 0$ for each rectifiable path γ in U, then

$\int_\gamma f(z)d\nu(z) = \lim_n \int_\gamma f^n(z)d\nu^n(z) = \lim_n \int_{\nu^n(\gamma)} p^n(y)dy = \int_{\nu(\gamma)} p(y)dy$,

where $p(y) = \lim_n p^n(y)$. This means that under rather general conditions the line integral of the type $\int_\gamma f(z)d\nu(z)$ relative to the function ν of bounded variation reduces to the usual non-commutative line integral $\int_\eta p(y)dy$, where $\eta = \nu(\gamma)$.

Take the branch of the non-commutative line integral prescribed by the left algorithm (see §2 in [23, 22]). The real algebra $\mathbf{g}_{k,l,s}$ formed from the generators i_j, i_k and i_s is alternative. Each rectifiable path can be presented as the limit of rectifiable paths consisting of joined segments parallel to the straight lines $i_j\mathbf{R}$ with respective j. We certainly have $(i_q i_p)i_p = -i_q$ for each $p \ge 1$ and $(i_q i_0)i_0 = i_q$ for each $q \ge 0$.

For each $j = 0, ..., 2^r - 1$ the $\mathbf{R}$- linear projection operator $\pi_j : \mathcal{A}_r \to \mathbf{R}i_j$ exists due to Formulas 1(1 − 3) so that $\pi_j(z) = i_j z_j = z_j i_j$:

$(P1)$ $\pi_j(z) = (-i_j(zi_j) - (2^r - 2)^{-1}\{-z + \sum_{k=1}^{2^r-1} i_k(zi_k^*)\})/2$

for each $j = 1, 2, ..., 2^r - 1$,

$$(P2) \quad \pi_0(z) = (z + (2^r - 2)^{-1}\{-z + \sum_{k=1}^{2^r-1} i_k(zi_k^*)\})/2,$$

where $2 \le r \in \mathbf{N}$.

4. Line anti-derivatives over Cayley-Dickson algebras.

Theorem. *Let a first order partial differential operator* Υ *be given by Equation 2(13) with real-valued continuous functions* $\psi_j(z) \in C^0(U, \mathcal{A}_v)$ *for each* j *such that* $\psi_j(z) \ne 0$ *for each* $z \in U$ *and each* $j = 0, ..., n$, *where a domain* U *satisfies Conditions 1(D1, D2),* ${}_0z$ *is a marked point in* U, $1 < n < 2^v$, $2 \le v$. *Then a line integral* $\mathcal{I}_\Upsilon : C^0(U, \mathcal{A}_v) \to C^1(U, \mathcal{A}_v)$, $\mathcal{I}_\Upsilon f(z) := {}_\Upsilon\int_{{}_0z}^z f(y)dy$ *on* $C^0(U, \mathcal{A}_v)$ *exists so that*

(1) $\Upsilon \mathcal{I}_\Upsilon f(z) = f(z)$

for each $z \in U$; *this anti-derivative is* $\mathbf{R}$*-linear (or* $\mathbf{H}$*-left-linear when* $v = 2$*):*

(2) $\mathcal{I}_\Upsilon[af(z) + bg(z)] = a\mathcal{I}_\Upsilon f(z) + b\mathcal{I}_\Upsilon g(z)$

for any real constants $a, b \in \mathbf{R}$ *(or* $a, b \in \mathbf{H}$ *for* $v = 2$*) and continuous functions* $f, g \in C^0(U, \mathcal{A}_v)$. *If there is a second anti-derivative* $\mathcal{I}_{\Upsilon,2}f(z)$, *then* $\mathcal{I}_\Upsilon f(z) - \mathcal{I}_{\Upsilon,2}f(z)$ *belongs to the kernel* $ker(\Upsilon)$ *of the operator* Υ.

Proof. Using the multiplication on the marked doubling generator i_{2^v} from the right we have

$$(3) \quad [\sum_{j=0}^{k-1} i_j(\partial g(z)/\partial z_j)]i_{2^v}\psi_j(z) = \sum_{j=0}^{k-1}(\partial g(z)/\partial z_j)i_{j+2^v}\psi_j(z),$$

where $i_j i_{2^v} =: i_{j+2^v}$ for each $0 \le j \le 2^v - 1$, $2 \le v$. On the other hand, $\sum_{j=0}^{k-1} i_j(\partial g(z)/\partial z_j)\psi_j(z) = [\sum_{j=0}^{k-1}(\partial g(z)/\partial z_j)^* i_j^*\psi_j(z)]^*$, since $\psi_j(z)$ is real for each j and z. Therefore, it is sufficient to consider the first-order partial differential operator of the form:

(4) $\Upsilon g(z) = \sum_{j=1}^n (\partial g/\partial z_j)i_j^*\psi_j(z)$

on the $\mathbf{R}$-linear space $C^1(U, \mathcal{A}_v)$ of all continuously differentiable functions $g : U \to \mathcal{A}_v$ by real variables $z_0, ..., z_{2^v-1}$, where $0 < n \le 2^v - 1$. The space of super-differentiable functions is everywhere dense in $C^0(U, \mathcal{A}_v)$ and the line integral has the continuous extension on $C^0(U, \mathcal{A}_v)$

along any continuous rectifiable path in U. Therefore, we take the space of super-differentiable functions and then take the continuous extension of $\mathcal{I}_\Upsilon$ on $C^0(U,\mathcal{A}_v)$ such that

$\lim_l \mathcal{I}_\Upsilon f^l = \mathcal{I}_\Upsilon \lim_l f^l = \mathcal{I}_\Upsilon f$

for a sequence f^l of super-differentiable functions uniformly converging to f on compact sub-domains V in U, where $\mathcal{I}_\Upsilon f^l$ is described below. Each function $\psi_j(z)$ is continuous and each function $\nu_j(z)$ is continuously differentiable on U (see also below), consequently, the integral $\int_\gamma f^l(y)d\nu_j(y)$ is continuously differentiable by $z=\gamma(1)$ (i.e. by each real variable z_k) and their sequence by l uniformly converges on each compact sub-domain V in U. Therefore, from $\Upsilon\mathcal{I}_\Upsilon f^l = f^l$ for each natural number $l\in \mathbf{N}$ we get

$\Upsilon\mathcal{I}_\Upsilon f = \Upsilon\mathcal{I}_\Upsilon \lim_l f^l = \lim_l \Upsilon\mathcal{I}_\Upsilon f^l = \lim_l f^l = f$,

since the sequence $\{\mathcal{I}_\Upsilon f^l(z)|_V :\ l\}$ is fundamental in $C^1(V,\mathcal{A}_v)$ for each compact sub-domain V in U and $\mathcal{I}_\Upsilon f(z)\in C^1(U,\mathcal{A}_v)$.

Consider the left algorithm of a calculation of the line integral over the Cayley-Dickson algebra $\mathcal{A}_v$ (see §3 and references therein). We shall seek an anti-derivative in the form:

$$(5)\quad {}_\Upsilon\int_{{}_0z}^{z} f(y)dy := n^{-1}\sum_{j=1}^{n}(\int_{{}_0z}^{z} f(y)d\nu_j(y))i_j$$

and use the homotopy theorem in the domain U satisfying conditions $1(D1,D2)$ so that γ is a continuous rectifiable path joining points ${}_0z=\gamma(0)$ and $z=\gamma(1)$ (see [22, 23, 20]). Moreover, a branch of the anti-derivative operator $\mathcal{I}_\Upsilon f(z)$ can be chosen such that it can be expressed with the help of a non-commutative line integral.

In view of Theorem 2.11 [27] and §3 we get

(6) $(\partial(\int_{{}_0z}^{z} f(y)d\nu_j(y))/\partial z_k) = (\hat{f}(z).[d\nu_j(z)/dz_k])$

(see also the chain rule over the Cayley-Dickson algebra in [22, 23, 20]).

Next we need some identities in the Cayley-Dickson algebra. Each Cayley-Dickson number has the decomposition: $z=z_0i_0+...+z_{2^v-1}i_{2^v-1}$, where $z_0,...,z_{2^v-1}\in\mathbf{R}$, $z\in\mathcal{A}_v$. To establish the identity

(7) $(ay)z^*+(az)y^* = a2Re(yz^*)$

for any $a,y,z\in\mathcal{A}_v$ it is sufficient to prove it for any three basic generators of the Cayley-Dickson algebra $\mathcal{A}_v$, since the real field $\mathbf{R}$ is its center, while the multiplication in $\mathcal{A}_v$ is distributive $(a+y)z=az+yz$ and $((\alpha a)(\beta y))(\gamma z^*) = (\alpha\beta\gamma)((ay)z^*)$ for all $\alpha,\beta,\gamma\in\mathbf{R}$ and $a,y,z\in\mathcal{A}_v$. If $a=i_0$, then (7) is evident, since $yz^*+zy^* = yz^*+(yz^*)^* = 2Re(yz^*)$. If either $y=i_0$, then $(ay)z^*+(az)y^* = az^*+az = a$

2 Re (z)= a 2 Re (yz*). Analogously for $z=i_0$. For three purely imaginary generators i_p,i_s,i_k consider the minimal Cayley-Dickson algebra $\Phi = alg_{\mathbf{R}}(i_p,i_s,i_k)$ over the real field generated by them. If it is associative, then it is isomorphic with either the complex field $\mathbf{C}$ or the quaternion skew field $\mathbf{H}$, so that $(ay)z^*+(az)y^* = a(yz^*+zy*) = a2Re(yz^*)$. If the algebra Φ is isomorphic with the octonion algebra, then we use Formulas $2(M1,M2)$ for either $a,y\in\mathbf{H}$ and $z=\mathbf{l}$ or $a,z\in\mathbf{H}$ and $y=\mathbf{l}$. This gives (7) in all cases, since the algebra $alg_{\mathbf{R}}(i_p,i_s)$ with two basic generators i_p and i_s is always associative. Particularly, if $y=i_s\ne z=i_k$, then the result is zero.

Using (7) we get more generally, that

(8) $((ay)z^*)b^*+((az)y^*)b^* = (a2Re(yz^*))b^* = (ab^*)2Re(yz^*)$,

consequently,

(9) $((ay)z^*)b^*+((az)y^*)b^*+((by)z^*)a^*+((bz)y^*)a^* = 4Re(ab^*)Re(yz^*)$

for any Cayley-Dickson numbers $a,b,y,z\in\mathcal{A}_v$.

We shall take unknown functions $\nu_j(z)\in\mathcal{A}_v$ as solutions of the system of linear partial differential equations by real variables z_k:

(10) $\partial\nu_j(z)/\partial z_j = 1/\psi_j(z)$ for all $1\le j\le n$ and $z\in U$;

(11) $\psi_k(z)\partial\nu_j(z)/\partial z_k = \psi_j(z)\partial\nu_k(z)/\partial z_j$ for all $1 \le j < k \le n$ and $z \in U$. Each function $\nu_j(z)$ can be written as $\nu_j(z) = \sum_{l=0}^{2^v-1} \nu_{j,l}(z)i_l$ with real-valued components $\nu_{j,l}(z)$. Practically, it is sufficient to consider non-zero $\nu_{j,l}(z)$ for $l = 1, ..., n$. Thus using the generators $i_0, ..., i_{2^v-1}$ the system can be written in the real form. This system has a non-trivial C^1 solution $\nu_j(z)$ for each j (see §12.2 [29], particularly, in the class of super-differentiable functions for super-differentiable $\psi_j(z)$ see also [?, 20]). In System $(10, 11)$ functions ψ_j are real and coordinates are real, consequently, a solution $\{\nu_j(z) :\ j\}$ may be chosen real-valued.

From Identities $3(2, 3)$ and $(6, 9-11)$ we infer that

$$(12)\quad \sum_{j\ne k\ge 1} [(\partial(\int_{0z}^{z} f(y)d\nu_j(y))/\partial z_k)i_j]i_k^*\psi_k(z) =$$

$$\sum_{1\le j<k\le n} \{[(\hat{f}(z).(\partial\nu_j(z)/\partial z_k))i_j]i_k^*\psi_k(z) + [(\hat{f}(z).(\partial\nu_k(z)/\partial z_j))i_k]i_j^*\psi_j(z)\} = 0 \text{ and}$$

$$(13)\quad \sum_{j=1}^{n}[(\partial(\int_{0z}^{z} f(y)d\nu_j(y))/\partial z_j)i_j]i_j^*\psi_j(z) = nf(z),$$

since $\sum_{j=1}^n i_j i_j^* = n$ and n is some fixed natural number for the domain U, $\hat{f}(z).x = f(z)x$ for each real number x, $(zi_j)i_j^* = z$ for each $z \in \mathcal{A}_v$, where $\hat{f}$ is the operator corresponding to $d\kappa(z)/dz$, when f is in the z-representation μ (see the notation in §3). Using Formulas $(4, 5, 12, 13)$ we get Formula (1).

From the identity $\int_\gamma a\mu dz = a\int_\gamma \mu dz$ for a suitable branch of the line integral given by the left algorithm and for each non-trivial phrase μ and constants $a, b \in \mathbf{R}$ for $v \ge 3$ or $a, b \in \mathbf{H}$ for $v = 2$ (see the rules in [22, 23, ?, 24]) we get Formula (2).

Since $\Upsilon(\mathcal{I}_\Upsilon f(z) - \mathcal{I}_{\Upsilon,2} f(z)) = 0$, the difference $(\mathcal{I}_\Upsilon f(z) - \mathcal{I}_{\Upsilon,2} f(z))$ belongs to the kernel $ker(\Upsilon) = \Upsilon^{-1}(0)$, where $\Upsilon : C^1(U, \mathcal{A}_v) \to C^0(U, \mathcal{A}_v)$.

4.1. Example. If ψ_j depends only on z_j for each j, there exists a C^1 differentiable change of variables $\zeta = \zeta(z)$ so that $\partial g(\zeta)/\partial\zeta_j = (\partial g(z)/\partial z_j)\psi_j(z)$ for each differentiable function $g : U \to \mathcal{A}_v$ by real variables $z_0, ..., z_{2^v-1}$ on U, where

(1) $(\partial z_k/\partial\zeta_j) = \delta_{j,k}\psi_k(z)$

for all j and k, $\delta_{j,j} = 1$, while $\delta_{j,k} = 0$ for each $j \ne k$. We take new functions ${}_jg$ satisfying the equation:

(2) ${}_jg(i_j^*z) = g(z)$ for each $z \in U$ and all j. We also put

(3) $\eta_j(z) = i_j^*z$.

The multiplication of generators implies that $i_j^*(i_jz) = z$ for all $j = 0, ..., 2^v - 1$ and $z \in \mathcal{A}_v$. Therefore, from Equations $(1, 2)$ we deduce that

(4) $(dg(z)/dz).i_j = (d_jg(\eta_j)/d\eta_j).[(d\eta_j/dz).i_j] = (d_jg(\eta_j)/d\eta_j).1 = (d_kg(\eta_k)/d\eta_k).[i_k^*i_j]$,

since $(d\eta_j/dz).i_j = i_j^*i_j = 1$ for each j. Then we take the integral

$$(5)\quad {}_\Upsilon\int_{0z}^{z} g(y)dy := n^{-1}\sum_{j=1}^{n}\int_{0z}^{z} {}_jg(\eta_j(y))i_jd\eta_j(y),$$

since $\int_{0z}^{z} {}_jg(\eta_j(y))i_jd\eta_j(y) = (\int_{0z}^{z} {}_jg(\eta_j(y))d\eta_j(y))i_j$.

Mention that generally $\Upsilon(f(z)b)$ may be not equal to $(\Upsilon f(z))b$ for a constant $b \in \mathcal{A}_v\setminus\mathbf{R}$ and a function $f \in C^1(U, \mathcal{A}_v)$ with $v \ge 2$, since the Cayley-Dickson algebra is non-commutative.

This theorem can be generalized in the following manner encompassing wider class of partial differential operators of the first order over Cayley-Dickson algebras.

5. Theorem. *Suppose that the first order partial differential operator* Υ *is given by the formula*

(1) $\Upsilon f = \sum_{j=0}^{n} (\partial f/\partial z_j)\phi_j^*(z)$,

where $\phi_j(z) \ne \{0\}$ *for each* $z \in U$ *and* $\phi_j(z) \in C^0(U, \mathcal{A}_v)$ *for each* $j = 0, ..., n$ *such that* $Re(\phi_j(z)\phi_k^*(z)) = 0$ *for each* $z \in U$ *and each* $0 \le j \ne k \le n$*, where a domain* U *satisfies Conditions 1*$(D1, D2)$*,* ${}_0z$ *is a marked point in* U*,* $1 < n < 2^v$*,* $2 \le v$*. Suppose also that the system* $\{\phi_0(z), ..., \phi_n(z)\}$ *is for* $n = 2^v - 1$*, or can be completed by Cayley-Dickson numbers* $\phi_{n+1}(z), ..., \phi_{2^v-1}(z)$*, such that* (α) $alg_{\mathbf{R}}\{\phi_j(z), \phi_k(z), \phi_l(z)\}$ *is alternative for all* $0 \le j, k, l \le 2^v - 1$ *and* (β) $alg_{\mathbf{R}}\{\phi_0(z), ..., \phi_{2^v-1}(z)\} = \mathcal{A}_v$ *for each* $z \in U$*. Then a line integral* $\mathcal{I}_\Upsilon : C^0(U, \mathcal{A}_v) \to C^1(U, \mathcal{A}_v)$*,* $\mathcal{I}_\Upsilon f(z) := {}_\Upsilon\int_{{}_0z}^{z} f(y)dy$ *on* $C^0(U, \mathcal{A}_v)$ *exists so that*

(2) $\Upsilon \mathcal{I}_\Upsilon f(z) = f(z)$

for each $z \in U$*; this anti-derivative is* **R***-linear (or* **H***-left-linear when* $v = 2$*). If there is a second anti-derivative* $\mathcal{I}_{\Upsilon,2} f(z)$*, then* $\mathcal{I}_\Upsilon f(z) - \mathcal{I}_{\Upsilon,2} f(z)$ *belongs to the kernel* $ker(\Upsilon)$ *of the operator* Υ*.*

Proof. We shall demonstrate that a branch of the anti-derivative operator $\mathcal{I}_\Upsilon f(z)$ can be chosen such that it can be expressed with the help of a non-commutative line integral from §3. Using the technique of §4 we can consider the case of purely imaginary $\phi_j(z)$ for all $z \in U$ and $j = 0, ..., n$. We seek an anti-derivative operator in the form:

$$(3) \quad {}_\Upsilon\int f(z)dz = (n+1)^{-1} \sum_{j=0}^{n} \int_{{}_0z}^{z} q(z)d\nu_j(z).$$

For finding unknown functions q and ν_j, $j = 0, ..., n$ we impose the following conditions:

(4) $(\hat{q}(z).[\partial\nu_j(z)/\partial z_j])\phi_j^*(z) = f(z)$ for each $j = 0, ..., n$ and

(5) $(\hat{q}(z).[\partial\nu_j(z)/\partial z_k])\phi_k^*(z) + (\hat{q}(z).[\partial\nu_k(z)/\partial z_j])\phi_j^*(z) = 0$ for all $0 \le j < k \le n$.

As in §4 it is sufficient to consider the case of a locally analytic (super-differentiable) function f using the limit transition. The function f is given on U and it defines the operator $\hat{f}$ on U, i.e. its phrase $\hat{\mu}$ is prescribed by the left algorithm for a given phrase μ of f (see [22, 23, ?, 24]). The operator $\hat{q}$ means that a function g and a phrase κ of g exist such that

$\hat{q}(z) = dg(z)/dz$, $\hat{q}(z).1 = q(z)$ for each $z \in U$.

In accordance with the conditions of this theorem the algebra $alg_{\mathbf{R}}(\phi_j(z), \phi_k(z))$ is alternative for all $0 \le j \le k \le n$ and $z \in U$. Therefore, due to Condition (β) Equations $(4, 5)$ take the form:

(6) $(dg(z)/dz).[\partial\nu_j(z)/\partial z_j] = f(z)(1/\phi_j^*(z))$ for each $j = 0, ..., n$ and

(7) $((dg(z)/dz).[\partial\nu_j(z)/\partial z_k])\phi_k^*(z) + ((dg(z)/dz).[\partial\nu_k(z)/\partial z_j])\phi_j^*(z) = 0$ for all $0 \le j < k \le n$.

Solutions of this system exist (see [?, 20]). To be more concrete we impose additional relations:

(8) $\partial\nu_j(z)/\partial z_j = \phi_j(z)$ for all $j = 0, ..., n$ and $z \in U$,

consequently, the system of partial differential equations (6) becomes:

(9) $(dg(z)/dz).\phi_j(z) = f(z)(1/\phi_j^*(z))$ for each $j = 0, ..., n$,

since $alg_{\mathbf{R}}\{\phi_j(z), \phi_k(z), \phi_l(z)\}$ is alternative for all $0 \le j, k, l \le 2^v - 1$ and $alg_{\mathbf{R}}\{\phi_0(z), ..., \phi_{2^v-1}(z)\} = \mathcal{A}_v$ for each $z \in U$ so that each Cayley-Dickson number $\xi \in \mathcal{A}_v$ has the decomposition $\xi = \xi_0\phi_0(z) + ... + \xi_{2^v-1}\phi_{2^v-1}(z)$ with real coefficients $\xi_0, ..., \xi_{2^v-1} \in \mathbf{R}$.

Solving the latter system (9) one gets the function $g(z)$ on U. Substituting the known function g in System $(6, 7)$ one gets a C^1 solution $\nu_0(z)$,...,$\nu_n(z)$ on U; or a super-differentiable solution, when $\phi_j(z)$ for each j and $f(z)$ are super-differentiable on U. Mention that the function g depends **R**-linearly on f, since the system of equations which was considered above is linear by f and g. Thus the operator $\hat{q}$ depends **R**-linearly on f.

Using Formulas $(4,5)$ and $4(6,9)$ we deduce that

$$(10)\quad \sum_{j\ne k\ge 0}[\partial(\int_{{}_0z}^z q(y)d\nu_j(y))/\partial z_k]\phi_k^*(z)=$$

$$\sum_{0\le j<k\le n}\{[\hat{q}(z).(\partial\nu_j(z)/\partial z_k)]\phi_k^*(z)+[\hat{q}(z).(\partial\nu_k(z)/\partial z_j)]\phi_j^*(z)\}=0 \text{ and}$$

$$(11)\quad \sum_{j=0}^n[\partial(\int_{{}_0z}^z q(y)d\nu_j(y))/\partial z_j]\phi_j^*(z)=\sum_{j=0}^n[\hat{q}(z).(\partial\nu_j(z)/\partial z_j)]\phi_j^*(z)=(n+1)f(z),$$

since $Re(\phi_j(z)\phi_k^*(z))=0$ for each $z\in U$ and each $0\le j\ne k\le n$.

The rest of the proof is analogous to that of Theorem 4.

6. Corollary. *Let suppositions of Theorem 5 be satisfied so that $\phi_j(z)=\omega(z;i_j)\psi_j(z)$ for each $z\in U$, where ω is an $\mathbf{R}$-linear automorphism $\omega:\mathcal{A}_v\to\mathcal{A}_v$ mapping the standard base of generators $\{i_j\}$ into a base of generators $\{\omega(z;i_j): j=0,...,2^v-1\}$, $|\omega(z;i_j)|=1$, where $\psi_j(z)$ satisfies conditions of theorem 4 for each $j=0,...,n$. Then the first order differential operator 5(1) has an anti-derivative $\mathcal{I}_\Upsilon$ on $C^0(U,\mathcal{A}_v)$. Two anti-derivatives of Theorems 4 and 5 under these suppositions are related with the help of the automorphism ω.*

Proof. This follows immediately from Theorem 5. It remains to find a relation between two anti-derivatives for two different partial differential operators:

(1) $\Upsilon_\omega f=\sum_{j=0}^n(\partial f/\partial z_j)\phi_j^*(z)$

and Υ given by equation 2(13).

For each Cayley-Dickson number $z=z_0i_0+...+z_{2^v-1}i_{2^v-1}\in\mathcal{A}_v$ its image is $\omega(y;z)=z_0N_0+z_1N_1+...+z_{2^v-1}N_{2^v-1}$, consequently, $\omega(y;z^*)=[\omega(y;z)]^*$, where $z_j\in\mathbf{R}$, $N_j=N_j(y):=\omega(y;i_j)$ for each j. Particularly, $N_0=i_0$, since $i_0i_j=i_j$ and $\omega(y;i_j)=\omega(y;i_0i_j)=\omega(y;i_0)\omega(y;i_j)$ for each j and y. Therefore, $\omega(y;x)=x$ for each real number $x\in\mathbf{R}$, since $\omega(y;1)=1$ and the mapping $\omega(y;*)$ is $\mathbf{R}$-linear by the second argument, $1=i_0$. Therefore, applying the automorphism ω we deduce that

(2) $\Upsilon_\omega f(z)=\omega(z;\Upsilon s(z))$,

where $\omega(z;s(z))=f(z)$ for each $z\in U$, that is $s(z)=\omega_2^{-1}(z;f(z))$, $\omega_2^{-1}(z;*)$ denotes the inverse automorphism by the second argument for $z\in U$. Let us take the function $f(z)={}_{\Upsilon_\omega}\int_{{}_0z}^z g(y)dy$, where $g(z)$ is a continuous function. Then $\Upsilon_\omega f(z)=g(z)$ for each $z\in U$ and from (2) and 5(1,2) one gets

(3) $\omega_2^{-1}(z;g(z))=\Upsilon\omega_2^{-1}(z;{}_{\Upsilon_\omega}\int_{{}_0z}^z g(y)dy)=\Upsilon{}_\Upsilon\int_{{}_0z}^z\omega_2^{-1}(y;g(y))dy$, consequently, applying ${}_\Upsilon\int$ and $\omega(z;*)$ one also gets

(4) ${}_{\Upsilon_\omega}\int_{{}_0z}^z g(y)dy=\omega(z;{}_\Upsilon\int_{{}_0z}^z\omega_2^{-1}(y;g(y))dy$

for each continuous function g on U.

6.1. Remark. If in Theorem 5 drop Conditions (α,β), then partial differential equations $5(4,5)$ will be hard to resolve.

To specify the anti-derivative operator $\mathcal{I}_\Upsilon$ in Theorems 4 and 5 more concretely it is possible to choose a family of rectifiable continuous paths (or C^1 paths) $\{\gamma^z: z\in U\}$ such that $\gamma^z(0)={}_0z$ and $\gamma^z(1)=z$ and $\lim_{z\to y}\sup_{\tau\in[0,1]}|\gamma^z(\tau)-\gamma^y(\tau)|=0$.

Another more rigorous procedure is in providing a foliation of a domain U by locally rectifiable paths $\{\gamma^\alpha: \alpha\in\Lambda\}$, where Λ is a set. We take for definiteness a canonical closed domain U in $\mathcal{A}_v$ satisfying Conditions $1(D1,D2)$.

A path $\gamma:<a,b>\to U$ is called locally rectifiable, if it is rectifiable on each compact segment $[c,e]\subset<a,b>$, where $<a,b>=[a,b]:=\{t\in\mathbf{R}: a\le t\le b\}$ or $<a,b>=[a,b):=\{t\in\mathbf{R}: a\le t<b\}$ or $<a,b>=(a,b]:=\{t\in\mathbf{R}: a<t\le b\}$ or $<a,b>=(a,b):=\{t\in\mathbf{R}: a<t<b\}$.

A domain U is called foliated by rectifiable paths $\{\gamma^\alpha : \alpha \in \Lambda\}$ if $\gamma :< a_\alpha, b_\alpha >\to U$ for each α and it satisfies the following three conditions:

$(F1)$ $\bigcup_{\alpha\in\Lambda} \gamma^\alpha(< a_\alpha, b_\alpha >) = U$ and

$(F2)$ $\gamma^\alpha(< a_\alpha, b_\alpha >) \cap \gamma^\beta(< a_\beta, b_\beta >) = \emptyset$ for each $\alpha \ne \beta \in \Lambda$.

Moreover, if the boundary $\partial U = cl(U) \setminus Int(U)$ of the domain U is non-void then

$(F3)$ $\partial U = (\bigcup_{\alpha\in\Lambda_1} \gamma^\alpha(a_\alpha)) \cup (\bigcup_{\beta\in\Lambda_2} \gamma^\beta(b_\beta))$,

where $\Lambda_1 = \{\alpha \in \Lambda :< a_\alpha, b_\beta >= [a_\alpha, b_\beta >\}$, $\Lambda_2 = \{\alpha \in \Lambda :< a_\alpha, b_\beta >=< a_\alpha, b_\beta]\}$. For the canonical closed subset U we have $cl(U) = U = cl(Int(U))$, where $cl(U)$ denotes the closure of U in $\mathcal{A}_v$ and $Int(U)$ denotes the interior of U in $\mathcal{A}_v$. For convenience one can choose C^1 foliation, i.e. each γ^α is of class C^1. When U is with non-void boundary we choose a foliation family such that $\bigcup_{\alpha\in\Lambda} \gamma(a_\alpha) = \partial U_1$, where a set ∂U_1 is open in the boundary ∂U and so that $w|_{\partial U_1}$ would be a sufficient initial condition to characterize a unique branch of an anti-derivative $w = \mathcal{I}_\Upsilon f$.

When $\partial U \ne \emptyset$ a marked point ${}_0z$ can be chosen on the boundary ∂U and each point on the boundary can be joined by a rectifiable path in U with ${}_0z$. This foliation is justified by the formula:

$\int_\gamma f(z)d\nu(z) = \int_{\gamma^1} f(z)d\nu(z) + \int_{\gamma^2} f(z)d\nu(z)$

for each continuous function f on U and each function ν of bounded variation on U, for any rectifiable paths $\gamma^1 : [a_1, b_1] \to U$ and $\gamma^2 : [a_2, b_2] \to U$ so that $a = a_1 < b_1 = a_2 < b_2 = b$ while $\gamma : [a, b] \to U$ is given piecewise as $\gamma(t) = \gamma^1(t)$ for each $t \in [a_1, b_1]$ and $\gamma(t) = \gamma^2(t)$ for each $t \in [a_2, b_2]$. Thus instead of $\int_{{}_0z}^z f(z)d\nu(z)$, i.e. $\int_\gamma f(z)d\nu(z)$ with $\gamma(a) = {}_0z$ and $\gamma(b) = z$, we take $\int_{\gamma^\alpha|_{[c,e]}} f(z)d\nu(z)$ for any $[c, e] \subset< a_\alpha, b_\alpha >$. If $\lim_{c\to a_\alpha, e\to b_\alpha} \int_{\gamma^\alpha|_{[c,e]}} f(z)d\nu(z)$ converges we denote it by $\int_{\gamma^\alpha} f(z)d\nu(z)$ and take instead of the family $\{\int_{\gamma^\alpha|_{[c,e]}} f(z)d\nu(z) : [c, e] \subset< a_\alpha, b_\beta >\}$. Therefore, a branch of the anti-derivation operator prescribed by the family $\{(\int_{\gamma^\alpha} \sum_j q(y)d\nu_j(y)) : \alpha \in \Lambda\}$ or $\{(\int_{\gamma^\alpha|_{[c,e]}} \sum_j q(y)d\nu_j(y)) : \alpha \in \Lambda;\ [c, e] \subset< a_\alpha, b_\beta >\}$ is defined up to a function defined on the boundary ∂U when it is non-void or by convergence to a definite limit at infinity along paths, when U is unbounded in certain directions $\mathbf{R}\eta$ in the Cayley-Dickson algebra $\mathcal{A}_v$, $\eta \in \mathcal{A}_v$.

Clearly, boundary conditions are necessary for specifying a concrete solution or a branch of an anti-derivative, since in the definition of the line integral $\int_\gamma f(z)d\nu(z)$ the operator $\hat{f}$ is restricted to the condition $\hat{f}(z).1 = f(z)$ for each $z \in U$ so it is defined up to a function of $2^v - 1$ independent real variables (see also §3). In accordance with the formulas of §§4 and 5 the anti-derivation operators are defined up to functions of $2^v - 1$ real variables after a suitable change of variables. For example, $\sum_{j=0}^n (\partial g(z)/\partial z_j)i_j^* = 0$ for $g(z) = nz_0 + z_1i_1^* + ... + z_ni_n^*$, or $\sum_{j=0}^n (\partial q(z)/\partial z_j)i_j^* = 0$ on the plane $z_0 - z_1 - ... - z_n = 0$ for $q(z) = z_0^2 + z_1^2i_1^* + ... + z_n^2i_n^*$. These functions can be written in the z-representation due to Formulas $1(1 - 3)$.

For concrete domains some concrete boundary conditions can be chosen (see also below). Mention, that a minimal necessary correct boundary conditions may be not on the entire boundary, but on its part. Otherwise, they may be on some hyper-surface S in U of real dimension $2^v - 1$ depending on the domain, for example, for an infinite cylinder C in both directions along its axis with S being the intersection of C with a hyper-plane perpendicular to its axis.

Mention that the homotopy theorem for domains satisfying Conditions $1(D1, D2)$ is accomplished for super-differentiable functions on U (see [23, 22]), but for a continuous function f on U it may certainly be not true. This is caused by several reasons. If a family of locally analytic functions f^n converges to f uniformly on a compact sub-domain V in U a radius r_x^n of local convergence of a power series of f^n in a neighborhood of a point $x \in V$ may tend to zero with n tending to the infinity. Phrases μ^n in the z-representation corresponding to f^n

may be inconsistent on the intersection $V_x \cap V_y$ of open neighborhoods V_x and V_y of different points $x, y \in V$, when $V_x \cap V_y \neq \emptyset$. Functions f^n or their phrases μ^n may be with branching points in the domain U. That is functions f^n accomplishing the approximation of f may have several branches on U and a slit of U by a $2^v - 1$ dimensional sub-manifold S^n over $\mathbf{R}$ may be necessary to specify branches of f^n. But the family S^n with different n may be inconsistent and S^n may depend of n.

For super-differentiable functions f^n operator valued functions $\hat{f}^n$ are also super-differentiable. If f is only continuous non super-differentiable function on the domain U, then the operator valued function $\hat{f}$ is defined only in the sense of distributions $[\hat{f}, \gamma; \nu) = \int_\gamma f(z) d\nu(z)$ for any rectifiable path γ in U and each function ν of bounded variation on U. Moreover, the homotopy theorem may be non true for generalized functions (see below).

7. Particular case. We consider a phrase ν which can be presented as

$(P3)$ $\nu = \rho(\mu)$ with a right $\mathcal{A}_v$-linear (super)-differentiable phrase μ and a projection operator ρ being an $\mathbf{R}$-linear combination of the projection operators π_j. Particularly, ρ may be the identity operator or one of the π_j.

For any z-differentiable phrase ψ and constants $a, b \in \mathcal{A}_v$ we have $\int_\gamma a(\psi(z)b)dz = a((\int_\gamma \psi(z)dz)b)$ and $\int_\gamma (a\psi(z))bdz = (a(\int_\gamma \psi(z)dz))b$. Then in view of the homotopy theorem [23, 22] Equation 3(2) implies for any such $\nu = \rho(\mu)$ that

$$(1) \quad \int_\gamma \Upsilon(\nu(z))dz = \rho(\int_\gamma [d\mu(z)/dz].\{\sum_j [(dz/dz).i_j](i_j^* \psi_j(z)dz)\}) =$$

$$\rho(\int_\gamma [d\mu(z)/dz].\sum_j \{i_j(i_j^*\psi_j(z))dz\}) = \rho(\int_\gamma [d\mu(z)/dz].[a(z)dz])$$

$$= \rho(\mu(z_a(\beta))) - \rho(\mu(z_a(\alpha))) = \nu(z_a(\beta)) - \nu(z_a(\alpha)),$$

since each $\psi_j(z)$ is the $\mathcal{A}_v$-valued function, where

(2) $z_a(x) = \int_\alpha^x a(t)dt + \phi_a(x', \alpha')$,

(3) $a(z) := \sum_{j=0}^{2^v-1} \psi_j(z)$, $\gamma(0) = \alpha$, $\gamma(1) = \beta$. In particular, if each function ψ_j is identically constant, then

(4) $\int_\gamma \sum_j [(dz/dz).i_j][(i_j^*\psi_j(z)dz] = t\beta - t\alpha - t\phi_1(\beta', \alpha')$,
where $t = \sum_j \psi_j$.

For non right $\mathcal{A}_v$-linear z-differentiable phrase μ Formulas $(1-3)$ may already be not valid. Certainly common line integrals of z-differentiable phrases (functions) can be calculated by the general algorithms (see [23, 22, 24, 27]). A result of the line integration along a rectifiable path γ in the domain U we denote as the composition of two functions

$$(5) \quad \sum_j \int_\gamma [(d\mu(z)/dz).i_j][i_j^*\psi_j(z)]dz_0 = \int_\gamma (d\nu(\xi)/d\xi).d\xi$$

$$= \lambda(\xi(\beta)) - \lambda(\xi(\alpha)),$$

where λ and ξ are two z-differentiable functions on their domains, $\gamma(0) = \alpha$, $\gamma(1) = \beta$. Frequently one can use a Cayley-Dickson subalgebra $\mathcal{G}$ isomorphic with either the quaternion skew field $\mathbf{H}$ or the octonion algebra $\mathbf{O}$ so that $\gamma(1) - \gamma(0) \in \mathcal{G}$ and use the homotopy theorem. On the other hand, each rectifiable continuous path γ in the domain U in the Cayley-Dickson algebra $\mathcal{A}_v$ can be presented as a uniform limit of rectifiable continuous paths γ^n in U composed of segments parallel to axes $\mathbf{R}i_k$, $k = 0, ..., 2^v - 1$. Therefore,

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \int_{\gamma^n} f(z)dz$$

for any continuous function on U (see [23, 22]). The functions λ and ξ depend on ψ so in more details we denote them by $\lambda = \lambda_\psi$ and $\xi = \xi_\psi$.

Thus the general integral of Equation 2(12) is:

$$(6) \quad \lambda_\psi(\xi_\psi(x)) = -\phi_{\lambda'}(Im\ \xi_\psi(x)) + \int_\alpha^x g_1(z)dz + \phi_{g_1}(x'),$$

where $Im(z) := z - Re(z)$, $Re(z) := (z + z^*)/2$. The term $\phi_{\lambda'}(Im(\xi))$ takes into account the non-commutativity for $2 \le v$ and non-associativity for $3 \le v$ of the Cayley-Dickson algebra $\mathcal{A}_v$, since its center is the real field $\mathbf{R} = Z(\mathcal{A}_v)$ for any $v \ge 2$. There is the bijective correspondence between $\lambda_\psi(\xi_\psi)$ and f which will be specified below.

8. Transformation of the first order partial differential operator over the Cayley-Dickson algebras.

To simplify the operator Υ and its particular variant σ one can use a change of variables. We consider this operator in the form:

(1) $\Upsilon f = \sum_{j=0}^{2^v-1} (\partial f/\partial z_j)\eta_j(z)$,
with either $\eta_j(z) = i_j^*\psi_j(z)$ or $\eta_j(z) = \phi_j^*(z) \in \mathcal{A}_v$ for each j (see Theorems 4 and 5 above). For it we seek the change of variables $x = x(z)$ so that

(2) $\sum_{j=0}^{2^v-1} (\partial x_l/\partial z_j)\omega_j(z) = t_l$,
where $t_l \in \mathcal{A}_v$ is a constant for each l, for η_j not being identically zero, while ω_j is chosen arbitrarily also z-differentiable so that the resulting matrix Ω will not be degenerate, i.e. its rows are real-independent as vectors (see below). Certainly $(\partial x_l/\partial z_j) \in \mathbf{R}$ are real partial derivatives, since x_l and z_j are real coordinates. We suppose that the functions $\eta_j(z)$ are linearly independent over the real field for each z in the domain U. Using the standard basis of generators $\{i_j :\ j = 0, ..., 2^v - 1\}$ of the Cayley-Dickson algebra $\mathcal{A}_v$ and the decompositions $\omega_j = \sum_k \omega_{j,k} i_k$ and $t_j = \sum_k t_{j,k} i_k$ with real elements $\omega_{j,k}$ and $t_{j,k}$ for all j and k we rewrite System (2) in the matrix form:

(3) $(\partial x_l/\partial z_j)_{l,j=0,...,2^v-1}\Omega = T$,
where $\Omega = (\omega_{j,k})_{j,k=0,...,2^v-1}$, $T = (t_{j,k})_{j,k=0,...,2^v-1}$. Suppose that the functions $\omega_j(z)$ are arranged into the family $\{\omega_j : j = 0, ..., 2^v - 1\}$ as above and are such that the matrix $\Omega(z)$ is non-degenerate for all z in the domain U. For example, this is always the case, when $|\omega_j(z)| > 0$ and $Re[\omega_j(z)\omega_k(z)^*] = 0$ for each $j \ne k$ for each $z \in U$. Here particularly $\omega_j(z) = \eta_j(z)$ can also be taken for all $j = 0, ..., 2^v - 1$ and $z \in U$. Therefore, Equality (3) becomes equivalent to

(4) $(\partial x_l/\partial z_j)_{l,j=0,...,2^v-1} = T\Omega^{-1}$.
We take the real matrix T of the same rank as the real matrix $(\omega_{j,k})_{j,k=0,...,2^v-1}$. Thus (4) is the linear system of partial differential equations of the first order over the real field. It can be solved by the standard methods [29].

We remind how each linear partial differential equation (3) or (4) can be resolved. Write it in the form:

(5) $X_1(x_1, ..., x_n, u)\partial u/\partial x_1 + ... + X_n(x_1, ..., x_n, u)\partial u/\partial x_n = R(x_1, ..., x_n, u)$
with u and $x_1, ..., x_n$ here instead of x_l and $z_0, ..., z_{2^v-1}$ in (3) seeking simultaneously suitable R corresponding to $t_{l,k}$. A function $u = u(x_1, ..., x_n)$ defined and continuous with its partial derivatives $\partial u/\partial x_1$, ...,$\partial u/\partial x_n$ in some domain V of variables $x_1, ..., x_n$ in $\mathbf{R}^n$ making (5) the identity is called a solution of this linear equation. If $R = 0$ identically, then the equation is called homogeneous. A solution $u = const$ of the homogeneous equation

(6) $X_1(x_1, ..., x_n, u)\partial u/\partial x_1 + ... + X_n(x_1, ..., x_n, u)\partial u/\partial x_n = 0$
is called trivial. Then one composes the equations:

(7) $dx_1/X_1(x) = dx_2/X_2(x) = ... = dx_n/X_n(x)$,
where $x = (x_1, ..., x_n)$. This system is called the system of ordinary differential equations in the symmetric form corresponding to the homogeneous linear equation in partial derivatives.

It is supposed that the coefficients $X_1, ..., X_n$ are defined and continuous together with their first order partial derivatives by $x_1, ..., x_n$ and that $X_1, ..., X_n$ are not simultaneously zero in a neighborhood of some point x^0. Such point x^0 is called non singular. For example when the function X_n is non-zero System (7) can be written as:

(8) $dx_1/dx_n = X_1/X_n$,...,$dx_{n-1}/dx_n = X_{n-1}/X_n$.

This system satisfies conditions of the theorem about an existence of integrals of the normal system. A system of n differential equations

(9) $dy_k/dx = f_k(x, y_1, ..., y_n)$, $k = 1, ..., n$,

is called normal of the n-th order. It is called linear if all functions f_k depend linearly on $y_1, ..., y_n$. Any family of functions $y_1, ..., y_n$ satisfying (9) in some interval (a, b) is called its solution. A function $g(x, y_1, ..., y_n)$ different from a constant identically and differentiable in a domain D and such that its partial derivatives $\partial g/\partial y_1$,...,$\partial/\partial y_n$ are not simultaneously zero in D is called an integral of System (9) in D if the complete differential $dg = (\partial g/\partial x)dx + (\partial g/\partial y_1)dy_1 + ... + (\partial g/\partial y_n)dy_n$ becomes identically zero, when the differentials dy_k are substituted on their values from (9), that is $(\partial g(x, y)/\partial x) + (\partial g/\partial y_1)f_1(x, y) + ... + (\partial g(x, y)/\partial y_n)f_n(x, y) = 0$ for each $(x, y) \in D$, where $y = (y_1, ..., y_n)$. The equality $g(x, y) = const$ is called the first integral of System (9).

It is supposed that each function $f_k(x, y)$ is continuous on D and satisfies the Lipschitz conditions by variables $y_1, ..., y_n$:

(L) $|f_k(x, y) - f_k(x, z)| \le C_k|y - z|$

for all (x, y) and $(x, z) \in D$, where C_k are constants. Then System (9) has exactly n independent integrals in some neighborhood D^0 of a marked point (x^0, y^0) in D, when the Jacobian $\partial(g_1, ..., g_n)/\partial(y_1, ..., y_n)$ is not zero on D^0 (see Section 5.3.3 [29]).

In accordance with Theorem 12.1,2 [29] each integral of System (7) is a non-trivial solution of Equation (6) and vice versa each non-trivial solution of Equation (6) is an integral of (7). If $g_1(x_1, ..., x_n)$,...,$g_{n-1}(x_1, ..., x_n)$ are independent integrals of (7), then the function

(10) $u = \Phi(g_1, ..., g_{n-1})$,

where Φ is an arbitrary function continuously differentiable by $g_1, ..., g_{n-1}$, is the solution of (6). Formula (10) is called the general solution of Equation (6).

To the non-homogeneous Equation (5) the system

(11) $dx_1/X_1 = ... = dx_n/X_n = du/R$

is posed. System (11) gives n independent integrals $g_1, ..., g_n$ and the general solution

(12) $\Phi(g_1(x_1, ..., x_n, u), ..., g_n(x_1, ..., x_n, u)) = 0$

of (5), where Φ is any continuously differentiable function by $g_1, ..., g_n$. If the latter equation is possible to resolve relative to u this gives the solution of (5) in the explicit form $u = u(x_1, ..., x_n)$ which generally depends on Φ and $g_1, ..., g_n$. Therefore, Formula (12) for different R and u and X_j corresponding to $t_{l,k}$ and x_l and $\omega_{j,k}$ respectively can be satisfied in (3) or (4), the variables x_j are used in (12) instead of z_j in $(3, 4)$, where $k = 0, ..., 2^v - 1$.

Thus after the change of the variables the operator Υ takes the form:

(13) $\Upsilon f = \sum_{j=0}^{2^v-1}(\partial f/\partial x_j)t_j$

with constants $t_j \in \mathcal{A}_v$. Undoubtedly, also the operator Υ with $j = 0, ..., n$, $2^{v-1} \le n \le 2^v - 1$ instead of $2^v - 1$ can also be reduced to the form $\Upsilon f = \sum_{j=0}^{n}(\partial f/\partial x_j)t_j$, when the rank is $rank(\omega_{j,k}) = n + 1$ in a basis of generators $N_0, ..., N_n$, where N_0,...,N_{2^v-1} is a generator basis of the Cayley-Dickson algebra $\mathcal{A}_v$. Particularly, if the rank is $rank(\omega_{j,k}) = m \le 2^v$ and T has the unit upper left $m \times m$ block and zeros outside it, then $t_j = N_j$ for each $j = 0, ..., m - 1$ can be chosen.

One can mention that direct algorithms of Theorems 4 and 5 may be simpler for finding the anti-derivative operator $\mathcal{I}_\Upsilon$, than this preliminary transformation of the partial differential operator Υ to the standard form (13).

9. Definitions.

Let X and Y be two $\mathbf{R}$ linear normed spaces which are also left and right $\mathcal{A}_r$ modules, where $1 \le r$. Let Y be complete relative to its norm. We put $X^{\otimes k} := X \otimes_{\mathbf{R}} ... \otimes_{\mathbf{R}} X$ is the k times ordered tensor product over $\mathbf{R}$ of X. By $L_{q,k}(X^{\otimes k}, Y)$ we denote a family of all continuous k times $\mathbf{R}$ poly-linear and $\mathcal{A}_r$ additive operators from $X^{\otimes k}$ into Y. Then $L_{q,k}(X^{\otimes k}, Y)$ is also a normed $\mathbf{R}$ linear and left and right $\mathcal{A}_r$ module complete relative to its norm. In particular, $L_{q,1}(X, Y)$ is denoted also by $L_q(X, Y)$.

We present X as the direct sum $X = X_0 i_0 \oplus ... \oplus X_{2^r-1} i_{2^r-1}$, where X_0,...,X_{2^r-1} are pairwise isomorphic real normed spaces. If $A \in L_q(X, Y)$ and $A(xb) = (Ax)b$ or $A(bx) = b(Ax)$ for each $x \in X_0$ and $b \in \mathcal{A}_r$, then an operator A we call right or left $\mathcal{A}_r$-linear respectively.

An $\mathbf{R}$ linear space of left (or right) k times $\mathcal{A}_r$ poly-linear operators is denoted by $L_{l,k}(X^{\otimes k}, Y)$ (or $L_{r,k}(X^{\otimes k}, Y)$ respectively).

As usually a support of a function $g : S \to \mathcal{A}_r$ on a topological space S is by the definition $supp(g) = cl\{t \in S : \ g(t) \ne 0\}$, where the closure is taken in S.

We consider a space of test function $\mathcal{D} := \mathcal{D}(\mathbf{R}^n, Y)$ consisting of all infinite differentiable functions $f : \mathbf{R}^n \to Y$ on $\mathbf{R}^n$ with compact supports. A sequence of functions $f_n \in \mathcal{D}$ tends to zero, if all f_n are zero outside some compact subset K in the Euclidean space $\mathbf{R}^n$, while on it for each $k = 0, 1, 2, ...$ the sequence $\{f_n^{(k)} : \ n \in \mathbf{N}\}$ converges to zero uniformly. Here as usually $f^{(k)}(t)$ denotes the k-th derivative of f, which is a k times $\mathbf{R}$ poly-linear symmetric operator from $(\mathbf{R}^n)^{\otimes k}$ to Y, that is $f^{(k)}(t).(h_1, ..., h_k) = f^{(k)}(t).(h_{\sigma(1)}, ..., h_{\sigma(k)}) \in Y$ for each $h_1, ..., h_k \in \mathbf{R}^n$ and every transposition $\sigma : \{1, ..., k\} \to \{1, ..., k\}$, σ is an element of the symmetric group S_k, $t \in \mathbf{R}^n$. For convenience one puts $f^{(0)} = f$. In particular, $f^{(k)}(t).(e_{j_1}, ..., e_{j_k}) = \partial^k f(t)/\partial t_{j_1} ... \partial t_{j_k}$ for all $1 \le j_1, ..., j_k \le n$, where $e_j = (0, ..., 0, 1, 0, ..., 0) \in \mathbf{R}^n$ with 1 on the j-th place.

Such convergence in $\mathcal{D}$ defines closed subsets in this space $\mathcal{D}$, their complements by the definition are open, that gives the topology on $\mathcal{D}$. The space $\mathcal{D}$ is $\mathbf{R}$ linear and right and left $\mathcal{A}_r$ module.

By a generalized function of class $\mathcal{D}' := [\mathcal{D}(\mathbf{R}^n, Y)]'$ is called a continuous $\mathbf{R}$-linear $\mathcal{A}_r$-additive function $g : \mathcal{D} \to \mathcal{A}_r$. The set of all such functionals is denoted by $\mathcal{D}'$. That is, g is continuous, if for each sequence $f_n \in \mathcal{D}$, converging to zero, a sequence of numbers $g(f_n) =: [g, f_n) \in \mathcal{A}_r$ converges to zero for n tending to the infinity.

A generalized function g is zero on an open subset V in $\mathbf{R}^n$, if $[g, f) = 0$ for each $f \in \mathcal{D}$ equal to zero outside V. By a support of a generalized function g is called the family, denoted by $supp(g)$, of all points $t \in \mathbf{R}^n$ such that in each neighborhood of each point $t \in supp(g)$ the functional g is different from zero. The addition of generalized functions g, h is given by the formula:

(1) $[g + h, f) := [g, f) + [h, f)$.

The multiplication $g \in \mathcal{D}'$ on an infinite differentiable function w is given by the equality:

(2) $[gw, f) = [g, wf)$ either for $w : \mathbf{R}^n \to \mathcal{A}_r$ and each test function $f \in \mathcal{D}$ with a real image $f(\mathbf{R}^n) \subset \mathbf{R}$, where $\mathbf{R}$ is embedded into Y; or $w : \mathbf{R}^n \to \mathbf{R}$ and $f : \mathbf{R}^n \to Y$.

A generalized function g' prescribed by the equation:

(3) $[g', f) := -[g, f')$ is called a derivative g' of a generalized function g, where $f' \in \mathcal{D}(\mathbf{R}^n, L_q(\mathbf{R}^n, Y))$, $g' \in [\mathcal{D}(\mathbf{R}^n, L_q(\mathbf{R}^n, Y))]'$.

Another space $\mathcal{B} := \mathcal{B}(\mathbf{R}^n, Y)$ of test functions consists of all infinite differentiable functions $f : \mathbf{R}^n \to Y$ such that the limit $\lim_{|t| \to +\infty} |t|^m f^{(j)}(t) = 0$ exists for each $m = 0, 1, 2, ...$, $j = 0, 1, 2, ...$. A sequence $f_n \in \mathcal{B}$ is called converging to zero, if the sequence $|t|^m f_n^{(j)}(t)$ converges to zero uniformly on $\mathbf{R}^n \setminus B(\mathbf{R}^n, 0, R)$ for each $m, j = 0, 1, 2, ...$ and each $0 < R < +\infty$, where $B(Z, z, R) := \{y \in Z : \ \rho(y, z) \le R\}$ denotes a ball with center at z of radius R in a metric space Z with a metric ρ. The family of all $\mathbf{R}$-linear and $\mathcal{A}_r$-additive functionals on $\mathcal{B}$ is denoted by $\mathcal{B}'$.

In particular we can take $X = \mathcal{A}_r^\alpha$, $Y = \mathcal{A}_r^\beta$ with $1 \le \alpha, \beta \in \mathbf{Z}$. Analogously spaces $\mathcal{D}(U,Y)$, $[\mathcal{D}(U,Y)]'$, $\mathcal{B}(U,Y)$ and $[\mathcal{B}(U,Y)]'$ are defined for domains U in $\mathbf{R}^n$. For definiteness we write $\mathcal{B}(U,Y) = \{f|_U: \ f \in \mathcal{B}(\mathbf{R}^n, Y)\}$ and $\mathcal{D}(U,Y) = \{f|_U: \ f \in \mathcal{D}(\mathbf{R}^n, Y)\}$.

A function $g: U \to \mathcal{A}_v$ is called locally integrable, if it is absolutely integrable on each bounded λ measurable sub-domain V in U, i.e.

$\int_V |g(z)|\lambda(dz) < \infty$, where λ denotes the Lebesgue measure on U.

A generalized function f is called regular if locally integrable functions ${}_{j,k}f^1, {}_lf^2 : U \to \mathcal{A}_v$ exist such that

$[f,\omega) = \int_U \{\sum_{j,k,l} {}_{j,k}f^1(z)_k\omega(z)_j f^2(z)\}_{q(3)}\lambda(dz)$,

for each test function $\omega \in \mathcal{B}(U,Y)$ or $\omega \in \mathcal{D}(U,Y)$ correspondingly, where $\omega = ({}_1\omega, ..., {}_\beta\omega)$, ${}_k\omega(z) \in \mathcal{A}_v$ for each $z \in U$ and all k, $q(3)$ is a vector indicating on an order of the multiplication in the curled brackets and it may depend on the indices $j, l = 1, ..., \alpha$, $k = 1, ..., \beta$.

We supply the space $\mathcal{B}(\mathbf{R}^n, Y)$ with the countable family of semi-norms

(4) $p_{\alpha,k}(f) := \sup_{x \in \mathbf{R}^n} |(1+|x|)^k \partial^\alpha f(x)|$

inducing its topology, where $k = 0, 1, 2, ...$; $\alpha = (\alpha_1, ..., \alpha_n)$, $0 \le \alpha_j \in \mathbf{Z}$. On this space we take the space $\mathcal{B}'(\mathbf{R}^n, Y)_l$ of all Y valued continuous generalized functions (functionals) of the form

(5) $f = f_0 i_0 + ... + f_{2^v-1} i_{2^v-1}$ and $g = g_0 i_0 + ... + g_{2^v-1} i_{2^v-1}$, where f_j and $g_j \in \mathcal{B}'(\mathbf{R}^n, Y)$, with restrictions on $\mathcal{B}(\mathbf{R}^n, \mathbf{R})$ being real- or $\mathbf{C_i} = \mathbf{R} \oplus \mathbf{iR}$- valued generalized functions $f_0, ..., f_{2^v-1}$, $g_0, ..., g_{2^v-1}$ respectively. Let $\phi = \phi_0 i_0 + ... + \phi_{2^v-1} i_{2^v-1}$ with $\phi_0, ..., \phi_{2^v-1} \in \mathcal{B}(\mathbf{R}^n, \mathbf{R})$, then

(6) $[f,\phi) = \sum_{k,j=0}^{2^v-1} [f_j, \phi_k) i_k i_j$. Let their convolution be defined in accordance with the formula:

(7) $[f * g, \phi) = \sum_{j,k=0}^{2^v-1} ([f_j * g_k, \phi) i_j) i_k$

for each $\phi \in \mathcal{B}(\mathbf{R}^n, Y)$. Particularly,

(8) $(f * g)(x) = f(x-y) * g(y) = f(y) * g(x-y)$

for all $x, y \in \mathbf{R}^n$ due to (7), since the latter equality is satisfied for each pair f_j and g_k.

10. The decomposition theorem of partial differential operators over the Cayley-Dickson algebras.

We consider a partial differential operator of order u:

$$(1) \quad Af(x) = \sum_{|\alpha| \le u} \mathbf{a}_\alpha(x) \partial^\alpha f(x),$$

where $\partial^\alpha f = \partial^{|\alpha|} f(x)/\partial x_0^{\alpha_0} ... \partial x_n^{\alpha_n}$, $x = x_0 i_0 + ... x_n i_n$, $x_j \in \mathbf{R}$ for each j, $1 \le n = 2^r - 1$, $\alpha = (\alpha_0, ..., \alpha_n)$, $|\alpha| = \alpha_0 + ... + \alpha_n$, $0 \le \alpha_j \in \mathbf{Z}$. By the definition this means that the principal symbol

$$(2) \quad A_0 := \sum_{|\alpha| = u} \mathbf{a}_\alpha(x) \partial^\alpha$$

has α so that $|\alpha| = u$ and $\mathbf{a}_\alpha(x) \in \mathcal{A}_r$ is not identically zero on a domain U in $\mathcal{A}_r$. As usually $C^k(U, \mathcal{A}_r)$ denotes the space of k times continuously differentiable functions by all real variables $x_0, ..., x_n$ on U with values in $\mathcal{A}_r$, while the x-differentiability corresponds to the super-differentiability by the Cayley-Dickson variable x.

Speaking about locally constant or locally differentiable coefficients we shall undermine that a domain U is the union of sub-domains U^j satisfying conditions $15(D1, i - vii)$ and $U^j \cap U^k = \partial U^j \cap \partial U^k$ for each $j \ne k$. All coefficients $\mathbf{a}_\alpha$ are either constant or differentiable of the same class on each $Int(U^j)$ with the continuous extensions on U^j. More generally it is up to a C^u or x-differentiable diffeomorphism of U respectively.

If an operator A is of the odd order $u = 2s - 1$, then an operator E of the even order $u + 1 = 2s$ by variables (t, x) exists so that

(3) $Eg(t,x)|_{t=0} = Ag(0,x)$ for any $g \in C^{u+1}([c,d] \times U, \mathcal{A}_r)$, where $t \in [c,d] \subset \mathbf{R}$, $c \le 0 < d$, for example, $Eg(t,x) = \partial(tAg(t,x))/\partial t$.

Therefore, it remains the case of the operator A of the even order $u = 2s$. Take $z = z_0 i_0 + ... + z_{2^v-1} i_{2^v-1} \in \mathcal{A}_v$, $z_j \in \mathbf{R}$. Operators depending on a less set $z_{l_1}, ..., z_{l_n}$ of variables can be considered as restrictions of operators by all variables on spaces of functions constant by variables z_s with $s \notin \{l_1, ..., l_n\}$.

Theorem. *Let $A = A_u$ be a partial differential operator of an even order $u = 2s$ with locally constant or variable (locally) $C^{s'}$ or x-differentiable on U coefficients $\mathbf{a}_\alpha(x) \in \mathcal{A}_r$ such that it has the form*

(4) $Af = c_{u,1}(B_{u,1}f) + ... + c_{u,k}(B_{u,k}f)$, *where each*

(5) $B_{u,p} = B_{u,p,0} + Q_{u-1,p}$

is a partial differential operator by variables $x_{m_{u,1}+...+m_{u,p-1}+1}, ..., x_{m_{u,1}+...+m_{u,p}}$ and of the order u, $m_{u,0} = 0$, $c_{u,k}(x) \in \mathcal{A}_r$ for each k, its principal part

(6) $B_{u,p,0} = \sum_{|\alpha|=s} \mathbf{a}_{p,2\alpha}(x) \partial^{2\alpha}$

is elliptic with real coefficients $\mathbf{a}_{p,2\alpha}(x) \ge 0$, either $0 \le r \le 3$ and $f \in C^u(U, \mathcal{A}_r)$, or $r \ge 4$ and $f \in C^u(U, \mathbf{R})$. Then three partial differential operators Υ^s and Υ^s_1 and Q of orders s and p with $p \le u-1$ with locally constant or variable (locally) $C^{s'}$ or x-differentiable correspondingly on U coefficients with values in $\mathcal{A}_v$ exist and coefficients of the third operator Q may be generalized functions, when coefficients of A are discontinuous locally constant or $C^{s'}$ discontinuous on the entire U or when $s' < s$, $r \le v$, such that

(7) $Af = \Upsilon^s(\Upsilon^s_1 f) + Qf$.

Proof. Certainly we have $ord Q_{u-1,p} \le u-1$, $ord(A - A_0) \le u-1$. We choose the following operators:

$$(8) \quad \Upsilon^s f(x) = \sum_{p=1}^{k} \sum_{|\alpha| \le s,\ \alpha_q = 0 \forall q < (m_{u,1}+...+m_{u,p-1}+1) \text{ and } q > (m_{u,1}+...+m_{u,p})} (\partial^\alpha f(x))[w_p^* \psi_{p,\alpha}] \text{ and}$$

$$(9) \quad \Upsilon^s_1 f(x) = \sum_{p=1}^{k} \sum_{|\alpha| \le s,\ \alpha_q = 0 \forall q < (m_{u,1}+...+m_{u,p-1}+1) \text{ and } q > (m_{u,1}+...+m_{u,p})} (\partial^\alpha f(x))[w_p \psi^*_{p,\alpha}],$$

where $w_p^2 = c_{u,p}$ for all p and $\psi^2_{p,\alpha}(x) = -\mathbf{a}_{p,2\alpha}(x)$ for each p and x, $w_p \in \mathcal{A}_r$, $\psi_{p,\alpha}(x) \in \mathcal{A}_{r,v}$ and $\psi_{p,\alpha}(x)$ is purely imaginary for $\mathbf{a}_{p,2\alpha}(x) > 0$ for all α and x, $Re(w_p Im(\psi_{p,\alpha})) = 0$ for all p and α, $Im(x) = (x - x^*)/2$, $v > r$. Here $\mathcal{A}_{r,v} = \mathcal{A}_v / \mathcal{A}_r$ is the real quotient algebra. The algebra $\mathcal{A}_{r,v}$ has the generators i_{j2^r}, $j = 0, ..., 2^{v-r} - 1$. A natural number v so that $2^{v-r} - 1 \ge \sum_{p=1}^k \sum_{q=0}^u \binom{m_p+q-1}{q}$ is sufficient, where $\binom{m}{q} = m!/(q!(m-q)!)$ denotes the binomial coefficient, $\binom{m+q-1}{q}$ is the number of different solutions of the equation $\alpha_1 + ... + \alpha_m = q$ in non-negative integers α_j. We have either $\partial^{\alpha+\beta} f \in \mathcal{A}_r$ for $0 \le r \le 3$ or $\partial^{\alpha+\beta} f \in \mathbf{R}$ for $r \ge 4$. Therefore, we can take $\psi_{p,\alpha}(x) \in i_{2^r q}\mathbf{R}$, where $q = q(p,\alpha) \ge 1$, $q(p^1,\alpha^1) \ne q(p,\alpha)$ when $(p,\alpha) \ne (p^1,\alpha^1)$.

Thus Decomposition (7) is valid due to the following. For $b = \partial^{\alpha+\beta} f(z)$ and $\mathbf{l} = i_{2^r p}$ and $w \in \mathcal{A}_r$ one has the identities:

(10) $(b(w\mathbf{l}))(w^*\mathbf{l}) = ((wb)\mathbf{l})(w^*\mathbf{l}) = -w(wb) = -w^2 b$ and

(11) $(((b\mathbf{l})w^*)\mathbf{l})w = (((bw)\mathbf{l})\mathbf{l})w = -(bw)w = -bw^2$ in the considered here cases, since $\mathcal{A}_r$ is alternative for $r \le 3$ while $\mathbf{R}$ is the center of the Cayley-Dickson algebra (see Formulas $2(M1, M2)$).

This decomposition of the operator A_{2s} is generally up to a partial differential operator of order not greater, than $(2s-1)$:

(12) $Qf(x) = \sum_{p=1}^k c_{u,p} Q_{u-1,p} +$

$\sum_{|\alpha|\le s,|\beta|\le s;\gamma\le\alpha,\epsilon\le\beta,|\gamma+\epsilon|>0}[\prod_{j=0}^{2^v-1}\binom{\alpha_j}{\gamma_j}\binom{\beta_j}{\epsilon_j}](\partial^{\alpha+\beta-\gamma-\epsilon}f(x))$
$[(\partial^\gamma\eta_\alpha(x))((\partial^\epsilon\eta^1_\beta(x)]$,
where operators Υ^s and Υ^s_1 are already written in accordance with the general form

(13) $\Upsilon^s f(x)=\sum_{|\alpha|\le s}(\partial^\alpha f(x))\eta_\alpha(x)$;

(14) $\Upsilon^s_1 f(x)=\sum_{|\beta|\le s}(\partial^\beta f(x))\eta^1_\beta(x)$.

The coefficients of Q may be generalized functions, since they are calculated with the participation of partial derivatives of the coefficients of the operator Υ^s_1, but the coefficients of the operators Υ^s and Υ^s_1 may be discontinuous locally constant or $C^{s'}$ discontinuous on the entire U or $s'<s$ when for the initial operator A they are such.

When A in (3) is with constant coefficients, then the coefficients w_p and $\psi_{p,\alpha}$ for Υ^m and Υ^m_1 can also be chosen constant and $Q-\sum_{p=1}^k c_{u,p}Q_{u-1,p}=0$.

11. Corollary. *Let suppositions of Theorem 10 be satisfied. Then a change of variables locally affine or variable C^1 or x-differentiable on U correspondingly on U exists so that the principal part $A_{2,0}$ of A_2 becomes with constant coefficients, when $\mathbf{a}_{p,2\alpha}>0$ for each p, α and x.*

12. Corollary. *If two operators $E=A_{2s}$ and $A=A_{2s-1}$ are related by Equation 10(3), and A_{2s} is presented in accordance with Formulas 10(4,5), then three operators Υ^s, Υ^{s-1} and Q of orders s, $s-1$ and $2s-2$ exist so that*

(1) $A_{2s-1}=\Upsilon^s\Upsilon^{s-1}+Q$.

Proof. It remains to verify that $ord(Q)\le 2s-2$ in the case of A_{2s-1}, where $Q=\{\partial(tA_{2s-1})/\partial t-\Upsilon^s\Upsilon^s_1\}|_{t=0}$. Indeed, the form $\lambda(E)$ corresponding to E is of degree $2s-1$ by x and each addendum of degree $2s$ in it is of degree not less than 1 by t, consequently, the product of forms $\lambda(\Upsilon_s)\lambda(\Upsilon^s_1)$ corresponding to Υ^s and Υ^s_1 is also of degree $2s-1$ by x and each addendum of degree $2s$ in it is of degree not less than 1 by t. But the principal parts of $\lambda(E)$ and $\lambda(\Upsilon_s)\lambda(\Upsilon^s_1)$ coincide identically by variables (t,x), hence $ord(\{E-\Upsilon^s\Upsilon^s_1\}|_{t=0})\le 2s-2$. Let $a(t,x)$ and $h(t,x)$ be coefficients from Υ^s_1 and Υ^s. Using the identities

$a(t,x)\partial_t\partial^\gamma tg(x)=a(t,x)\partial^\gamma g(x)$ and

$h(t,x)\partial^\beta\partial_t[a(t,x)\partial^\gamma g(x)]=h(t,x)\partial^\beta[(\partial_t a(t,x))\partial^\gamma g(x)]$

for any functions $g(x)\in C^{2s-1}$ and $a(t,x)\in C^s$,

$ord[(h(t,x)\partial^\beta),(a(t,x)\partial^\gamma)]|_{t=0}\le 2s-2$,

where $\partial_t=\partial/\partial t$, $|\beta|\le s-1$, $|\gamma|\le s$, $[A,B]:=AB-BA$ denotes the commutator of two operators, we reduce $(\Upsilon^s\Upsilon^s_1+Q_1)|_{t=0}$ from Formula 10(7) to the form prescribes by equation (1).

13. We consider operators of the form:

(1) $(\Upsilon^k+\beta I_r)f(z):=\{\sum_{0<|\alpha|\le k}(\partial^\alpha f(z)\eta_\alpha(z)\}+f(z)\beta(z)$,

with $\eta_\alpha(z)\in\mathcal{A}_v$, $\alpha=(\alpha_0,...,\alpha_{2^r-1})$, $0\le\alpha_j\in\mathbf{N}$ for each j, $|\alpha|=\alpha_0+...+\alpha_{2^r-1}$, $\beta I_r f(z):=f(z)\beta$,

$\partial^\alpha f(z):=\partial^{|\alpha|}f(z)/\partial z_0^{\alpha_0}...\partial z_{2^r-1}^{\alpha_{2^r-1}}$, $2\le r\le v<\infty$, $\beta(z)\in\mathcal{A}_v$, $z_0,...,z_{2^r-1}\in\mathbf{R}$, $z=z_0i_0+...+z_{2^r-1}i_{2^r-1}$.

Proposition. *The operator $(\Upsilon^k+\beta)^*(\Upsilon^k+\beta)$ is elliptic on the space $C^{2k}(\mathbf{R}^{2^r},\mathcal{A}_v)$, where $(\Upsilon^k+\beta)^*$ denotes the adjoint operator (i.e. with adjoint coefficients).*

Proof. In view of Formulas (1) and 4(8) the form corresponding to the principal symbol of the operator $(\Upsilon^k+\beta)^*(\Upsilon^k+\beta)$ is with real coefficients, of degree $2k$ and non-negative definite, consequently, the operator $(\Upsilon^k+\beta)^*(\Upsilon^k+\beta)$ is elliptic.

14. Example. Let Υ^* be the adjoint operator defined on differentiable $\mathcal{A}_v$ valued functions f given by the formula:

$$(1)\quad (\Upsilon+\beta)^*f=[\sum_{j=0}^n(\partial f(z)/\partial z_j)\phi_j(z)]+f(z)\beta(z)^*.$$

Thus we can consider the operator

(2) $\Xi_\beta := (\Upsilon + \beta)(\Upsilon + \beta)^*$.

From Proposition (13) we have that the operator Ξ_β is elliptic as classified by its principal symbol with real coefficients. Put $\Xi = \Xi_0$. In the x coordinates from §8 it has the simpler form:

$$(3) \quad (\Upsilon + \beta)(\Upsilon + \beta)^* f = \sum_{j=0}^{n} (\partial^2 f/\partial x_j^2)|t_j|^2$$

$$+2 \sum_{0\le j<k\le n} (\partial^2 f/\partial x_j \partial x_k) Re(t_j t_k^*) + 2\sum_{j=0}^{n} (\partial f/\partial x_j) Re(t_j^* \beta) + \{f|\beta|^2 + \sum_{j=0}^{n} [f(\partial\beta^*/\partial x_j)] t_j\},$$

because the coefficients t_j are already constant. After a change of variables reducing the corresponding quadratic form to the sum of squares $\sum_j \epsilon_j s_j^2$ we get the formula:

(4) $\Upsilon\Upsilon^* f = \sum_{j=1}^m (\partial^2 f/\partial s_j^2)\epsilon_j$,

where $s_j \in \mathbf{R}$, $\epsilon_j = 1$ for $1 \le j \le p$ and $\epsilon_j = -1$ for each $p < j \le m$, $m \le 2^v$, $1 \le p \le m$ depending on the signature $(p, m-p)$.

Generally (see Formula 5(1)) we have

(5) $A = (\Upsilon + \beta)(\Upsilon_1 + \beta^1) f(z) = B_0 f(z) + Q f(z)$, where

(6) $B_0 f(z) = \sum_{j,k} [(\partial^2 f(z)/\partial z_j \partial z_k)\phi_j^1(z)^*]\phi_k^*(z) + [f(z)\beta^1(z)]\beta(z)$ and

(7) $Qf(z) = \sum_{j,k}[(\partial f(z)/\partial z_j)(\partial\phi_j^1(z)^*/\partial z_k)]\phi_k^*(z) + \sum_j [(\partial f(z)/\partial z_j)\phi_j^1(z)^*]\beta(z) + \sum_k [f(z)(\partial\beta^1(z)/\partial z_k)]\phi_k^*(z)$,

(8) $(\Upsilon_1 + \beta^1) f(z) = [\sum_j (\partial f(z)/\partial z_j)\phi_j^1(z)^*] + f(z)\beta^1(z)$.

The latter equations show that coefficients of the operator Q may be generalized functions, when $\phi_j^1(z)$ for some j or $\beta^1(z)$ are locally C^0 or C^0 or locally C^1 functions, while $\phi_k(z)$ for each k and $\beta(z)$ are locally C^0 or C^0 functions on U. We consider this in more details in the next section.

15. Partial differential operators with generalized coefficients.

Let an operator Q be given by Formula 14(7) on a domain U. Initially it is considered as a domain in the Cayley-Dickson algebra $\mathcal{A}_v$. But in the case when Q and f depend on smaller number of real coordinates $z_0, ..., z_{n-1}$ we can take the real shadow of U and its sub-domain V of variables $(z_0, ..., z_{n-1})$, where z_k are marked for example being zero for all $n \le k \le 2^v - 1$. Thus we take a domain V which is a canonical closed subset in the Euclidean space $\mathbf{R}^n$, $2^{v-1} \le n \le 2^v - 1$, $v \ge 2$.

A canonical closed subset P of the Euclidean space $X = \mathbf{R}^n$ is called a quadrant if it can be given by the condition $P := \{x \in X : q_j(x) \ge 0\}$, where $(q_j : j \in \Lambda_P)$ are linearly independent elements of the topologically adjoint space X^*. Here $\Lambda_P \subset \mathbf{N}$ (with $card(\Lambda_P) = k \le n$) and k is called the index of P. If $x \in P$ and exactly j of the q_i's satisfy $q_i(x) = 0$ then x is called a corner of index j.

That is P is affine diffeomorphic with $P^n = \prod_{j=1}^n [\mathsf{a}_j, b_j]$, where $-\infty \le \mathsf{a}_j < b_j \le \infty$, $[\mathsf{a}_j, b_j] := \{x \in \mathbf{R} : \ \mathsf{a}_j \le x \le b_j\}$ denotes the segment in $\mathbf{R}$. This means that there exists a vector $p \in \mathbf{R}^n$ and a linear invertible mapping C on $\mathbf{R}^n$ so that $C(P) - p = P^n$. We put $t^{j,1} := (t_1, ..., t_j, ..., t_n : \ t_j = \mathsf{a}_j)$, $t^{j,2} := (t_1, ..., t_j, ..., t_n : \ t_j = b_j)$. Consider $t = (t_1, ..., t_n) \in P^n$.

This permits to define a manifold M with corners. It is a metric separable space modelled on $X = \mathbf{R}^n$ and is supposed to be of class C^s, $1 \le s$. Charts on M are denoted (U_l, u_l, P_l), that is, $u_l : U_l \to u_l(U_l) \subset P_l$ is a C^s-diffeomorphism for each l, U_l is open in M, $u_l \circ {u_j}^{-1}$ is of C^s class of smoothness from the domain $u_j(U_l \cap U_j) \ne \emptyset$ onto $u_l(U_l \cap U_j)$, that is, $u_j \circ u_l^{-1}$ and $u_l \circ u_j^{-1}$ are bijective, $\bigcup_j U_j = M$.

A point $x \in M$ is called a corner of index j if there exists a chart (U, u, P) of M with $x \in U$ and $u(x)$ is of index $ind_M(x) = j$ in $u(U) \subset P$. A set of all corners of index $j \ge 1$ is called a border ∂M of M, x is called an inner point of M if $ind_M(x) = 0$, so $\partial M = \bigcup_{j\ge 1} \partial^j M$, where $\partial^j M := \{x \in M : ind_M(x) = j\}$ (see also [30]). We consider that

$(D1)$ V is a canonical closed subset in the Euclidean space $\mathbf{R}^n$, that is $V = cl(Int(V))$, where $Int(V)$ denotes the interior of V and $cl(V)$ denotes the closure of V.

Particularly, the entire space $\mathbf{R}^n$ may also be taken.

Let a manifold W be satisfying the following conditions $(i-v)$.

(i). The manifold W is continuous and piecewise C^α, where C^l denotes the family of l times continuously differentiable functions. This means by the definition that W as the manifold is of class $C^0 \cap C^\alpha_{loc}$. That is W is of class C^α on open subsets $W_{0,j}$ in W and $W \setminus (\bigcup_j W_{0,j})$ has a codimension not less than one in W.

(ii). $W = \bigcup_{j=0}^m W_j$, where $W_0 = \bigcup_k W_{0,k}$, $W_j \cap W_k = \emptyset$ for each $k \ne j$, $m = dim_{\mathbf{R}} W$, $dim_{\mathbf{R}} W_j = m - j$, $W_{j+1} \subset \partial W_j$.

(iii). Each W_j with $j = 0, ..., m-1$ is an oriented C^α-manifold, W_j is open in $\bigcup_{k=j}^m W_k$. An orientation of W_{j+1} is consistent with that of ∂W_j for each $j = 0, 1, ..., m-2$. For $j > 0$ the set W_j is allowed to be void or non-void.

(iv). A sequence W^k of C^α orientable manifolds embedded into $\mathbf{R}^n$, $\alpha \ge 1$, exists such that W^k uniformly converges to W on each compact subset in $\mathbf{R}^n$ relative to the metric $dist$.

For two subsets B and E in a metric space X with a metric ρ we put

(1) $dist(B, E) := \max\{\sup_{b\in B} dist(\{b\}, E), \sup_{e\in E} dist(B, \{e\})\}$, where $dist(\{b\}, E) := \inf_{e\in E} \rho(b, e)$, $dist(B, \{e\}) := \inf_{b\in B} \rho(b, e)$, $b \in B$, $e \in E$.

Generally, $dim_{\mathbf{R}} W = m \le n$. Let $(e_1^k(x), ..., e_m^k(x))$ be a basis in the tangent space $T_x W^k$ at $x \in W^k$ consistent with the orientation of W^k, $k \in \mathbf{N}$.

We suppose that the sequence of orientation frames $(e_1^k(x_k), ..., e_m^k(x_k))$ of W^k at x_k converges to $(e_1(x), ..., e_m(x))$ for each $x \in W_0$, where $\lim_k x_k = x \in W_0$, while $e_1(x)$,...,$e_m(x)$ are linearly independent vectors in $\mathbf{R}^n$.

(v). Let a sequence of Riemann volume elements λ_k on W^k (see §XIII.2 [38]) induce a limit volume element λ on W, that is, $\lambda(B \cap W) = \lim_{k\to\infty}(B \cap W^k)$ for each compact canonical closed subset B in $\mathbf{R}^n$, consequently, $\lambda(W \setminus W_0) = 0$. We shall consider surface integrals of the second kind, i.e. by the oriented surface W (see (iv)), where each W_j, $j = 0, ..., m-1$ is oriented (see also §XIII.2.5 [38]).

Suppose that a boundary ∂U of U satisfies Conditions $(i-v)$ and

(vii) let the orientations of ∂U^k and U^k be consistent for each $k \in \mathbf{N}$ (see Proposition 2 and Definition 3 [38]).

Particularly, the Riemann volume element λ_k on ∂U^k is consistent with the Lebesgue measure on U^k induced from $\mathbf{R}^n$ for each k. This induces the measure λ on ∂U as in (v). This consideration certainly encompasses the case of a domain U with a C^α boundary ∂U as well.

Suppose that U_1,...,U_l are domains in $\mathbf{R}^n$ satisfying conditions $(D1, i-vii)$ and such that $U_j \cap U_k = \partial U_j \cap \partial U_k$ for each $j \ne k$, $U = \bigcup_{j=1}^l U_j$. Consider a function $g : U \to \mathcal{A}_v$ such that each its restriction $g|_{U_j}$ is of class C^s, but g on the entire domain U may be discontinuous or not C^k, where $0 \le k \le s$, $1 \le s$. If $x \in \partial U_j \cap \partial U_k$ for some $j \ne k$, $x \in Int(U)$, such that x is of index $m \ge 1$ in U_j (and in U_k also). Then there exists canonical C^α local coordinates $(y_1, ..., y_n)$ in a neighborhood V_x of x in U such that $S := V_x \cap \partial^m U_j = \{y : y \in V_x;\ y_1 = 0, ..., y_m = 0\}$. Using locally finite coverings of the locally compact topological space $\partial U_j \cap \partial U_k$ without loss of generality we suppose that C^α functions $P_1(z), ..., P_m(z)$ on $\mathbf{R}^n$ exist with $S = \{z : z \in \mathbf{R}^n;\ P_1(z) = 0, ..., P_m(z) = 0\}$. Therefore, on the surface S the delta-function $\delta(P_1, ..., P_m)$ exists, for $m = 1$ denoting them $P = P_1$ and $\delta(P)$ respectively (see §III.1 [6]). It is possible to choose $y_j = P_j$ for $j = 1, ..., m$. Using generalized functions with definite supports, for example compact supports, it is possible without loss of generality consider that $y_1, ..., y_n \in \mathbf{R}$ are real variables. Let $\theta(P_j)$ be the characteristic function of the domain

$\mathcal{P}_j := \{z : P_j(z) \ge 0\}$, $\theta(P_j) := 1$ for $P_j \ge 0$ and $\theta(P_j) = 0$ for $P_j < 0$. Then the generalized function $\theta(P_1, ..., P_m) := \theta(P_1)...\theta(P_m)$ can be considered as the direct product of generalized functions $\theta(y_1)$,...,$\theta(y_m)$, $1(y_{m+1}, ..., y_n) \equiv 1$, since variables $y_1, ..., y_n$ are independent. Then in the class of generalized functions the following formulas are valid:

(2) $\partial\theta(P_j)/\partial z_k = \delta(P_j)(\partial P_j/\partial z_k)$ for each $k = 1, ..., n$, consequently,

(3) $grad[\theta(P_1, ..., P_m)] = \sum_{j=1}^m [\theta(P_1)...\theta(P_{j-1})\delta(P_j)(grad(P_j))\theta(P_{j+1})...\theta(P_m)]$,

where , $grad\ g(z) := (\partial g(z)/\partial z_1, ..., \partial g(z)/\partial z_n)$ (see Formulas III.1.3$(1, 7, 7', 9)$ and III.1.9(6) [6]).

Let for the domain U in the Euclidean space $\mathbf{R^n}$ the set of internal surfaces $cl_U[Int_{\mathbf{R}^n}(U) \cap \bigcup_{j\ne k}(\partial U_j \cap \partial U_k)]$ in U on which a function $g : U \to \mathcal{A}_v$ or its derivatives may be discontinuous is presented as the disjoint union of surfaces Γ_j, where each surface Γ^j is the boundary of the sub-domain

(4) $\mathcal{P}^j := \{P_{j,1}(z) \ge 0, ..., P_{j,m_j}(z) \ge 0\}$, $\Gamma^j = \partial\mathcal{P}^j = \bigcup_{k=1}^{m_j} \partial^k\mathcal{P}^j$,

$m_j \in \mathbf{N}$ for each j, $cl_X(V)$ denotes the closure of a subset V in a topological space X, $Int_X(V)$ denotes the interior of V in X. By its construction $\{\mathcal{P}^j : j\}$ is the covering of U which is the refinement of the covering $\{U_k : k\}$, i.e. for each $\mathcal{P}^j$ a number k exists so that $\mathcal{P}^j \subset U_k$ and $\partial\mathcal{P}^j \subset \partial U_k$ and $\bigcup_j \mathcal{P}^j = \bigcup_k U_k = U$. We put

(5) $h_j(z(x)) = h(x)|_{x\in\Gamma^j} :=$

$\lim_{y_{j,1}\downarrow 0,...,y_{j,n}\downarrow 0} g(z(x + y)) - \lim_{y_{j,1}\downarrow 0,...,y_{j,n}\downarrow 0} g(z(x - y))$

in accordance with the supposition made above that g can have only discontinuous of the first kind, i.e. the latter two limits exist on each Γ^j, where x and y are written in coordinates in $\mathcal{P}^j$, $z = z(x)$ denotes the same point in the global coordinates z of the Euclidean space $\mathbf{R}^n$. We take new continuous function

(6) $g^1(z) = g(z) - \sum_j h_j(z)\theta(P_{j,1}(z), ..., P_{j,m_j}(z))$.

Let the partial derivatives and the gradient of the function g^1 be calculated piecewise one each U_k. Since g^1 is the continuous function, it is the regular generalized function by the definition, consequently, its partial derivatives exist as the generalized functions. If $g^1|_{U_j} \in C^1(U_j, \mathcal{A}_v)$, then $\partial g^1(z)/\partial z_k$ is the continuous function on U_j, i.e. in this case $\partial g^1(z)\chi_{U_j}(z)/\partial z_k$ is the regular generalized function on U_j for each k, where $\chi_G(z)$ denotes the characteristic function of a subset G in $\mathcal{A}_v$, $\chi_G(z) = 1$ for each $z \in G$, while $\chi(z) = 0$ for $z \in \mathcal{A}_v \setminus G$. Therefore, $g^1(z) = \sum_j g^1(z)\chi_{U_j \setminus \bigcup_{k<j} U_k}(z)$, where $U_0 = \emptyset$, $j, k \in \mathbf{N}$.

On the other hand, the function $g(z)$ is locally continuous on U, consequently, it defines the regular generalized function on the space $\mathcal{D}(U, \mathcal{A}_v)$ of test functions as

$[g, \omega) := \int_U g(z)\omega(z)\lambda(dz)$,

where λ is the Lebesgue measure on U induced by the Lebesgue measure on the real shadow $\mathbf{R}^{2^v}$ of the Cayley-Dickson algebra $\mathcal{A}_v$, $\omega \in \mathcal{D}(U, \mathcal{A}_v)$. Thus partial derivatives of g exist as generalized functions.

In accordance with Formulas $(2, 3, 6)$ we infer that the gradient of the function $g(z)$ on the domain U is the following:

(7) $grad\ g(z) = grad\ g^1(z) + \sum_j h_j(z) grad\ \theta(P_{j,1}, ..., P_{j,m_j})$.

Thus Formulas $(3, 7)$ permit calculations of coefficients of the partial differential operator Q given by Formula 14(7).

16. Line generalized functions.

Let U be a domain satisfying Conditions $1(D1, D2)$ and $15(D1, i - vii)$. We embed the Euclidean space $\mathbf{R}^n$ into the Cayley-Dickson algebra $\mathcal{A}_v$, $2^{v-1} \le n \le 2^v - 1$, as the $\mathbf{R}$ affine sub-space putting $\mathbf{R}^n \ni x = (x_1, ..., x_n) \mapsto x_1 i_{j_1} + ... + x_n i_{j_n} + x^0 \in \mathcal{A}_v$, where $j_k \ne j_l$ for each $k \ne l$, x^0 is a marked Cayley-Dickson number, for example, $j_k = k$ for each k, $x^0 = 0$. Moreover, each z_j can be written in the z-representation in accordance with Formulas $1(1-3)$.

We denote by $\mathbf{P} = \mathbf{P}(U)$ the family of all rectifiable paths $\gamma : [a_\gamma, b_\gamma] \to U$ supplied with the metric

(1) $\rho(\gamma,\omega) := |\gamma(a)-\omega(a_\omega)| + \inf_\phi V_a^b(\gamma(t)) - \omega(\phi(t))$
where the infimum is taken by all diffeomorphisms $\phi : [a_\gamma, b_\gamma] \to [a_\omega, b_\omega]$ so that $\phi(a_\gamma) = a_\omega$, $a = a_\gamma < b_\gamma = b$ (see §3).

Let us introduce a continuous mapping $g : \mathcal{B}(U,\mathcal{A}_v)\times \mathbf{P}(U)\times \mathcal{V}(U,\mathcal{A}_v) \to Y$ such that its values are denoted by $[g;\omega,\gamma;\nu]$, where Y is a module over the Cayley-Dickson algebra $\mathcal{A}_v$, $\omega\in\mathcal{B}(U,\mathcal{A}_v)$, $\gamma\in\mathbf{P}(U)$, $\mathcal{V}(U,\mathcal{A}_v)$ denotes the family of all functions on U with values in the Cayley-Dickson algebra of bounded variation (see §3), $\nu\in\mathcal{V}(U,\mathcal{A}_v)$. For the identity mapping $\nu(z) = id(z) = z$ values of this functional will be denoted shortly by $[g;\omega,\gamma]$. Suppose that this mapping g satisfies the following properties $(G1-G5)$:

$(G1)$ $[g;\omega,\gamma;\nu]$ is bi- $\mathbf{R}$ homogeneous and $\mathcal{A}_v$ additive by a test function ω and by a function of bounded variation ν;

$(G2)$ $[g;\omega,\gamma;\nu] = [g;\omega,\gamma^1;\nu] + [g;\omega,\gamma^2;\nu]$ for each γ,γ^1 and $\gamma^2\in\mathbf{P}(U)$ such that $\gamma(t) = \gamma^1(t)$ for all $t\in[a_{\gamma^1},b_{\gamma^1}]$ and $\gamma(t)=\gamma^2(t)$ for any $t\in[a_{\gamma^2},b_{\gamma^2}]$ and $a_{\gamma^1}=a_\gamma$ and $a_{\gamma^2}=b_{\gamma^1}$ and $b_\gamma = b_{\gamma^2}$.

$(G3)$ If a rectifiable curve γ does not intersect a support of a test function ω or a function of bounded variation ν, $\gamma([a,b]\cap(supp(\omega)\cap supp(\nu)) = \emptyset$, then $[g;\omega,\gamma;\nu]=0$, where $supp(\omega) := cl\{z\in U:\ \omega(z)\ne 0\}$.

Further we put

$(G4)$ $[\partial^{|m|}g(z)/\partial z_0^{m_0}...\partial z_{2^v-1}^{m_{2^v-1}};\omega,\gamma] := (-1)^{|m|}[g;\partial^{|m|}\omega(z)/\partial z_0^{m_0}...\partial z_{2^v-1}^{m_{2^v-1}},\gamma]$
for each $m=(m_0,...,m_{2^v-1})$, m_j is a non-negative integer $0\le m_j\in\mathbf{Z}$ for each j, $|m| := m_0+...+m_{2^v-1}$. In the case of a super-differentiable function ω and a generalized function g we also put

$(G5)$ $[(d^kg(z)/dz^k).(h_1,...,h_k);\omega,\gamma] := (-1)^k[g;(d^k\omega(z)/dz^k).(h_1,...,h_k),\gamma]$
for any natural number $k\in\mathbf{N}$ and Cayley-Dickson numbers $h_1,..,h_k\in\mathcal{A}_v$.

Then g is called the Y valued line generalized function on $\mathcal{B}(U,\mathcal{A}_v)\times\mathbf{P}(U)\times\mathcal{V}(U,\mathcal{A}_v)$. Analogously it can be defined on $\mathcal{D}(U,\mathcal{A}_v)\times\mathbf{P}(U)\times\mathcal{V}(U,\mathcal{A}_v)$ (see also §9). If $Y=\mathcal{A}_v$ we call it simply the line generalized function, while for $Y=L_q(\mathcal{A}_v^k,\mathcal{A}_v^l)$ we call it the line generalized operator valued function, $k,l\ge 1$, omitting "on $\mathcal{B}(U,\mathcal{A}_v)\times\mathbf{P}(U)\times\mathcal{V}(U,\mathcal{A}_v)$" or "line" for short, when it is specified. Their spaces we denote by $L_q(\mathcal{B}(U,\mathcal{A}_v)\times\mathbf{P}(U)\times\mathcal{V}(U,\mathcal{A}_v);Y)$. Thus if g is a generalized function, then Formula $(G5)$ defines the operator valued generalized function $d^kg(z)/dz^k$ with $k\in\mathbf{N}$ and $l=1$.

If g is a continuous function on U (see §3), then the formula

$(G6)$ $[g;\omega,\gamma;\nu] = \int_\gamma g(y)\omega(y)d\nu(y)$

defines the generalized function. If $\hat{f}(z)$ is a continuous $L_q(\mathcal{A}_v,\mathcal{A}_v)$ valued function on U, then it defines the generalized operator valued function with $Y=L_q(\mathcal{A}_v,\mathcal{A}_v)$ such that

$(G7)$ $[\hat{f};\omega,\gamma;\nu] = \int_\gamma\{\hat{f}(z).\omega(z)\}d\nu(z)$.
Particularly, for $\nu=id$ we certainly have $d\nu(z)=dz$.

We consider on $L_q(\mathcal{B}(U,\mathcal{A}_v)\times\mathbf{P}(U)\times\mathcal{V}(U,\mathcal{A}_v);Y)$ the strong topology:

$(G8)$ $\lim_l f^l = f$ means that for each marked test function $\omega\in\mathcal{B}(U,\mathcal{A}_v)$ and rectifiable path $\gamma\in\mathbf{P}(U)$ and function of bounded variation $\nu\in\mathcal{V}(U,\mathcal{A}_v)$ the limit relative to the norm in Y exists

$\lim_l[f^l;\omega,\gamma;\nu] = [f;\omega,\gamma;\nu]$.

17. Line integration of generalized functions.

Let $C^m_{ab}(V,\mathcal{A}_v)$ denote the $\mathbf{R}$ linear space and right and left $\mathcal{A}_v$ module of all functions $\gamma : V\to\mathcal{A}_v$ such that $\gamma(z)$ and each its derivative $\partial^{|k|}g(z)/\partial z_1^{m_1}...\partial z_n^{m_n}$ for $1\le|k|\le m$ is absolutely continuous on V (see §§3 and 16). This definition means that $C^{m+1}(V,\mathcal{A}_v)\subset C^m_{ab}(V,\mathcal{A}_v)$, where $C^m(V,\mathcal{A}_v)$ denotes the family of all m times continuously differentiable functions on a domain V either open or canonical closed in $\mathbf{R}^n$, which may be a a real shadow of U as well.

17.1. Lemma. *Let $\gamma \in C^m_{ab}([a,b],\mathcal{A}_v)\cap \mathbf{P}(U)$ and $\omega \in \mathcal{B}(U,\mathcal{A}_v)$ and $\nu \in C^0_{ab}(U,\mathcal{A}_v)$ for $m=0$ or $\nu = id$ for $m\ge 1$, where $0\le m\in \mathbf{Z}$, then a line generalized function $[g;\omega,\gamma|_{[a,x]};\nu]$ is continuous for $m=0$ or of class C^m by the parameter $x\in[a,b]$ for $m\ge 1$.*

Proof. For absolutely continuous functions $\gamma(t)$ and ν (i.e. when $m=0$) the continuity by the parameter x follows from the definition of the line generalized function, since $\lim_{\Delta x\to 0}\rho(\gamma|_{[a,x]},\gamma|_{[a,x+\Delta x]})=0$ and

$\lim_{\Delta x\to 0}\rho(\nu\circ\gamma|_{[a,x]},\nu\circ\gamma|_{[a,x+\Delta x]})=0$.

Consider now the case $m\ge 1$ and $\nu=id$. In view of properties $16(G1,G2)$ for any $\Delta x\ne 0$ so that $x\in(a,b]:=\{t\in\mathbf{R}:\ a<t\le b\}$ and $x+\Delta x\in(a,b):=\{t\in\mathbf{R}:\ a<t<b\}$ the difference quotient satisfies the equalities:

(1) $\{[g;\omega,\gamma|_{[a,x+\Delta x]}]-[g;\omega,\gamma|_{[a,x]}]\}/\Delta x=[g;\omega/\Delta x,\gamma\circ\phi|_{[a,x]}]-[g;\omega/\Delta x,\gamma|_{[a,x]}]$,

where $\phi:[a,x]\to[a,x+\Delta x]$ is a diffeomorphism of $[a,x]$ onto $[a,x+\Delta x]$ with $\phi(a)=a$. Therefore, $\Delta\omega:=\omega(z+\Delta z)-\omega(z)$ for $z=\gamma(t)$ and $z+\Delta z=\gamma(\phi(t))$, $t\in[a,x]$ in the considered case. Using Conditions $(G1,G3)$ one can mention that if $\omega=\omega^1$ on an open neighborhood V of γ in U, then

(2) $[g;\omega,\gamma]=[g;\omega^1,\gamma]$,

since $\omega-\omega^1=0$ on V and $\gamma\cap supp(\omega-\omega^1)=0$.

From Conditions $16(G1,G4)$ and Formula (2) we deduce that

(3) $\lim_{\Delta x\to 0}\{[g;\omega,\gamma|_{[a,x+\Delta x]}]-[g;\omega,\gamma|_{[a,x]}]\}/\Delta x$

$=\sum_{j=0}^{2^v-1}[g;(\partial\omega(z)/\partial z_j),(d\gamma_j(t)/dt)\gamma|_{[a,x]}]$,

where $z_j'=d\gamma_j(t)/dt$ for $z=\gamma(t)$, $t\in[a,b]$, since each partial derivative of the test function ω is again the test function. From the first part of the proof we get that $[g;\omega,\gamma|_{[a,x]}]$ is of class C^1 by the parameter x, since the product $(d\gamma_j(t)/dt)\gamma(t)$ of absolutely continuous functions $(d\gamma_j(t)/dt)$ and $\gamma(t)$ is absolutely continuous for each j. Repeating this proof by induction for $k=1,...,m$ one gets the statement of this lemma for $\gamma\in C^m_{ab}([a,b],\mathcal{A}_v)\cap\mathbf{P}(U)$.

17.2. Lemma. *If γ is a rectifiable path, then a line generalized function $[g;\omega,\gamma|_{[a,x]}]$ is of bounded variation by the parameter $x\in[a,b]$.*

Proof. Let $\gamma\in\mathbf{P}(U)$ be a rectifiable path in U, $\gamma:[a,b]\to U$. We can present γ in the form

(1) $\gamma(t)=\sum_{j=0}^{2^v-1}\gamma_j(t)i_j$,

where each function $\gamma_j(t)$ is real-valued. Therefore, $\gamma_j(t)$ is continuous and of bounded variation for each j, since γ is such. Thus the function $\omega(\gamma(t))$ is of bounded variation $V^b_a\omega(\gamma))<\infty$, since ω is infinite differentiable and $\gamma([a,b])$ is compact.

On the other hand, each function $f:[a,b]\to\mathbf{R}$ of bounded variation can be written as the difference $f=f^1-f^2$ of two monotone non-decreasing functions f^1 and f^2 of bounded variations: $f^1(t):=V^t_af$ and $f^2(t)=f^1(t)-f(t)$ for each $t\in[a,b]$ (see [5, 16]). This means that $f^k=g^k+h^k$, where a function g^k is continuous monotone and of bounded variation, while h^k is a monotone step function, where $k=1,2$. When the function f is continuous one gets $f=g^1-g^2$. For a monotone non-decreasing function p one has $V^t_ap=p(t)-p(a)$.

In view of Property $17(G1)$ we infer that

(2) $[g;\omega,\gamma|_{[a,x]}]=\sum_{j=0}^{2^v-1}[g_j;\omega,\gamma|_{[a,x]}]i_j$,

where the function $[g_j;\omega,\gamma|_{[a,x]}]$ by x is real-valued for any $\omega\in\mathcal{B}(U,\mathcal{A}_v)$ and $\gamma\in\mathbf{P}(U)$ for all $j=0,...,2^v-1$.

The metric space $\mathbf{P}(\bar U)$ is complete, where $\bar U=cl(U)$. Indeed, let g^n be a sequence of rectifiable paths in $\bar U$ fundamental relative to the metric ρ given by Formula 16(1). Using diffeomorphism preserving orientations of segments we can consider without loss of generality that each path g^n is defined on the unit segment $[0,1]$, $a=0$, $b=1$. It is lightly to mention that

(3) $|g(a) - f(a))| + V_a^b(g - f) \ge \sup_{t\in[a,b]} |g(t) - f(t)|$
for any two functions of bounded variation, $f,\ g : [a, b] \to \bar{U}$. For each $\epsilon > 0$ a natural number $n_0 = n_0(\epsilon)$ exists so that $\rho(g^n, g^m) < \epsilon/2$ for all $n, m \ge n_0$. That is $\phi^n : [0, 1] \to [0, 1]$ diffeomorphisms exist such that

$|g^n(a) - g^m(a))| + V_a^b(g^n \circ \phi^n - g^m \circ \phi^m) < \epsilon$ for all $n, m \ge n_0$, since $\phi^m \circ (\phi^n)^{-1}$ is also the diffeomorphism preserving the orientation of the segment. Using induction by $\epsilon = 1/l$ with $l \in \mathbf{N}$ one chooses a sequence of diffeomorphisms ϕ^n such that for each $l \in \mathbf{N}$ a natural number $n_0 = n_0(l)$ exists such that

$|g^n(a) - g^m(a))| + V_a^b(g^n \circ \phi^n - g^m \circ \phi^m) < 1/l$ for all $n, m \ge n_0(l)$, consequently,

$\sup_{t\in[a,b]} |g^n(\phi^n(t)) - g^m(\phi^m(t))| < 1/l$ for all $n, m \ge n_0(l)$.

Thus the sequence $g^n \circ \phi^n$ is fundamental in $C^0([a, b], \bar{U})$. The latter metric space is complete relative to the metric

$d(f, g) := \sup_{t\in[a,b]} |f(t) - g(t)|$,

since from the completeness of the Cayley-Dickson algebra $\mathcal{A}_v$ considered as the normed space over the real field the completeness of the closed subset $\bar{U}$ follows (see also Chapter 8 in [3]). Therefore, the sequence $g^n \circ \phi^n$ converges to a continuous function $f : [a, b] \to \bar{U}$. On the other hand, $\lim_{m\to\infty} \rho(g^n \circ \phi^n, g^m \circ \phi^m) = \rho(g^n \circ \phi^n, f) \le 1/l$ for each $n > n_0(l)$, $l \in \mathbf{N}$. The function $g^n \circ \phi^n$ is of bounded variation, consequently, the function f is also of bounded variation. That is $f \in \mathbf{P}(\bar{U})$. Thus $\mathbf{P}(\bar{U})$ is complete.

Take any sequence γ^n of $C^2_{ab}([a, b], \mathcal{A}_v)$ paths in U converging to γ relative to the metric ρ on $\mathbf{P}(\bar{U})$ and the latter metric space is complete as it was demonstrated above. In view of Formula 17.1(3) and Property 16$(G3)$ the sequence $[g; \omega, \gamma^n|_{[a,x]}]$ is fundamental in $\mathbf{P}(\bar{U})$. On the other hand, the generalized function g is continuous on $\mathcal{B}(U, \mathcal{A}_r) \times \mathbf{P}(\bar{U})$, consequently, the sequence $[g; \omega, \gamma^n|_{[a,x]}]$ converges in $\mathcal{B}(U, \mathcal{A}_r) \times \mathbf{P}(\bar{U})$ to $[g; \omega, \gamma^n|_{[a,x]}]$ for each $a < x \le b$, hence $[g; \omega, \gamma|_{[a,x]}] = \lim_n [g; \omega, \gamma^n|_{[a,x]}]$ in $\mathbf{P}(\bar{U})$. By the conditions of this lemma $[g; \omega, \gamma|_{[a,x]}] \in \mathbf{P}(U)$, since $\gamma([a, b]) \subset U$. Thus the function $[g; \omega, \gamma|_{[a,x]}]$ by $x \in [a, b]$ is of bounded variation:

$V_a^b[g; \omega, \gamma|_{[a,x]}] < \infty$.

18. Definition. Let f and η be two line generalized functions on $\mathcal{B}(U, \mathcal{A}_v) \times \mathbf{P}(U) \times \mathcal{V}(U, \mathcal{A}_v)$. We define a line functional with values denoted by

$[\int_\gamma f d\eta, \omega^1 \otimes \omega) := [\hat{f}; \omega^1, \gamma; [\eta; \omega, \kappa]]|_{\kappa=\gamma} = [\hat{f}; \omega^1, *; [\eta; \omega, *]](\gamma)$,

where $\gamma \in \mathbf{P}(U)$ is a rectifiable path in U, $\omega, \omega^1 \in \mathcal{B}(U, \mathcal{A}_v)$ are any test functions. The functional $\int_\gamma f d\eta$ is called the non-commutative line integral over the Cayley-Dickson algebra $\mathcal{A}_v$ of line generalized functions f by η. Quite analogously such integral is defined for line generalized functions f and η on $\mathcal{D}(U, \mathcal{A}_v) \times \mathbf{P}(U) \times \mathcal{V}(U, \mathcal{A}_v)$.

19. Theorem. *Let F and Ξ be two generalized functions on U, $F,\ \Xi \in \mathcal{B}'(U, \mathcal{A}_v)$ or $F,\ \Xi \in \mathcal{D}'(U, \mathcal{A}_v)$, then the the non-commutative line integral over the Cayley-Dickson algebra $\mathcal{A}_v$ of line generalized functions f by ξ exists, where f is induced by F and ξ by Ξ.*

Proof. At first it easy to mention that Definition 18 is justified by Definition 16 and Lemma 17.2, since the function $[\eta; \omega, \kappa|_{[a,x]}]$ is of bounded variation by the variable x for each rectifiable path $\kappa \in \mathbf{P}(U)$ and any test function ω (see Properties 16$(G1 - G3)$), while the operator $\hat{f}$ always exists in the class of generalized line operators, $\hat{f} = dg/dz$, $(dg(z)/dz).1 = f(z)$ (see Property 16$(G5)$).

Each generalized function $f \in \mathcal{B}(U, \mathcal{A}_v)$ can be written in the form:

(1) $[f, \omega) = \sum_{j,k=0}^{2^v-1} [f_{j,k}, \omega_k) i_j$,

where each $f_{j,k}$ is a real valued generalized function, $f_{j,k} \in \mathcal{B}'(U, \mathbf{R})$, $\omega = \sum_k \omega_k i_k$, $\omega_k \in \mathcal{B}(U, \mathbf{R})$ is a real valued test function, $[f_{j,k}, \omega_k) = [f_j, \omega_k i_k)$, $[f, \omega) = \sum_j [f_j, \omega) i_j$, $[f_j, \omega) \in \mathbf{R}$ for each $j = 0, ..., 2^v - 1$ and $\omega \in \mathcal{B}(U, \mathcal{A}_v)$, $i_0, ..., i_{2^v-1}$ is the standard base of generators of the Cayley-Dickson algebra $\mathcal{A}_v$. It is well-known that in the space $\mathcal{B}'(U, \mathbf{R})$ of generalized functions the space $\mathcal{B}(U, \mathbf{R})$ of test functions is everywhere dense (see [6] and §9 above). In view of the decomposition given by Formula (1) we get that $\mathcal{B}(U,\mathcal{A}_v)$ is everywhere dense in $\mathcal{B}'(U,\mathcal{A}_v)$. Thus sequences of test functions F^l and Ξ^l exist converging to F and Ξ correspondingly.

Without loss of generality we can embed U into $\mathcal{A}_v$ taking its ϵ-enlargement (open neighborhood) in case of necessity. So it is sufficient to treat the case of a domain U in $\mathcal{A}_v$. In view of the analog of the Stone-Weierstrass theorem (see [22, 23]) in $C^0(Q,\mathcal{A}_v)$ for each compact canonical closed subset in $\mathcal{A}_v$ the family of all super-differentiable on Q functions is dense, consequently, the space $\mathcal{H}(U,\mathcal{A}_v)$ of all super-differentiable functions on U is everywhere dense in $\mathcal{D}(U,\mathcal{A}_v)$. For each rectifiable path γ in the domain U a compact canonical closed domain Q exists $Q\subset U$ so that $\gamma([a,b])\subset Q$. Therefore, it is sufficient to consider test functions with compact supports in Q. Thus we take super-differentiable functions F^n and Ξ^n.

Let γ^l be a sequence of rectifiable paths continuously differentiable, $\gamma^l\in C^1([a,b],\mathcal{A}_v)$, converging to γ in $\mathbf{P}(U)$ relative to the metric ρ.

Then for any super-differentiable functions p and q we have

(2) $\int_{\gamma^l} p(z)dq(z) = \int_a^b (d\zeta(z)/dz).[(dq(z)/dz).d\gamma^l(t)]|_{z=\gamma^l(t)}$

$= \int_a^b \sum_{k=0}^{2^v-1} (\partial\zeta(z)/\partial z_k)[\sum_{j=0}^{2^v-1}(\partial q_k(z)/\partial z_j)d\gamma^l_j(t)]$,

since each super-differentiable function is Fréchet differentiable, $d\gamma^l_j(t) = {\gamma^l_j}'(t)dt$, where $(d\zeta(z)/dz).1 = p(z)$ and for the corresponding phrases of them for each $z\in U$. On the other hand, the functional

(3) $\int_a^b \sum_{k=0}^{2^v-1} (\partial\zeta(z)/\partial z_k)[\sum_{j=0}^{2^v-1}(\partial q_k(z)/\partial z_j)d\gamma^l_j(t)]$ is continuous on $\mathcal{B}(U,\mathcal{A}_v)^2\times\mathbf{P}(U)$, i.e. for $\zeta,p\in\mathcal{B}(U,\mathcal{A}_v)$ and $\gamma\in\mathbf{P}(U)$ as well.

For a rectifiable path γ in U it is possible to take a sequence of open ϵ neighborhoods $\Gamma^\epsilon := \bigcup_{z\in\gamma([a,b])} \breve{B}(\mathcal{A}_v,z,\epsilon)$, $\epsilon=\epsilon(l)=1/l$, where $\breve{B}(\mathcal{A}_v,z,\epsilon) := \{y:\ y\in\mathcal{A}_v;\ |y-z|<\epsilon\}$. Therefore, for each function ν of bounded variation on U and each rectifiable path γ in U a sequence of test functions θ^l with supports contained in $\Gamma^{1/l}$ exists such that

$\lim_l \int_U [(d\zeta(z)/dz).\theta^l(z)]\lambda(dz) = \int_\gamma p(z)d\nu(z)$

for each super-differentiable test functions $p,\zeta\in\mathcal{H}(U,\mathcal{A}_v)$ with $(d\zeta(z)/dz).1 = p(z)$ on U, where λ denotes the Lebesgue measure on U induced by the Lebesgue measure on the real shadow $\mathbf{R}^{2^v}$ of the Cayley-Dickson algebra $\mathcal{A}_v$, where $\mathcal{H}(U,\mathcal{A}_v)$ denotes the family of all super-differentiable functions on the domain U with values in the Cayley-Dickson algebra $\mathcal{A}_v$.

Using the latter property and in accordance with Formulas $(1-3)$ and $16(G6,G7)$ we put:

(4) $[\xi;\omega,\gamma] := \lim_l[\Xi^l;\omega,\gamma] = \lim_l \int_\gamma \Xi^l(y)\omega(y)dy$ and

(5) $[\hat{f};\omega^1,\gamma;\nu] = \lim_l[dG^l/dz;\omega^1,\gamma;\nu] = \lim_l \int_\gamma \{(dG^l(z)/dz).\omega^1(z)\}d\nu(z)$

for any $\nu\in\mathcal{V}(U,\mathcal{A}_v)$, where $(dG^l/dz).1 = F^l(z)$ on U.

Therefore Ξ^l converges to ξ and dG^l/dz converges to $\hat{f}$, where $[\xi;\omega,*](\kappa|_{[a,x]}) = [\xi;\omega,\kappa|_{[a,x]}]$ for each $\kappa\in\mathbf{P}(U)$, $a<x\le b$ (see Lemma 17.2). Therefore, from Formulas $(2-5)$ and Lemmas 17.1 and 17.2 we infer that

(6) $[\int_\gamma f d\eta, \omega^1\otimes\omega) = \lim_l[dG^l/dz;\omega^1,*;[\Xi^l;\omega,*]](\gamma^l)$

$= \lim_l \int_{\gamma^l}[dG^l/dz;\omega^1,*;d[\Xi^l;\omega,*](z)]$,

where $z=\gamma^l(t)$, $a\le t\le b$.

19.1. Corollary. *If $F:U\to\mathcal{A}_v$ is a continuous function on U and Ξ is a generalized function on U, then the non-commutative line integral over the Cayley-Dickson algebra $\mathcal{A}_v$ of line generalized functions f by ξ*

(1) $[\int_\gamma f d\xi, \omega^1\otimes\omega)$

exists, where f is induced by F and ξ by Ξ.

Proof. This follows from Theorem 19 and the fact that each continuous function F on U gives the corresponding regular line operator valued generalized function on the space of test functions ω^1 in $\mathcal{B}(U,\mathcal{A}_v)$ or $\mathcal{D}(U,\mathcal{A}_v)$:

$[\hat{F};\omega^1,\gamma] = \int_\gamma (\hat{F}(z).\omega^1(z))dz$.

In this case one can take the marked function $\omega^1 = \chi_V$, where V is a compact canonical closed sub-domain in U, since $\gamma([a,b])$ is compact for each rectifiable path γ in U so that $\gamma([a,b]) \subset V$ for the corresponding compact sub-domain V. This gives $\hat{F}.\chi_V(z) = F(z)$ for each $z \in V$ and $\hat{F}.\chi_V(z) = 0$ for each $z \in U \setminus V$.

19.2. Corollary. *If $F \in \mathcal{B}'(U, \mathcal{A}_v)$ or $F \in \mathcal{D}'(U, \mathcal{A}_v)$ is a generalized function on U and Ξ is a function of bounded variation on U, then the non-commutative line integral over the Cayley-Dickson algebra $\mathcal{A}_v$ of line generalized functions f by ξ*

(1) $[\int_\gamma f d\xi, \omega^1 \otimes \omega)$

exists, where f is induced by F and ξ by Ξ.

Proof. In this case we put

$[\xi; \omega, \kappa] := \int_\kappa \omega(z) d\Xi(z)$

for each test function ω and each rectifiable path κ in U. It is sufficient to take marked test function $\omega(z) = 1$ for each $z \in U$ that gives $d[\xi; 1, *] = d\Xi$. Thus this corollary follows from Theorem 19.

19.3. Corollary. *If F is a continuous function on U and Ξ is a function of bounded variation on U, then the non-commutative line integral over the Cayley-Dickson algebra $\mathcal{A}_v$ of line generalized functions f by ξ*

(1) $[\int_\gamma f d\xi, \omega^1 \otimes \omega)$

exists, where f is induced by F and ξ by Ξ. Moreover, this integral coincides with the non-commutative line integral from §3 for the unit test functions $\omega(z) = \omega^1(z) = 1$ for each $z \in U$:

(2) $[\int_\gamma f d\xi, 1 \otimes 1) = \int_\gamma f d\xi$.

Proof. This follows from the combination of two preceding corollaries, since for a rectifiable path γ its image in U is contained in a compact sub-domain V in U, i.e. $\gamma([a,b]) \subset V$.

19.4. Convolution formula for solutions of partial differential equations.

Using convolutions of generalized functions a solution of the equation

$(C1)$ $(\Upsilon^s + \beta) f = g$ in $\mathcal{B}(\mathbf{R}^n, Y)$ or in the space $\mathcal{B}'(\mathbf{R}^n, Y)_l$ is:

$(C2)$ $f = \mathcal{E}_{\Upsilon^s+\beta} * g$,

where $\mathcal{E}_{\Upsilon^s+\beta}$ denotes a fundamental solution of the equation

$(C3)$ $(\Upsilon^s + \beta)\mathcal{E}_{\Upsilon+\beta} = \delta$,

$(\delta, \phi) = \phi(0)$ (see §9). The fundamental solution of the equation

$(C4)$ $A_0 \mathcal{V} = \delta$ with $A_0 = (\Upsilon^s + \beta)(\Upsilon_1^{s_1} + \beta_1)$

can be written as the convolution

$$(C5) \quad \mathcal{V} =: \mathcal{V}_{A_0} = \mathcal{E}_{\Upsilon^s+\beta} * \mathcal{E}_{\Upsilon_1^{s_1}+\beta_1}.$$

In view of Formulas $4(7-9)$ each generalized function $\mathcal{E}_{\Upsilon^s+\beta}$ can also be found from the elliptic partial differential equation

$(C6)$ $\Xi_\beta \Psi_{\Upsilon^s+\beta} = \delta$ by the formula:

$(C7)$ $\mathcal{E}_{\Upsilon^s+\beta} = [(\Upsilon^s + \beta)^*]\Psi_{\Upsilon^s+\beta}$, where

$(C8)$ $\Xi_\beta := (\Upsilon^s + \beta)(\Upsilon^s + \beta)^*$

(see §33 [28]).

20. Poly-functionals. Let $\mathbf{a}_k : \mathcal{B}(U, \mathcal{A}_r)^k \to \mathcal{A}_r$ or $\mathbf{a}_k : \mathcal{D}(U, \mathcal{A}_r)^k \to \mathcal{A}_r$ be a continuous mapping satisfying the following three conditions:

$(P1)$ $[\mathbf{a}_k, \omega^1 \otimes ... \otimes \omega^k)$ is $\mathbf{R}$ homogeneous

$[\mathbf{a}_k, \omega^1 \otimes ... \otimes (\omega^l t) \otimes ... \otimes \omega^k) = [\mathbf{a}_k, \omega^1 \otimes ... \otimes \omega^l \otimes ... \otimes \omega^k)t = [\mathbf{a}_k t, \omega^1 \otimes ... \otimes \omega^k)$

for each $t \in \mathbf{R}$ and $\mathcal{A}_r$ additive

$[\mathbf{a}_k, \omega^1 \otimes ... \otimes (\omega^l + \kappa^l) \otimes ... \otimes \omega^k) = [\mathbf{a}_k, \omega^1 \otimes ... \otimes \omega^l \otimes ... \otimes \omega^k) + [\mathbf{a}_k, \omega^1 \otimes ... \otimes \kappa^l \otimes ... \otimes \omega^k)$

by any $\mathcal{A}_r$ valued test functions ω^l and κ^l, when other are marked, $l = 1, ..., k$, i.e. it is k $\mathbf{R}$ linear and k $\mathcal{A}_r$ additive, where $[\mathbf{a}_k, \omega^1 \otimes ... \otimes \omega^k)$ denotes a value of $\mathbf{a}_k$ on given test $\mathcal{A}_r$ valued functions $\omega^1, ..., \omega^k$;

$(P2)$ $[\mathbf{a}_k\alpha, \omega^1\otimes ...\otimes(\omega^l\beta)\otimes ...\otimes\omega^k) = ([\mathbf{a}_k, \omega^1\otimes ...\otimes\omega^l\otimes ...\otimes\omega^k)\alpha)\beta = [(\mathbf{a}_k\alpha)\beta, \omega^1\otimes ...\otimes\omega^l\otimes ...\otimes\omega^k)$ for all real-valued test functions and $\alpha, \beta\in\mathcal{A}_r$;

$(P3)$ $[\mathbf{a}_k, \omega^{\sigma(1)}\otimes ...\otimes\omega^{\sigma(k)}) = [\mathbf{a}_k, \omega^1\otimes ...\otimes\omega^k)$ for all real-valued test functions and each transposition σ, i.e. bijective surjective mapping $\sigma : \{1, ..., k\}\to\{1, ..., k\}$.

Then $\mathbf{a}_k$ will be called the symmetric k $\mathbf{R}$ linear k $\mathcal{A}_r$ additive continuous functional, $1\le k\in\mathbf{Z}$. The family of all such symmetric functionals is denoted by $\mathcal{B}'_{k,s}(U, \mathcal{A}_v)$ or $\mathcal{D}'_{k,s}(U, \mathcal{A}_r)$ correspondingly. A functional satisfying Conditions $(P1, P2)$ is called a continuous k-functional over $\mathcal{A}_r$ and their family is denoted by $\mathcal{B}'_k(U, \mathcal{A}_r)$ or $\mathcal{D}'_k(U, \mathcal{A}_r)$. When a situation is outlined we may omit for short "continuous" or "k $\mathbf{R}$ linear k $\mathcal{A}_v$ additive".

The sum of two k-functionals over the Cayley-Dickson algebra $\mathcal{A}_r$ is prescribed by the equality:

$(P4)$ $[\mathbf{a}_k+\mathbf{b}_k, \omega^1\otimes ...\otimes\omega^k) = [\mathbf{a}_k, \omega^1\otimes ...\otimes\omega^k) + [\mathbf{b}_k, \omega^1\otimes ...\otimes\omega^k)$
for each test functions. Using Formula $(P4)$ each k-functional can be written as

(1) $[\mathbf{a}_k, \omega^1\otimes ...\otimes\omega^k) = [\mathbf{a}_{k,0}i_0, \omega^1\otimes ...\otimes\omega^k) + ... + [\mathbf{a}_{k,2^r-1}i_{2^r-1}, \omega^1\otimes ...\otimes\omega^k)$,
where $[\mathbf{a}_{k,j}, \omega^1\otimes ...\otimes\omega^k)\in\mathbf{R}$ is real for all real-valued test functions $\omega^1, ..., \omega^k$ and each j; i_0,...,i_{2^r-1} denote the standard generators of the Cayley-Dickson algebra $\mathcal{A}_r$.

The direct product $\mathbf{a}_k\otimes\mathbf{b}_p$ of two functionals $\mathbf{a}_k$ and $\mathbf{b}_p$ for the same space of test functions is a $k+p$-functional over $\mathcal{A}_r$ given by the following three conditions:

$(P5)$ $[\mathbf{a}_k\otimes\mathbf{b}_p, \omega^1\otimes ...\otimes\omega^{k+p}) = [\mathbf{a}_k, \omega^1\otimes ...\otimes\omega^k)[\mathbf{b}_p, \omega^{k+1}\otimes ...\otimes\omega^{k+p})$
for any real-valued test functions $\omega^1, ..., \omega^{k+p}$;

$(P6)$ if $[\mathbf{b}_p, \omega^{k+1}\otimes ...\otimes\omega^{k+p})\in\mathbf{R}$ is real for any real-valued test functions, then

$[(\mathbf{a}_kN_1)\otimes(\mathbf{b}_pN_2), \omega^1\otimes ...\otimes\omega^{k+p}) = ([\mathbf{a}_k\otimes\mathbf{b}_p, \omega^1\otimes ...\otimes\omega^{k+p})N_1)N_2$
for any real-valued test functions $\omega^1, ..., \omega^{k+p}$ and Cayley-Dickson numbers $N_1, N_2\in\mathcal{A}_r$;

$(P7)$ if $[\mathbf{a}_k, \omega^1\otimes ...\otimes\omega^k)\in\mathbf{R}$ and $[\mathbf{b}_p, \omega^{k+1}\otimes ...\otimes\omega^{k+p})\in\mathbf{R}$ are real for any real-valued test functions, then

$[\mathbf{a}_k\otimes\mathbf{b}_p, \omega^1\otimes ...\otimes(\omega^lN_1)\otimes ...\otimes\omega^{k+p}) = [\mathbf{a}_k\otimes\mathbf{b}_p, \omega^1\otimes ...\otimes\omega^{k+p})N_1$
for any real-valued test functions $\omega^1, ..., \omega^{k+p}$ and each Cayley-Dickson number $N_1\in\mathcal{A}_r$ for each $l = 1, ..., k+p$.

Therefore, we can now consider a partial differential operator of order u acting on a generalized function $f\in\mathcal{B}'(U, \mathcal{A}_r)$ or $f\in\mathcal{D}'(U, \mathcal{A}_r)$ and with generalized coefficients either $\mathbf{a}_\alpha\in\mathcal{B}'_{|\alpha|}(U, \mathcal{A}_r)$ or all $\mathbf{a}_\alpha\in\mathcal{D}'_{|\alpha|}(U, \mathcal{A}_r)$ correspondingly:

$$(1)\quad Af(x) = \sum_{|\alpha|\le u}(\partial^\alpha f(x))\otimes[(\mathbf{a}_\alpha(x))\otimes 1^{\otimes(u-|\alpha|)}],$$

where $\partial^\alpha f = \partial^{|\alpha|}f(x)/\partial x_0^{\alpha_0}...\partial x_n^{\alpha_n}$, $x = x_0i_0 + ...x_ni_n$, $x_j\in\mathbf{R}$ for each j, $1\le n = 2^r-1$, $\alpha = (\alpha_0, ..., \alpha_n)$, $|\alpha| = \alpha_0 + ... + \alpha_n$, $0\le\alpha_j\in\mathbf{Z}$, $[1, \omega) := \int_U\omega(y)\lambda(dy)$, λ denotes the Lebesgue measure on U, for convenience $1^{\otimes 0}$ means the multiplication on the unit $1\in\mathbf{R}$. The partial differential equation

(2) $Af = g$ in terms of generalized functions has a solution f means by the definition that

(3) $[Af, \omega^{\otimes(u+1)}) = [g, \omega^{\otimes(u+1)})$
for each real-valued test function ω on U, where $\omega^{\otimes k} = \omega\otimes ...\otimes\omega$ denotes the k times direct product of a test functions ω.

21. Theorem. *Let $A = A_u$ be a partial differential operator with generalized over the Cayley-Dickson algebra $\mathcal{A}_r$ coefficients of an even order $u = 2s$ on U such that each $\mathbf{a}_\alpha$ is symmetric for $|\alpha| = u$ and A has the form*

(4) $Af = (B_{u,1}f)c_{u,1} + ... + (B_{u,k}f)c_{u,k}$, where each

(5) $B_{u,p} = B_{u,p,0} + Q_{u-1,p}$
is a partial differential operator by variables $x_{m_{u,1}+...+m_{u,p-1}+1}, ..., x_{m_{u,1}+...+m_{u,p}}$ and of the order u, $m_{u,0} = 0$, $c_{u,k}(x)\in\mathcal{A}_r$ for each k, its principal part

(6) $B_{u,p,0}f = \sum_{|\alpha|=s} (\partial^{2\alpha} f) \otimes \mathbf{a}_{p,2\alpha}(x)$
is elliptic, i.e. $\sum_{|\alpha|=s} y^{2\alpha}[\mathbf{a}_{p,2\alpha}, \omega^{\otimes 2s}) \geq 0$ *for all* $y_{k(1)},...,y_{k(m_{u,p})}$ *in* $\mathbf{R}$ *with* $k(1) = m_{u,1} + ... + m_{u,p-1} + 1,...,k(m_{u,p}) = m_{u,1} + ... + m_{u,p}$, $y^\beta := y_{k(1)}^{\beta_{k_1}} ... y_{k(m_{u,p})}^{\beta_{k(m_{u,p})}}$ *and* $[\mathbf{a}_{p,2\alpha}, \omega^{\otimes 2s}) \geq 0$ *for each real test function* ω, *either* $0 \leq r \leq 3$ *and* f *is with values in* $\mathcal{A}_r$, *or* $r \geq 4$ *and* f *is real-valued on real-valued test functions. Then three partial differential operators* Υ^s *and* Υ_1^s *and* Q *of orders* s *and* p *with* $p \leq u-1$ *with generalized on* U *coefficients with values in* $\mathcal{A}_v$ *exist such that*

(7) $[Af, \omega^{\otimes(u+1)}) = [\Upsilon^s(\Upsilon_1^s f) + Qf, \omega^{\otimes(u+1)})$ *for each real-valued test function* ω *on* U.

Proof. If a_{2s} is a symmetric functional and $[\mathbf{c}_s, \omega^{\otimes s}) = [\mathbf{a}_{2s}, \omega^{\otimes 2s})^{1/2}$ for each real-valued test function ω, then by Formulas $20(P1, P2)$ this functional $\mathbf{c}_s$ has an extension up to a continuous s-functional over the Cayley-Dickson algebra $\mathcal{A}_r$. This is sufficient for Formula (7), where only real-valued test functions ω are taken.

Consider a continuous p-functional $\mathbf{c}_p$ over $\mathcal{A}_v$, $p \in \mathbf{N}$. Supply the domain U with the metric induced from the corresponding Euclidean space or the Cayley-Dickson algebra in which U is embedded. It is possible to take a sequence of non-negative test functions $_l\omega$ on U with a support $supp(_l\omega)$ contained in the ball $B(U, z, 1/l)$ with center z and radius $1/l$ and $_l\omega$ positive on some open neighborhood of a marked point z in U so that $\int_U {_l\omega}(z)\lambda(dz) = 1$ for each $l \in \mathbf{N}$. If the p-functional $\mathbf{c}_p$ is regular and realized by a continuous $\mathcal{A}_v$ valued function g on U^p, then $\lim_l [\mathbf{c}_p, \omega^{\otimes p}) = g(z, ..., z)$. Thus the partial differential equation $20(2)$ for regular functionals and their derivatives implies the classical partial differential equation $2(1)$.

Therefore, the statement of this theorem follows from Theorem 10, and §§14, 15 and 20, since the spaces of test functions are dense in the spaces of generalized functions (see §19).

22. Corollary. *If* $Af = \sum_{j,k} (\partial^2 f(z)/\partial z_k \partial z_j) \otimes a_{j,k}(z) + \sum_j (\partial f(z)/\partial z_j) \otimes b_j(z) \otimes 1 + f(z) \otimes \eta(z) \otimes 1$ *is a second order partial differential operator with generalized coefficients in* $\mathcal{B}'(U, \mathcal{A}_r)$ *or* $\mathcal{D}'(U, \mathcal{A}_r)$, *where each* $a_{j,k}$ *is symmetric,* f *and* $\mathcal{A}_r$ *are as in §20, then three partial differential operators* $\Upsilon + \beta$, $\Upsilon_1 + \beta_1$ *and* Q *of the first order with generalized coefficients with values in* $\mathcal{A}_v$ *for suitable* $v \geq r$ *of the same class exist such that*

(1) $[Af, \omega^{\otimes 3}) = [(\Upsilon + \beta)((\Upsilon_1 + \beta_1)f + Qf), \omega^{\otimes 3})$ *for each real-valued test function* ω *on* U.

Proof. This follows from Theorem 21 and Corollary 12 and §§2 and 8.

23. Anti-derivatives of first order partial differential operators with generalized coefficients.

Theorem. *Let* Υ *be a first order partial differential operator given by the formula*

(1) $\Upsilon f = \sum_{j=0}^n (\partial f/\partial z_j) \otimes [i_j^* \psi_j(z)]$ *or*

(2) $\Upsilon f = \sum_{j=0}^n (\partial f/\partial z_j) \otimes \phi_j^*(z)$,
where $supp(\psi_j(z)) = U$ *or* $supp(\phi_j(z)) = U$ *for each* j *respectively,* f *and* $\psi_j(z)$ *or* $\psi_j(z)$ *are* $\mathcal{A}_v$*-valued generalized functions in* $\mathcal{B}'(U, \mathcal{A}_r)$ *or* $\mathcal{D}'(U, \mathcal{A}_r)$ *on the domain* U *satisfying Conditions* $1(D1, D2)$, $alg_{\mathbf{R}}\{[\phi_j, \omega), [\phi_k, \omega), [\phi_l, \omega)\}$ *is alternative for all* $0 \leq j, k, l \leq 2^v - 1$ *and* $alg_{\mathbf{R}}\{[\phi_0, \omega), ..., [\phi_{2^v-1}, \omega)\} \subset \mathcal{A}_v$ *for each real-valued test function* ω *on* U. *Then its anti-derivative operator* $\mathcal{I}_\Upsilon$ *exists such that* $\Upsilon \mathcal{I}_\Upsilon f = f$ *for each continuous generalized function* $f : U \to \mathcal{A}_v$ *and it has an expression through line integrals of generalized functions.*

Proof. When an operator with generalized coefficients is given by Formula (1), we shall take unknown generalized functions $\nu_j(z) \in \mathcal{A}_v$ as solutions of the system of partial differential equations by real variables z_k:

(3) $[(\partial \nu_j(z)/\partial z_j) \otimes \psi_j(z), \omega^{\otimes 2}) = [1, \omega^{\otimes 2})$ for all $1 \leq j \leq n$;

(4) $[\psi_k(z) \otimes (\partial \nu_j(z)/\partial z_k), \omega^{\otimes 2}) = [\psi_j(z) \otimes (\partial \nu_k(z)/\partial z_j), \omega^{\otimes 2})$ for all $1 \leq j < k \leq n$ and and real-valued test functions ω on U.

If the operator is given by Formula (2) we consider the system of partial differential equations:

(5) $[((dg(z)/dz).[\partial\nu_j(z)/\partial z_k]) \otimes \phi^*_k(z) + ((dg(z)/dz).[\partial\nu_k(z)/\partial z_j]) \otimes \phi^*_j(z), \omega^{\otimes 2}) = 0$ for all $0 \le j < k \le n$;

(6) $\partial\nu_j(z)/\partial z_j = \phi_j(z)$ for all $j = 0, ..., n$;

(7) $[((dg(z)/dz).\phi_j(z)) \otimes \phi^*_j(z), \omega^{\otimes 2}) = [f(z) \otimes 1, \omega^{\otimes 2})$ for each $j = 0, ..., n$ and every real-valued test function ω.

Certainly the system of differential equations given by Formulas $(3, 4)$ or $(5-7)$ have solutions in the spaces of test functions $\mathcal{B}(U, \mathcal{A}_r)$ or $\mathcal{D}(U, \mathcal{A}_r)$, when all functions ψ_j or ϕ_j are in the same space respectively. Applying §§4 or 5 we find generalized functions ν_j resolving these system of partial differential equations correspondingly, when all functions ψ_j or ϕ_j are generalized functions, since the spaces of test functions are dense in the spaces of generalized functions (see §19). Substituting line integrals $\int_\gamma q(y)d\nu_j(y)$ from §§4 and 5 on line integrals $[\int_\gamma q(y)d\nu_j(y), \omega^1 \otimes \omega)$ from §19 one gets the statement of this theorem, since test functions ω^1 and ω in the line integrals of generalized functions can also be taken real-valued and the real field is the center of the Cayley-Dickson algebra $\mathcal{A}_v$. Therefore, we infer that

(8) $\partial[\int_{\gamma^\alpha|_{<a_\alpha, t_z]}} f(y)d\nu_j(y), \omega \otimes \omega)/\partial z_k = [\hat{f}(z).[d\nu_j(z)/dz_k], \omega \otimes \omega)$

for each real-valued test function ω and $z \in U$, where $\gamma^\alpha(t_z) = z$, $t_z \in < a_\alpha, b_\alpha >$, $\alpha \in \Lambda$. Equality (8), Theorem 19 and Corollaries 19.1-19.3 and Conditions $20(P1-P7)$ give the formula for an anti-derivative operator:

$$(9) \quad [\mathcal{I}_\Upsilon f, \omega \otimes \omega) = [_\Upsilon \int f(z)dz, \omega \otimes \omega) = (n+1)^{-1} \sum_{j=0}^{n} \{ [\int_{\gamma^\alpha|_{[a_\alpha, t]}} q(y)d\nu_j(y), \omega \otimes \omega)$$

for each real-valued test-function ω, where $\alpha \in \Lambda$, $a_\alpha \le t \le b_\alpha$, $t = t_z$, $z = \gamma(t)$, consequently,

(10) $[\Upsilon_\Upsilon \int f(y)dy, \omega^{\otimes 3}) = [f \otimes 1 \otimes 1, \omega^{\otimes 3})$.

23.1. Note. Certainly, the case of the partial differential operator

(1) $\Upsilon f = \sum_{j=0}^n (\partial f/\partial z_{k(j)}) \otimes \phi^*_{k(j)}(z)$,

where $0 \le k(0) < k(1) < ... < k(n) \le 2^v - 1$ reduces to the considered in §23 case by a suitable change of variables $z \mapsto y$ so that $z_{k(j)} = y_j$.

24. Example. We consider a consequence of Formulas $15(2-6)$. If $q(t)$ is a differentiable function on the real field $\mathbf{R}$ having simple zeros $q(t_j) = 0$ (i.e. zeros of the first order), then

(1) $\delta(q(t)) = \sum_j \frac{1}{q(t_j)} \delta(t - t_j)$,

where the sum is accomplished by all zeros t_j of the equation $q(t) = 0$ (see Formula II.2.6(IV) [6]). Therefore, if $\gamma(\tau)$ is a C^1 path in U intersecting the surface $\partial U_s \cap \partial U_p$ at the marked point x of index $l = l_{s,p}(x)$, $\gamma(\tau_0) = x$, $0 < \tau_0 < 1$, such that $\gamma(\tau) \in U_s$ for each $\tau < \tau_0$ and $\gamma(\tau) \in U_p$ for each $\tau > \tau_0$ then

(2) $dg(\gamma(\tau))/d\tau = dg^1(\gamma(\tau))/d\tau + h \sum_{j=1}^l \delta(P_j) dP_j(\gamma(\tau))/d\tau$,

where $\theta(t) = 0$ for $t < 0$ and $\theta(t) = 1$ for $t \ge 0$, $g^1(\gamma(\tau)) = g(\gamma(\tau)) - h\theta(\tau - \tau_0)$,

(2.1) $h = \lim_{\tau \downarrow \tau_0} g(\gamma(\tau)) - \lim_{\tau \uparrow \tau_0} g(\gamma(\tau))$,

(2.2) $dP_j(\gamma(\tau))/d\tau = \sum_{k=1}^n (\partial P_j(z)/\partial z_k)(\partial \gamma_k(\tau)/d\tau)|_{z=\gamma(\tau)}$ (see Example I.2.2.2 [6]). Particularly, if a point x is of index 1, then Formula (2) simplifies:

(3) $dg(\gamma(\tau))/d\tau = dg^1(\gamma(\tau))/d\tau + h\delta(P)[dP(\gamma(\tau))/d\tau]$. Particularly, these formulas can be applied to $d\nu_j$.

Let a partial differential differential operator Q be given by Formula 14(7) and functions ν_k are found (see Theorems 5 and 21 above). We put in accordance with Formula 15(6)

(4) $\nu^1_k(z) = \nu_k(z) - \sum_{s,p} h_{k;s,p}(z) \theta(P_{s,p;1}(z), ..., P_{s,p;m_j}(z))$,

where

(5) $h_{k;s,p}(z(x)) = h_k(x)|_{x \in \Gamma^{s,p}} :=$

$\lim_{y_{1;s,p} \downarrow 0, ..., y_{n;s,p} \downarrow 0} \nu_k(z(x+y)) - \lim_{y_{1;s,p} \downarrow 0, ..., y_{n;s,p} \downarrow 0} \nu_k(z(x-y))$,

where the sum is by s and p with $\partial U_s \cap \partial U_p \neq \emptyset$.

Let a domain W be a canonical closed compact set in the Euclidean space $\mathbf{R}^{n+1}$ embedded into $\mathcal{A}_v$ and contained in a canonical closed compact domain U so that $W = \{z \in U: z_j = 0\ \forall n < j \le 2^v - 1\}$. Thus Υ from test and generalized functions on W is extended on test and generalized functions on U. We can put $\nu_j = 0$ for $n < j \le 2^v - 1$, when $n < 2^v - 1$. Then for the rectifiable path γ (see above) we get

$$(6)\quad (n+1)^{-1}\sum_{k=0}^{n}[\int_{\gamma|_{[a,t]}} q(y)d\nu_k(y), \omega^1 \times \omega)$$

$$= (n+1)^{-1}\sum_{k=0}^{n}\{[\int_{\gamma} q(y)d\nu_k^1(y), \omega^1 \times \omega)$$

$$+[\hat{q};\omega^1,*;[\sum_{s,p} h_{k;s,p}(z)\sum_{j=1}^{l}\delta(P_j)\sum_{m=1}^{n}(\partial P_j(z)/\partial z_m)(\partial\gamma_m(\tau)/d\tau)|_{z=\gamma(\tau)};\omega,*]](\gamma)\},$$

where $\gamma \in \{\gamma^\alpha: \alpha \in \Lambda\}$ is taken from the foliation C^1 family of paths (see §6.1 above and also Theorem 2.13 [27]), $z = \gamma(t_z)$, $t_z \in [a_\alpha, b_\alpha]$, $[a,t] = [a_\alpha, t_z]$, $l = l_{s,p}(z)$ denotes an index of a point z in the intersection of boundaries $\Gamma^{s,p} := \partial U_s \cap \partial U_p \ne \emptyset$, ω^1 and ω are real-valued test functions. Since ω^1 is real-valued, we get $\hat{f}(z).\omega^1 = f(z)\omega^1(z)$ and

(7) $[\hat{q};\omega^1,*;[\sum_{s,p} h_{k;s,p}(z)\sum_{j=1}^{l}\delta(P_j)\sum_{m=1}^{n}(\partial P_j(z)/\partial z_m)(\partial\gamma_m(\tau)/d\tau)|_{z=\gamma(\tau)};\omega,*]](\gamma)$
$= \sum_{s,p}[q(z)\omega^1(z);[\theta(\tau-\tau_{s,p})h_{k;s,p}(z),\omega))|_{z=\gamma(\tau)}$,
where $\tau_{s,p}$ corresponds to the intersection point $\gamma(\tau_{s,p})$ of γ with $\Gamma^{s,p} \ne \emptyset$. Here the expression $[q,\omega)|_{z=\gamma(\tau)} := \lim_j[q\circ\kappa^j, \omega\circ\kappa^j)$ denotes the restriction of the generalized function from U onto $\gamma([a,b])$, $\kappa^j \in \mathcal{D}(U,\mathcal{A}_v)$ is a sequence of test functions and $\kappa^j(\phi([a,b])) \subset U$ for each $j \in \mathbf{N}$, $\phi \in \mathcal{D}([a,b],\mathcal{A}_v)$, $\bigcap_{j=1}^\infty supp(\kappa^j) = \phi([a,b])$, $\lim_j \kappa^j\circ\phi = \gamma$ in $\mathbf{P}(U)$. Therefore, the derivative of the operator $[(n+1)^{-1}\sum_{k=0}^n[\int_{\gamma|_{[a,t]}} q(y)d\nu_k(y), \omega^1\times\omega)$ by the parameter $\tau \in [a,b]$ for the real test functions ω^1 and ω is the following:

(8) $\partial(n+1)^{-1}\sum_{k=0}^n[\int_{\gamma|_{[a,t]}} q(y)d\nu_k(y), \omega^1\times\omega)/\partial\tau =$
$[(n+1)^{-1}\sum_{k=0}^n\{1\otimes\hat{q}^1(z).(d\nu_k^1(\gamma(\tau))/d\tau) + \sum_{s,p}(h^{g'}_{s,p}(z).h_{k;s,p}(z) + \hat{q}^1(z).h_{k;s,p}(z) + h^{g'}_{s,p}(z).(d\nu^1_k(\gamma(\tau))/d\tau))|_{z=\gamma(\tau)})\otimes\delta(\tau-\tau_{s,p}),\omega^1\otimes\omega)$,
where $dg(z)/dz = \hat{q}(z)$ on U in the class of generalized operator $L_q(\mathcal{A}_v,\mathcal{A}_v)$ valued functions, $(dg(z)/dz).1 = q(z)$ on U, $h^{g'}_{s,p}(z) = h(z)$ is given by Formula (2.1) for the derivative operator $dg(z)/dz = g'$ instead of g on each $\Gamma^{s,p} \ne \emptyset$, $\hat{q}^1$ is given by Formula 15(6) for the function $\hat{q}(z)$ with values in $L_q(\mathcal{A}_v,\mathcal{A}_v)$ instead of $g(z)$. The terms like $\hat{q}^1(z).(d\nu^1_k(\gamma(\tau))/d\tau)$ correspond to the action of the operator valued generalized function $\hat{q}^1(z)$ on the generalized function $(d\nu^1_k(\gamma(\tau))/d\tau)$ which gives a generalized function.

Using Formulas $(6-8)$ for n constant on U and $\psi_j(z)$ or $\phi_j(z)$ respectively non-zero for each $z \in U$ and all $j = 0,...,n$ we infer that for a continuous or generalized function f

(9) $\Upsilon\mathcal{I}_\Upsilon f(z) = f(z)$, where

$$(10)\quad {}_\Upsilon\int f(z)dz := \{(n+1)^{-1}\sum_{j=0}^{n}\int_{\gamma^\alpha|_{[a_\alpha,t]}} q(z)d\nu^1_j(z),\quad \alpha\in\Lambda,\ a_\alpha \le t \le b_\alpha\},$$

where $q = (dg/dz).1$ and g is given by the Equation 5(9), since $f^1 = f$ and $h^{g'}_{s,p} = 0$ in the class of generalized functions f and in the class of continuous functions f, also $h_{k;s,p} = 0$ for $\nu_k = \nu^1_k$ on U.

Formulas $(9,10)$ show what sort of boundary conditions is sufficient to specify a unique solution for a given domain U with sub-domains U_s. If U is C^1 diffeomorphic to the half-space

$H_p := \{z \in \mathcal{A}_v : z_0p_0+...+z_{2^v-1}p_{2^v-1} \ge 0\}$, where $p = p_0i_0+...+p_{2^v-1}i_{2^v-1}$ is a marked Cayley-Dickson number, $p_0, ..., p_{2^v-1} \in \mathbf{R}$, and sub-domains U_s are not prescribed, then it is sufficient to give the boundary condition $F|_{\partial U} = G$ when a solution is in the class of continuous or generalized functions with the corresponding f and ψ_j or ϕ_j. Indeed, if the functions ν_k along γ^α are defined up to constants μ_k, the differentials are the same $d(\nu_k + \mu_k)(z)|_{z=\gamma^\alpha(\tau)} = d\nu_k|_{z=\gamma^\alpha(\tau)}$ in the anti-derivative operator, when $d\mu_k|_{z=\gamma^\alpha(\tau)} = 0$ for each $\alpha \in \Lambda$ and $\tau \in [a_\alpha, b_\alpha]$.

The operator $\mathcal{I}_\Upsilon$ may be applied also piecewise on each U_s. If a solution F is locally continuous on U and continuous on each sub-domain U_s, then boundary conditions $F|_{\partial U_s} = G^s$ for all $s = 1, ..., m$ may be necessary to specify a solution F. Without boundary conditions the anti-derivative operator applied to f gives the general solution $\mathcal{I}_\Upsilon f$ of the differential equation $\Upsilon F = f$.

If each ν_k is continuously differentiable, which is possible, when each function ψ_k or ϕ_k is continuous, and f is continuous on U, then a solution $F = \{_\Upsilon \int_{\gamma^\alpha|_{[a_\alpha,\tau]}} f(y)dy : \alpha \in \Lambda, a_\alpha \le \tau \le b_\alpha\}$ is continuously differentiable by each z_k, $z \in U$, $z = \gamma^\alpha(t_z)$.

One can also mention that the sequence $_m\omega(z_0, ..., z_n) = (2\pi m)^{-(n+1)/2} \exp\{-(z_0^2 + ... + z_n^2)/(2m)\}$ converges to the delta-function on $\mathbf{R}^{n+1}$ embedded into $\mathcal{A}_v$, $m \in \mathbf{N}$. Then the sequence $_m\theta(z_0) = \int_{-m}^{z_0} (2\pi m)^{-1/2} \exp\{-t^2/(2m)\}dt$ converges to the θ function, while each function $_m\theta(z_0)$ is analytic by z_0, since $m \in \mathbf{N}$, $\exp\{-t^2/(2m)\}$ is the analytic function with the infinite radius of convergence of its power series, while $\lim_{m\to+\infty} \int_{-\infty}^{-m} (2\pi m)^{-1/2} \exp\{-t^2/(2m)\}dt = 0$. Then each $_m\omega(z_0, ..., z_n)$ and $\prod_{j=0}^n {_m\theta(z_j)}$ can be written in the z-representation over $\mathcal{A}_v$ as the analytic function with the help of Formulas $1(1-3)$, where $n \le 2^v - 1$, $z_j \in \mathbf{R}$ for each j, $z = z_0i_0 + ... + z_{2^v-1}i_{2^v-1}$. Thus $\lim_{m\to\infty} {_m\omega(z_0, ..., z_n)} = \delta(z_0, ..., z_n)$ and $\lim_{m\to\infty} \prod_{j=0}^n {_m\theta(z_j)} = \prod_{j=0}^n \theta(z_j)$, $d\theta(z_0)/dz_0 = \delta(z_0)$.

25. Boundary conditions.

If U is a domain as in §15, we put $\mathcal{B}(\partial U, Y) = \{f|_{\partial U} : f \in \mathcal{B}(U, Y)\}$ and $\mathcal{D}(\partial U, Y) = \{f|_{\partial U} : f \in \mathcal{D}(U, Y)\}$ when a boundary ∂U is non-void so that the topologically adjoint linear over $\mathbf{R}$ spaces, left and right $\mathcal{A}_r$ modules, of generalized functions are $\mathcal{B}'(\partial U, Y)$ and $\mathcal{D}'(\partial U, Y)$.

Let us consider a generalized function $f \in \mathcal{B}'(\partial U, Y)$ or $\mathcal{D}'(\partial U, Y)$ and a test function $h \in \mathcal{B}(\partial U, Y)$ or $\mathcal{D}(\partial U, Y)$ respectively. One can take $g \in \mathcal{B}(U, Y)$ or $\mathcal{D}(U, Y)$ and a sequence $q^m \in \mathcal{B}(U, Y)$ or $\mathcal{D}(U, Y)$ with supports non intersecting with the boundary $supp(q^m) \cap \partial U = \emptyset$ such that $(g - q^m)$ tends to zero in $\mathcal{B}(V, Y)$ or $\mathcal{D}(V, Y)$ for each compact subset V in the interior $Int(U)$, when m tends to the infinity, while $\lim_m (g - q^m) = h$ in $\mathcal{B}(\partial U, Y)$ or $\mathcal{D}(\partial U, Y)$ respectively. Here as usually the interior $Int(U)$ is taken in the corresponding topological space $\mathbf{R}^n$ or $\mathcal{A}_r$. Each generalized function is a limit of test functions, consequently, a generalized function $\xi \in \mathcal{B}'(U, Y)$ or $\mathcal{D}'(U, Y)$ exists so that

$(B1)$ $\lim_m [\xi, (g - q^m)) = [f, h)$.

Vise versa if $\xi \in \mathcal{B}'(U, Y)$ or $\mathcal{D}'(U, Y)$ is a generalized function on U, then Formula $(B1)$ defines a generalized function $f \in \mathcal{B}'(\partial U, Y)$ or $\mathcal{D}'(\partial U, Y)$, which we consider as the restriction of ξ on $\mathcal{B}(\partial U, Y)$ or $\mathcal{D}(\partial U, Y)$ correspondingly. In view of the definition of convergence of test and generalized functions Formula $(B1)$ defines the unique restriction f for the given generalized function ξ.

A subsequent use of decomposition of operators into compositions of first order partial differential operators and their anti-derivation operators permits to integrate partial differential equations with locally continuous or generalized coefficients.

The results and definitions of previous sections show that for the differential equation

(1) $Af = g$,

where a partial differential operator is written in accordance with Formulas $10(1, 2)$. When ∂U is a C^1-manifold without corner points of index greater than one, the following boundary conditions may be used:

(2) $f(t)|_{\partial U} = f_0(t')$, $(\partial^{|q|} f(t)/\partial s_1^{q_1}...\partial s_n^{q_n})|_{\partial U} = f_{(q)}(t')$ for $|q| \le \alpha - 1$, where $s = (s_1, ..., s_n) \in \mathbf{R}^n$, $(q) = (q_1, ..., q_n)$, $|q| = q_1 + ... + q_n$, $0 \le q_k \in \mathbf{Z}$ for each k, $t \in \partial U$ is denoted by t', f_0, $f_{(q)}$ are given functions. Generally these conditions may be excessive, so one uses some of them or their linear combinations (see (4) below). Frequently, the boundary conditions

(3) $f(t)|_{\partial U} = f_0(t')$, $(\partial^l f(t)/\partial \nu^l)|_{\partial U} = f_l(t')$ for $1 \le l \le \alpha - 1$ are also used, where ν denotes a real variable along a unit external normal to the boundary ∂U at a point $t' \in \partial U_0$. Using partial differentiation in local coordinates on ∂U and (3) one can calculate in principle all other boundary conditions in (2) almost everywhere on ∂U.

It is possible to describe as an example a particular class of domains and boundary conditions. Suppose that a domain U_1 and its boundary ∂U_1 satisfy Conditions $(D1, i - vii)$ and $g_1 = g\chi_{U_1}$ is a regular or generalized function on $\mathbf{R}^n$ with its support in U_1. Then any function g on $\mathbf{R}^n$ gives the function $g\chi_{U_2} =: g_2$ on $\mathbf{R}^n$, where $U_2 = \mathbf{R}^n \setminus U_1$. Take now new domain U satisfying Conditions $(D1, i - vii)$ and $(D2 - D5)$:

$(D2)$ $U \supset U_1$ and $\partial U \subset \partial U_1$;

$(D3)$ if a straight line ξ containing a point w_1 (see $15(vi)$) intersects ∂U_1 at two points y_1 and y_2, then only one point either y_1 or y_2 belongs to ∂U, where $w_1 \in U_1$, $U - w_1$ and $U_1 - w_1$ are convex; if ξ intersects ∂U_1 only at one point, then it intersects ∂U at the same point;

$(D4)$ any straight line ξ through the point w_1 either does not intersect ∂U or intersects the boundary ∂U only at one point.

Take now g with $supp(g) \subset U$, then $supp(g\chi_{U_1}) \subset U_1$. Therefore, any problem (1) on U_1 can be considered as the restriction of the problem (1) defined on U, satisfying $(D1 - D4, i - vii)$. Any solution f of (1) on U with the boundary conditions on ∂U gives the solution as the restriction $f|_{U_1}$ on U_1 with the boundary conditions on ∂U.

Henceforward, we suppose that the domain U satisfies Conditions $(D1, D4, i - vii)$, which are rather mild and natural.

Thus the sufficient boundary conditions are:

(4) $(\partial^{|\beta|} f(t^{(lj)})/\partial \tau_{\gamma(1)}^{\beta_1}...\partial \tau_{\gamma(m)}^{\beta_m})|_{\partial U_{(lj)}} = \phi_{\beta,(lj)}(t^{(lj)})$

for $|\beta| = |q|$, where $m = |h(lj)|$, $|j| \le \alpha$, $|(lj)| \ge 1$, $\mathbf{a}_j \ne 0$, $q_k = 0$ for $l_k j_k = 0$, $m_k + q_k + h_k = j_k$, $h_k = sign(l_k j_k)$, $0 \le q_k \le j_k - 1$ for $k > n - \kappa$; $\phi_{\beta,(l)}(t^{(l)})$ are known functions on $\partial U_{(l)}$, $t^{(l)} \in \partial U_{(l)}$. In the half-space $t_n \ge 0$ only the partial derivatives by t_n

(5) $\partial^\beta f(t)/\partial t_n^\beta|_{t_n=0}$

are necessary for $\beta = |q| < \alpha$ and q as above.

Depending on coefficients of the operator A and the domain U some boundary conditions may be dropped, when the corresponding terms vanish.

Conditions in (5) are given on disjoint for different (l) sub-manifolds $\partial U_{(l)}$ in ∂U and partial derivatives are along orthogonal to them coordinates in $\mathbf{R}^n$, so they are correctly posed.

We recall, that a characteristic surface of a partial differential operator given by Formula 10(1) is a surface defined as a zero of C^u differentiable function $\phi(x_1, ..., x_n) = 0$ in the Euclidean space or in the Cayley-Dickson algebra such that at each point x of it the condition is satisfied

(CS) $\sum_{|j|=u} \mathbf{a}_j(t(x))(\partial\phi/\partial x_1)^{j_1}...(\partial\phi/\partial x_n)^{j_n} = 0$

and at least one of the partial derivatives $(\partial\phi/\partial x_k) \ne 0$ is non-zero. Generally a domain U is worthwhile to choose with its interior $Int(U)$ non-intersecting with a characteristic surface $\phi(x_1, ..., x_n) = 0$ (see also [32, 37]).

26. Solutions of second order partial differential equations with the help of the line integration over the Cayley-Dickson algebras.

Mention that the operator $(\Upsilon + \beta)(z_0, ..., z_n)$ acting on a function depending on variables $z_0, ..., z_n$ only can be written as

(1) $\Psi(z_0, ..., z_{n+1})(f(z)z_{n+1})|_{z_{n+1}=1} = \Upsilon(z_0, ..., z_n)f(z) + f(z)\beta(z)$

$= [\sum_{j=0}^{n+1}(\partial(f(z)z_{n+1})/\partial z_j)\phi_j^*(z)]|_{z_{n+1}=1}$,
where $\phi_{n+1}^*(z) = \beta(z)$, each function $\phi_j(z)$ and $f(z)$ may depend on $z_0, ..., z_n$ only, omitting for short the direct product $\otimes$ in the case of generalized coefficients in Formula (1) and henceforth. The operator $\Psi(z_0, ..., z_{n+1})(f(z)z_{n+1})$ may be reduced to the form satisfying conditions of Theorems 5 or 23 using a suitable change of variables. This procedure gives an anti-derivative operator

(1.1) $\mathcal{I}_{\Upsilon+\beta} = \mathcal{I}_\Psi|_{z_{n+1}=1}$ such that

(1.2) $(\Upsilon + \beta)\mathcal{I}_{\Upsilon+\beta} f = f$

for a continuous function or a generalized function f. Therefore, we shall consider operators of the form Υ and their compositions and sums.

We take the partial differential equation with piecewise continuous or generalized coefficients

(2) $A = \Upsilon\Upsilon_1 f(z) + \Upsilon_2 f(z) = g$,

where

(3) $\Upsilon_k f(z) = [\sum_{j=0}^{n}(\partial f(z)/\partial z_j)\phi_j^k(z)^*]$

for $k = 1, 2$ or without this index k, $\phi_j^k(z) = i_j\psi_j^k(z)$ (see §§4, 5 and 23). For solving it we write the system:

(4) $\Upsilon_1 f = g_1$, $\Upsilon g_1 = g - \Upsilon_2 f$.

In accordance with Equation (2) we have:

(5) $\Upsilon g_1 + \Upsilon_2(\Upsilon_1^{-1} g_1) = g$,

where the inverse operator Υ_1^{-1} is the anti-derivation operator $\mathcal{I}_{\Upsilon_1}$ described above in Theorems 4, 5 and 23. If $\Upsilon_2 \neq 0$ we suppose that either (i) g_1 or $[g_1, \omega)$ is real-valued or the algebra $alg_{\mathbf{R}}\{g_1(z), \phi_j^1(z), \phi_k^2(z)\}$ or $alg_{\mathbf{R}}\{[g_1, \omega), [\phi_j^1, \omega), [\phi_k^2, \omega)\}$ for all $j, k = 0, ..., n$ is associative for each $z \in U$ in the case of functions or for every real-valued test function ω in the case of generalized functions correspondingly. Calculating the expression

(6) $\Upsilon g_1 + \Upsilon_2(\mathcal{I}_{\Upsilon_1} g_1) = (\Upsilon + \beta^3)g_1 = g$

we get a new operator $(\Upsilon + \beta^3) = \Psi$ in accordance with Formulas 4(5, 6), 5(6 − 8) and 23(3 − 7) omitting $\otimes$ and $\otimes\omega$ and $[*, \omega^{\otimes 3})$ for short in the case of generalized coefficients, where

$\beta^3 = \sum_{j,k=0}^{n}(\partial\nu_j^1/\partial z_k)(\phi_k^2)^*$,

when a solution $\{\nu_j(z) : j\}$ is chosen real, i.e. each function ν_j is real-valued or a real-valued generalized function on real-valued test functions (see System 4(10, 11) and §23).

Generally without supposition (i) we deduce that

(6.1) $\Upsilon g_{1,l} + \Upsilon_2(\mathcal{I}_{\Upsilon_1} g_{1,l}) = (\Upsilon + \beta_l^3)g_{1,l} = i_l^* g^l$

for each $l = 0, ..., n$, where $\beta_l^3 = i_l^*[\sum_{j,k=0}^{n}(i_j(\partial\nu_j^1/\partial z_k))(\phi_k^2)^*]$,

(6.2) $\sum_{l=0}^{n} g^l = g$,

(6.3) $\sum_{l=0}^{n} g_{1,l} i_l = g_1$,

each $g_{1,l}(z)$ or $[g_{1,l}, \omega)$ is real-valued for each $z \in U$ or every real-valued test function ω respectively. Solving the system (6.1 − 3) with the help of known anti-derivative operators $\mathcal{I}_{\Psi_l}$ one finds g_1, where $\Psi_l = \Upsilon + \beta_l^3$. Thus an anti-derivative operator $\mathcal{J} := \mathcal{J}_{\Upsilon+\Upsilon_2\mathcal{I}_{\Upsilon_1}}$ exists so that

(6.4) $(\Upsilon + \Upsilon_2\mathcal{I}_{\Upsilon_1})\mathcal{J}_{\Upsilon+\Upsilon_2\mathcal{I}_{\Upsilon_1}} g = g$.

In the particular case (i) the equality $\mathcal{J}_{\Upsilon+\Upsilon_2\mathcal{I}_{\Upsilon_1}} = \mathcal{I}_\Psi$ is satisfied.

Therefore, in the case of either continuous coefficients of operators and g or generalized coefficients and g the general solution is:

(7) $f = \mathcal{I}_{\Upsilon_1} g_1 = \mathcal{I}_{\Upsilon_1}\mathcal{J} g$, where

(8) $g_1 = \mathcal{J} g$.

If $\Upsilon_2 = 0$ the formula simplifies to

(9) $f = \mathcal{I}_{\Upsilon_1}\mathcal{I}_\Upsilon g$.

Examples of boundary conditions and domains permitting to specify a unique solution are given in §25 above.

27. Example. Let us consider a function and its phrase satisfying Condition 7($P3$). Therefore, we get

(1) $\Upsilon_1^{-1}(g_1)(z_{a^1}(x)) = -\phi_{f'}(Im(z_{a^1}(x))) + \int_\alpha^x g_1(z)dz + \phi_{g_1}(x')$
in accordance with Formula 7(6), where x' and $Im(z_{a^1}(x))$ are written in the x and z-representations respectively using Identities $1(1-3)$. In particular, for $\Upsilon^1 = \sigma_1$ we have the coefficients $\psi_l^1(z) = i_l(\mathbf{a}_k(z)w_j i_{2^r k}^*)$ for each $l = 2^r k$ with $k \in \{m_1+...+m_{j-1}+1, ..., m_1+...+m_j\}$, while $\psi_l^1(z) = 0$ for all others l for each z. A function a^1 is given by Formula 7(3) for ψ^1 instead of ψ. Let the first order operator Q be written in its standard form:

$$(2)\quad Qf = \sum_{j=1}^{m} \sum_{k=m_1+...+m_{j_1-1}+1}^{m_1+...+m_{j_1}} \mathbf{s}_k(z)(\partial f/\partial z_{2^r k})(u_j i_{2^r k}),$$

since $i_{2^r k} w_j = w_j^* i_{2^r k}$ for each $w_j \in \mathcal{A}_r$ and $k \ge 1$, when $r \ge 1$, where $w_j \in \mathcal{A}_r$ and $u_j = u_j(z) \in \mathcal{A}_v$ for each j, $\mathbf{s}_k(z)$ is the real-valued (super-)differentiable function for each k. If $\xi = z_a(y)$, then $(d\xi/dy).[(dy/d\xi).h] = h$ for each Cayley-Dickson number $h \in \mathcal{A}_v$. This implies that these two $\mathbf{R}$-linear $\mathcal{A}_v$-additive operators are related by the equality $(dy/d\xi).h = (d\xi/dy)^{-1}.h$. On the other hand, $(dz_a(y)/dy).1 = a(y) \in \mathcal{A}_v$ and $y \in U \subset \mathcal{A}_v$ in the considered case. The function $z_a(y)$ is defined up to the addendum $z_a(\alpha)$, where $\alpha \in H_{\alpha_0} \cap U$, $H_{\alpha_0} := \{z \in \mathcal{A}_v : Re(z) = Re(\alpha_0)\}$. We can choose $\phi_a(y')$ so that $(dx/dy).(1/a) = 1$ for each y for which $a = a(y) \ne 0$ and inevitably we get $(dy/dx).1 = 1/a(y)$.

In the particular case of σ, σ_1 and Q accomplishing the differentiation with the help of the latter identities we infer that:

$$(3)\quad Q(\sigma_1^{-1}g_1)(x) = -\sum_{j=1}^{m} \sum_{k=m_1+...+m_{j_1-1}+1}^{m_1+...+m_{j_1}} [(d\phi_{f'}(x')/dx).i_{2^r k}$$

$$+\{\hat{g}_1(y) + (d\phi_{g_1}(y')/dy)\}.[(dz_{a^1}(y)/dy)^{-1}.i_{2^r k}](\mathbf{s}_k(y)u_j(y)i_{2^r k}),$$

where $\hat{g}_1 = d\zeta_1(y)/dy$ for a (super-)differentiable function ζ_1 such that $(d\zeta_1(y)/dy).1 = g_1(y')$, $\psi_l^1(z) = i_l(\mathbf{a}_k(z)w_j i_{2^r k}^*)$ for each $l = 2^r k$ with $k \in \{m_1 + ... + m_{j-1} + 1, ..., m_1 + ... + m_j\}$, while $\psi_l^1(z) = 0$ for all others l for each z. Also $\psi_l(z) = i_l(\mathbf{a}_k(z)w_j^* i_{2^r k}^*)$ for each $l = 2^r k$ with $k \in \{m_1 + ... + m_{j-1} + 1, ..., m_1 + ... + m_j\}$, while $\psi_l(z) = 0$ for all others l and for each z. We introduce the notation:

(4) $\theta_l(y) = i_l[(dz_a(y)/dy)^{-1}.i_{2^r k}](\mathbf{s}_k(y)u_j(y)i_{2^r k})$ for $l = 2^r k$ and $k \in \{m_1 + ... + m_{j_1-1} + 1, ..., m_1 + ... + m_{j_1}\}$, $\theta_l(y) = 0$ for all others l and for each y;

$$(5)\quad a(x) = -\sum_{j=1}^{m} w_j \sum_{k=m_1+...+m_{j-1}+1}^{m_1+...+m_j} \mathbf{a}_k(x) = \sigma(x) \text{ and}$$

(6) $\kappa_l(x) = i_l[s_k(x)u_j(x)i_{2^r k}]$ for $l = 2^r k$ with $k \in \{m_1 + ... + m_{j_1-1} + 1, ..., m_1 + ... + m_{j_1}\}$, $\kappa_l(x) = 0$ for every other l and for each x.

In the general case

$$(7)\quad \Upsilon_2(\Upsilon_1^{-1}g_1)(x) = -\sum_{j=0}^{2^v-1} [(d\phi_{f'}(x')/dx).i_j$$

$$+\{\hat{g}_1(y) + (d\phi_{g_1}(y')/dy)\}.[(dz_{a^1}(y)/dy)^{-1}.i_j](i_j^*\psi_j(x)),$$

where $\hat{g}_1 = d\zeta_1(y)/dy$ for a (super-)differentiable function ζ_1 such that $(d\zeta_1(y)/dy).1 = g_1(y')$. We shall use the notation:

(8) $\theta_j(y) = i_j[(dz_a(y)/dy)^{-1}.i_j](i_j^*\psi_j(y))$ and for each y and each j;

$$(9) \quad a(x) = \Upsilon(x) \text{ and}$$

(10) $\kappa_j(x) = i_j[i_j^*\psi_j(x)]$ for each x and j. Substituting (7) into 26(5) we deduce that:
(11) $g_1(z_a(x)) + g_1(z_\theta(x)) = \zeta(x)$, where

$$(12) \quad \zeta(x) = -\phi_{g_1}(Im(z_a(x))) - \phi_{g_1}(Im(z_\theta(x))) + \phi_{f'}(Im(z_\kappa(x))) + \int_\alpha^x g(z)dz.$$

For $x = \alpha$ we certainly have $z_a(\alpha) = z_\theta(\alpha)$. Suppose that $a(x) \ne \theta(x)$ identically. The dimension of the Cayley-Dickson algebra $\mathcal{A}_v$ over the real field is not less, than four. Therefore, we can choose a path γ so that γ is orthogonal to θ and κ at each point on γ, that is $\gamma'(t) \perp \theta'(\gamma(t))$ and $\gamma'(t) \perp \kappa'(\gamma(t))$ relative to the real-valued scalar product (RS) for each $t \in [0,1]$, where $\gamma'(t) := d\gamma(t)/dt$. Then $g_1(z_\theta(x)) = g_1(\alpha)$ and $\phi_{f'}(Im(z_\kappa(x))) = \phi_{f'}(\alpha')$ for each $x = \gamma(t)$. Therefore, along such path γ one has

$$(13) \quad g_1(z_a(x)) + g_1(\alpha) = \zeta(x) = -\phi_{g_1}(Im(z_a(x)) - \phi_{g_1}(\alpha') + \phi_{f'}(\alpha') + \int_\alpha^x g(z)dz$$

for each $x = \gamma(t)$. Expressing $g_1(z)$ from Equation (11), substituting into 26(4) and integrating one gets:

$$(14) \quad f(z_{a^1}(x)) = -\phi_f(Im(z_{a^1}(x))) + \int_\alpha^x g_1(z)dz.$$

Particularly, if the operator A is with constant coefficients, then $s_k(x) = 0$ identically for each k, consequently, $\theta = 0$ and $\kappa = 0$ identically and $g_1(z_a(x)) = g_1(\alpha) = \phi_{g_1}(Im(z_a(x)))$ for each x, when f has a right linear derivative by z. Arbitrary integration terms in $(11, 14)$ can be specified from the boundary conditions.

Finally, the restriction from the domain in $\mathcal{A}_v$ onto the initial domain of real variables in the real shadow and the extraction of $\pi_r^v \circ f \in \mathcal{A}_r$ with the help of Formulas $1(1-3)$ gives the reduction of a solution from $\mathcal{A}_v$ to $\mathcal{A}_r$, where $\pi_r^v : \mathcal{A}_v \to \mathcal{A}_r$ is the $\mathbf{R}$-linear projection operator defined as the sum of projection operators $\pi_0 + ... + \pi_{2^r-1}$ given by Formulas $3(P1, P2)$ on $\mathbf{R}i_j$ for $j = 0, ..., 2^r - 1$.

28. Laplace's operator. When
(1) $A_0 = \Delta_n = \sum_{j=1}^n \partial^2/\partial z_j^2$,
is Laplace's operator, then
(2) $\Upsilon f(z) = \sum_{j=1}^n (\partial f(z)/\partial z_j)i_j^*$, so that
(3) $\Delta_n = \Upsilon\Upsilon^* = -\Upsilon\Upsilon$, $\Upsilon^1 = -\Upsilon$,
where $2 \le n \le 2^r - 1$, $z_1, ..., z_n \in \mathbf{R}$, in accordance with §2. Consider the fundamental solution Ψ of the following equation
(4) $\Xi\Psi(z_1, ..., z_n) = \delta(z_1, ..., z_n)$ with $\Xi = \Delta_n$ satisfies the identity:
(5) $\Psi = -(\Upsilon\Psi) * (\Upsilon\Psi)$ (see the convolution of generalized functions and this formula in §§9, 19.4).
We seek the real fundamental solution $\Psi = \Psi_n$, since the Laplace operator is real. The Fourier transform with the generator $\mathbf{i}$ (see §33 [28]) by real variables $z_1, ..., z_n$ gives
(6) $F(\Psi_n)(x) = -[F(\Upsilon\Psi_n)(x)]^2 = -[-\mathbf{i}(\sum_{j=1}^n x_j i_j^*)F(\Psi_n)(x)]^2$, since
$F(\Upsilon\Psi_n)(x) = \sum_{j=1}^n F(\partial\Psi_n/\partial z_j)i_j^* = -\mathbf{i}(\sum_{j=1}^n x_j i_j^*)F(\Psi_n)(x)$,
where $x = (x_1, ..., x_n)$, $x_1, ..., x_n \in \mathbf{R}$ (see also §2). Thus we get the identity
(7) $F(\Psi_n)(x) = -(\sum_{j=1}^n x_j^2)[F(\Psi_n)(x)]^2$ or
(8) $F(\Psi_n)(x) = -(1/(\sum_{j=1}^n x_j^2))$ for $n \ge 3$ is the regular generalized function (functional), while

(9) $F(\Psi_2)(x) = -\mathcal{P}(1/(\sum_{j=1}^2 x_j^2))$ for $n = 2$.
We recall that the generalized function $\mathcal{P}(1/(\sum_{j=1}^2 x_j^2))$ on $\phi \in \mathcal{D}(\mathbf{R}^2, \mathbf{R})$ is defined as the regularization:

$$(\mathcal{P}(1/(\sum_{j=1}^2 x_j^2)), \phi) = \int_{|x|<1} [\phi(x) - \phi(0)]|x|^{-2}dx + \int_{|x|>1} \phi(x)|x|^{-2}dx,$$

where $x = (x_1, x_2)$, $|x|^2 = x_1^2 + x_2^2$, $x_1, x_2 \in \mathbf{R}$.

The inverse Fourier transform $(F^{-1}g)(x) = (2\pi)^{-n}(Fg)(-x)$ of the functions $1/(\sum_{j=1}^n z_j^2)$ for $n \ge 3$ and $\mathcal{P}(1/(\sum_{j=1}^2 z_j^2))$ for $n = 2$ in the class of the generalized functions is known (see [6] and §§9.7 and 11.8 [37]) and gives

(10) $\Psi_n(z_1, ..., z_n) = C_n(\sum_{j=1}^n z_j^2)^{1-n/2}$ for $3 \le n$, where $C_n = -1/[(n-2)\sigma_n]$, $\sigma_n = 4\pi^{n/2}/\Gamma((n/2)-1)$ denotes the surface of the unit sphere in $\mathbf{R}^n$, $\Gamma(x)$ denotes Euler's gamma-function, while

(11) $\Psi_2(z_1, z_2) = C_2 \ln(\sum_{j=1}^2 z_j^2)$ for $n = 2$, where $C_2 = 1/(4\pi)$.
Thus the technique of convolutions over the Cayley-Dickson algebra has permitted to get the solution of the Laplace operator.

Another method is with the line integration over the Cayley-Dickson algebras. In accordance with Formula 26(9) we get

$\Psi_n(z_1, ..., z_n) = -\mathcal{I}_\Upsilon \mathcal{I}_\Upsilon \delta$.

Laplace's operator and the delta-function are invariant under any orthogonal transform $T \in \mathsf{O}_n(\mathbf{R})$ of $\mathbf{R}^n$. Therefore, a fundamental solution Ψ_n also is invariant relative to the orthogonal group $\mathsf{O}_n(\mathbf{R})$. That is Ψ_n depends on $|z|$ and is independent of spherical angles in spherical system of coordinates. Thus we choose the corresponding branches of the anti-derivative $\mathcal{I}_\Upsilon \mathcal{I}_\Upsilon \delta$. The volume element in the Euclidean space $\mathbf{R}^n$ can be written as $\lambda(dz) = x^{n-1}dxds$, where $x = |z|$ and ds is the surface element (measure) on the unit sphere S^{n-1}. For each orthogonal transform its Jacobian is unit.

One can take the family of test functions $\eta_\epsilon = \frac{1}{(2\pi)^{n/2}\epsilon^n} \exp\{-(z_1^2 + ... + z_n^2)/(2\epsilon^2)\}$ tending to the delta-function, when $\epsilon > 0$ tends to zero. These functions can be written in the z-representation due to Formulas $1(1-3)$. On the other hand, for each z-analytic function η with real expansion coefficients into a power series each line integral over the Cayley-Dickson algebra $\mathcal{A}_v$ restricted on any complex plane $\mathbf{R} \oplus M\mathbf{R}$ coincides with the usual complex integral, where M is a purely imaginary Cayley-Dickson number. Therefore,

$\int_a^t [\int_a^y \eta(z)dz] y^k dy = \frac{1}{k+1} \int_a^t (t^{k+1} - z^{k+1})\eta(z)dz$ for $k \ne -1$ and

$\int_a^t [\int_a^y \eta(z)dz] \frac{1}{y} dy = \int_a^t (\ln(t) - \ln(z))\eta(z)dz$.

For the characteristic function $\chi_{B(\mathbf{R}^n,0,x)}$ of the ball $B(\mathbf{R}^n, 0, x)$ of radius $x > 0$ with the center at zero in the Euclidean space $\mathbf{R}^n$ embedded into the Cayley-Dickson algebra $\mathcal{A}_v$ one can take a sequence of test functions ${}_l\omega^1$ converging to the regular generalized function $\chi_{B(\mathbf{R}^n,0,x)}$, when l tends to the infinity, consequently, $\lim_l \int_{\mathbf{R}^n} {}_l\omega^1(z)\lambda(dz) = \sigma_n x^n$. Integrating twice with the anti-derivative operator these test functions η_ϵ in accordance with Example 4.1 and Theorems 19 and 23 and taking the limit with ϵ tending to zero from the right one gets Formulas $(10, 11)$.

This can also be deduced with the help of the Fourier transform with the generator $\mathbf{i}$:

(12) $F(\Psi_n)(x) = F(-\mathcal{I}_\Upsilon \mathcal{I}_\Upsilon \delta) = (\sum_{j=1}^n x_j^2)^{-1} F(\delta) = (\sum_{j=1}^n x_j^2)^{-1}$.
Applying the inverse Fourier transform to both sides of Equation (12) we again get Formulas $(10, 11)$.

29. The hyperbolic operators with constant coefficients.

Consider now the hyperbolic operator

(1) $A_0 = L_{p,q} = \sum_{j=1}^{p} \partial^2/\partial z_j^2 - \sum_{j=p+1}^{n} \partial^2/\partial z_j^2$,
where $p+q=n$, $1 \le p$ and $1 \le q$, (p,q) is the signature of this operator, $z_1,...,z_n \in \mathbf{R}$. Take two operators Υ and Υ_1 with constant $\mathcal{A}_v$ coefficients so that

(2) $\Upsilon f(z) = \sum_{j=1}^{p} (\partial f(z)/\partial z_j) i_{2j}^* + \sum_{j=p+1}^{n} (\partial f(z)/\partial z_j)[i_1^* i_{2j}^*]$ and

(3) $\Upsilon_1 f(z) = \sum_{j=1}^{p} (\partial f(z)/\partial z_j) i_{2j}^* + \sum_{j=p+1}^{n} (\partial f(z)/\partial z_j)[i_1 i_{2j}^*]$, so that

(4) $L_{p,q} = \Upsilon \Upsilon_1$,
where $2 \le n \le 2^{v-r}-1$, $r = 1 < v$, in accordance with Formulas 2(7−9). Then the fundamental solution Ψ of the partial differential equation

$\Xi\Psi(z_1,...,z_n) = \delta(z_1,...,z_n)$ with $\Xi = L_{p,q}$ satisfies the identity:

(5) $\Psi = (\Upsilon^*\Psi) * (\Upsilon_1^*\Psi)$.

We seek the real fundamental solution $\Psi = \Psi_{p,q}$, since the hyperbolic operator $L_{p,q}$ is real. Using the Fourier transform with the generator $\mathbf{i}$ by real variables $z_1,...,z_n$ we infer that

(6) $F(\Psi_{p,q})(x) = [F(\Upsilon^*\Psi_{p,q})(x)][F(\Upsilon_1^*\Psi_{p,q})(x)]$
$= [-\mathbf{i}(\sum_{j=1}^{p} x_j i_{2j} + \sum_{j=p+1}^{n} x_j i_{2j} i_1) F(\Psi_{p,q})(x)][-\mathbf{i}(\sum_{j=1}^{p} x_j i_{2j} + \sum_{j=p+1}^{n} x_j i_{2j} i_1^*) F(\Psi_{p,q})(x)]$, since

$F(\Upsilon^*\Psi_{p,q})(x) = \sum_{j=1}^{p} F(\partial\Psi_{p,q}/\partial z_j) i_{2j} + \sum_{j=p+1}^{n} F(\partial\Psi_{p,q}/\partial z_j) i_{2j} i_1$
$= -\mathbf{i}(\sum_{j=1}^{p} x_j i_{2j} + \sum_{j=p+1}^{n} x_j i_{2j} i_1) F(\Psi_{p,q})(x)$
and analogously for Υ_1^*, where $x = (x_1,...,x_n)$, $x_1,...,x_n \in \mathbf{R}$ (see also §2). For the function

(7) $P(x) = \sum_{j=1}^{p} x_j^2 - \sum_{j=p+1}^{n} x_j^2$ with $p \ge 1$ and $q \ge 1$ the generalized functions $(P(x)+\mathbf{i}0)^\lambda$ and $(P(x)-\mathbf{i}0)^\lambda$ are defined for any complex number $\lambda \in \mathbf{C} = \mathbf{R} \oplus \mathbf{i}\mathbf{R}$ (see Chapter 3 in [6]). The function P^λ has the cone surface $P(z_1,...,z_n) = 0$ of zeros, so that for the correct definition of generalized functions corresponding to P^λ the generalized functions

(8) $(P(x)+c\mathbf{i}0)^\lambda = \lim_{0<c\epsilon,\epsilon\to 0} (P(x)^2+\epsilon^2)^{\lambda/2} \exp(\mathbf{i}\lambda\, arg(P(x)+\mathbf{i}c\epsilon))$
with either $c=-1$ or $c=1$ were introduced. Therefore, the identity

(9) $F(\Psi_{p,q})(x) = -(\sum_{j=1}^{p} x_j^2 - \sum_{j=p+1}^{n} x_j^2)[F(\Psi_{p,q})(x)]^2$ or

(10) $F(\Psi) = -1/(P(x)+c\mathbf{i}0)$ follows, where $c=-1$ or $c=1$.

The inverse Fourier transform in the class of the generalized functions is:

(11) $F^{-1}((P(x)+c\mathbf{i}0)^\lambda)(z_1,...,z_n) = \exp(-\pi c q\mathbf{i}/2) 2^{2\lambda+n} 2^{2\lambda+n} \pi^{n/2} \Gamma(\lambda+n/2)(Q(z_1,...,z_n) - c\mathbf{i}0)^{-\lambda-n/2)}/[\Gamma(-\lambda)|D|^{1/2}]$
for each $\lambda \in \mathbf{C}$ and $n \ge 3$ (see §IV.2.6 [6]), where $D = \det(g_{j,k})$ denotes a discriminant of the quadratic form $P(x) = \sum_{j,k=1}^{n} g_{j,k} x_j x_k$, while $Q(y) = \sum_{j,k=1}^{n} g^{j,k} x_j x_k$ is the dual quadratic form so that $\sum_{k=1}^{n} g^{j,k} g_{k,l} = \delta_l^j$ for all j,l; $\delta_l^j = 1$ for $j=l$ and $\delta_l^j = 0$ for $j \ne l$. In the particular case of $n=2$ the inverse Fourier transform is given by the formula:

(12) $F^{-1}((P(x)+c\mathbf{i}0)^{-1})(z_1,z_2) = -4^{-1}|D|^{-1/2} \exp(-\pi c q\mathbf{i}/2) \ln(Q(z_1,...,z_n) - c\mathbf{i}0)$.

Making the inverse Fourier transform F^{-1} of the function $-1/(P(x)+\mathbf{i}0)$ in this particular case of $\lambda=-1$ we get two complex conjugated fundamental solutions

(13) $\Psi_{p,q}(z_1,...,z_n) = -\exp(\pi c q\mathbf{i}/2)\Gamma((n/2)-1)(P(z_1,...,z_n)+c\mathbf{i}0)^{1-(n/2)}/(4\pi^{n/2})$ for $3 \le n$ and $1 \le p$ and $1 \le q$ with $n = p+q$, while

(14) $\Psi_{1,1}(z_1,z_2) = 4^{-1}\exp(\pi c q\mathbf{i}/2)\ln(P(z_1,z_2)+c\mathbf{i}0)$ for $n=2$, where either $c=1$ or $c=-1$.

Another approach consists in using the anti-derivative operators. The hyperbolic operator $L_{p,q}$ and the delta-function are invariant under the Lie group $\mathsf{O}_{p,q}(\mathbf{R})$ or all linear transforms of the Euclidean space $\mathbf{R}^n$, $n=p+q$, preserving the scalar product $(x,y)_{p,q} = \sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{p+q} x_j y_j$ invariant. Thus $\Psi_{p,q}$ can be written as a composition $\xi(P(x))$ of some function $\xi(y)$ and of $P(x)$ given by Formula (7). Therefore, we take the corresponding branch of the anti-derivative in the form $\mathcal{I}_\Upsilon \mathcal{I}_{\Upsilon_1}\delta = \xi(P(x))$. Applying the Fourier transform with the generator $\mathbf{i}$ we infer that

(15) $F(\Psi_{p,q})(x) = F(\mathcal{I}_\Upsilon \mathcal{I}_{\Upsilon_1}\delta) = (P(x)+c\mathbf{i}0)^{-1}F(\delta) = (P(x)+c\mathbf{i}0)^{-1}$.

Applying the inverse Fourier transform to both sides of Equation (15) one gets Formulas (13, 14).

Thus the results of §§2-25 over the Cayley-Dickson algebra $\mathcal{A}_v$ lead to the fundamental solution of the hyperbolic operator $L_{p,q}$. This means that the approach of §§2-25 over the Cayley-Dickson algebras leads to the effective solution of any hyperbolic partial differential equation with constant coefficients. Thus Formulas of §§2, 8 with the known function $\Psi = \Psi_n$ from Formulas 28(10, 11) and (13, 14) of this section give the fundamental solution of any first and second order linear partial differential equation with variable z-differentiable $\mathcal{A}_v$-valued coefficients, $z \in U \subset \mathcal{A}_v$.

30. Example. The heat kernel. Each function of the type $f(z) = P_n(z)\exp(-t|z|^2)$ with a marked positive parameter can be written in the z-representation due to Formulas $1(1-3)$, where $P_n(z)$ denotes the polynomial by z of degree n. Therefore, $f(z)$ in the z-representation is z-differentiable, consequently, infinite z-differentiable (see [23, 22]) and

$$\lim_{|z|\to\infty} f^{(m)}(z).(h_1, ..., h_m)(1 + |z|^k) = 0$$

for each $0 \le m, k \in \mathbf{Z}$ and Cayley-Dickson numbers $h_1, ..., h_m \in \mathcal{A}_v$. Therefore, the space E of infinite z-differentiable tending to zero at infinity functions together with their derivatives multiplied on the weight factor $(1 + |z|^k)$ is infinite-dimensional. Thus it is worthwhile to consider the topologically adjoint space E'_q of $\mathbf{R}$-linear $\mathcal{A}_v$-additive continuous $\mathcal{A}_v$-valued functionals on E. Elements of E'_q are also called the generalized functions. A function or a generalized function is called finite if its support is a bounded set.

The heat partial differential equation reads as

(1) $\partial v(z)/\partial z_0 = a^2 \Delta v(z) + f(z)$,

where $z = z_0 i_0 + ... + z_m i_m$, $z_0, ..., z_m \in \mathbf{R}$, $1 \le m \le 2^v - 1$, $2 \le v$, where $a > 0$, $f(z)$ is a real-valued generalized finite function so that $f(z)$ is zero for $z_0 < 0$ (see §16 [37]). We shall seek the generalized solution $\mathcal{E}$ of this equation using the technique given above. The generalized function $v = \mathcal{E} * f$ is the solution of (1), where

(2) $\partial \mathcal{E}(z)/\partial z_0 - a^2 \Delta \mathcal{E}(z) = \delta(z)$,

$$(3) \quad (\mathcal{E} * f)(x) = \int_0^{x_0} \int_{\mathbf{R^m}} \mathcal{E}(x - z) f(z) dz_0....dz_m.$$

As usually δ denotes the δ generalized function so that

$$(4) \quad (\delta * f)(x) = \int_0^{x_0} \int_{\mathbf{R^m}} \delta(x - z) f(z) dz_0....dz_m = f(x)$$

for each continuous bounded function f. If f is (super-)differentiable and bounded in each domain $\{z : 0 \le z_0 \le T\}$ for $0 < T < \infty$, $f(z) = 0$ for $z_0 < 0$, then the solution v is also (super-)differentiable in the domain $z_0 > 0$ as it will be seen from the formulas given below. Let us seek the generalized solution $\mathcal{E}$ in the form $\mathcal{E}(z) = w(z_0)e^{u(z)}$, where w and u are unknown real-valued functions to be determined. Calculating derivatives of $\mathcal{E}$ and substituting into Equation (2) one gets:

(5) $e^{u(z)}\{w'(z_0) + w(z_0)\partial u(z)/\partial z_0\} - a^2 e^{u(z)} w(z_0) \sum_{j=1}^m [(\partial u(z)/\partial z_j)^2 + \partial^2 u(z)/\partial z_j^2] = \delta(z)$,

consequently,

(6) $(1/w(z_0))w'(z_0) = -\partial u(z)/\partial z_0 + a^2 \sum_{j=1}^m [(\partial u(z)/\partial z_j)^2 + \partial^2 u(z)/\partial z_j^2] + e^{-u(z)}(1/w(z_0))\delta(z)$.

Take now any sequence of continuous non-negative functions η_n with compact supports U_n such that $U_{n+1} \subset U_n$ for each n, with $\bigcap_n U_n = \{0\}$,

$$(7) \quad \int_{\mathbf{R}^{m+1}} \eta_n(z) dz_0...dz_m = 1$$

for all n, tending to δ on the space of continuous functions $p(z)$ on $\mathbf{R}i_0 \oplus ... \oplus \mathbf{R}i_m$ with the converging integral $\int_{\mathbf{R}^{m+1}} |p(z)|^2 dz_0...dz_m < \infty$:

(8) $\lim_{n\to\infty} \int_{\mathbf{R}^{m+1}} p(z)\eta_n(z)dz_0...dz_m = p(0)$.

Therefore, we get that on $\mathbf{R}i_0 \oplus ... \oplus \mathbf{R}i_m \setminus \{0\}$ for $z_0 > 0$ the following equation

$$(9) \quad (1/w(z_0))w'(z_0) = -\partial u(z)/\partial z_0 + a^2 \sum_{j=1}^m [(\partial u(z)/\partial z_j)^2 + \partial^2 u(z)/\partial z_j^2]$$

need to be satisfied. The left side of (9) is independent of $z - z_0$, hence the right side is also independent of $z - z_0$. The partial differential operator

$\{\partial u(z)/\partial z_0 + a^2 \sum_{j=1}^m [(\partial u/\partial z_j)^2 + \partial^2 u(z)/\partial z_j^2]\}$ acting on u is of the second order. For each Cayley-Dickson number $z \in \mathcal{A}_r$ the identity $z^2 = 2z\ Re(z) - |z|^2$ is satisfied, particularly, $M^2 = -|M|^2$ for each purely imaginary number $M \in \mathcal{A}_r$. Therefore, a function u may be only a polynomial by real variables $z_1, ..., z_m$ of degree not higher than two. On the other hand, the Laplace operator Δ and the δ function are invariant relative to all elements C of the orthogonal group $\mathsf{O}_m(\mathbf{R})$ acting on variables $z_1, ..., z_m$. Each $\mathsf{O}_m(\mathbf{R})$ invariant real polynomial P of the second order has the form $\alpha(z_1^2 + ... + z_m^2) + \beta$, where α and β are two constants independent of $z_1, ..., z_m$. Thus u as the polynomial of $z_1, ..., z_m$ may depend on $|z - z_0|^2 = z_1^2 + ... + z_m^2$ only. The latter sum of squares can be written in the z-representation with the help of Formulas $1(1-3)$. This means that $\mathcal{E}$ has the form:

(10) $\mathcal{E}(z) = w(z_0)\exp\{\alpha(z_0)(z_1^2 + ... + z_m^2) + \beta(z_0)\}$ and Equation (9) simplifies:

(11) $(1/w(z_0))w'(z_0) = -[d\alpha(z_0)/dz_0](z_1^2 + ... + z_m^2) - [d\beta(z_0)/dz_0]$

$$+a^2\alpha(z_0)\{2m + \alpha(z_0)\sum_{j=1}^m 4z_j^2\}.$$

We can denote $w(z_0)e^{\beta(z_0)}$ by $w(z_0)$ again and take without loss of generality that $\beta = 0$. The left side of (11) is independent of $z_1, ..., z_m$, hence terms with $|z - z_0|^2$ in (11) are canceling: $\alpha^{-2}(z_0)[d\alpha(z_0)/dz_0] = 4a^2$. The latter differential equation gives $\alpha(z_0) = -1/(c_0 + 4a^2 z_0)$, where c_0 is the real constant. Substituting this α into (11) one gets:

(12) $(1/w(z_0))w'(z_0) = a^2\alpha(z_0)2m$.

Together with Condition (2) this gives $C_0 = 0$ and the heat kernel $\mathcal{E}$:

(13) $\mathcal{E}(z) = \theta(z_0)[2a(\pi z_0)^{1/2}]^{-m}\exp\{-|z - z_0|^2/[4a^2 z_0]\}$

and the solution

(14) $v = \mathcal{E} * f$,

where $\theta(z_0) = 1$ for $z_0 \ge 0$ and $\theta(z_0) = 0$ for $z_0 < 0$.

If use anti-derivation operators the solution has the form $26(6-8)$ supposing that a solution $\mathcal{E}$ is real-valued on real-valued test functions ω, $[\mathcal{E}, \omega) \in \mathbf{R}$, where $\Upsilon_1 = \Upsilon$, $\Upsilon\Upsilon = -a^2\Delta_m$ (see §) and $\Upsilon_2 = \partial/\partial z_0$. Therefore,

(14) $a^2\mathcal{E} = -\mathcal{I}_\Upsilon\mathcal{I}_\Upsilon(\partial\mathcal{E}/\partial z_0 - \delta)$.

Making the Fourier transform $F = F_{z_1,...,z_m}$ by the variables $z_1, ..., z_m$ with the generator $\mathbf{i}$ of both sides of Equation (14) one gets for suitable branches of the anti-derivatives

(15) $a^2F(\mathcal{E})(z_0, x_1, ..., x_m) = [a^2\sum_{j=1}^m x_j^2]^{-1}(\partial F(\mathcal{E})/\partial z_0 - \delta(z_0))$.

Solving the latter ordinary differential equation one finds $F(\mathcal{E})(z_0, x_1, ..., x_m)$ and making the inverse Fourier transform by the variables $x_1, ..., x_m \in \mathbf{R}$ one gets Formula (13).

31. Example. Wave operator. In this section the fundamental solution $\mathcal{E} = \mathcal{E}_n$ of the wave operator is considered:

(1) $\overline{\coprod}\mathcal{E}(t, x) = \delta(t, x)$, where

(2) $\overline{\coprod} f = \partial^2 f/\partial t^2 - \Delta f$ denotes the wave (d'Alambert) operator with

$$(3) \quad \Delta f = \sum_{j=1}^{n} \partial^2 f/\partial x_j^2,$$

where $t \ge 0$, $x_1, ..., x_n \in \mathbf{R}$. We make the change of variables putting $t = z_2$, $x_j = z_{2j+2}$ for each $j = 1, ..., n$, $z = z_0 i_0 + z_2 i_2 + ... + z_{2^v-2} i_{2^v-2} \in \mathcal{A}_{1,v}$, $z_0, ..., z_{2^v-1} \in \mathbf{R}$, $r = 1$. We consider the case $n = 3$ and $v = 4$ so that $\mathcal{A}_{1,4}$ is isomorphic with the octonion algebra $\mathcal{A}_3 = \mathbf{O}$. Let us seek $\mathcal{E}$ in the class of the generalized functions in the form $\mathcal{E}(z) = \theta(z_2) f(z)$, where θ and f are some generalized functions to be calculated, f may depend only on $z_2, z_4, ..., z_{2n+2}$. D'Alambert's operator $\overline{\coprod}$ is invariant relative to any $\mathbf{R}$-linear transformations A from the Lie group $\mathsf{O}_{1,n}(\mathbf{R})$. Elements of the group $\mathsf{O}_{1,n}(\mathbf{R})$ are characterized by the condition $A^t G A = G$, where G denotes the square $(n+1) \times (n+1)$ diagonal matrix $G = diag(1, -1, ..., -1)$, the transposed matrix A is denoted by A^t. This means that the wave operator $\overline{\coprod}$ is invariant under change of variables $\xi = (z_2, z_4, ..., z_{2n+2})A$ for any $A \in \mathsf{O}_{1,n}(\mathbf{R})$. Making the differentiation of $\mathcal{E}$ one gets the differential equation:

(4) $\overline{\coprod}\mathcal{E} = (\partial^2\theta/\partial z_2^2) f + 2(\partial\theta/\partial z_2)(\partial f/\partial z_2) + \theta \overline{\coprod} f = \delta(z)$.

The δ-function $\delta(z)$ is also invariant relative to all transformations of the Lie group $\mathsf{O}_{1,n}(\mathbf{R})$, since $\delta g = g(0)$ for each continuous function g with $\int_{\mathbf{R}^{n+1}} |g(z)|^2 dz_2 ... dz_{2n+2} < \infty$. On the other hand, Equation (4) implies that

(5) $\partial^2\theta/\partial z_2^2 = -[2(\partial\theta/\partial z_2)(\partial f/\partial z_2) + \theta \overline{\coprod} f - \delta(z)]/f(z)$

for each $z \in \mathcal{A}_{r,v}$, when $f(z) \ne 0$. The left side of Equation (5) may depend only on z_2, consequently, the right side of (5) is independent of $z_4, ..., z_{2n+2}$. In view of Formulas 29(2, 3) with $p = 1$ and $q = n$ we get the operators $\sigma = \Upsilon$ and $\sigma_1 = \Upsilon_1$ with $Q = 0$ up to the enumeration of the variables. Therefore, one gets the functions $\Psi_{1,n}$ (see Formulas 29(13, 14)) over the Cayley-Dickson algebra $\mathcal{A}_v$. But due to the $\mathsf{O}_{1,n}(\mathbf{R})$ invariance of the generalized function $\mathcal{E}$ we infer that it is necessary to take the $\mathsf{O}_{1,n}(\mathbf{R})$ invariant polynomial $P(y) = (y_2^2 - \sum_{j=1}^n y_{2j+2}^2)$. Thus we put $\mathcal{E} = \theta(z_2) f(z)$ with $f(z) = u(z_2^2 - \sum_{j=1}^n z_{2j+2}^2)$, where u is some generalized function. Substituting u instead of f into (5) one gets the simplified differential equation. If suppose that $\partial\theta/\partial z_2 = 0$ for $z_2 > 0$, then $\partial^2\theta/\partial z_2^2 = 0$ and Equation (5) leads to the differential equation

(7) $4u^{(2)}.(1, 1)(\eta)\eta - 4u'.1(\eta) = \delta(z)/c$,

where $\eta = \eta(z) = z_2^2 - z_4^2 - z_6^2 - z_8^2$, $\theta(z_2) = c = const$ for $z_2 > 0$. Choose any sequence of z-differentiable functions $g_n(z)$ with compact supports converging to the δ-function as in §§24 and 30, when n tends to the infinity. Each function $g_n(z)/\eta(z)$ has poles of the first order at points $z_2 = [z_4^2 + z_6^2 + z_8^2]^{1/2}$ and $z_2 = -[z_4^2 + z_6^2 + z_8^2]^{1/2}$. Making the substitution $p = u'.1$ in (7) and Formula 3(10) [20] with the right side $Q(z) = g_n(z)/\eta(z)$ and $b(z) = -1/\eta(z)$ we obtain the integral expression for the solution p_n of the differential equation

(8) ${p'}_n.1(\eta) - p_n(\eta)/\eta = g_n(z)/(4c\eta)$.

To evaluate the appearing integrals it is possible to use Jordan's Lemma (see §2 in [24]) over the octonion algebra isomorphic with $\mathcal{A}_{1,4}$. The evaluation of the integrals (see §3 also) with the given functions can be reduced to the complex case, when α and x belong to the same complex plane $\mathbf{C}_M$. Applying Jordan's lemma one deduces the expression for p_n and the limit function $p(\eta) = \delta'(\eta)/(2\pi c) + K$, where K is a constant, since $\eta \in \mathbf{R}$. Therefore, $u(\eta) = \int p(\eta) d\eta$. Thus one infers the fundamental solution

(9) $\mathcal{E}_3(z) = \theta(t)\delta(t^2 - |x|^2)/(2\pi)$

and the generalized solution $\mathcal{E}_3 * s$ of the wave equation

$\overline{\coprod} f = s$, where $s = s(z)$ is a generalized function or particularly a z-differentiable function. The delta generalized function $\delta(P)$ of the quadratic form $P(x) = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2$ is described in details in §IV.2 [6].

32. Helmholtz' operator.

When $\beta \neq 0$ with $Re(t_j^* \beta) = 0$ for each j (see §8), for example, $\beta = \beta_0 + i_k \beta_1$ with real β_0 and β_1 and $k > n$, then

(1) $A_0 = \Delta_n + |\beta|^2$ is Helmholtz' operator. The corresponding operator Υ is given by Formula 28(2).

For an arbitrary real non-degenerate quadratic form $P(x)$ generalized functions $(c^2 + P + b\mathbf{i}0)^\lambda$ with $c > 0$, $b = 1$ or $b = -1$, $\lambda \in \mathbf{C} = \mathbf{R} \oplus \mathbf{Ri}$, are defined as:

(2) $(c^2 + P + b\mathbf{i}0)^\lambda = \lim_{0<\epsilon\to 0}(c^2 + P + b\epsilon\mathbf{i}P_1)^\lambda$,
where P_1 is a positive definite quadratic form.

Some special functions are useful for such equations. Bessel's functions are solutions of the differential equation

$(S1)$ $z^2 d^2u/dz^2 + zdu/dz + (z^2 - \lambda^2)u = 0$,
where λ and z are complex. Bessel's function of the first kind is given by the series:

$$(S2) \quad J_\lambda(z) = \sum_{m=0}^{\infty} (-1)^m (z/2)^{2m+\lambda} / [m!\Gamma(\lambda + m + 1)],$$

where z and $\lambda \in \mathbf{C_i}$. Then

$(S3)$ $I_\lambda(z) = \exp(-\pi\lambda\mathbf{i}/2) J_\lambda(\mathbf{i}z)$
is called Bessel's function of the imaginary argument. Other needed functions for non-integer λ are:

$(S4)$ $N_\lambda(z) = [J_\lambda(z)\cos(\pi\lambda) - J_{-\lambda}(z)]/\sin(\pi z)$,

$(S5)$ $H^{(1)}(z) = J_\lambda(z) + \mathbf{i}N_\lambda(z)$,

$(S6)$ $H^{(2)}(z) = J_\lambda(z) - \mathbf{i}N_\lambda(z)$,

$(S7)$ $K_\lambda(z) = \pi[I_{-\lambda}(z) - I_\lambda(z)]/[2\sin(\pi\lambda)] = \pi \exp(\mathbf{i}\pi(\lambda+1)/2) H_\lambda^{(1)}(\mathbf{i}z)/2$
with the complex variable z and non-integer complex parameter λ. For integer λ values of these functions $(S4 - 7)$ are defined as limits by $\lambda \in \mathbf{C_i} \setminus \mathbf{Z}$. The functions $H^{(1)}$ and $H^{(2)}$ are also solutions of Bessel's differential equation $(S1)$ and they are called Hankel's functions of the first and the second kind respectively, $K_\lambda(z)$ is known as Mcdonald's function. The functions $I_\lambda(z)$ and $K_\lambda(z)$ are linearly independent solutions of the differential equation:

$(S8)$ $z^2 d^2u/dz^2 + zdu/dz - (z^2 + \lambda^2)u = 0$
(see about special functions in [18, 31]).

Analogously to §28 using Formulas 19.4$(C4 - C7)$ and 14$(3, 4)$ or Theorem 23 for a fundamental solution Ψ_n of the equation

(3) $A_0 \Psi_n = \delta$,
where A_0 is Helmholtz' operator, we get the identity

(4) $F(\Psi_n)(x) = [c^2 - (\sum_{j=1}^n x_j^2)][F(\Psi_n)(x)]^2$ or

(5) $F(\Psi_n)(x) = 1/[c^2 - (\sum_{j=1}^n x_j^2) + b\mathbf{i}0]$,
where $c > 0$, $c = |\beta|$. The Fourier transform of the generalized function $(c^2 + P(x) + b\mathbf{i}0)^\lambda$ by the real variables $x = (x_1, ..., x_n)$ with the generator $\mathbf{i}$ is:

(6) $F[(c^2 + P(x) + b\mathbf{i}0)^\lambda](y) =$
$2^{\lambda+1}(2\pi)^{n/2} c^{\lambda+(n/2)} K_{\lambda+(n/2)}[c(Q(y) - b\mathbf{i}0)]/[\Gamma(-\lambda) D^{1/2} (Q(y) - b\mathbf{i}0)^{(\lambda/2+n/4)}]$,
where $D = \det(g_{j,k})$ denotes a discriminant of the quadratic form $P(x) = \sum_{j,k=1}^n g_{j,k} x_j x_k$, $Q(y) = \sum_{j,k=1}^n g^{j,k} x_j x_k$ is the dual quadratic form so that $\sum_{k=1}^n g^{j,k} g_{k,l} = \delta_l^j$, $\delta_l^j = 1$ for $j = l$ and $\delta_l^j = 0$ for $j \neq l$ (see §IV.8.2 [6]). Mention that $D^{1/2} = |D|^{1/2} \exp(-q\pi\mathbf{i}/2)$ if the canonical representation of the quadratic form P has q negative terms.

Another formula is:

(7) $F[(c^2+P(x)+b\mathbf{i}0)^\lambda](y)=$
$2^{\lambda+1}(2\pi)^{n/2}\exp(-bq\pi\mathbf{i}/2)c^{\lambda+(n/2)}\{K_{\lambda+(n/2)}[c(Q_+(y))^{1/2}]/[\Gamma(-\lambda)|D|^{1/2}(Q_+(y))^{\lambda/2+n/4}]$
$+(b\pi\mathbf{i}/2)H^{(j(b))}_{-\lambda-(n/2)}[c(Q_-(y))^{\lambda/2+n/4}]/[\Gamma(-\lambda)|D|^{1/2}(Q_-(y))^{\lambda/2+n/4}]\}$,
where $j(1)=1$, $j(-1)=2$,

$(P^\lambda_+,\phi)=\int_{P>0}P^\lambda\phi dx_1...dx_n$,

$(P^\lambda_-,\phi)=\int_{P<0}P^\lambda\phi dx_1...dx_n$.

The functions $(P+b\mathbf{i}0)^\lambda$ and P^λ_+ and P^λ_- by the complex variable λ are regularized as it is described in [6] with the help of their Laurent series in neighborhoods of singular isolated points λ such that after the regularization only the regular part of the Laurent series remains. The functions $(P+b\mathbf{i}0)^\lambda$ with $b=1$ or $b=-1$ have only simple poles at the points $\lambda=-n/2$, -(n/2)-1,...,$-(n/2)-k$,..., where $k=1,2,...$ is a natural number. Using formula (6) with $\lambda=-1$ and $P(x)=-(x_1^2+...+x_n^2)$ one gets the fundamental solution Ψ_n, where $(F^{-1}g)(x)=(2\pi)^{-n}(Fg)(-x)$. Particularly, $\Psi_3(x)=-\exp(bc\mathbf{i}|x|)/(4\pi|x|)$, $\Psi_2(x)=-\mathbf{i}H^{(1)}_0(c|x|)/4$ or its complex conjugate $\mathbf{i}H^{(2)}_0(c|x|)/4$, where $H^{(j)}_0$ denotes Hankel's function, $j=1,2$.

33. Klein-Gordon's operator.

Consider β and t_j as in §8 with $Re(t_j^*\beta)=0$ for each j, $c=|\beta|>0$. Take the operator

(1) $A_0=L_{p,q}+c^2$,

where $L_{p,q}$ is the hyperbolic operator as in §29. For $p=1$ and $q=3$ the operator A_0 is called Klein-Gordon's operator. From Formulas 32$(C4-C7)$ and 14$(3,4)$ or Theorem 23 we infer that

(2) $F(\Psi_n)(x)=[c^2-(\sum_{j=1}^p x_j^2-\sum_{j=p+1}^n x_j^2)][F(\Psi_n)(x)]^2$ or

(3) $F(\Psi_n)(x)=1/[c^2-(\sum_{j=1}^p x_j^2-\sum_{j=p+1}^n x_j^2)+b\mathbf{i}0]$.

Then Formulas 19.4(6) or 32(7) with $\lambda=-1$ and $P(x)=-x_1^2-...-x_p^2+x_{p+1}^2+...+x_n^2$, $n=p+q$, give the fundamental solution $\Psi_{p,q}$ of the equation

(4) $(L_{p,q}+c^2)\Psi_{p,q}=\delta$,

where $(F^{-1}g)(x)=(2\pi)^{-n}(Fg)(-x)$. There are two $\mathbf{R}$-linearly independent fundamental solutions, so their $\mathbf{R}$-linear combination with real coefficients α_1 and α_2 such that $\alpha_1+\alpha_2=1$ is also a fundamental solution.

34. Remark. Certainly, more general partial differential equations as 30(1), but with $\partial^l v/\partial z_0^l$, $l\ge 2$, instead of $\partial v/\partial z_0$ can be considered. It is worth to mention, that alternative deductions using Formulas 7(1) and 27(11,14) can be used instead of 8(1) and 19.4$(C1-C7)$ in §§30 and 31 providing also $u(z)=\alpha(z_1^2+...+z_m^2)+\beta$ and $f(z)=u(z_2^2-\sum_{j=1}^n z_{2j+2}^2)$ with the help of the symmetry Lie groups $\mathsf{O}_m(\mathbf{R})$ and $\mathsf{O}_{1,n}(\mathbf{R})$. Indeed, Functions $P(x)^\lambda$ satisfy Condition 7$(P3)$ for any real λ, where $P(z)=z_1^2+...+z_p^2-z_{p+1}^2-...-z_n^2$, $1\le p\le n\le 2^{v-1}-1$, $1\le v$, since $(dP(z)^\lambda/dz).h=P(z)^{\lambda-1}\lambda 2Re(\eta(z)h)$, where $z\in\mathcal{A}_v$, $\eta(z)=z_1i_1+...+z_ni_n$ for $p=n$, while $\eta(z)=z_1i_2+...+z_pi_{2p}+z_{p+1}i_1^*i_{2(p+1)}+...+z_ni_1^*i_{2n}$ for $p<n$. The function $\eta(z)$ can be written in the z-representation due to Formulas $1(1-3)$.

Formally the case of the hyperbolic operators $L_{p,q}+c^2$ and their fundamental solutions is obtained from the elliptic operators Δ_n+c^2 with $c\ge 0$ by the change of variables $(x_1,...,x_n)\mapsto(x_1,...,x_p,x_{p+1}\mathbf{i},...,x_{p+q}\mathbf{i})$, where $n=p+q$, $x_j\in\mathbf{R}$ and $x_j\mathbf{i}\in\mathbf{C}=\mathbf{R}\oplus\mathbf{R}\mathbf{i}$ for each j, since the quadratic forms P may be with complex coefficients and their Fourier transforms can be considered as in [6]. At the same operators σ or Υ for these particular operators $L_{p,q}+c^2$ and Δ_n+c^2 can be written over the complexified algebra $(\mathcal{A}_r)_\mathbf{C}$ instead of the Cayley-Dickson algebra $\mathcal{A}_v$, $2\le r<v$ (see §2 above). For this we take in Formula 2(8) $\mathbf{i}$ instead of w_j^* so that

(1) $\sigma f(z)=\sum_{j=1}^p(\partial f(z)/\partial z_j)i_j^*+\sum_{j=p+1}^n(\partial f(z)/\partial z_j)\mathbf{i}i_j^*$, consequently,

(2) $(c+\sigma)(c-\sigma)f=(L_{p,q}+c^2)f$ and

(3) $(c+\sigma)(c+\sigma)^*f=(c+\sigma)^*(c+\sigma)f=(\Delta_n+c^2)f$.

Let

(4) $\Xi_{c,p,q} = L_{p,q} + c^2$, where $c \in \mathbf{R}$, $L_{n,0} = \Delta_n$, $1 \le p \le n$, $q = n - p$, and let $\Psi_{c,p,q}$ be a fundamental solution of the equation

(5) $\Xi_{c,p,q}\Psi_{c,p,q} = \delta$.

Then due to Identities $(2,3)$ a fundamental solution $\mathcal{E} = \mathcal{E}_{\beta+\sigma}$ of the equation

(6) $(\sigma + \beta)\mathcal{E} = \delta$ can be written in the form:

(7) $\mathcal{E}_{\beta+\sigma} = (\sigma + \beta)^*\Psi_{c,n,0}$, where $\beta \in \mathcal{A}_r$, $|\beta| = c$, $Re(t_j^*\beta) = 0$ for each j, $t_j = i_j^*$ for $1 \le j \le p$, $t_j = \mathbf{i}i_j^*$ for $p < j \le n$. Moreover,

(8) $\mathcal{E}_{c+\sigma} = (c - \sigma)\Psi_{c,p,q}$.

Therefore, we infer a solution of the equation

(9) $(\sigma + \beta)f = g$ in $\mathcal{D}(\mathbf{R}^m, \mathcal{A}_r)$ or in the space $\mathcal{D}(\mathbf{R}^m, \mathcal{A}_r)^*_l$:

(10) $f = \mathcal{E}_{\beta+\sigma} * g$. From $(2,3)$ we deduce a fundamental solution $\mathcal{V}$ of the equation

(10) $A_0\mathcal{V} = (\sigma + \beta)(\sigma_1 + \beta_1)\mathcal{V} = \delta$ in the convolution form:

(11) $\mathcal{V} = \mathcal{E}_{\sigma+\beta} * \mathcal{E}_{\sigma_1+\beta_1}$,

since

(12) $A_0\mathcal{V} = (\sigma + \beta)(\sigma_1 + \beta_1)(((\sigma + \beta)^*\Psi_{c,n,0}) * ((\sigma_1 + \beta_1)^*\Psi_{c,n,0})$
$= (((\sigma + \beta)(\sigma + \beta_1)^*\Psi_{c,n,0}) * (((\sigma_1 + \beta_1)(\sigma_1 + \beta_1)^*\Psi_{c,n,0})) = \delta * \delta = \delta$. Particularly,

$\Psi_{c,p,q} = -\mathcal{E}_{c+\sigma} * \mathcal{E}_{-c+\sigma} = ((c-\sigma)\Psi_{c,n,0}) * ((c+\sigma)\Psi_{c,n,0})$, that can be lightly verified after the Fourier transform by real variables with the generator $\mathbf{i}$, since by Formulas $(1,2)$ the operator σ and its anti-derivative operator $\mathcal{I}_\sigma$ correspond to the signature (p,q) and $F(\sigma\sigma^* g) = -|z|^2 F(g)$ for any $g \in \mathcal{D}(\mathbf{R}^n, \mathcal{A}_v)^*_l$.

Knowing a fundamental solution it is possible to consider then a boundary problem (see also [13, 37]).

35. Partial differential equations of order higher than two.

The fundamental solution $\Psi_{\Upsilon^m+\beta}$ of the equation

(1) $\Xi_{2m,\beta}\Psi_{\Upsilon^m+\beta} = \delta$,

where

(2) $\Xi_{2m,\beta} = (\Upsilon^m + \beta)(\Upsilon^m + \beta)^*$

can be obtained using decompositions with the help of operators of the first order $\Upsilon_k + \beta_k$ by induction, if such decomposition exists (see §§10-14 above). Suppose that this decomposition is constructed

(3) $(\Upsilon^m+\beta)f(z) = (\Upsilon_m+\beta_m)[...[(\Upsilon_2+\beta_2)[(\Upsilon_m+\beta_m)f(z)]]...]$, then the fundamental solution can be written as the iterated convolution

(4) $\Psi_{\Upsilon^m+\beta} = [...[[(\Upsilon_m + \beta_m)^*\mathcal{E}_m] * [(\Upsilon_{m-1} + \beta_{m-1})^*\mathcal{E}_{m-1}]] * ...] * [(\Upsilon_1 + \beta_1)^*\mathcal{E}_1]$,

where $\mathcal{E}_j$ denotes the fundamental solution of the elliptic second order partial differential equation

(5) $A_j\mathcal{E}_j = \delta$, with

(6) $A_j = (\Upsilon_j + \beta_j)(\Upsilon_j + \beta_j)^*$.

The fundamental solutions $\mathcal{E}_j$ were written above in §§2 and 28. Indeed, using Equalities $4(7-9)$ by induction we have

(7) $\sum_s(...(a^*_{m,k_m}a^*_{m-1,k_{m-1}})a^*_{m-2,k_{m-2}})...)a^*_{1,k_1})a_{1,l_1})a_{2,l_2})...)a_{m,l_m}$
$= Re(a^*_{m,k_m}a_{m,l_m})...Re(a^*_{1,k_1}a_{1,l_1})$,

where $\sum_s$ denotes the sum by all pairwise transpositions (k_1,l_1),...,(k_m,l_m), $a_{j,k} \in \mathcal{A}_v$. Therefore,

(8) $\Xi_{2m,\beta}\Psi_{\Upsilon^m+\beta} =$
$[...[[(\Upsilon_m+\beta_m)(\Upsilon_m+\beta_m)^*\mathcal{E}_m]*[(\Upsilon_{m-1}+\beta_{m-1})(\Upsilon_{m-1}+\beta_{m-1})^*\mathcal{E}_{m-1}]]*...]*[(\Upsilon_1+\beta_1)(\Upsilon_1+\beta_1)^*\mathcal{E}_1]$
$= [...[\delta * \delta] * ...] * \delta = \delta$.

Vice versa if the fundamental solution $\Psi_{\Upsilon^m+\beta}$ is known, then we get the fundamental solution $\mathcal{E}^m_\beta$ of the equation

(9) $(\Upsilon^m + \beta)\mathcal{E}^m_\beta = \delta$ as

(10) $\mathcal{E}^m_\beta = (\Upsilon^m + \beta)^* \Psi_{\Upsilon^m+\beta}$ in accordance with $(2,7)$. Moreover, the equation

(11) $A_{m+k} f = g$ with $A_{m+k} = (\Upsilon^m_1 + \beta_1)(\Upsilon^k_2 + \beta_2)$

in $\mathcal{D}(\mathbf{R}^n, \mathcal{A}_v)$ or in the space $\mathcal{D}(\mathbf{R}^n, \mathcal{A}_v)^*_l$, where n is a number of real variables, $2 \le n \le 2^v$, has the fundamental solution $\mathcal{V}_{m+k}$:

(12) $\mathcal{V}_{m+k} = \mathcal{E}_{\Upsilon^m_1+\beta_1} * \mathcal{E}_{\Upsilon^k_2+\beta_2}$, where

(13) $\mathcal{E}_{\Upsilon^m_1+\beta_1} = (\Upsilon^m_1 + \beta_1)^* \Psi_{\Upsilon^m_1+\beta_1}$

denotes the fundamental solution of the equation

(14) $(\Upsilon^m_1 + \beta_1)\mathcal{E}_{\Upsilon^m_1+\beta_1} = \delta$, consequently,

(15) $f = \mathcal{V}_{m+k} * g$ is the solution of Equation (11).

For example, the fourth order partial differential operator

$A_4 f(z) = \sum_{j=1}^p \partial^4 f(z)/\partial z_j^4 - \sum_{j=p+1}^n \partial^4 f(z)/\partial z_j^4$

can be decomposed as the composition of two operators of the second order Υ^2 and Υ^2_1 formally as σ and σ_1 in $2(8,9)$ with the substitution of $\partial f/\partial z_{2^r j}$ on $\partial^2 f/\partial z^2_{2^r j}$ so that in accordance with Theorem 10 this operator A_4 can be presented in the form given by Formulas $(2,3,11)$.

On the other hand, fundamental solutions of Δ^k_n and $L^k_{p,q}$ and A^k_2 for certain other second order partial differential operators are known. So combining them with operators of the form $\Upsilon^{m_1}_1 ... \Upsilon^{m_k}_k$ permits to consider fundamental solutions of many partial differential operators of order higher than two as well.

Thus knowing fundamental solutions of the corresponding first or second order operators permits to write fundamental solutions of higher order partial differential operators considered above with the help of the iterated convolutions in a definite order prescribed by the induction process.

36. Non-linear partial differential equations.

We consider the differential equation

(1) $(\Upsilon^m + \beta + \hat{f}(y)\Upsilon)y = g$,

where $\Upsilon^m + \beta$ is a partial differential operator as in Formula 10(13) of order m, $f(y)$ is a $\mathcal{A}_v$ differentiable function, $y = y(z)$ is an unknown function, $\hat{f}(y)\Upsilon y := \sum_{j=0}^n [\hat{f}(y).(\partial y(z)/\partial z_j)]\phi^*_j(z)$. Suppose that a fundamental solution $\mathcal{E}_{\Upsilon^m+\beta}$ of Equation 35(9) for the operator $(\Upsilon^m + \beta)$ is known. Find at first a fundamental solution $y = \mathcal{V}$ of (1) with $g = \delta$. Then

(2) $(\Upsilon^m + \beta)\mathcal{V} = \delta - \mu$,

where $\mu(z) = \hat{f}(y(z))\Upsilon y(z)$. The anti-derivative gives

(3) $w(y(z)) = {}_\Upsilon\int (\hat{f}(y)\Upsilon y)dz = \int f(y(x))dy(x) = \int f(y)dy = {}_\Upsilon\int \mu(z)dz$,

then

(4) $y = w^{-1}({}_\Upsilon\int_{\gamma^\alpha|_{[a,t_z]}} \mu(x)dx)$,

where w^{-1} denotes the inverse function. On the other hand,

(5) $y = \mathcal{E}_{\Upsilon^m+\beta} * (\delta - \mu) = \mathcal{I}_{\Upsilon^m+\beta}(\delta - \mu)$, when Υ^m is either of the first order for $m = 1$ or is expressed as a composition of operators of the first order,

(5.1) $\Upsilon^m + \beta = (\Upsilon_1 + \beta^1)...(\Upsilon_m + \beta^m)$ so that

(5.2) $\mathcal{I}_{\Upsilon^m+\beta} = \mathcal{I}_{\Upsilon_m+\beta^m}...\mathcal{I}_{\Upsilon_1+\beta^1}$,

consequently, $(4,5)$ imply the equation:

(6) $\mathcal{E}_{\Upsilon^m+\beta} * (\delta - \mu) = \mathcal{I}_{\Upsilon^m+\beta}(\delta - \mu) = w^{-1}({}_\Upsilon\int_{\gamma^\alpha|_{[a,t_z]}} \mu(x)dx)$ or

(7) $w(\mathcal{E}_{\Upsilon^m+\beta} * (\delta - \mu)) = {}_\Upsilon\int_{\gamma^\alpha|_{[a,t_z]}} \mu(x)dx$.

We have the identity

$(\partial(\mathcal{E} * \Psi)/\partial z_p), \phi) = -((\mathcal{E} * \Psi), \partial\phi/\partial z_p) = ((\mathcal{E} * (\partial\Psi/\partial z_p)), \phi)$

with a generalized function Ψ.

Therefore, differentiating (6) by z_0,...,z_n, we infer that:

(8) $\{\sum_{j=0}^n [\mathcal{E}_{\Upsilon^m+\beta} * (\partial\mu(z)/\partial z_j)]\phi^*_j(z)\} +$

$(n+1)^{-1} \sum_{j,k=0}^n [(dw^{-1}(\xi)/d\xi).(\hat{\theta}(z).(\partial\nu_j(z)/\partial z_k))]\phi^*_k(z) = \Upsilon\mathcal{E}_{\Upsilon^m+\beta}$ or

(9) $\Upsilon(\mathcal{I}_{\Upsilon^m+\beta}\mu)+$
$(n+1)^{-1}\sum_{j,k=0}^n[(dw^{-1}(\xi)/d\xi).(\hat{\theta}(z).(\partial\nu_j(z)/\partial z_k))]\phi_k^*(z)=\Upsilon(\mathcal{I}_{\Upsilon^m+\beta}\delta)$,
where

(9.1) $[((dg(z)/dz).\phi_j(z))\otimes\phi_j^*(z),\omega^{\otimes 2})=[\mu(z)\otimes 1,\omega^{\otimes 2})$

for each real-valued test function ω and each j, $\hat{\theta}(z)=dg(z)/dz$, $\xi=_{\Upsilon}\int_{\gamma^\alpha|_{[a,t_z]}}\mu(x)dx$ (see also §§4, 5, 17, 23 and 26). If Υ and β are independent of z_j, i.e. $\phi_j(z)=0$ is zero identically on U, then $\partial(\mathcal{I}_{\Upsilon^m+\beta}\mu)/\partial z_j=\mathcal{I}_{\Upsilon^m+\beta}(\partial\mu/\partial z_j)$ (see also Note 23.1). Otherwise the derivative $\partial(\mathcal{I}_{\Upsilon^m+\beta}\mu)/\partial z_j$ is given by Formulas 4(6) and 23(8) and 26(1). The function w is known from (3) after the line integration by the variable y, so Equation (8) is linear by $(\partial\mu(z)/\partial z_j)$, $j=0,...,n$. It can be solved as in [20]. Calculating μ from (8) or (9) we get the fundamental solution:

(10) $\mathcal{V}=\mathcal{E}_{\Upsilon^m+\beta}*(\delta-\mu)$

and the (particular) solution of (1) is:

(11) $y=\mathcal{V}*g$.

When $[\mathcal{E}_{\Upsilon^m+\beta},\omega)$ is real for each real-valued test function ω or $\Upsilon f=(\partial f/\partial z_0)\phi_0(z)$ with a real-valued function $\phi_0(z)$ and the inverse relative to the convolution generalized function is known $\mathcal{E}^{-1}_{\Upsilon^m+\beta}$ such that

(12) $\mathcal{E}^{-1}_{\Upsilon^m+\beta}*\mathcal{E}_{\Upsilon^m+\beta}=\delta$,

then Equation (8) simplifies:

(13) $\Upsilon\mu(z)+(n+1)^{-1}\sum_{j,k=0}^n\mathcal{E}^{-1}_{\Upsilon^m+\beta}*[[(dw^{-1}(\xi)/d\xi).(\hat{\theta}(z).(\partial\nu_j(z)/\partial z_k))]\phi_k^*(z)]=\sum_{j=0}^n[\mathcal{E}^{-1}_{\Upsilon^m+\beta}*(\partial\mathcal{E}_{\Upsilon^m+\beta}/\partial z_j)]\phi_j^*(z)$, consequently,

(14) $\Upsilon\mu(z)+\nu(\mu)=b(z)$,

where $\nu(\mu):=(n+1)^{-1}\sum_{j,k=0}^n\mathcal{E}^{-1}_{\Upsilon^m+\beta}*[[(dw^{-1}(\xi)/d\xi).(\hat{\theta}(z).(\partial\nu_j(z)/\partial z_k))]\phi_k^*(z)]$ and $b(z)=\mathcal{E}^{-1}_{\Upsilon^m+\beta}*(\Upsilon\mathcal{E}_{\Upsilon^m+\beta})$.

If equation (1) is solved, then it provides a solution of more general equation:

(15) $(\Upsilon^m+\beta+\hat{f}((\Upsilon)^{k-1}\xi)(\Upsilon)^k)\xi=g$

finding ξ from the equation $(\Upsilon)^{k-1}\xi=y$, where $(\Upsilon)^k$ denotes the k-th power of the operator Υ.

If $\phi_j(z)=i_j\psi_j(z)$ for each j, then functions $\{\nu_j(z):\ j\}$ can be chosen real-valued or real-valued generalized functions on real valued test functions (see System 4(10, 11) and §23). In such case the equality

$\sum_{j,k=0}^n[(dw^{-1}(\xi)/d\xi).(\hat{\theta}(z).(\partial\nu_j(z)/\partial z_k))]\phi_k^*(z)$
$=\sum_{j,k=0}^n[(dw^{-1}(\xi)/d\xi).(\mu(z)(\partial\nu_j(z)/\partial z_k)i_j)]i_k^*\psi_k(z)]$

is satisfied. For $\Upsilon f=(\partial f/\partial z_0)\phi_0(z)$ with a real-valued function $\phi_0(z)$ these equations simplify, since $\hat{\theta}.h=\mu(z)h$ for each $h\in\mathbf{R}$ and $z\in U$ and $(n+1)^{-1}\sum_{j,k=0}^n\mathcal{E}^{-1}_{\Upsilon^m+\beta}*[[(dw^{-1}(\xi)/d\xi).(\hat{\theta}(z).(\partial\nu_j(z)/\partial z_k))]\phi_k^*(z)]=\mathcal{E}^{-1}_{\Upsilon^m+\beta}*[(dw^{-1}(\xi)/d\xi).\mu(z)]$.

Thus the results of this paper over the Cayley-Dickson algebras enrich the technique of integration of partial differential equations in comparison with the complex field.

It is planned to present in the next paper solutions of some types of non-linear partial differential equations with the help of non-linear mappings and non-commutative line integration over the Cayley-Dickson algebras.

References

[1] Baez J.C. The octonions // *Bull. Amer. Mathem. Soc.* 39, 2, pp. 145-205, 2002.

[2] Dickson L.E. The collected mathematical papers. Volumes 1-5, Chelsea Publishing Co., New York, 1975.

[3] Engelking R. General topology, Mir, Moscow, 1986.

[4] Emch G. Méchanique quantique quaternionienne et Relativitè restreinte. // *Helv. Phys. Acta* 36, pp. 739-788, 1963.

[5] Fihtengolz G.M. Course of differential and integral calculus. V. 1-3, Nauka, Moscow, 1966.

[6] Gelfand I.M., Shilov G.E. Generalized functions and operations with them. Fiz.-Mat. Lit., Moscow, 1958.

[7] Grubb G. Functional calculus of pseudodifferential boundary problems, Birkhäuser, Boston, 1996.

[8] Gürsey F., Tze C.-H. On the role of division, Jordan and related algebras in particle physics. World Scientific Publ. Co., Singapore, 1996.

[9] Gürlebeck K., Habetha K., Sprössig W. Holomorphic functions in the plane and the n-dimensional space. Birkhäuser, Basel, 2008.

[10] Gürlebeck K., Sprössig W. Quaternionic and Clifford calculus for physicists and engineers. John Wiley and Sons, Inc., Chichester, 1997.

[11] Gürlebeck K., Sprössig W. Quaternionic analysis and elliptic boundary value problem. Birkhäuser, Basel, 1990.

[12] Hamilton W.R. Selected works. Optics. Dynamics. Quaternions. Nauka, Moscow, 1994.

[13] Hörmander L. The analysis of linear partial differential operators. V. 1-4. Springer-Verlag, Berlin, 1985.

[14] Kantor I.L., Solodovnikov A.S. Hypercomplex numbers. Springer-Verlag, Berlin, 1989.

[15] Kneschke A. Differentialgleichungen und Randwertprobleme. Bande 1-3. B.G. Teubner Verlaggesellschaft, Leipzig, 1965.

[16] Kolmogorov A.N., Fomin S.V. Elements of theory of functions and functional analysis. Nauka, Moscow, 1989.

[17] Kurosh A.G. Lectures on general algebra. Nauka, Moscow, 1973.

[18] Lavrentjev M.A., Shabat B.V. Methods of functions of the complex variable. Nauka, Moscow, 1987.

[19] Lawson H.B., Michelson M.-L. Spin geometry. Princ. Univ. Press, Princeton, 1989.

[20] Ludkovsky S.V. Differential equations over octonions. // *Adv. Appl. Clifford Alg. AACA*, in press 2010 (previous variant, Los Alam. Nat. Lab. math/1003.2620).

[21] Ludkovsky S.V. Functions of several Cayley-Dickson variables and manifolds over them. // *J. Mathem. Sci.* 141, 3, pp. 1299-1330, 2007 (*Sovrem. Mathem. i ee Pril.* 28 (2005); previous variant, Los Alamos Nat. Lab. math.CV/0302011).

[22] Ludkovsky S.V. Differentiable functions of Cayley-Dickson numbers and line integration. // *J. Mathem. Sciences* 141, 3, pp. 1231-1298, 2007 (*Sovrem. Mathem. i ee Pril.* 28, 2005; previous version, Los Alam. Nat. Lab. math.NT/0406048; math.CV/0406306; math.CV/0405471).

[23] Lüdkovsky S.V., F. van Oystaeyen. Differentiable functions of quaternion variables.// *Bull. Sci. Math.* 127, pp. 755-796, 2003.

[24] Ludkovsky S.V. The two-sided Laplace transformation over the Cayley-Dickson algebras and its applications. // *J. of Mathem. Sciences* 151, 5, pp. 3372-3430, 2008 (earlier version, Los Alamos Nat. Lab. math.CV/0612755).

[25] Ludkovsky S.V. Normal families of functions and groups of pseudoconformal diffeomorphisms of quaternion and octonion variables. // *J. Mathem. Sci.* 150, 4, pp. 2224-2287, 2008 [transl. from, Sovrem. Mathem. Fundam. Napravl.V. 18, pp. 101-164, 2006.] (previous variant, Los Alam. Nat. Lab. math.DG/0603006).

[26] Ludkovsky S.V. Quasi-conformal functions of quaternion and octonion variables, their integral transformations. // *Far East Journal of Mathematical Sciences (FJMS)* 28, 1, pp. 37-88, 2008.

[27] Ludkovsky S.V. Residues of functions of octonion variables // *Far East Journal of Mathematical Sciences (FJMS)*, bf 39, 1, pp. 65-104, 2010 (previous version, Los Alam. Nat. Lab. math.CV/0902.2495 42 pages).

[28] Ludkovsky S.V. Multidimensional Laplace transforms over Cayley-Dickson algebras and partial differential equations. // *Los Alam. Nat. Lab.* math/1004.4355, 2010.

[29] Matveev N.M. Methods of integration of ordinary differential equations. Lan, St. Petersbourg, 2003.

[30] Michor P.W. Manifolds of differentiable mappings. Boston, Shiva, 1980.

[31] Nikiforov A.F., Uvarov V.B. Special functions of mathematical physics. Nauka, Moscow, 1978.

[32] Petrovsky I.G. Lectures on partial differential equations. Gos. Izd. Techn. and Theoret. Liter., Moscow, 1953.

[33] Rothe H. Systeme Geometrischer Analyse. In Encyklopädie der Mathematischen Wissenschaften. Band 3. Geometrie, 1277-1423. Teubner, Leipzig, 1914-1931.

[34] Rubinstein I., Rubinstein L. Partial differential equations in classical mathematical Physics. Cambridge Univ. Press, Cambridge, 1998.

[35] Solovjev M.A. A structure of a space of non-abelian gauge fields // *Proceed. Lebedev Phys. Inst.* 210, pp. 112-155, 1993.

[36] Spanier E.H. Algebraic topology. New York, Academic Press, 1966.

[37] Vladimirov V.S. Equations of Mathematical Physics. Nauka, Moscow, 1971.

[38] Zorich. Mathematical Analysis. V. 2. Nauka, Moscow, 1984.

ИНТЕГРИРОВАНИЕ ВДОЛЬ ПУТЕЙ И ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ ВТОРОГО ПОРЯДКА НАД АЛГЕБРАМИ КЭЛИ-ДИКСОНА

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Изучается интегрирование вдоль путей обобщенных функций. Исследуются дифференциальные уравнения в частных производных второго порядка с кусочно непрерывными и обобщенными переменными коэффициентами над алгебрами Кэли-Диксона. Выведены формулы для их интегрирования. Для этой цели используется некоммутативное интегрирование вдоль путей. Даются примеры решений дифференциальных уравнений в частных производных.

Ключевые слова: алгебра Кэли-Диксона, дифференциальное уравнение в частных производных, интегрирование вдоль пути, обобщенная функция.