BIANCHI TYPE IDENTITIES IN GENERALIZED FINSLER SPACE[1]

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In the work [4, 14] we have studied a generalized Finsler space ($\mathbb{GF}_N$) (with non-symmetric basic tensor) and have obtained four curvature tensors by using four kinds derivatives in the sense of Rund's δ-differentiation.

In the present work we study Bianchi type identities, related to mentioned curvature tensors in $\mathbb{GF}_N$, generalizing the known Bianchi identity from the usual Finsler space.

Key Words: generalized Finsler spaces, non-symmetric connection, Bianchi type identities.
AMS Subj. Class.: 53A45, 53B05, 53B40.

1 Introduction

The Finsler space and it's generalizations were investigated by many authors, for example: I. Čomić [1], L. Xin and C. Zhe [3], A. Moór [5, 6], S. I. Vacaru [9], M. I. Wanas [10], I. Yoshihoro, L. Il-Yong, and P. Hong-Suh [12] N. L. Youssef and Amr M. Sid- Ahmed [13] and many others. Some of them found very appropriate applications of this theory in physics. A lot of investigations are concerned with spaces which are not torsion free.

The *generalized Finsler space* ($\mathbb{GF}_N$) is a differentiable manifold with non-symmetric basic tensor

$$g_{ij}(x^1,\dots,x^N,\dot{x}^1,\dots,\dot{x}^N)\equiv g_{ij}(x,\dot{x}),$$

where

$$g_{ij}(x,\dot{x})\neq g_{ji}(x,\dot{x}),\quad (g=det(g_{ij})\neq 0,\ \dot{x}=dx/dt). \tag{1.1}$$

Based on (1.1), one can defined the symmetric and anti-symmetric part of g_{ij}

$$g_{\underline{ij}}=\frac{1}{2}(g_{ij}+g_{ji}),\qquad g_{\underset{\vee}{ij}}=\frac{1}{2}(g_{ij}-g_{ji}), \tag{1.2}$$

where are

$$a)\ g_{\underline{ij}}(x,\dot{x})=\frac{1}{2}\frac{\partial^2F^2(x,\dot{x})}{\partial\dot{x}^i\partial\dot{x}^j},\qquad b)\ \frac{\partial g_{\underset{\vee}{ij}}(x,\dot{x})}{\partial\dot{x}^k}=0. \tag{1.3}$$

The function $F(x,\dot{x})$ is a metric function in $\mathbb{GF}_N$, having the properties known from the theory of usual Finsler space ($\mathbb{F}_N$) (see e.g. [8]), the following conditions are valid:

1. $F(x,\dot{x})$ is continuously differentiable at least four times in its $2N$ arguments.

2. $F(x,\dot{x})>0$ providing all dx^i are not 0.

3. $F(x,x)$ is positively homogeneous of the 1^{st} degree in $\dot{x}$, i.e. $F(x,k\dot{x})=kF(x,\dot{x}),\quad k>0.$

[1]The authors gratefully acknowledge for support from the research projects 144032 of the Serbian Ministry of Science.

4. $\dfrac{\partial^2 F^2(x,\dot{x})}{\partial \dot{x}^i \partial \dot{x}^j}\xi^i\xi^j > 0$ for any given $\dot{x}$, and $\sum\limits_i (\xi^i)^2 > 0,\ \xi^i \in \mathbb{R}$.

The lowering and the raising of indices one defines by the tensors $g_{\underline{ij}}$ and h^{ij} respectively, where h^{ij} is defined as following

$$g_{\underline{ij}}\, h^{jk} = \delta_i^k, \quad (\underline{g} = det(g_{\underline{ij}}) \neq 0). \tag{1.4}$$

We can define *generalized Cristoffel symbols* of the 1^{st} and the 2^{nd} kind:

$$\gamma_{i.jk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}) \neq \gamma_{i.kj}, \tag{1.5}$$

$$\gamma^i_{jk} = h^{ip}\gamma_{p.jk} = \frac{1}{2}h^{ip}(g_{jp,k} - g_{jk,p} + g_{pk,j}) \neq \gamma^i_{kj}, \tag{1.6}$$

where, e.g., $g_{ji,k} = \partial g_{ji}/\partial x^k$.

Then we have

$$\gamma^p_{jk} g_{\underline{ip}} = \gamma_{s.jk} h^{ps} g_{\underline{ip}} = \gamma_{s.jk}\delta^{*s}_i = \gamma_{i.jk}. \tag{1.7}$$

Introducing a tensor C_{ijk} as in $\mathbb{F}_N$, we have

$$C_{ijk}(x,\dot{x}) \overset{def}{=} \frac{1}{2} g_{ij,\dot{x}^k} \underset{(1.3b)}{=} \frac{1}{2} g_{\underline{ij},\dot{x}^k} = \frac{1}{4} F^2_{\dot{x}^i\dot{x}^j\dot{x}^k}, \tag{1.8}$$

where " $\underset{(1.3b)}{=}$ " signifies "equal based on (1.3b)". We see that C_{ijk} is symmetric in relation to each pair of indices. Also, we have

$$C^i_{jk} \overset{def}{=} h^{ip} C_{pjk} \underset{(1.8)}{=} h^{ip} C_{jpk} = h^{ip} C_{jkp}. \tag{1.9}$$

With help of coefficients

$$P^i_{jk} = \gamma^i_{jk} - C^i_{jp}\gamma^p_{sk}\dot{x}^{*s} \neq P^i_{kj} \tag{1.10}$$

one obtains coefficients of non-symmetric affine connections in the Rund's sense (see [8]):

$$P^{*i}_{jk} = \gamma^i_{jk} - h^{iq}(C_{jqp}P^p_{ks} + C_{kqp}P^p_{js} - C_{jkp}P^p_{qs})\dot{x}^{*s} \underset{(1.6)}{\neq} P^{*i}_{kj}, \tag{1.11}$$

$$P^*_{i.jk} = P^{*r}_{jk} g_{\underline{ir}} = \gamma_{i.jk} - (C_{ijp}P^p_{ks} + C_{ikp}P^p_{js} - C_{jkp}P^p_{is})\dot{x}^{*s} \neq P^*_{i.kj}. \tag{1.12}$$

In $\mathbb{GF}_N$ we denote anti-symmetric and symmetric part for connection P^* respectively:

$$a)\ T^{*i}_{jk}(x,\dot{x}) = P^{*i}_{\underset{\vee}{jk}} = \frac{1}{2}(P^{*i}_{jk} - P^{*i}_{kj}) = \frac{1}{2}(\gamma^i_{jk} - \gamma^i_{kj}), \quad b)\ P^{*i}_{\underline{jk}} = \frac{1}{2}(P^{*i}_{jk} + P^{*i}_{kj}), \tag{1.13}$$

where T^{*i}_{jk} is the double *torsion tensor*.

It is possible to define four kinds of covariant derivative of a tensor in the space $\mathbb{GF}_N$. For example, for a tensor $a^i_j(x,\dot{x})$ we have

$$a^i_{j\underset{1}{|}m}(x,\dot{x}) = \delta_m a^i_j + P^{*i}_{pm} a^p_j - P^{*p}_{jm} a^i_p, \quad a^i_{j\underset{2}{|}m}(x,\dot{x}) = \delta_m a^i_j + P^{*i}_{mp} a^p_j - P^{*p}_{mj} a^i_p, \tag{1.14}$$

$$a^i_{j\underset{3}{|}m}(x,\dot{x}) = \delta_m a^i_j + P^{*i}_{pm} a^p_j - P^{*p}_{mj} a^i_p, \quad a^i_{j\underset{4}{|}m}(x,\dot{x}) = \delta_m a^i_j + P^{*i}_{mp} a^p_j - P^{*p}_{jm} a^i_p, \tag{1.15}$$

where

$$\delta_m = \frac{\partial}{\partial x^m} + \frac{\partial \dot{x}^p}{\partial x^m}\frac{\partial}{\partial \dot{x}^p}. \tag{1.15'}$$

Remark 1.1 Let $\mathbb{GF}_N$ be an N-dimensional differentiable manifold, with non-symmetric connection P^{*i}_{jk} given by (1.11). Because of the non-symmetry of the connection P^{*i}_{jk}, another connection can be defined by $\widetilde{P}^{*i}_{jk} = P^{*i}_{kj}$.

In the work [4] we obtained 10 Ricci type identities in general case for a tensor $a^{r_1\ldots r_u}_{t_1\ldots t_v}(x,\dot{x})$ and got fourth curvature tensors in $\mathbb{GF}_N$:

$$\underset{1}{\widetilde{K}}{}^i_{\ jmn} = \delta_n P^{*i}_{jm} - \delta_m P^{*i}_{jn} + P^{*p}_{jm} P^{*i}_{pn} - P^{*p}_{jn} P^{*i}_{pm}, \tag{1.16}$$

$$\underset{2}{\widetilde{K}}{}^i_{\ jmn} = \delta_n P^{*i}_{mj} - \delta_m P^{*i}_{nj} + P^{*p}_{mj} P^{*i}_{np} - P^{*p}_{nj} P^{*i}_{mp}, \tag{1.17}$$

$$\underset{3}{\widetilde{K}}{}^i_{\ jmn} = \delta_n P^{*i}_{jm} - \delta_m P^{*i}_{nj} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{nj} P^{*i}_{pm} + P^{*p}_{nm} P^{*i}_{[pj]}, \tag{1.18}$$

$$\underset{4}{\widetilde{K}}{}^i_{\ jmn} = \delta_n P^{*i}_{jm} - \delta_m P^{*i}_{nj} + P^{*p}_{jm} P^{*i}_{np} - P^{*p}_{nj} P^{*i}_{pm} + P^{*p}_{mn} P^{*i}_{[pj]}. \tag{1.19}$$

The magnitudes $\underset{t}{\widetilde{K}}{}^i_{\ jmn}$, $t = 1,2,3,4$ are tensors and we call them curvature tensors of the first, the second, the third kind and the fourth kind, respectively. Some properties for the mentioned tensors (the antisymmetry with respect of two indices, the cyclic symmetry, the symmetry with respect of pairs of indices) are given in the work [14].

Applying four kinds of covariant derivatives (1.14, 1.15) we get

$$\begin{aligned} a^i_{j\underset{\frac{1}{2}}{|}m}(x,\xi) &= \delta_m a^i_j + P^{*i}_{\substack{pm\\mp}} a^p_j - P^{*p}_{\substack{jm\\mj}} a^i_p \\ &= \delta_m a^i_j + (P^{*i}_{\underline{\substack{pm\\mp}}} + T^{*i}_{\substack{pm\\mp}}) a^p_j - (P^{*p}_{\underline{\substack{jm\\mj}}} + T^{*p}_{\substack{jm\\mj}}) a^i_p \\ &= a^i_{j;m} + T^{*i}_{\substack{pm\\mp}} a^p_j - T^{*p}_{\substack{jm\\mj}} a^i_p, \quad \text{i.e.} \end{aligned} \tag{1.20}$$

$$\begin{aligned} a^i_{j\underset{\theta}{|}m}(x,\dot{x}) &= a^i_{j;m} + (-1)^{\theta-1}(T^{*i}_{pm} a^p_j - T^{*p}_{jm} a^i_p), \quad \theta = 1,2, \quad \text{and also} \\ a^i_{j\underset{\theta}{|}m}(x,\dot{x}) &= a^i_{j;m} + (-1)^{\theta-1}(T^{*i}_{pm} a^p_j - T^{*p}_{mj} a^i_p), \quad \theta = 3,4. \end{aligned} \tag{1.21}$$

Remark 1.2 By (;) we denoted covariant derivative with respect to the symmetric connection $P^{*i}_{\underline{jm}} = \frac{1}{2}(P^{*i}_{jk} + P^{*i}_{kj})$.

Here we demonstrated how the tensor $\underset{1}{\widetilde{K}}$, given by equation (1.16) can be presented:

$$\begin{aligned} \underset{1}{\widetilde{K}}{}^i_{\ jmn} &= \delta_n P^{*i}_{\underline{jm}} + \delta_n P^{*i}_{\underset{\vee}{jm}} - \delta_m P^{*i}_{\underline{jn},m} - \delta_m P^{*i}_{\underset{\vee}{jn}} \\ &+ P^{*p}_{\underline{jm}} P^{*i}_{\underline{pn}} + P^{*p}_{\underset{\vee}{jm}} P^{*i}_{\underline{pn}} + P^{*p}_{\underline{jm}} P^{*i}_{\underset{\vee}{pn}} + P^{*p}_{\underset{\vee}{jm}} P^{*i}_{\underset{\vee}{pn}} \\ &- P^{*p}_{\underline{jn}} P^{*i}_{\underline{pm}} - P^{*p}_{\underset{\vee}{jn}} P^{*i}_{\underline{pm}} - P^{*p}_{\underline{jn}} P^{*i}_{\underset{\vee}{pm}} - P^{*p}_{\underset{\vee}{jn}} P^{*i}_{\underset{\vee}{pm}} \\ &= \widetilde{K}^i_{jmn} + T^{*i}_{jm;n} - T^{*i}_{jn;m} + T^{*p}_{jm} T^{*i}_{pn} - T^{*p}_{jn} T^{*i}_{pm}, \end{aligned} \tag{1.22}$$

where $\widetilde{K}^i_{jmn}$ is curvature tensor given by

$$\widetilde{K}^i_{\ jmn} = \delta_n P^{*i}_{\underline{jm}} - \delta_m P^{*i}_{\underline{jn}} + P^{*p}_{\underline{jm}} P^{*i}_{\underline{pn}} - P^{*p}_{\underline{jn}} P^{*i}_{\underline{pm}}. \tag{1.23}$$

In this way, following this procedure, we obtain:

$$\underset{2}{\widetilde{K}}{}^i{}_{jmn} = \widetilde{K}^i_{jmn} + T^{*i}_{mj;n} - T^{*i}_{nj;m} + T^{*p}_{mj} T^{*i}_{np} - T^{*p}_{nj} T^i_{mp}, \tag{1.24}$$

$$\underset{3}{\widetilde{K}}{}^i{}_{jmn} = \widetilde{K}^i_{jmn} + T^{*i}_{jm;n} - T^{*i}_{nj;m} + T^{*p}_{jm} T^{*i}_{np} - T^{*p}_{nj} T^i_{pm} + T^{*p}_{mn} T^{*i}_{jp}, \tag{1.25}$$

$$\underset{4}{\widetilde{K}}{}^i{}_{jmn} = \widetilde{K}^i_{jmn} + T^{*i}_{jm;n} - T^{*i}_{nj;m} + T^{*p}_{jm} T^{*i}_{np} - T^{*p}_{nj} T^{*i}_{pm} - T^{*p}_{mn} T^{*i}_{jp}. \tag{1.26}$$

2 Bianchi identities in $\mathbb{GF}_N$

In [13] the authors consider some Bianchi type for different kinds of covariant derivatives. We are extending these results and continuing investigations from [4, 14].

In the space $\mathbb{F}_N$ of symmetric connection (and in $\mathbb{R}_N$) the following Bianchi identity [8]

$$\underset{mnv}{\mathfrak{S}} \widetilde{K}^i{}_{jmn;v} = \widetilde{K}^i{}_{jmn;v} + \widetilde{K}^i{}_{jnv;m} + \widetilde{K}^i{}_{jvm;n} = 0 \tag{2.1}$$

is valid, where $\underset{mnv}{\mathfrak{S}}$ denotes a cyclic permutation of the indices m, n, v. In the $\mathbb{GF}_N$ one can consider 16 cases for

$$\underset{mnv}{\mathfrak{S}} \underset{\theta}{\widetilde{K}}{}^i{}_{jmn\underset{\omega}{|}v}, \quad \theta \in \{1, \dots, 4\}, \quad \omega \in \{1, \dots, 4\}. \tag{2.2}$$

Here we obtained 16 new Bianchi type identities in $\mathbb{GF}_N$, so we can state the following

Theorem 2.1. *In a space $\mathbb{GF}_N$ of non-symmetric connection P^{*i}_{jk}, and torsion tensor T^{*i}_{jk} for the curvature tensor $\underset{1}{\widetilde{K}}{}^i{}_{jmn}$ are valid next identities*

$$\underset{mnv}{\mathfrak{S}} \underset{1}{\widetilde{K}}{}^i{}_{jmn\underset{1}{|}v} = 2 \underset{mnv}{\mathfrak{S}} T^{*p}_{mn} \underset{1}{\widetilde{K}}{}^i{}_{jpv}, \tag{2.3}$$

$$\underset{mnv}{\mathfrak{S}} \underset{1}{\widetilde{K}}{}^i{}_{jmn\underset{2}{|}v} = 2 \underset{mnv}{\mathfrak{S}} (T^{*p}_{jm} \underset{1}{\widetilde{K}}{}^i{}_{pnv} + T^{*p}_{mn} \underset{1}{\widetilde{K}}{}^i{}_{jvp} + T^{*i}_{mp} \underset{1}{\widetilde{K}}{}^p{}_{jnv}), \tag{2.4}$$

$$\underset{mnv}{\mathfrak{S}} \underset{1}{\widetilde{K}}{}^i{}_{jmn\underset{3}{|}v} = 2 \underset{mnv}{\mathfrak{S}} (T^{*p}_{jm} \underset{1}{\widetilde{K}}{}^i{}_{pnv} + T^{*p}_{mn} \underset{1}{\widetilde{K}}{}^i{}_{jvp}), \tag{2.5}$$

$$\underset{mnv}{\mathfrak{S}} \underset{1}{\widetilde{K}}{}^i{}_{jmn\underset{4}{|}v} = 2 \underset{mnv}{\mathfrak{S}} (T^{*i}_{mp} \underset{1}{\widetilde{K}}{}^p{}_{jnv} + T^{*p}_{mn} \underset{1}{\widetilde{K}}{}^i{}_{jpv}). \tag{2.6}$$

Proof. By applying (1.21), (1.22) and using cyclic permutation of indices m, n, v we get

$$\begin{aligned}
\underset{mnv}{\mathfrak{S}} \underset{1}{\widetilde{K}}{}^i{}_{jmn\underset{1}{|}v} &= \underset{mnv}{\mathfrak{S}} (\underset{1}{\widetilde{K}}{}^i{}_{jmn;v} + T^{*i}_{pv} \underset{1}{\widetilde{K}}{}^p{}_{jmn} - T^{*p}_{jv} \underset{1}{\widetilde{K}}{}^i{}_{pmn} - T^{*p}_{mv} \underset{1}{\widetilde{K}}{}^i{}_{jpn} - T^{*p}_{nv} \underset{1}{\widetilde{K}}{}^i{}_{jmp}) \\
&= \underset{mnv}{\mathfrak{S}} (\widetilde{K}^i{}_{jmn;v} + T^{*i}_{jm;nv} - T^{*i}_{jn;mv} + T^{*p}_{jm;v} T^{*i}_{pn} + T^{*p}_{jm} T^{*i}_{pn;v} - T^{*p}_{jn;v} T^{*i}_{pm} \\
&\quad - T^{*p}_{jn} T^{*i}_{pm;v} + T^{*i}_{pv} \underset{1}{\widetilde{K}}{}^p{}_{jmn} - T^{*p}_{jv} \underset{1}{\widetilde{K}}{}^i{}_{pmn} - T^{*p}_{mv} \underset{1}{\widetilde{K}}{}^i{}_{jpn} - T^{*p}_{nv} \underset{1}{\widetilde{K}}{}^i{}_{jmp}).
\end{aligned}$$

Let us consider the following difference

$$T^{*i}_{jm;nv} - T^{*i}_{jm;vn} = \widetilde{K}^i{}_{pnv} T^{*p}_{jm} - \widetilde{K}^p{}_{jnv} T^{*i}_{pm} - \widetilde{K}^p{}_{mnv} T^{*i}_{jp}.$$

Using the non-symmetry of the tensors T^{*i}_{mn}, $\underset{1}{\widetilde{K}}{}^i{}_{jmn}$ with respect to indices m, n, symmetry properties for $\widetilde{K}^i{}_{jmn}$ and appropriate Ricci identities we obtain

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{1}{\widetilde{K}}{}^i{}_{jmn\underset{1}{|}v} =&\widetilde{K}^i{}_{pnv}T^{*p}_{jm}-\widetilde{K}^p{}_{jnv}T^{*i}_{pm}-\widetilde{K}^p{}_{mnv}T^{*i}_{jp}+\widetilde{K}^i{}_{pvm}T^{*p}_{jn}-\widetilde{K}^p{}_{jvm}T^{*i}_{pn}-\widetilde{K}^p{}_{nvm}T^{*i}_{jp}\\ &+\widetilde{K}^i{}_{pmn}T^{*p}_{jv}-\widetilde{K}^p{}_{jmn}T^{*i}_{pv}-\widetilde{K}^p{}_{vmn}T^{*i}_{jp}+T^{*i}_{pn}(\underset{1}{\widetilde{K}}{}^p{}_{jvm}+T^{*p}_{jm;v}-T^{*p}_{jv;m})\\ &+T^{*p}_{jm}(\underset{1}{\widetilde{K}}{}^i{}_{pvn}+T^{*i}_{pn;v}-T^{*i}_{pv;n})+T^{*i}_{pm}(\underset{1}{\widetilde{K}}{}^p{}_{jnv}+T^{*p}_{jv;n}-T^{*p}_{jn;v})\\ &+T^{*p}_{jn}(\underset{1}{\widetilde{K}}{}^i{}_{pmv}+T^{*i}_{pv;m}-T^{*i}_{pm;v})+T^{*i}_{pv}(\underset{1}{\widetilde{K}}{}^p{}_{jmn}+T^{*p}_{jn;m}-T^{*p}_{jm;n})\\ &+T^{*p}_{jv}(\underset{1}{\widetilde{K}}{}^i{}_{pnm}+T^{*i}_{pm;n}-T^{*i}_{pn;m})+2(T^{*p}_{vm}\underset{1}{\widetilde{K}}{}^i{}_{jpn}+T^{*p}_{nv}\underset{1}{\widetilde{K}}{}^i{}_{jpm}+T^{*p}_{mn}\underset{1}{\widetilde{K}}{}^i{}_{jpv})\end{aligned}$$

Using (2.1) and(1.22) we get

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{1}{\widetilde{K}}{}^i{}_{jmn\underset{1}{|}v} =&\widetilde{K}^i{}_{pnv}T^{*p}_{jm}-\widetilde{K}^p{}_{jnv}T^{*i}_{pm}+\widetilde{K}^i{}_{pvm}T^{*p}_{jn}-\widetilde{K}^p{}_{jvm}T^{*i}_{pn}+\widetilde{K}^i{}_{pmn}T^{*p}_{jv}-\widetilde{K}^p{}_{jmn}T^{*i}_{pv}\\ &+T^{*i}_{pn}(\widetilde{K}^p{}_{jvm}+T^{*s}_{jv}T^{*p}_{sm}-T^{*s}_{jm}T^{*p}_{sv})+T^{*p}_{jm}(\widetilde{K}^i{}_{pvn}+T^{*s}_{pv}T^{*i}_{sn}-T^{*s}_{pn}T^{*i}_{sv})\\ &+T^{*i}_{pm}(\widetilde{K}^p{}_{jnv}+T^{*s}_{jn}T^{*p}_{sv}-T^{*s}_{jv}T^{*p}_{sn})+T^{*p}_{jn}(\widetilde{K}^i{}_{pmv}+T^{*s}_{pm}T^{*i}_{sv}-T^{*s}_{pv}T^{*i}_{sm})\\ &+T^{*i}_{pv}(\widetilde{K}^p{}_{jmn}+T^{*s}_{jm}T^{*p}_{sn}-T^{*s}_{jn}T^{*p}_{sm})+T^{*p}_{jv}(\widetilde{K}^i{}_{pnm}+T^{*s}_{pn}T^{*i}_{sm}-T^{*s}_{pm}T^{*i}_{sn})\\ &+2(T^{*p}_{vm}\underset{1}{\widetilde{K}}{}^i{}_{jpn}+T^{*p}_{nv}\underset{1}{\widetilde{K}}{}^i{}_{jpm}+T^{*p}_{mn}\underset{1}{\widetilde{K}}{}^i{}_{jpv})\\ =&2\underset{mnv}{\mathfrak{S}}T^{*p}_{mn}\underset{1}{\widetilde{K}}{}^i{}_{jpv}\Rightarrow (2.3).\end{aligned}$$

Using (1.21) for second kind of covariant derivative and (1.22), we obtain

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{1}{\widetilde{K}}{}^i{}_{jmn\underset{2}{|}v} &=\underset{mnv}{\mathfrak{S}}(\underset{1}{\widetilde{K}}{}^i{}_{jmn;v}+T^{*i}_{vp}\underset{1}{\widetilde{K}}{}^p{}_{jmn}-T^{*p}_{vj}\underset{1}{\widetilde{K}}{}^i{}_{pmn}-T^{*p}_{vm}\underset{1}{\widetilde{K}}{}^i{}_{jpn}-T^{*p}_{vn}\underset{1}{\widetilde{K}}{}^i{}_{jmp})\\ &=\underset{mnv}{\mathfrak{S}}(\widetilde{K}^i{}_{jmn;v}+T^{*i}_{jm;nv}-T^{*i}_{jn;mv}+T^{*p}_{jm;v}T^{*i}_{pn}+T^{*p}_{jm}T^{*i}_{pn;v}-T^{*p}_{jn;v}T^{*i}_{pm}\\ &\quad -T^{*p}_{jn}T^{*i}_{pm;v}+T^{*i}_{vp}\underset{1}{\widetilde{K}}{}^p{}_{jmn}-T^{*p}_{vj}\underset{1}{\widetilde{K}}{}^i{}_{pmn}-T^{*p}_{vm}\underset{1}{\widetilde{K}}{}^i{}_{jpn}-T^{*p}_{vn}\underset{1}{\widetilde{K}}{}^i{}_{jmp}).\end{aligned}$$

By help of (1.22) and Ricci identity in symmetric case and the non-symmetry of the tensors T^{*i}_{mn}, $\underset{1}{\widetilde{K}}{}^i{}_{jmn}$ with respect to indices m,n we get

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{1}{\widetilde{K}}{}^i{}_{jmn\underset{2}{|}v} =&\widetilde{K}^i{}_{pnv}T^{*p}_{jm}-\widetilde{K}^p{}_{jvn}T^{*i}_{mp}+\widetilde{K}^i{}_{pvm}T^{*p}_{jn}-\widetilde{K}^p{}_{jmv}T^{*i}_{np}+\widetilde{K}^i{}_{pmn}T^{*p}_{jv}-\widetilde{K}^p{}_{jnm}T^{*i}_{vp}\\ &+T^{*i}_{pn}(\underset{1}{\widetilde{K}}{}^p{}_{jmv}+T^{*p}_{jm;v}-T^{*p}_{jv;m})+T^{*p}_{jm}(\underset{1}{\widetilde{K}}{}^i{}_{pnv}+T^{*i}_{pn;v}-T^{*i}_{pv;n})\\ &+T^{*i}_{pm}(\underset{1}{\widetilde{K}}{}^p{}_{jvn}+T^{*p}_{jv;n}-T^{*p}_{jn;v})+T^{*p}_{jn}(\underset{1}{\widetilde{K}}{}^i{}_{pvm}+T^{*i}_{pv;m}-T^{*i}_{pm;v})\\ &+T^{*i}_{pv}(\underset{1}{\widetilde{K}}{}^p{}_{jnm}+T^{*p}_{jn;m}-T^{*p}_{jm;n})+T^{*p}_{jv}(\underset{1}{\widetilde{K}}{}^i{}_{pnm}+T^{*i}_{pm;n}-T^{*i}_{pn;m})\\ &+2(T^{*p}_{mv}\underset{1}{\widetilde{K}}{}^i{}_{jpn}+T^{*p}_{nv}\underset{1}{\widetilde{K}}{}^i{}_{jmp}+T^{*p}_{mn}\underset{1}{\widetilde{K}}{}^i{}_{jvp})\end{aligned}$$

It is easy to prove that

$$\widetilde{K}^i{}_{pnv}+T^{*i}_{pn;v}-T^{*i}_{pv;n}=\underset{1}{\widetilde{K}}{}^i{}_{pnv}-T^{*s}_{pn}T^{*i}_{sv}+T^{*s}_{pv}T^{*i}_{sn} \tag{2.7}$$

and the previous equation becomes

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{1}{\widetilde{K}}{}^i_{jmn\underset{2}{|}v} =&2(T^{*p}_{jm}\underset{1}{\widetilde{K}}{}^i_{pnv}+T^{*p}_{jn}\underset{1}{\widetilde{K}}{}^i_{pvm}+T^{*p}_{jv}\underset{1}{\widetilde{K}}{}^i_{pmn})\\ &+2(T^{*i}_{mp}\underset{1}{\widetilde{K}}{}^p_{jnv}+T^{*i}_{np}\underset{1}{\widetilde{K}}{}^p_{jvm}+T^{*i}_{vp}\underset{1}{\widetilde{K}}{}^p_{jmn})\\ &+2(T^{*p}_{mn}\underset{1}{\widetilde{K}}{}^i_{jvp}+T^{*p}_{nv}\underset{1}{\widetilde{K}}{}^i_{jmp}+T^{*p}_{vm}\underset{1}{\widetilde{K}}{}^i_{jnp})\\ =&2\underset{mnv}{\mathfrak{S}}(T^{*p}_{jm}\underset{1}{\widetilde{K}}{}^i_{pnv}+T^{*p}_{mn}\underset{1}{\widetilde{K}}{}^i_{jvp}+T^{*i}_{mp}\underset{1}{\widetilde{K}}{}^p_{jnv})\Rightarrow(2.4).\end{aligned}$$

Applying to $\underset{1}{\widetilde{K}}$ the 3^{rd} kind of covariant derivative, we get

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{1}{\widetilde{K}}{}^i_{jmn\underset{3}{|}v} =&\underset{mnv}{\mathfrak{S}}(\underset{1}{\widetilde{K}}{}^i_{jmn;v}+T^{*i}_{pv}\underset{1}{\widetilde{K}}{}^p_{jmn}-T^{*p}_{vj}\underset{1}{\widetilde{K}}{}^i_{pmn}-T^{*p}_{vm}\underset{1}{\widetilde{K}}{}^i_{jpn}-T^{*p}_{vn}\underset{1}{\widetilde{K}}{}^i_{jmp})\\ =&\underset{mnv}{\mathfrak{S}}(\widetilde{K}^i_{jmn;v}+T^{*i}_{jm;nv}-T^{*i}_{jn;mv}+T^{*p}_{jm;v}T^{*i}_{pn}+T^{*p}_{jm}T^{*i}_{pn;v}-T^{*p}_{jn;v}T^{*i}_{pm}\\ &-T^{*p}_{jn}T^{*i}_{pm;v}+T^{*i}_{pv}\underset{1}{\widetilde{K}}{}^p_{jmn}-T^{*p}_{vj}\underset{1}{\widetilde{K}}{}^i_{pmn}-T^{*p}_{vm}\underset{1}{\widetilde{K}}{}^i_{jpn}-T^{*p}_{vn}\underset{1}{\widetilde{K}}{}^i_{jmp}).\end{aligned}$$

Using the non-symmetry of the tensors T^{*i}_{mn}, $\underset{1}{\widetilde{K}}{}^i_{jmn}$ with respect to indices m,n, symmetry properties for $\widetilde{K}^i_{jmn}$ and appropriate Ricci identities we obtain

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{1}{\widetilde{K}}{}^i_{jmn\underset{3}{|}v} =&T^{*i}_{pn}(\widetilde{K}^p_{jmv}+\underset{1}{\widetilde{K}}{}^p_{jvm}+T^{*p}_{jm;v}-T^{*p}_{jv;m})+T^{*p}_{jm}(\widetilde{K}^i_{pnv}+\underset{1}{\widetilde{K}}{}^i_{pnv}+T^{*i}_{pn;v}-T^{*i}_{pv;n})\\ &+T^{*i}_{pm}(\widetilde{K}^p_{jvn}+\underset{1}{\widetilde{K}}{}^p_{jnv}+T^{*p}_{jv;n}-T^{*p}_{jn;v})+T^{*p}_{jn}(\widetilde{K}^i_{pvm}+\underset{1}{\widetilde{K}}{}^i_{pvm}+T^{*i}_{pv;m}-T^{*i}_{pm;v})\\ &+T^{*i}_{pv}(\widetilde{K}^p_{jnm}+\underset{1}{\widetilde{K}}{}^p_{jmn}+T^{*p}_{jn;m}-T^{*p}_{jm;n})+T^{*p}_{jv}(\widetilde{K}^i_{pmn}+\underset{1}{\widetilde{K}}{}^i_{pmn}+T^{*i}_{pm;n}-T^{*i}_{pn;m})\\ &+2(T^{*p}_{mv}\underset{1}{\widetilde{K}}{}^i_{jpn}+T^{*p}_{vn}\underset{1}{\widetilde{K}}{}^i_{jpm}+T^{*p}_{mn}\underset{1}{\widetilde{K}}{}^i_{jpv}).\end{aligned}$$

According to (2.7) the previous equation becomes

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{1}{\widetilde{K}}{}^i_{jmn\underset{3}{|}v} =&2\underset{mnv}{\mathfrak{S}}T^{*p}_{mn}\underset{1}{\widetilde{K}}{}^i_{jvp}+T^{*i}_{pn}(\underset{1}{\widetilde{K}}{}^p_{jvm}+\underset{1}{\widetilde{K}}{}^p_{jmv}-T^{*s}_{jm}T^{*p}_{sv}+T^{*s}_{jv}T^{*p}_{sm})\\ &+T^{*p}_{jm}(2\underset{1}{\widetilde{K}}{}^i_{pnv}-T^{*s}_{pn}T^{*i}_{sv}+T^{*s}_{pv}T^{*i}_{sn})+T^{*i}_{pm}(\underset{1}{\widetilde{K}}{}^p_{jnv}+\underset{1}{\widetilde{K}}{}^p_{jvn}-T^{*s}_{jv}T^{*p}_{sn}+T^{*s}_{jn}T^{*p}_{sv})\\ &+T^{*p}_{jn}(2\underset{1}{\widetilde{K}}{}^i_{pvm}-T^{*s}_{pv}T^{*i}_{sm}+T^{*s}_{pm}T^{*i}_{sv})+T^{*i}_{pv}(\underset{1}{\widetilde{K}}{}^p_{jmn}+\underset{1}{\widetilde{K}}{}^p_{jnm}-T^{*s}_{jn}T^{*p}_{sm}+T^{*s}_{jm}T^{*p}_{sn})\\ &+T^{*p}_{jv}(2\underset{1}{\widetilde{K}}{}^i_{pmn}-T^{*s}_{pm}T^{*i}_{sn}+T^{*s}_{pn}T^{*i}_{sm})\\ =&2\underset{mnv}{\mathfrak{S}}(T^{*p}_{jm}\underset{1}{\widetilde{K}}{}^i_{pnv}+T^{*p}_{mn}\underset{1}{\widetilde{K}}{}^i_{jvp})\Rightarrow(2.5).\end{aligned}$$

Analogously to previous case, by applying (1.22) and fourth kind of covariant derivative, we have

$$\underset{mnv}{\mathfrak{S}}\underset{1}{\widetilde{K}}{}^i_{jmn\underset{4}{|}v} = 2\underset{mnv}{\mathfrak{S}}(T^{*i}_{mp}\underset{1}{\widetilde{K}}{}^p_{jnv}+T^{*p}_{mn}\underset{1}{\widetilde{K}}{}^i_{jpv})\Rightarrow(2.6).$$

□

Theorem 2.2. *For the curvature tensor* $\underset{2}{\widetilde{K}}{}^i_{jmn}$ *given by (1.17) and (1.24) the next four Bianchi* *type identities:*

$$\underset{mnv}{\mathfrak{S}}\underset{2}{\widetilde{K}}{}^i_{jmn\underset{1}{|}v} = 2\underset{mnv}{\mathfrak{S}}(T^{*p}_{mj}\underset{2}{\widetilde{K}}{}^i_{pnv} + T^{*p}_{mn}\underset{2}{\widetilde{K}}{}^i_{jpv} + T^{*i}_{pm}\underset{2}{\widetilde{K}}{}^p_{jnv}), \tag{2.8}$$

$$\underset{mnv}{\mathfrak{S}}\underset{2}{\widetilde{K}}{}^i_{jmn\underset{2}{|}v} = 2\underset{mnv}{\mathfrak{S}}T^{*p}_{mn}\underset{2}{\widetilde{K}}{}^i_{jvp}, \tag{2.9}$$

$$\underset{mnv}{\mathfrak{S}}\underset{2}{\widetilde{K}}{}^i_{jmn\underset{3}{|}v} = 2\underset{mnv}{\mathfrak{S}}(T^{*i}_{pm}\underset{2}{\widetilde{K}}{}^p_{jnv} + T^{*p}_{mn}\underset{2}{\widetilde{K}}{}^i_{jvp}), \tag{2.10}$$

$$\underset{mnv}{\mathfrak{S}}\underset{2}{\widetilde{K}}{}^i_{jmn\underset{4}{|}v} = 2\underset{mnv}{\mathfrak{S}}(T^{*p}_{mj}\underset{2}{\widetilde{K}}{}^i_{pnv} + T^{*p}_{mn}\underset{2}{\widetilde{K}}{}^i_{jpv}) \tag{2.11}$$

are valid.

Proof. The proof is analogous to previous one. □

Theorem 2.3. *For the curvature tensor* $\underset{3}{\widetilde{K}}{}^i_{jmn}$ *given by (1.18) and (1.25) the next Bianchi type identities:*

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{3}{\widetilde{K}}{}^i_{jmn\underset{1}{|}v} = 2\underset{mnv}{\mathfrak{S}}[&T^{*i}_{jm;nv} + T^i_{jp}T^{*p}_{mn;v} + T^{*i}_{pm}(\underset{3}{\widetilde{K}}{}^p_{jnv} + T^{*p}_{vj;n}) + T^{*p}_{mj}(\underset{3}{\widetilde{K}}{}^i_{pnv} + T^{*i}_{vp;n})\\ &+ T^{*p}_{mn}(\underset{3}{\widetilde{K}}{}^i_{jpv} - T^{*i}_{jv;p} - T^{*s}_{jp}T^{*i}_{sv} + T^{*s}_{jv}T^{*i}_{sp})],\end{aligned} \tag{2.12}$$

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{3}{\widetilde{K}}{}^i_{jmn\underset{2}{|}v} = 2\underset{mnv}{\mathfrak{S}}[&T^{*i}_{jm;nv} + T^i_{jp}T^{*p}_{mn;v} + T^{*i}_{mp}(\underset{3}{\widetilde{K}}{}^p_{jnv} - \widetilde{K}^p_{jnv} - T^{*p}_{jv;n})\\ &+ T^{*p}_{jm}(\underset{3}{\widetilde{K}}{}^i_{pnv} - \widetilde{K}^i_{pnv} - T^{*i}_{pn;v}) + T^{*p}_{mn}(\underset{3}{\widetilde{K}}{}^i_{jvp} - T^{*i}_{jv;p} - T^{*s}_{jp}T^{*i}_{sv} + T^{*s}_{jv}T^{*i}_{sp})],\end{aligned} \tag{2.13}$$

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{3}{\widetilde{K}}{}^i_{jmn\underset{3}{|}v} = 2\underset{mnv}{\mathfrak{S}}[&T^{*i}_{jm;nv} + T^i_{jp}T^{*p}_{mn;v} + T^{*i}_{mp}(\underset{3}{\widetilde{K}}{}^p_{jnv} + T^{*p}_{jn;v})\\ &+ T^{*p}_{jm}(\widetilde{K}^i_{pnv} - \underset{1}{\widetilde{K}}{}^i_{pnv} - T^{*i}_{pn;v}) + T^{*p}_{mn}(2\widetilde{K}^i_{jvp} - \underset{1}{\widetilde{K}}{}^i_{jvp} - T^{*i}_{jv;p} + T^s_{jp}T^{*i}_{sv} + T^s_{jv}T^{*i}_{sp})],\end{aligned} \tag{2.14}$$

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{3}{\widetilde{K}}{}^i_{jmn\underset{4}{|}v} = 2\underset{mnv}{\mathfrak{S}}[&T^{*i}_{jm;nv} + T^i_{jp}T^{*p}_{mn;v} + T^{*i}_{pm}(\widetilde{K}^p_{jnv} + T^{*p}_{jn;v}) + T^{*p}_{mj}(\widetilde{K}^i_{pnv} + T^{*i}_{pn;v})\\ &+ T^{*p}_{mn}(2\widetilde{K}^i_{jvp} - \underset{1}{\widetilde{K}}{}^i_{jpv} - T^{*i}_{jv;p} - T^s_{jp}T^{*i}_{sv} - T^s_{jv}T^{*i}_{sp})]\end{aligned} \tag{2.15}$$

are valid.

Proof. Firstly, we verify identities

$$\underset{mnv}{\mathfrak{S}}T^{*i}_{mp}T^{*p}_{sj}T^{*s}_{nv} = \underset{mnv}{\mathfrak{S}}T^{*i}_{vp}T^{*p}_{sj}T^{*s}_{mn} = \underset{mnv}{\mathfrak{S}}T^{*p}_{mn}T^{*s}_{pj}T^{*i}_{vs}, \tag{2.16}$$

$$\underset{mnv}{\mathfrak{S}}T^{*p}_{jm}T^{*i}_{sp}T^{*s}_{nv} = \underset{mnv}{\mathfrak{S}}T^{*p}_{jv}T^{*i}_{sp}T^{*s}_{mn} = \underset{mnv}{\mathfrak{S}}T^{*p}_{mn}T^{*i}_{ps}T^{*s}_{jv} \tag{2.17}$$

$$\underset{mnv}{\mathfrak{S}}T^{*i}_{pj}T^{*p}_{ns}T^{*s}_{mv} = \underset{mnv}{\mathfrak{S}}T^{*i}_{pj}T^{*p}_{ms}T^{*s}_{vn} = \underset{mnv}{\mathfrak{S}}T^{*i}_{pj}T^{*p}_{sv}T^{*s}_{mn} = \underset{mnv}{\mathfrak{S}}T^{*i}_{sj}T^{*p}_{mn}T^{*s}_{pv}. \tag{2.18}$$

By virtue of equations (1.21, 1.25) we get

$$\underset{3}{\widetilde{K}}{}^i_{jmn} = 2\widetilde{K}^i_{jmn} - \underset{1}{\widetilde{K}}{}^i_{jmn} + 2T^{*i}_{jm;n} - 2T^{*i}_{pj}T^{*p}_{mn} \tag{2.19}$$

and based on the third kind of covariant derivative and (2.1), (2.3), (2.18) we have

$$\underset{mnv}{\mathfrak{S}}\underset{3}{\widetilde{K}}{}^i_{jmn\underset{1}{|}v} = 2\underset{mnv}{\mathfrak{S}}(\widetilde{K}^i_{jmn} + T^{*i}_{jm;n} - T^{*i}_{pj}T^{*p}_{mn})_{\underset{1}{|}v} - \underset{mnv}{\mathfrak{S}}\underset{1}{\widetilde{K}}{}^i_{jmn\underset{1}{|}v} \Rightarrow (2.12).$$

To prove (2.13), we have

$$\begin{aligned}\underset{mnv}{\mathfrak{S}}\underset{3}{\widetilde{K}}{}^i_{jmn\underset{2}{|}v} = 2\underset{mnv}{\mathfrak{S}}[&T^{*i}_{jm;nv} + T^i_{jp}T^{*p}_{mn;v} + T^{*i}_{mp}(\underset{3}{\widetilde{K}}{}^p_{jnv} - \widetilde{K}^p_{jnv} - T^{*p}_{jn;v})\\ &+ T^{*p}_{jm}(\underset{3}{\widetilde{K}}{}^i_{pnv} - \widetilde{K}^i_{pnv} - T^{*i}_{pn;v}) + T^{*p}_{mn}(\underset{3}{\widetilde{K}}{}^i_{jvp} - T^{*i}_{jv;p}) + T^s_{jp}T^{*i}_{sv} - T^s_{jv}T^{*i}_{sp}],\end{aligned}$$

and using (2.16), (2.17), the previous equation becomes

$$\underset{mnv}{\mathfrak{S}}\underset{3}{\widetilde{K}}{}^{i}{}_{jmn\underset{2}{|}v}=2\underset{mnv}{\mathfrak{S}}[T^{*i}_{jm;nv}+T^{i}_{jp}T^{*p}_{mn;v}+T^{*i}_{mp}(\underset{3}{\widetilde{K}}{}^{p}{}_{jnv}-\widetilde{K}^{p}_{jnv}-T^{*p}_{jv;n}) \tag{2.20}$$
$$+T^{*p}_{jm}(\underset{3}{\widetilde{K}}{}^{i}{}_{pnv}-\widetilde{K}^{i}_{pnv}-T^{*i}_{pn;v})+T^{*p}_{mn}(\underset{3}{\widetilde{K}}{}^{i}{}_{jvp}-T^{*i}_{jv;p}-T^{*s}_{jp}T^{*i}_{sv}+T^{*s}_{jv}T^{*i}_{sp})],\Rightarrow (2.13).$$

By virtue of (2.19) and based on (2.1), (2.5), (2.18) we have

$$\underset{mnv}{\mathfrak{S}}\underset{3}{\widetilde{K}}{}^{i}{}_{jmn\underset{3}{|}v}=2\underset{mnv}{\mathfrak{S}}(\widetilde{K}^{*i}_{jmn}+T^{*i}_{jm;n}-T^{*i}_{pj}T^{*p}_{mn})_{\underset{1}{|}v}-\underset{mnv}{\mathfrak{S}}\underset{1}{\widetilde{K}}{}^{i}{}_{jmn\underset{3}{|}v},$$

i.e. equation (2.14) is valid.

In the same manner we have (2.15). □

Taking into account (1.25, 1.26) we get

$$\underset{4}{\widetilde{K}}{}^{i}{}_{jmn}=\underset{3}{\widetilde{K}}{}^{i}{}_{jmn}+4T^{*i}_{pj}T^{*p}_{mn}$$

based on which we shall investigate

$$\underset{mnv}{\mathfrak{S}}\underset{4}{\widetilde{K}}{}^{i}{}_{jmn\underset{\theta}{|}v},\quad \theta\in\{1,\dots,4\}.$$

Following procedure in the previous case for $\underset{3}{\widetilde{K}}$, we can formulate the next theorem:

Theorem 2.4. *For the curvature tensor $\underset{4}{\widetilde{K}}{}^{i}{}_{jmn}$, given by (1.19) and (1.26), the next Bianchi type identities are valid:*

$$\underset{mnv}{\mathfrak{S}}\underset{4}{\widetilde{K}}{}^{i}{}_{jmn\underset{1}{|}v}=2\underset{mnv}{\mathfrak{S}}[T^{*i}_{jm;nv}+T^{i}_{pj}T^{*p}_{mn;v}+3T^{p}_{mn}T^{s}_{pj}T^{*i}_{sv}+3T^{p}_{mn}T^{s}_{jv}T^{*i}_{sp}+T^{*i}_{pm}(\underset{3}{\widetilde{K}}{}^{p}{}_{jnv}+T^{*p}_{vj;n}) \tag{2.21}$$
$$+T^{*p}_{mj}(\underset{3}{\widetilde{K}}{}^{i}{}_{pnv}+T^{*i}_{vp;n})+T^{*p}_{mn}(\underset{3}{\widetilde{K}}{}^{i}{}_{jvp}+T^{*i}_{vj;p})+2(T^{p}_{mn}T^{*i}_{pj;v}+T^{i}_{pj}T^{p}_{ns}T^{*s}_{mv}+T^{i}_{pj}T^{p}_{ms}T^{*s}_{vn})],$$

$$\underset{mnv}{\mathfrak{S}}\underset{4}{\widetilde{K}}{}^{i}{}_{jmn\underset{2}{|}v}=2\underset{mnv}{\mathfrak{S}}[T^{*i}_{jm;nv}+T^{i}_{pj}T^{*p}_{mn;v}+T^{*i}_{mp}(\underset{3}{\widetilde{K}}{}^{p}{}_{jnv}-\widetilde{K}^{*p}_{jnv}-T^{*p}_{jn;v})+T^{*p}_{jm}(\underset{3}{\widetilde{K}}{}^{i}{}_{pnv}-\widetilde{K}^{*i}_{pnv}-T^{*i}_{pn;v})$$
$$+T^{*p}_{mn}(\underset{3}{\widetilde{K}}{}^{i}{}_{jvp}-T^{*i}_{jv;p}-2T^{*i}_{jp;v}+3T^{s}_{jp}T^{*i}_{sv}-3T^{s}_{jv}T^{*i}_{sp}+4T^{i}_{sj}T^{*s}_{vp})], \tag{2.22}$$

$$\underset{mnv}{\mathfrak{S}}\underset{4}{\widetilde{K}}{}^{i}{}_{jmn\underset{3}{|}v}=2\underset{mnv}{\mathfrak{S}}[T^{*i}_{jm;nv}+T^{i}_{pj}T^{*p}_{mn;v}+T^{*i}_{pm}(\widetilde{K}^{*p}_{jnv}+T^{*p}_{jn;v})+T^{*p}_{jm}(\widetilde{K}^{*i}_{pnv}-\underset{1}{\widetilde{K}}{}^{i}{}_{pnv}+T^{*i}_{pn;v}) \tag{2.23}$$
$$+T^{*p}_{mn}(2\widetilde{K}^{*i}_{jvp}-\underset{1}{\widetilde{K}}{}^{i}{}_{jvp}+T^{*i}_{jv;p}-2T^{*i}_{jp;v}-T^{s}_{jv}T^{*i}_{sp}-T^{s}_{jp}T^{*i}_{sv})],$$

$$\underset{mnv}{\mathfrak{S}}\underset{4}{\widetilde{K}}{}^{i}{}_{jmn\underset{4}{|}v}=2\underset{mnv}{\mathfrak{S}}[T^{*i}_{jm;nv}+T^{i}_{pj}T^{*p}_{mn;v}+T^{*i}_{mp}(\widetilde{K}^{*p}_{jnv}-\underset{1}{\widetilde{K}}{}^{p}{}_{jnv}+T^{*p}_{jn;v})+T^{*p}_{mj}(\widetilde{K}^{*i}_{pnv}+T^{*i}_{pn;v}) \tag{2.24}$$
$$+T^{*p}_{mn}(2\widetilde{K}^{*i}_{jvp}-\underset{1}{\widetilde{K}}{}^{i}{}_{jvp}-T^{*i}_{jv;p}+T^{s}_{jv}T^{*i}_{sp}+T^{s}_{jp}T^{*i}_{sv})].$$

Proof. Knowing that

$$\underset{mnv}{\mathfrak{S}}T^{*i}_{pj}T^{*p}_{ns}T^{*s}_{mv}=\underset{mnv}{\mathfrak{S}}T^{*i}_{pj}T^{*p}_{ms}T^{*s}_{vn}=\underset{mnv}{\mathfrak{S}}T^{*i}_{pj}T^{*p}_{sv}T^{*s}_{mn}=\underset{mnv}{\mathfrak{S}}T^{*i}_{sj}T^{*p}_{mn}T^{*s}_{pv},$$

equation (2.18) becomes

$$\underset{mnv}{\mathfrak{S}}\underset{4}{\widetilde{K}}{}^{i}{}_{jmn\underset{1}{|}v}=2\underset{mnv}{\mathfrak{S}}[T^{*i}_{jm;nv}+T^{i}_{pj}T^{*p}_{mn;v}+T^{*i}_{pm}(\underset{3}{\widetilde{K}}{}^{p}{}_{jnv}-T^{*p}_{jv;n})+T^{*p}_{mj}(\underset{3}{\widetilde{K}}{}^{i}{}_{pnv}-T^{*i}_{pv;n}) \tag{2.25}$$
$$+T^{*p}_{mn}(\underset{3}{\widetilde{K}}{}^{i}{}_{jvp}-2T^{*i}_{jp;v}-T^{*i}_{jv;p}-3T^{s}_{jp}T^{*i}_{sv}+3T^{s}_{jv}T^{*i}_{sp}+4T^{s}_{pv}T^{*i}_{sj})].$$

□

3 Concluding remarks

1. For $g_{ij}(x,\dot{x}) = g_{ji}(x,\dot{x})$ we obtain usual Finsler space $\mathbb{F}_N$. All Bianchi type identities given in this work reduce to one in the symmetric case

$$\widetilde{K}^i_{jmn;v} + \widetilde{K}^j_{ivm;n} + \widetilde{K}^i_{jnv;m} = 0.$$

2. If $g_{ij}(x) \neq g_{ji}(x)$ one obtains a generalized Riemannian space $\mathbb{GR}_N$ (see [2]). For $g_{ij}(x) = g_{ji}(x)$ $\mathbb{GR}_N$ reduces to the Riemannian space $\mathbb{R}_N$.

In this paper we proved some Bianchi type identities for the tensors $\underset{\theta}{\widetilde{K}}$, $\theta = \overline{1,4}$. In the future work we will employ some properties for the tensors $\underset{\theta}{\widetilde{K}}$, $\theta = \overline{1,4}$ (the antisymmetry with respect of two indices, the cyclic symmetry, the symmetry with respect of pairs of indices) given in the work [14] and also proved Bianchi type identities.

In this way, we complement the properties of the four curvature tensors in $\mathbb{GF}_N$. All these tensors are interesting in constructions of new mathematical and physical structures.

Acknowledgement:

The authors gratefully acknowledge the support from research projects 144032 and 174012 of the Serbian Ministry of Science. Also authors wish to thank to anonymous referee for valuable comments improving the quality of the paper.

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ТОЖДЕСТВА ТИПА БЬЯНКИ В ОБОБЩЕННОМ ФИНСЛЕРОВОМ ПРОСТРАНСТВЕ

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В работах [4,14] мы исследовали обобщенное финслерово пространство (с несимметричным базисным тензором) и, используя четыре типа производных в смысле дельта-дифференцирования Рунда, получили четыре тензора кривизны. В настоящей работе, обобщая известные тождества Бьянки обычного финслерова пространства, мы исследуем тождества типа Бьянки связанные с упомянутыми тензорами кривизны в обобщенном финслеровом пространстве.

Ключевые слова: обобщенные финслеровы пространства, несимметричная связность, тождества типа Бьянки.
AMS: 53A45, 53B05, 53B40.