AXIALLY SYMMETRIC GENERALIZATION OF THE CAUCHY-RIEMANN SYSTEM AND MODIFIED CLIFFORD ANALYSIS

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The main goal of this paper is to describe the most adequate generalization of the Cauchy-Riemann system fixing properties of classical functions in the octonionic case. An octonionic generalization of the Laplace transform is introduced. Octonionic generalizations of the inversion transformation, of the Euler gamma function and of the Riemann zeta-function are given.

Key Words: generalizations of the Cauchy-Riemann system, functions of the octonionic variable, octonionic Laplace transform.
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1 Introduction

The theory of holomorphic functions $f(z) = u + iv$ of a complex variable $z = x + iy$ has been developed on the basis of the classical Laplace equation in the plane $\mathbf{R}^2 = \{(x, y)\}$

$$\Delta h = \text{div grad } h = \frac{\partial h^2}{\partial x^2} + \frac{\partial h^2}{\partial y^2} = 0 \tag{1}$$

where solutions $h = h(x, y)$ have been called harmonic potential functions.

The Cauchy-Riemann system is the following first order system of equations

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \tag{2}$$

where $u(x, y) = \dfrac{\partial h}{\partial x}$, $v(x, y) = -\dfrac{\partial h}{\partial y}$ (see, e.g. [1]). The main goal of Clifford analysis [2] and of the related hypercomplex methods is an in-depth research of the multidimensional Laplace equation and multidimensional harmonic functions. In the physical formulation, problems for isotropic media in the case of constant density in space are considered, in particular.

Leutwiler [3] in 1992 started an in-depth research of a remarkable hyperbolic version of the Laplace equation in $\mathbf{R}^{n+1} = \{(x_0, x_1, ..., x_n)\}$

$$x_n \Delta h - (n-1)\frac{\partial h}{\partial x_n} = 0 \qquad \left(\Delta = \frac{\partial^2}{\partial {x_0}^2} + \frac{\partial^2}{\partial {x_1}^2} + ... + \frac{\partial^2}{\partial {x_n}^2}\right), \tag{3}$$

where C^3-solutions $h = h(x_0, x_1, ..., x_n)$ were called hyperbolically harmonic functions.

Remark 1.1. *As is easily seen, if* $x_n \neq 0$, *then*

$$x_n \Delta h - (n-1)\frac{\partial h}{\partial x_n} = x_n^n \text{div } (x_n^{1-n} \text{grad } h) = 0.$$

The first advances in modified Clifford analysis [3] were connected with the transition from the multidimensional Laplace equation for the multidimensional Laplace-Beltrami equations. These equations describe physical problems in isotropic media in the case of variable density in space, in particular. There have been established entirely new properties of many functions of the quaternionic variable. Leutwiler [3, 4] has constructed an important class of functions of the quaternionic variable associated with classical holomorphic functions.

The system of (n+1) real twice continuously differentiable (C^2–) functions

$$u_0 = u_0(x_0, x_1, \dots, x_n),\ u_1 = u_1(x_0, x_1, ..., x_n), \dots, u_n = u_n(x_0, x_1, ..., x_n),$$

where $u_0 = \dfrac{\partial h}{\partial x_0},\ u_1 = -\dfrac{\partial h}{\partial x_1}, \dots, u_n = -\dfrac{\partial h}{\partial x_n}$, in this case satisfies the following asymmetric system of equations (H_{n+1})

$$\begin{cases} x_n\left(\dfrac{\partial u_0}{\partial x_0} - \dfrac{\partial u_1}{\partial x_1} - ... - \dfrac{\partial u_n}{\partial x_n}\right) + (n-1)u_n = 0 \\ \dfrac{\partial u_0}{\partial x_m} = -\dfrac{\partial u_m}{\partial x_0} \qquad (m = 1, ..., n) \\ \dfrac{\partial u_l}{\partial x_m} = \dfrac{\partial u_m}{\partial x_l} \qquad (l, m = 1, ..., n) \end{cases} \tag{4}$$

Leutwiler investigated various classes of solutions (4) connected with the (universal) Clifford algebra $\mathbf{Cl_n}$, especially connected with the associative quaternionic algebra $\mathbf{H} = \mathbf{Cl_2}$.

Remark 1.2. *The associative Clifford algebra* $\mathbf{Cl_3}$ *(without division) and the alternative octonionic algebra* $\mathbf{O}$ *(with division) are not equivalent.*

Leutwiler focused his attention in $\mathbf{R}^3$ [4, 5, 6, 7] and introduced new terms, the reduced quaternionic variables $f = u + iv + jw$ and $z = x + iy + jt$. The Laplace-Beltrami equation in $\mathbf{R}^3 = \{(x, y, t)\}$

$$t\Delta h - \frac{\partial h}{\partial t} = 0 \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2}\right) \tag{5}$$

has been applied as a basis to investigate solutions in the reduced quaternionic form of an asymmetric system of equations (H)

$$\begin{cases} t\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} - \dfrac{\partial w}{\partial t}\right) + w = 0 \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x},\ \dfrac{\partial u}{\partial t} = -\dfrac{\partial w}{\partial x},\ \dfrac{\partial v}{\partial t} = \dfrac{\partial w}{\partial y}, \end{cases} \tag{6}$$

where $u = \dfrac{\partial h}{\partial x}, \quad v = -\dfrac{\partial h}{\partial y}, \quad w = -\dfrac{\partial h}{\partial t}.$

Then the Laplace-Beltrami equation in $\mathbf{R}^4 = \{(x, y, t, s)\}$

$$s\Delta h - 2\frac{\partial h}{\partial t} = 0 \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}\right) \tag{7}$$

has been applied [8] to investigate solutions of an asymmetric system of equations (H_4)

$$\begin{cases} s\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} - \dfrac{\partial w}{\partial t} - \dfrac{\partial r}{\partial s}\right) + 2r = 0 \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}, \ \dfrac{\partial u}{\partial t} = -\dfrac{\partial w}{\partial x}, \ \dfrac{\partial u}{\partial s} = -\dfrac{\partial r}{\partial x} \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial w}{\partial y}, \ \dfrac{\partial v}{\partial s} = \dfrac{\partial r}{\partial y}, \ \dfrac{\partial w}{\partial s} = \dfrac{\partial r}{\partial t} \end{cases} \tag{8}$$

in the form, in particular, of nontrivial quaternionic polynomials with quaternionic coefficients, where the quaternionic variables have been denoted $f = u+iv+jw+kr$ and $z = x+iy+jt+ks$ (see also [9]).

Interesting papers on octonion analysis [11, 10] and on functions of the octonionic variable [12] have appeared. However generalizations of the Cauchy-Riemann system having solutions in the form of functions of the octonionic variable have not been obtained there.

The question naturally arises — can we apply the methods of modified Clifford analysis, to move to the octonionic variable? Clifford algebras do not imply such development of theory.

It is well known that axially symmetric models are widely used for solving spatial problems, in particular in the framework of function theory of a complex variable [1]. But it is unexpectedly that an axially symmetric generalization of the 2-dimensional Laplace equation in $\mathbf{R}^8$ can be a good basis to construct the octonionic generalizations of many classical holomorphic functions.

The physical meaning of functions of the octonionic variable is unclear. Research of the physical meaning and properties of the octonionic Laplace transform is interesting for real-valued originals in the first place. The Laplace transform in this case automatically entails the determination of classical special functions in various octonionic areas. Refinements of the properties of the Euler gamma function and of the Riemann zeta-function may naturally lead to reform of methods of analytic number theory (see, e.g. [1, 13]).

The first electronic version of this paper was presented in 2003 [14].

2 On axial symmetry and on solutions associated with holomorphic functions in $\mathbf{R}^{n+1}$

Leutwiler [3, 4] has introduced an important class of solutions of the system (H_{n+1}) associated with classical holomorphic functions (in particular x^k, where $k \in \mathbf{N}$, e^x, $\ln x$) and has given the special axially symmetric conditions

$$x_l u_m = x_m u_l \qquad (l, m = 1, ..., n), \tag{9}$$

characterizing this class, at least locally.

Let us consider a second order elliptic equation in $\mathbf{R}^{n+1} = \{(x_0, x_1, ..., x_n)\}$

$$(x_1^2 + ... + x_n^2)\Delta h - (n-1)\left(x_1 \frac{\partial h}{\partial x_1} + ... + x_n \frac{\partial h}{\partial x_n}\right) = 0 \tag{10}$$

Definition 2.1. *C^3-solutions $h = h(x_0, x_1, ..., x_n)$ of the equation (10) are called ϕ-harmonic functions in $\mathbf{R}^{n+1}$.*

Remark 2.2. *If* $(x_1^2 + ... + x_n^2) \neq 0$, *then*

$$(x_1^2+...+x_n^2)\Delta h-(n-1)\left(x_1\frac{\partial h}{\partial x_1}+...+x_n\frac{\partial h}{\partial x_n}\right) = (x_1^2+...+x_n^2)^{\frac{n+1}{2}} \operatorname{div}\left[(x_1^2+...+x_n^2)^{\frac{1-n}{2}} \operatorname{grad} h\right] = 0.$$

The system of (n+1) real twice continuously differentiable (C^2–) functions

$$u_0 = u_0(x_0, x_1, \ldots, x_n),\ u_1 = u_1(x_0, x_1, \ldots, x_n), \ldots,\ u_n = u_n(x_0, x_1, \ldots, x_n)$$

where $u_0 = \frac{\partial h}{\partial x_0}$, $u_1 = -\frac{\partial h}{\partial x_1}, \ldots,$ $u_n = -\frac{\partial h}{\partial x_n}$, in this case satisfies the following axially symmetric system of equations (A_{n+1})

$$\begin{cases} (x_1^2 + ... + x_n^2)\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_n}{\partial x_n}\right) + (n-1)(x_1 u_1 + ... + x_n u_n) = 0 \\ \frac{\partial u_0}{\partial x_m} = -\frac{\partial u_m}{\partial x_0} \qquad (m = 1, ..., n) \\ \frac{\partial u_l}{\partial x_m} = \frac{\partial u_m}{\partial x_l} \qquad (l, m = 1, ..., n) \end{cases} \tag{11}$$

Singular hyperplanes play an essential role in modified Clifford analysis [3].

Definition 2.3. *The subspace* $\mathbf{R}^n = \{(x_0, x_1, ..., x_{n-1})\}$ *of the Euclidean space* $\mathbf{R}^{n+1} = \{(x_0, x_1, ..., x_n)\}$ *is called a singular hyperplane (*$[\mathbf{x_n} = \mathbf{0}]$*).*

Theorem 2.4. *(On solutions associated with classical holomorphic functions in* $\mathbf{R}^{n+1}$*). In any point in* $\mathbf{R}^{n+1} \setminus [\mathbf{x_n} = \mathbf{0}]$ *every system of (n+1)* C^2*-functions* $(u_0, u_1, ..., u_n)$ *with conditions (9) is a solution of the system of equations* (H_{n+1}) *if and only if the system of functions* $(u_0, u_1, ..., u_n)$ *is a solution of the system of equations* (A_{n+1})*.*

Proof. Let $x_n \neq 0$.
If a solution $(u_0, u_1, ..., u_n)$ of the system of equations (4) satisfies conditions (9), then

$$x_n\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_n}{\partial x_n}\right) + (n-1)u_n =$$

$$\left(\sum_{m=1}^{n} x_m^2\right) x_n\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_n}{\partial x_n}\right) + (n-1)u_n\left(\sum_{m=1}^{n} x_m^2\right) =$$

$$\left(\sum_{m=1}^{n} x_m^2\right) x_n\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_n}{\partial x_n}\right) + (n-1)x_n\left(\sum_{m=1}^{n} x_m u_m\right) = 0$$

and we obtain the first equation of the system (11).

If a solution $(u_0, u_1, ..., u_n)$ of the system of equations (11) satisfies conditions (9), then

$$(x_1^2 + \ldots + x_n^2)\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_n}{\partial x_n}\right) + (n-1)(x_1 u_1 + ... + x_n u_n) =$$

$$x_n(x_1^2 + ... + x_n^2)\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_n}{\partial x_n}\right) + (n-1)(x_1 u_1 + ... + x_n u_n)x_n =$$

$$x_n(x_1^2 + ... + x_n^2)\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_n}{\partial x_n}\right) + (n-1)u_n(x_1^2 + ... + x_n^2) = 0$$

and we obtain the first equation of the system (4). □

Corollary 2.5. *All solutions associated with classical holomorphic functions in singular hyperplane* $[\mathbf{x_n = 0}]$ *have the special axially symmetric condition*

$$(x_1^2 + ... + x_{n-1}^2)\left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_{n-1}}{\partial x_{n-1}}\right) + (n-2)(x_1 u_1 + ... + x_{n-1} u_{n-1}) = 0, \qquad (12)$$

except lower singular hyperplane $[\mathbf{x_{n-1}}] \equiv \mathbf{R}^{n-1}$.

Proof. In according with the previous theorem solutions associated with classical holomorphic functions on $[\mathbf{x_n = 0}] \equiv \mathbf{R}^n$ (except the subspace $\mathbf{R}^{n-1} = \{(x_0, x_1, ..., x_{n-2})\}$) satisfy the system (A_n). Then the first equation of the system (A_n) coincides with (12). □

Remark 2.6. *All solutions associated with classical holomorphic functions in the singular hyperplane* $[\mathbf{x_n = 0}]$ *have* $u_n = 0$.

Remark 2.7. *All solutions associated with classical holomorphic functions for every plane* $\mathbf{R}^2 = \{(x_0, x_m)\}$ $(m = 1, ..., n)$ *in* $\mathbf{R}^2 \setminus \mathbf{R}$ *have the special condition*

$$x_m \left(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - ... - \frac{\partial u_n}{\partial x_n}\right) + (n-1) u_m = 0 \qquad (13)$$

Remark 2.8. *The second order elliptic equation (10) (see, e.g. [15]) and as a consequence the first order system of equations* (A_{n+1}) *in this case are linear.*

We see that the system of equations (A_{n+1}) can be interpreted as a natural axially symmetric generalization of the Cauchy-Riemann system having class of solutions associated with classical holomorphic functions in $\mathbf{R}^{n+1} \setminus \mathbf{R}$.

3 On Real-Valued Originals and on the Octonionic Generalization of the Laplace Transform

Recall the octonionic algebra $\mathbf{O}$ is alternative, non-associative normed division algebra over $\mathbf{R}$ with $e_0 = 1$; the imaginary units of octonions are e_1, e_2, e_3, e_4, e_5, e_6, e_7 $(e_1^2 = \ldots = e_7^2 = -1)$, where $e_i e_j = -e_j e_i$, $i, j = 1, ..., 7$ $(i \neq j)$ and $e_3 = e_1 e_2$, $e_5 = e_1 e_4$, $e_6 = e_2 e_4$, $e_7 = e_3 e_4$ (see, e.g. [16]). Thus

$$x = x_0 + \sum_{m=1}^{7} x_m e_m = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + (x_4 + x_5 e_1 + x_6 e_2 + x_7 e_3) e_4.$$

If $x \notin \mathbf{R}$ then we can use the polar form

$$x = x_0 + \sum_{m=1}^{7} x_m e_m = |x|(\cos\varphi + I(x)\sin\varphi) = |x| e^{I(x)\varphi}, \qquad (14)$$

where $$I(x) = \frac{x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2}} \qquad (I(x)^2 = -1),$$

$$\varphi = \arccos \frac{x_0}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2}} \qquad (0 < \varphi < \pi).$$

For any $x \notin \mathbf{R}$

$$\ln x = \ln|x| + I(x)\varphi \quad (\textit{principal value}) \qquad (15)$$

and for any $n \in \mathbf{N}$

$$x^n = |x|^n(\cos n\varphi + I(x)\sin n\varphi). \tag{16}$$

Similarly [3], formula

$$e^{I(x)\rho} = \cos\rho + I(x)\sin\rho,$$

where $\rho \in \mathbf{R}$,

$$\rho = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2}$$

and

$$I(x)\rho = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7,$$

has as a consequence the beautiful formula

$$e^x = e^{x_0}e^{I(x)\rho} = e^{x_0}(\cos\rho + I(x)\sin\rho). \tag{17}$$

The octonionic inversion is described by the simple relation

$$x^{-1} = \frac{\bar{x}}{|x|^2} = \frac{x_0 - \sum\limits_{m=1}^{7} x_m e_m}{|x|^2} = |x|(\cos\varphi - I(x)\sin\varphi) = |x|e^{-I(x)\varphi}. \tag{18}$$

Then

$$x^{-n} = |x|^{-n}(\cos n\varphi - I(x)\sin n\varphi) = |x|^{-n}e^{-I(x)n\varphi}. \tag{19}$$

Remark 3.1. *As is easily seen, for example, elementary functions x^n, $\ln x$, e^x, x^{-n} of the octonionic variable x satisfy the special conditions $u_l x_m = u_m x_l$ $(l, m = 1, \ldots, 7)$, and for every $m = 1, \ldots, 7$ the condition $x_m = 0$ implies $u_m = 0$.*

It is not difficult to verify that functions x^n, $\ln x$, e^x, x^{-n} generate solutions of the system (A_8). In addition the system (A_8) is linear, therefore, all linear combinations with real coefficients of these elementary functions generate solutions as well.

Definition 3.2. *A real-valued function $\tilde{\eta} = \tilde{\eta}(\tau)$ with a real argument τ is called an real-valued original, if*

1. $\tilde{\eta}(\tau)$ complies with the Hölder's condition for every τ except some points $\tau = \tau_{\tilde{\eta}}^1, \tau_{\tilde{\eta}}^2, \ldots$ (there exists a finite quantity or zero of such points for every finite interval), where the function $\tilde{\eta}(\tau)$ has gaps of the first kind,

2. $\tilde{\eta}(\tau) = 0$ for all $\tau < 0$,

3. there exist constants $B_{\tilde{\eta}} > 0, \alpha_{\tilde{\eta}} \geq 0:$ for all τ $|\tilde{\eta}(\tau)| < B_{\tilde{\eta}}e^{\alpha_{\tilde{\eta}}\tau}$.

The Hölder's condition for the function $\tilde{\eta} = \tilde{\eta}(\tau)$ has the form: for every τ, there exist constants $A_{\tilde{\eta}} > 0$, $0 < \lambda_{\tilde{\eta}} \leq 1$, $\delta_{\tilde{\eta}} > 0$ so that $|\tilde{\eta}(\tau+\delta) - \tilde{\eta}(\tau)| \leq A_{\tilde{\eta}}|\delta|^{\lambda_{\tilde{\eta}}}$ for every δ, $|\delta| \leq \delta_{\tilde{\eta}}$.

Remark 3.3. *It is important that the Laplace transform exists in complex areas for every original $\tilde{\eta} = \tilde{\eta}(\tau)$, if Re $z = x > \alpha_{\tilde{\eta}}$ (see, e.g. [1, 17]). It is not difficult to verify that similar property is important in octonionic areas as well.*

Definition 3.4. *For every real-valued original $\tilde{\eta} = \tilde{\eta}(\tau)$ a function of the octonionic variable*

$$\mathcal{L}[\tilde{\eta}](x) = \int_0^\infty e^{-x\tau}\tilde{\eta}(\tau)d\tau \tag{20}$$

is called the octonionic generalization of the Laplace transform (the octonionic image or the octonionic Laplace transform for $\tilde{\eta} = \tilde{\eta}(\tau)$).

Remark 3.5. *It is clear that* $\mathcal{L}[\tilde{\eta}](x) = \int\limits_0^{\infty} e^{-x\tau}\tilde{\eta}(\tau)d\tau = \int\limits_{-\infty}^{+\infty} e^{-x\tau}\tilde{\eta}(\tau)d\tau$.

Proposition 3.6. *The octonionic Laplace transform* $\mathcal{L}[\tilde{\eta}](x)$ *for every real-valued original* $\tilde{\eta} = \tilde{\eta}(\tau)$ *defines a solution associated with a classical holomorphic function.*

Proof. Let $\mathcal{L}[\tilde{\eta}](x) = u_0 + \sum\limits_{m=1}^{7} u_m e_m$.
The octonionic exponential function defines a solution $(u_0, u_1, ..., u_7)$ of the system (A_8) associated with the classical exponential function.

Besides,

$$\frac{\partial}{\partial x_m}\int\limits_0^{\infty} e^{-x\tau}\tilde{\eta}(\tau)d\tau = \int\limits_0^{\infty} \frac{\partial}{\partial x_m} e^{-x\tau}\tilde{\eta}(\tau)d\tau \qquad (m = 0, 1, ..., 7).$$

Thus we obtain (A_8)

$$\begin{cases} (x_1^2 + ... + x_7^2)\left(\dfrac{\partial u_0}{\partial x_0} - \dfrac{\partial u_1}{\partial x_1} - ... - \dfrac{\partial u_7}{\partial x_7}\right) + 6(x_1u_1 + ... + x_7u_7) = 0 \\ \dfrac{\partial u_0}{\partial x_m} = -\dfrac{\partial u_m}{\partial x_0} \qquad (m = 1, ..., 7) \\ \dfrac{\partial u_l}{\partial x_m} = \dfrac{\partial u_m}{\partial x_l} \quad (l, m = 1, ..., 7). \end{cases} \tag{21}$$

□

Example 1. *The real-valued original* $\tilde{\eta} = \tilde{\eta}(\tau) = \begin{cases} 1, \ \tau \geq 0 \\ 0, \ \tau < 0 \end{cases}$
implies the octonionic image $\mathcal{L}[\tilde{\eta}](x) = x^{-1}$.

Example 2. *The real-valued original* $\tilde{\eta} = \tilde{\eta}(\tau) = \begin{cases} cos\,\omega\tau, \tau \geq 0 \\ 0, \ \tau < 0 \end{cases}$
implies the octonionic image $\mathcal{L}[\tilde{\eta}](x) = x(x^2 + \omega^2)^{-1}$.

Example 3. *The real-valued original* $\tilde{\eta} = \tilde{\eta}(\tau) = \begin{cases} \tau^a, \tau \geq 0 \\ 0, \ \tau < 0 \end{cases}$
for every $a > 0$ *implies the octonionic image* $\mathcal{L}[\tilde{\eta}](x) = \Gamma(a+1)x^{-a-1}$,
where $\Gamma(a+1)$ *denotes the Euler gamma function with a real argument [1].*

Remark 3.7. *Examples are not correct in terms of the associative Clifford algebra* $\mathbf{Cl_3}$ *, where imaginary units* $e_1, ..., e_7$ $(e_1^2 = ... = e_7^2 = -1)$ *satisfy to conditions* $e_ie_j = -e_je_i, \ i, j = 1, ..., 7 \ (i \neq j)$ *[3, 9].*

However we can consider the octonionic generalization of the two-sided (or bilateral) Laplace transform, if a real-valued original $\tilde{\eta} = \tilde{\eta}(\tau)$ not identically equal to 0 for $\tau < 0$ (see, e.g. [18]).

Definition 3.8. *For every real-valued original* $\tilde{\eta} = \tilde{\eta}(\tau)$ *a function of the octonionic variable*

$$\mathcal{L}_{-\infty}^{+\infty}[\tilde{\eta}](x) = \int_{-\infty}^{\infty} e^{-x\tau}\tilde{\eta}(\tau)d\tau = \int\limits_0^{+\infty} e^{-x\tau}\tilde{\eta}(\tau)d\tau + \int\limits_{-\infty}^{0} e^{-x\tau}\tilde{\eta}(\tau)d\tau \tag{22}$$

is called the octonionic generalization of the two-sided Laplace transform (the octonionic image or the octonionic two-sided Laplace transform for $\tilde{\eta} = \tilde{\eta}(\tau)$ *).*

Natural octonionic generalizations of many classical functions (see, e.g. [1, 13, 19, 20]) can be characterized in this way.

Example 4. *The octonionic generalization of the Euler gamma function.*

Let $x = x_0 + \sum_{m=1}^{7} x_m e_m, \quad x_0 > 0.$

The original $\tilde{\eta} = \tilde{\eta}(\tau) = e^{-e^{\tau}}$ *implies the octonionic image*

$$\mathcal{L}_{-\infty}^{+\infty}[\tilde{\eta}](x) = \int_{-\infty}^{+\infty} e^{-x\tau} e^{-e^{\tau}} d\tau = \int_{0}^{\infty} {\tau_1}^{-x-1} e^{-\tau_1} d\tau_1, \ \textit{where}\ \tau_1 = e^{\tau},\ d\tau_1 = e^{\tau} d\tau.$$

We can introduce the appropriate definition

$$\Gamma(x) = \int_{0}^{\infty} {\tau_1}^{x-1} e^{-\tau_1} d\tau_1 = \int_{-\infty}^{+\infty} e^{x\tau} e^{-e^{\tau}} d\tau = \mathcal{L}_{-\infty}^{+\infty}[\tilde{\eta}](-x). \tag{23}$$

Example 5. *The octonionic generalization of the Riemann zeta-function.*

Let $x = x_0 + \sum_{m=1}^{7} x_m e_m, \quad x_0 > 1.$

The original $\tilde{\eta} = \tilde{\eta}(\tau) = (e^{e^{\tau}} - 1)^{-1}$ *implies the octonionic image*

$$\begin{aligned}
\mathcal{L}_{-\infty}^{+\infty}[\tilde{\eta}](x) &= \int_{-\infty}^{+\infty} e^{-x\tau} (e^{e^{\tau}} - 1)^{-1} d\tau = \int_{0}^{\infty} {\tau_1}^{-x-1} (e^{\tau_1} - 1)^{-1} d\tau_1 \\
&= \int_{0}^{\infty} {\tau_1}^{-x-1} \left(\sum_{n=1}^{\infty} e^{-n\tau_1} \right) d\tau_1 = \sum_{n=1}^{\infty} \int_{0}^{\infty} {\tau_1}^{-x-1} e^{-n\tau_1} d\tau_1 \\
&= \sum_{n=1}^{\infty} \left(n^{x} \int_{0}^{\infty} {\tau_2}^{-x-1} e^{-\tau_2} \right) d\tau_2 = \left(\sum_{n=1}^{\infty} n^{x} \right) \int_{0}^{\infty} {\tau_2}^{-x-1} e^{-\tau_2} d\tau_2 \\
&= \left(\sum_{n=1}^{\infty} n^{x} \right) \Gamma(-x),
\end{aligned}$$

where $\tau_1 = e^{\tau},\ d\tau_1 = e^{\tau} d\tau; \quad \tau_2 = n\tau_1.$

We can introduce the appropriate definition

$$\zeta(x) = \sum_{n=1}^{\infty} n^{-x} = \mathcal{L}_{-\infty}^{+\infty}[\tilde{\eta}](-x)\ \Gamma^{-1}(x). \tag{24}$$

4 On Boundary Value Problems and on Functions of the Octonionic Variable

Second order elliptic equations in divergence form have various interesting applications in mathematical physics (see, e.g. [15]). For a stationary temperature field h the function $\overline{f} = \operatorname{grad} h$ can be interpreted, in particular, as the temperature gradient in $\mathbf{R}^{n+1}$. If χ is the coefficient of thermal conductivity, then the heat equation has the form $\operatorname{div}(\chi \operatorname{grad} h) = 0$.

The equation in the case of axially symmetric distribution of the coefficient $\chi(x) = (x_1^2 + \ldots + x_n^2)^{\frac{1-n}{2}}$

$$\operatorname{div}\Big[(x_1^2 + \ldots + x_n^2)^{\frac{1-n}{2}} \operatorname{grad} h\Big] = 0 \tag{25}$$

is equivalent to the system (A_{n+1}), at least in simply connected domains $\Lambda \subset \Omega$ $(\Lambda \subset \mathbf{R}^{n+1}$, $x_1^2 + ... + x_n^2 \neq 0)$.

Example 6. *The function of the octonionic variable, conjugated to the octonionic inversion* $\overline{f(x)} = \overline{x^{-1}} = \text{grad } h$, $x \neq 0$, *describes the inversion transformation in* $\mathbf{R}^8$ *(see, e.g. [21]). We can interprete it, in particular, as the axially symmetric generalization of the plane potential field of single source in the case of variable coefficient of thermal conductivity* χ.

Remark 4.1. *This example of application in mathematical physics is not realized in modified Clifford analysis using* $\mathbf{Cl_3}$ *[3, 9].*

Theorem 4.2 (On the uniqueness of solutions of the Dirichlet problem for the system (A_{n+1}))**.** *Assume that a simply connected domain* $\Lambda \subset \mathbf{R}^{n+1}$ $(\Lambda \cap \mathbf{R} = \emptyset)$ *has a* C^2*-smooth boundary* $\partial\Lambda$. *Let* $P = (P_0, P_1, ..., P_n)$, $|P| = 1$, *is outer unit normal to* $\partial\Lambda$. *Assume that there exist two functions* $\hat{f} = \hat{f}(x) = \hat{u}_0 + e_1\hat{u}_1 + ... + e_n\hat{u}_n$ *and* $\check{f} = \check{f}(x) = \check{u}_0 + e_1\check{u}_1 + ... + e_n\check{u}_n$ *determining* C^2*-solutions in* Λ *of the first boundary value problem for the system* (A_{n+1})

$$u_0|_{\partial\Lambda} = \psi_0, \ u_1|_{\partial\Lambda} = -\psi_1, \ ..., \ u_n|_{\partial\Lambda} = -\psi_n, \quad \psi = (\psi_0, \psi_1, ..., \psi_n) \in C^0(\partial\Lambda).$$

If there does not exist a point $x^0 \in \partial\Lambda$, *where* $(P, \ \psi) = \sum\limits_{m=0}^{n} P_m\psi_m = 0$, *then* $\hat{f} = \check{f}$.

Proof. The first boundary value problem

$$u_0|_{\partial\Lambda} = \psi_0, \ u_1|_{\partial\Lambda} = -\psi_1, \ ..., \ u_n|_{\partial\Lambda} = -\psi_n$$

for the system (11) is equivalent to the third boundary value problem

$$\left.\frac{\partial h}{\partial x_0}\right|_{\partial\Lambda} = \psi_0, \ \left.\frac{\partial h}{\partial x_1}\right|_{\partial\Lambda} = \psi_1, \ ..., \ \left.\frac{\partial h}{\partial x_n}\right|_{\partial\Lambda} = \psi_n$$

for the equation (25). Let us have

$$\begin{aligned} \hat{u}_0 &= \frac{\partial \hat{h}}{\partial x_0}, \quad \hat{u}_1 = -\frac{\partial \hat{h}}{\partial x_1}, \ ..., \quad \hat{u}_n = -\frac{\partial \hat{h}}{\partial x_n}, \\ \check{u}_0 &= \frac{\partial \check{h}}{\partial x_0}, \quad \check{u}_1 = -\frac{\partial \check{h}}{\partial x_1}, \ ..., \quad \check{u}_n = -\frac{\partial \check{h}}{\partial x_n}. \end{aligned}$$

Then for the function $h = \hat{h} - \check{h}$ we obtain

$$\hat{u}_0 - \check{u}_0 = \frac{\partial h}{\partial x_0}, \ \hat{u}_1 - \check{u}_1 = -\frac{\partial h}{\partial x_1}, \ \ldots, \ \hat{u}_n - \check{u}_n = -\frac{\partial h}{\partial x_n}$$

and

$$\left.\frac{\partial h}{\partial x_0}\right|_{\partial\Lambda} = 0, \ \left.\frac{\partial h}{\partial x_1}\right|_{\partial\Lambda} = 0, \ \ldots, \ \left.\frac{\partial h}{\partial x_n}\right|_{\partial\Lambda} = 0.$$

If there does not exist a point $x^0 \in \partial\Lambda$, where $(P, \ \psi) = 0$, then the homogeneous boundary value problem

$$\left.\frac{\partial h}{\partial x_0}\right|_{\partial\Lambda} = 0, \quad \left.\frac{\partial h}{\partial x_1}\right|_{\partial\Lambda} = 0, \ ..., \quad \left.\frac{\partial h}{\partial x_n}\right|_{\partial\Lambda} = 0$$

for the equation (25) can only have a constant solution (see, e.g. [22]). Hence $h = const$ and $\hat{u}_0 - \check{u}_0 = 0, \ \hat{u}_1 - \check{u}_1 = 0, \ ..., \ \hat{u}_n - \check{u}_n = 0.$ ∎

Remark 4.3. *We can see that functions of the octonionic variable associated with classical holomorphic functions can define (under suitable conditions in simply connected domains* $\Lambda \subset \mathbf{R}^8$ $(\Lambda \cap \mathbf{R} = \emptyset)$ *with a* C^2*-smooth boundary* $\partial\Lambda$*) solutions* $h = h(x_0, x_1, ..., x_7)$ *of the third boundary value problem for the following elliptic equation in divergence form*

$$\operatorname{div}\Big[(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2)^{-3} \operatorname{grad} h\Big] = 0 \tag{26}$$

to within arbitrary constant.

We can consider ϕ-harmonic functions $h = h(x_0, x_1, ..., x_n)$ as new potential functions in $\mathbf{R}^{n+1}$.

5 Conclusions

It is shown how effectively the application of second order elliptic equations as the basis for multidimensional generalizations of the Cauchy-Riemann system.

The axially symmetric system (A_{n+1}) takes up an intermediate place between the Stein-Weiss conjugate harmonic system in Clifford analysis [2] and the Leutwiler asymmetric system (H_{n+1}) in modified Clifford analysis [3] .

The results, obtained in this paper, demonstrate the following specifics

- the axially symmetric generalization of the Cauchy-Riemann system closely connected with an important class of the octonionic generalization of classical holomorphic functions,
- there exist transitions between lower and higher spatial dimensions for the octonionic generalization of classical holomorphic functions,
- the generalization of the Laplace transform is not realized in modified Clifford analysis using $\mathbf{Cl_3}$, at least for every real-valued original, but generates generalizations of classical holomorphic functions in various octonionic areas,
- the octonionic generalizations of classical objects of analytic number theory, such as the Euler gamma function and the Riemann zeta-function, are easily built in the framework of the octonionic generalization of the Laplace transform
- the first physical applications are naturally constructed, the classical boundary problems in some inhomogeneous isotropic media are correct for the related spatial potential fields. The axially symmetric potential fields in $\mathbf{R}^3$ are of particular interest, since they are not described by the well-known hypercomplex methods.

How many various generalizations of the Cauchy-Riemann system in $\mathbf{R}^8$ having solutions in the form of functions of the octonionic variable exist? This is an open question (see also the remarkable paper [23]).

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АКСИАЛЬНО-СИММЕТРИЧНОЕ ОБОБЩЕНИЕ СИСТЕМЫ КОШИ-РИМАНА И МОДИФИЦИРОВАННЫЙ КЛИФФОРДОВ АНАЛИЗ

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Главной целью статьи является описание наиболее адекватного обобщения системы Коши-Римана, фиксирующего свойства классических функций в октонионном случае. Вводится октонионное обобщение преобразования Лапласа. Даются октонионные обобщения преобразования инверсии, гамма-функции Эйлера и дзета-функции Римана.

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