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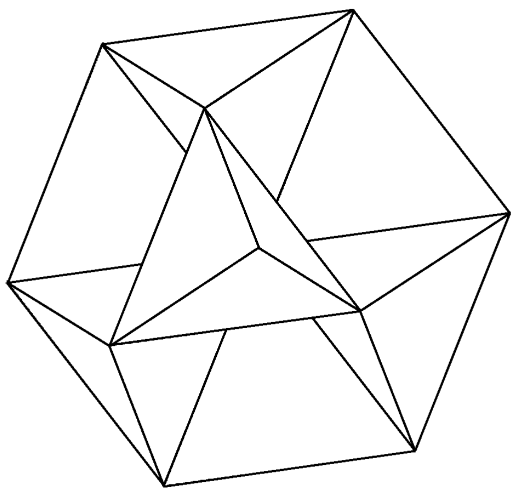

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Оглавление

Gladyshev V. O., Pavlov D. G. Conference "Number, time, relativity" – 2004 3
Garasko G. И., Pavlov D. G. Normal conjugation on the poly-number manifold . . 6
Garas'ko G. I. Generalized analytical functions and the congruence of the geodetic . 14
Eliovich A. A. On the norm of biquaternions and other algebras with central conjugation 23
Solovey L. G. On some distributive universal algebras 51
Rylov Y. A. Deformation principle as foundation of physical geometry and its application to space-time geometry 69
Rowlands P. The Nilpotent Vacuum 96
Toppan F. Division Algebra, Generalized Supersymmetries and Octonionic M-Theory 111
Information for authors 128

CONFERENCE "NUMBER, TIME, RELATIVITY" – 2004

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On the 10 – 13th august of 2004 the International scientific conference "Number, time, relativity"took place in MSTU n. a. N. E. Bauman.

The purposes of the conference were: to attract the attention of foreign and Russian physicists to Finslerian generalizations of the relativistic theory, to gather the leading specialists in the field of hyper-complex numbers, Finslerian geometry (that generalize the Riemannian manifolds), and the specialists in the field of the relativistic theory.

The conference was devoted to 175th anniversary of MSTU n. a. N. E. Bauman. The conference was performed by: the Bauman University's cathedra of physics, the theoretical physics cathedra of Moscow State University of M.V. Lomonosov and the United Physics Society of Russian Federation. The main sponsor of the conference was the Fund of 175th anniversary of MSTU n. a. N. E. Bauman.

На торжественном открытии Конференции с приветствиями к участникам выступили проректор по международным связям MSTU n. a. N. E. Bauman Г. П. Павлихин, заведующий кафедрой физики А. Н. Морозов, научный сотрудник Лаборатории им. Оливера Лоджа физического факультета Ливерпульского университета П. Роуландс.

Основные направления программы Конференции:

1. Финслерова геометрия.
2. Гиперкомплексные числа и связанные с ними пространства.
3. Поличисла и полипространства.
4. Геометрические аспекты понятия времени.
5. Финслеровы обобщения теории относительности.

Финслерова геометрия, на принципиальную возможность построения которой обратил внимание еще Риман, и гиперкомплексные числа, чья история берет свое начало в трудах Гамильтона, вплоть до конца XX века существовали как бы в не пересекающихся плоскостях. Организаторы Конференции ставили перед собой цель найти пути синтеза этих областей знания, что, возможно, приведет к созданию нового теоретического аппарата, удобного для более полного описания многообразия физических явлений. В связи с этим ряд докладов на Конференции был посвящен собственно Финслеровой геометрии, некоторые доклады были посвящены гиперкомплексным числам, но особый интерес вызвали доклады, нацеленные на объединение понятий числа и геометрии.

На Конференции были представлены свыше 50 докладов с результатами исследований, проводимых в MSU n. a. M. V. Lomonosov, Институте общей физики РАН, Институте механики сплошных сред РАН, Акустическом институте им. акад. Н. Н. Андреева РАН, Институте механики и машиностроения КазНЦ РАН, Объединенном институте ядерных исследований и в других университетах и академических институтах России, стран ближнего и дальнего зарубежья.

Среди участников – физики и математики из России, Азербайджана, Алжира, Бангладеш, Бразилии, Великобритании, Греции, Индии, Казахстана, Канады, Португалии, США, Украины, Швеции.

Основными языками Конференции являлись русский и английский. Работа Конференции освещалась на Web-сайте http://www.hypercomplex.ru. Этот сайт, созданный по инициативе Д. Г. Павлова, содержит большое количество информации о научных работах, посвященных связи финслеровой геометрии и гиперкомплексных алгебр, о научных семинарах и конкурсах по этой тематике (в том числе студенческих).

Перед началом работы Конференции был издан сборника тезисов докладов на русском и английском языках под редакцией Д. Г. Павлова (председатель оргкомитета, MSTU n. a. N. E. Bauman) и Г. С. Асанова (сопредседатель, MSU n. a. M. V. Lomonosov). Полные тексты наиболее интересных докладов будут опубликованы в настоящем журнале "Гиперкомплексные числа в геометрии и физике" на русском и английском языках.

Работа Конференции проходила в зале заседаний Ученого совета MSTU n. a. N. E. Bauman. Для участников Конференции были организованы экскурсия в музей МГТУ, экскурсии по центру Москвы и в Сергиев Посад.

Конференция, посвященная юбилею Бауманского Университета, позволила участникам Конференции познакомиться с историей развития науки и техники в старейшем Российском техническом университете, в котором работали такие выдающиеся ученые как Д. И. Менделеев, Н. Е. Жуковский, П. Л. Чебышев, С. А. Чаплыгин, А. С. Ершов, Д. К. Советкин, Ф. М. Дмитриев, А. В. Летников, А. П. Гавриленко и многие другие, а также узнать о тесной исторической и научной связи двух главных Университетов страны (MSU n. a. M. V. Lomonosov и MSTU n. a. N. E. Bauman).

* * *

Основной задачей Оргкомитета было собрать физиков, математиков и философов, пытающихся взглянуть на наиболее глубокие проблемы естествознания с самых общих позиций, среди которых одно из первых мест занимает идея связи алгебраических структур, геометрии и физики.

The recently ended century is marked out by two fundamental scientific revolutions. One of them was made by the Einstein Special and General theory of relativity: changing the concepts of the space-time they made possible the theory of gravitation.

Another revolution in physics happened to be less popular, but more radical for the physical world view: Bohr, Heisenberg, and Dirac have created the quantum mechanics. As a result of that, the fundament of the modern physics consists of two non-related bases, in a sense, two whales, floating in the absolutely unknown sea. The theory of relativity have satisfactorily described the picture of the world. The quantum theory have completely and consistently described the material word for it's circle of phenomena. In spite of the fact that the there can be no two truths, in a modern nature science there are. Therefore a lot of outstanding scientists seek after more general theoretical conception, since the middle of the XX-th century.

To create the theory of relativity, Einstein had to exceed the traditional limits of the classic Euclidean geometry, replacing it with the Riemannian geometry. We may suppose, that the future physics development requires a new geometry. The same could be the Finsler geometry, which is more general than the Minkowski one.

As is known, the point of a straight line and a plane are the geometrical images of the real an the complex numbers. The points of the n-dimensional Finsler spaces in many cases may be expressed as the hyper-complex numbers, the algebras with their own specific properties.

В последние 20 лет число научных публикаций в области исследований гиперкомплексных чисел растет экспоненциально. Международная научная Конференция

"Число, время и относительность" была призвана обсудить наиболее важные и интересные результаты последних лет в этой области.

Среди представленных на Конференции докладов наибольший интерес вызвали доклады Д. Г. Павлова "Число, геометрия пространства времени и относительность", Г. С. Асанова "Геометрия, основанная на финслероиде", Г. И. Гарасько "Нормальное сопряжение на множестве поличисел", П. Роуландса (Великобритания) "Нильпотентный вакуум", А. Ф. Турбина "Алгебры гиперкомплексных чисел: от алгебры к геометрии и анализу", Ф. Топпана (Бразилия) "Алгебры с делением, обобщенные суперсимметрии и октонионная М-теория", Э. Г. Мычелкина "Неизбежность антискалярной гравитации", Х. Б. Альмейда (Португалия) "Альтернативная формулировка Общей теории относительности в терминах четырехмерной оптики, новое определение времени", Р. В. Михайлова "Особая роль четырехмерных пространств в топологии", Ю. А. Рылова "Принцип деформации как основа физической геометрии" и ряд других.

Помимо нового взгляда на основания физики, исследования гиперкомплексных чисел могут приводить к важным прикладным результатам. Так, геофизик из Тюмени В. Кутрунов обнаружил, что с помощью кватернионов многие геофизические задачи, в том числе столь актуальные, как поиск новых нефтегазовых месторождений Сибири, решаются эффективней, чем векторными методами. Перспективы исследования гиперкомплексных чисел и Финслеровой геометрии, как следует из докладов, прозвучавших на Конференции, просматриваются в таких разных областях, как расчет электрических цепей, представление о природе гравитации, изучение феномена времени.

Это свидетельствует, что данное направление в науке продолжит свое развитие и завоюет признание научного сообщества. Однако в настоящее время оно развивается лишь благодаря энтузиазму исследователей. Поэтому, как отметил председатель оргкомитета Конференции Д. Г. Павлов, особенно важно поддержать молодых ученых и студентов, учредить специальные стипендии, организовать конкурсы работ, обеспечить возможность участия в Конференциях студентов и аспирантов. Первые шаги в этом направлении уже сделаны. Не один год проводится конкурс научных работ по проблемам, связанным с финслеровой геометрией, в 2004 году впервые при поддержке Объединенного физического общества состоялся конкурс рефератов студентов и школьников. За лучший реферат по гиперкомплексным числам студент из Саратова А. В. Малыгин получил возможность принять участие в Конференции и ежемесячную стипендию.

Подводя итоги Конференции, можно сказать, что она подтвердила существование тесной связи между гиперкомплексными алгебрами и некоторыми выделенными финслеровыми пространствами. Организаторам Конференции удалось поддержать и, в известном смысле, стимулировать рост интереса к этой тематике. Поэтому можно считать, что основная цель Конференции была достигнута.

NORMAL CONJUGATION ON THE POLY-NUMBER MANYFOLD

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The poly-number space is a linear space with several poly-linear forms. We introduce the notion of the normal conjugation on the non-degenerated n-numbers manyfold. The normal conjugation is an $(n-1)$-nary operation which is commutative for each argument but, in general, is not associative. Such operation is equivalent to the usual conjugation for complex and hyperbolic numbers. The normal conjugation may be applied to scrutinize the algebraic and geometric structure of the n-numbers coordinate space. It is also useful to introduce the notions of the scalar product and angular characteristics of two and more numbers(vectors).

Introduction

The poly-number spaces are the examples of vector spaces, where the poly-forms of several arguments play the role of the fundamental metric forms. [1]. Such spaces are principally different from the habitual Euclidian and pseudo-Euclidian spaces. Therefore they demand the development of the notions of the angle, orthogonality, scalar product etc. The necessity of the proper investigations is caused by the frequent attempts to consider the Finslerian spaces (the poly-number spaces as a rule are the ones) as the geometrical fundament of physics. [2, 3]. The physics progress strongly depends on the adecuasy of it's mathematical apparatus and geometrical ideas.

Surprisingly, the first known mention about such spaces belongs to Rieman. In 1854th while entering the professor post of the Goettingen university he read a famous lecture, in which he noticed that beside the usual quadratic metric forms the linear element may be represented as a fourth degree root from the differential expression of the same degree [4]. Per se he described the particular case of the spaces which later were named the Finslerian spaces.

Finslerian metric functions are very multifarious even for the linear spaces. Therefore they require the individual approach for every single case. However, if the hypercomplex numbers stand behind the Finslerian spaces it is possible to suggest the unified algorithm, some elements of it are represented bellow.

The exclusive role of the poly-number spaces is beyond any doubt. Despite of it they are very rarely mentioned in the modern geometrical literature. Obviously, it is explained by the seaming simplicity of the poly-number algebraic structure. It does not encourage neither the scrutiny of the poly-numbers them selves, nor the scrutiny of the spaces related to them. However, even the thoroughly examined complex numbers recently brought the surprise to the mathematicians. It turned out that the fractals may be built on the ground of the complex numbers. This fact makes us think that we may expect something similar from others of the hypercomplex numbers. The simplicity of the fractal construction algorithm underlines the potential variety hiding behind the most trivial number structures.

Such notions as the scalar product, orthogonality, angle between two vectors are the essential parts of the Euclidean space theory apparatus. These notions are naturally generalized for the pseudo-Euclidean spaces. The approach given bellow allows the similar generalization of the concerned notions for the poly-number spaces.

The poly-number spaces P_n with $n > 2$ are not Euclidean or pseudo-Euclidean. Thus, if $e_1, e_2, \dots, e_n \in P_n$ – the basis and

$$e_i e_j = p_{ij}^k e_k, \tag{1}$$

$$P_n \ni X = x^1 e_1 + x^2 e_2 + \dots + x^n e_n, \tag{2}$$

than n-th degree of the number X norm may be expressed with the n-linear symmetric form

$$(X, Y, \dots, Z) = \omega_{i_1 i_2 \dots i_n} x^{i_1} y^{i_2} \dots z^{i_n} \tag{3}$$

of one argument X. When $n > 2$ with two arguments X and Y we obtain $(n-1)$ different forms , therefore we can introduce the scalar product and the angle between two vectors (numbers) in several ways.

Besides the metric form (3) we may take other invariant forms in the P_n-space, the bilinear for example.

$$((X, Y)) = q_{ij} x^i y^j, \tag{4}$$

where

$$q_{ij} = C p_{im}^k p_{kj}^m, \tag{5}$$

$C \neq 0$ – some real number. For every concrete poly-number system this number may be chosen according to the simplicity and symmetry of the obtained formulas. As it follows from the definition, the given form is symmetric, i.e. $((X, Y)) = ((Y, X))$.

Thus, the P_n-space is n-dimensional space with several poly-linear forms. Two of the forms are dedicated: the metric form of the n-th order and the bilinear form.

The notion of the conjugated number is related (complex numbers, quaternions) with the changing of the sign of imaginary (symbolic) units. This makes us introduce $(n-1)$ conjugations in general and use the number itself and it's $(n-1)$ conjugations to construct of them the poly-number $(|X|^n \cdot 1 + 0e)$.

Normal conjugation

We shall call the n-numbers *nondegenerated*, if the matrix (q_{ij}) (5) is nondegenerated, i.e.

$$det(q_{ij}) \neq 0. \tag{6}$$

In this case, besides the two-times covariant tensor q_{ij}, the two times contravariant tensor q^{ij} is defined in the P_n-space.

Let us define the $(n-1)$-nary operation of the *normal conjugation of a complex* $\{X_{(1)}, X_{(2)}, \dots, X_{(n-1)}\}$ with the following way:

$$[X_{(1)}, X_{(2)}, \dots, X_{(n-1)}] = \omega_{i_1 i_2 \dots i_{n-1} i_n} q^{i_n k} x_{(1)}^{i_1} \dots x_{(n-1)}^{i_{n-1}} e_k. \tag{7}$$

It is obvious from this formula that the normal conjugation operation is commutative for every argument, but, generally, is not associative. The constant C in the formula (5) may be chosen with the following condition: $[1, 1, \dots, 1] = 1$.

We shall say that the number $Z = [X_{(1)}, X_{(2)}, ..., X_{(n-1)}]$ *is normally conjugated* to the complex of numbers $\{X_{(1)}, X_{(2)}, ..., X_{(n-1)}\}$.

Let's define the *scalar product of the number* X *and the complex* $\{X_{(1)}, X_{(2)}, ..., X_{(n-1)}\}$ with the bilinear form

$$((X, Z)) = (X, X_{(1)}, X_{(2)}, ..., X_{(n-1)}). \tag{8}$$

Let us introduce the designation

$$\tilde{X} = [X, X, ..., X], \tag{9}$$

than

$$((X, \tilde{X})) = |\, X|^n, \tag{10}$$

If in the given poly-number system the n-th degree of the number X norm may be expressed as

$$|\, X|^n = (X, X, ..., X). \tag{11}$$

According to the definition, the number $\tilde{X}$ is normally conjuncted to the number X.

Now shall we illustrate the introduced notions with some examples.

Complex Numbers

$$X = x^1 + ix^2, \qquad i^2 = -1, \tag{12}$$

$$(X, Y) = x^1y^1 + x^2y^2, \tag{13}$$

$$(\omega_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{14}$$

$$(q_{ij}) = 2C \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

Let's take $C = \frac{1}{2}$, than

$$(\omega_{ik}q^{kj}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{16}$$

$$\tilde{X} = x^1 - ix^2, \tag{17}$$

i.e. the normal conjugation for complex numbers is the usual conjugation. The scalar product of the numbers X and Y is

$$((X, \widetilde{Y})) = x^1y^1 + x^2y^2 = (X, Y). \tag{18}$$

Thus

$$((X, \tilde{X})) = |\, X|^2, \tag{19}$$

$$X \cdot \tilde{X} = |\, X|^2 \cdot 1 + 0 \cdot i. \tag{20}$$

Hyperbolic numbers, H_2

$$X = x^1 + jx^2, \qquad j^2 = 1, \tag{21}$$

$$(X,Y) = x^1y^1 - x^2y^2, \tag{22}$$

$$(\omega_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{23}$$

$$(q_{ij}) = 2C \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{24}$$

Let us take $C = \frac{1}{2}$, than

$$(\omega_{ik}q^{kj}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{25}$$

$$\tilde{X} = x^1 - jx^2, \tag{26}$$

i.e. the normal conjugation for hyperbolic numbers is the usual conjugation. The scalar product of the numbers X and Y is

$$((X, \tilde{Y})) = x^1y^1 - x^2y^2 = (X,Y). \tag{27}$$

Thus

$$((X, \tilde{X})) = |\,X|^2, \tag{28}$$

$$X \cdot \tilde{X} = |\,X|^2 \cdot 1 + 0 \cdot j. \tag{29}$$

Hypercomplex Numbers H_3

The most easy way is to work in ψ-basis:

$$X = x^1\psi_1 + x^2\psi_2 + x^3\psi_3, \tag{30}$$

$$p^k_{ij} = \begin{cases} 1, & \text{if} \quad i = j = k, \\ \\ 0, & \text{in all other cases}, \end{cases} \tag{31}$$

$$(q_{ij}) = C \cdot diag(1,1,1), \tag{32}$$

$$(X,Y,Z) = \frac{1}{6}(x^1y^2z^3 + x^1y^3z^2 + x^2y^1z^3 + x^2y^3z^1 + x^3y^1z^2 + x^3y^2z^1). \tag{33}$$

Let us take $C = \frac{1}{3}$, than

$$[X,Y] = \frac{1}{2}[(x^2y^3 + x^3y^2)\psi_1 + (x^1y^3 + x^3y^1)\psi_2 + (x^1y^2 + x^2y^1)\psi_3], \tag{34}$$

$$[1,1] = 1, \tag{35}$$

$$\tilde{X} = x^2x^3\psi_1 + x^1x^3\psi_2 + x^1x^2\psi_3, \tag{36}$$

$$X \cdot \tilde{X} = |\, X|^3 \cdot 1 + 0 \cdot e, \tag{37}$$

if the norm $X \in H_3$ is defined with

$$|\, X|^3 = x^1 x^2 x^3. \tag{38}$$

The scalar product of the complex $\{X, Y\}$ and the number Z is the scalar

$$((Z, [X, Y])) = (X, Y, Z). \tag{39}$$

The bilinear form (4) from two numbers X and Y looks like

$$((X, Y)) = \frac{1}{3}(x^1 y^1 + x^2 y^2 + x^3 y^3). \tag{40}$$

Lets fine all numbers of H_3, that satisfy the equation

$$\tilde{X} = X. \tag{41}$$

Solving the system of three quadratic equations with three unknowns we have five roots: $(0, 0, 0)$, $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$. The four latter numbers (if we consider them radius-vectors) constitute the regular tetrahedron while the first number is it's center.

If $X, Y \in H_3$ are the divisors of zero for the normal conjugation (i. e. $[X, Y] = 0$, with $X \neq 0,\ Y \neq 0$), they have to be the divisors of zero for the poly-number multiplication.

Any number $Y \in H_3$ may be represented as

$$Y = [1, Z], \quad \text{где} \quad Z = (-y^1 + y^2 + y^3, y^1 - y^2 + y^3, y^1 + y^2 - y^3). \tag{42}$$

Let's scrutinize the eigenvectors and eigenvalues problem that is

$$[1, Y] = \lambda Y, \tag{43}$$

where λ – some real or complex number. All eigenvalues are real: $\lambda_1 = 1$, $\lambda_{2,3} = -\frac{1}{2}$, because the matrix of the linear transformation in the right side of the formula (43) is symmetric. Eigenvectors appropriate to the first eigenvalue constitute a straight line $1t$, where t – parameter along the straight line. The eigenvectors appropriate to the eigenvalue $(-\frac{1}{2})$, constitute a plain, which is Euclid-perpendicular to the straight line along the unity and contains the coordinate zero. I.e. this plain is strained on two radius-vectors. For example: $(2, -1, -1)$, $(0, 1, 1)$.

Formulas (30) – (40) may be automatically generalized for polynumbers [1] H_n with replacement $3 \to n$, $C = \frac{1}{n}$.

The examples given above makes us suppose (while comparing the formulas (20), (29) и (37))that for the complex and H_n numbers the following formula is true.

$$X \cdot \tilde{X} = |\, X|^n \cdot 1 + 0 \cdot e. \tag{44}$$

It is also possible, that it is true for any non-degenerated poly-numbers, but this requires further prove.

We may say that X is *"orthogonal"* for Y, if

$$((X, \tilde{Y})) = 0. \tag{45}$$

[1] H_n – the hypercomplex numbers isomorphous to the real square diagonal matrixes algebra $n \times n$.

Notice that this notion in general is not symmetric for $n > 2$, i. e. the fact that X is orthogonal for Y, does not mean that Y is orthogonal for X. Is these two are orthogonal to each other than X and Y are *mutually orthogonal.*

If we have $(n-1)$-number complex (some of numbers may coincide), and Z is a normally conjugated for this complex, than X is *"orthogonal" for the given complex,* if

$$((X, Z)) = 0. \tag{46}$$

Angular parameters of several numbers

In the poly-number spaces $n > 2$ we can introduce the angle between two numbers (vectors) with several ways. In this paper we use the algebraic approach based on the triangle-formula analog form the Euclid space.

Let us illustrate this on the H_3 example. If X и Y are such that

$$x^i > 0, \quad y^i > 0, \quad i = 1, 2, 3. \tag{47}$$

In this case they are not the divizors of zero. Shall we find the expression for the norm of the cube of their summ $Z = X + Y$

$$|Z|^3 = (X+Y, X+Y, X+Y) = |X|^3 + 3(X, X, Y) + 3(X, Y, Y) + |Y|^3. \tag{48}$$

Let's introduce two hyperbolic angles β_X, β_Y according to the formulas:

$$\cosh\beta_X = \frac{(X, X, Y)}{|X|^2|Y|}, \quad \cosh\beta_Y = \frac{(X, Y, Y)}{|X||Y|^2}, \tag{49}$$

than

$$|Z|^3 = |X|^3 + |Y|^3 + 3|X|^2|Y|\cosh\beta_X + 3|X||Y|^2\ \cosh\beta_Y. \tag{50}$$

These two hyperbolic angles β_X, β_Y we shall call the *angular characteristics* of the pair of numbers X, Y.

Let us elucidate the meaning of the forms that appear in formulas (48), (49). For this let us consider the complex $\{X, Y\}$ and the normal conjugated number for this complex $W = [X, Y]$. The form (X, X, Y) is a scalar product of X and complex $\{X, Y\}$, and the form (X, Y, Y) is a scaler product of Y and the same complex.

If X, Y are not divisors of zero, but also they do not satisfy the (47) conditions, than the right sides of (49) may take negative values. If we want to preserve the formula (50), than the angular characteristics β_X, β_Y become, in general, complex numbers β_X, β_Y. Opposite, if we want to have real angular characteristics, we have to change the formulas (49) и (50). For example, if the right side in the first formula (49) is lesser than zero, than we can replace $\cosh\beta_X$ by $\sinh\beta_X$ in this formula and in (50).

Why do we need two angular characteristics for two numbers(vectors) in three-dimensional H_3 instead of one angle in the three-dimensional Euclid space? It is related with the fact that H_3-space and all the poly-number spaces of the dimension greater than two has marked out directions and planes., i.e. they are anisotropic.

Fractals

Over the last thirty years there was an impetuous progress of the direction of the dynamic systems theory related with complex fractals. [5]. The most brilliant representatives of latter are the Julia and Mandelbrot sets. Lots of beautiful and useful results have shaded the important fact that they all were obtained from the complex numbers and the Euclid plane basis. Opposite, the construction of multi-dimensional fractals based on quaternions was not impressive after all.

The deepest cause of the problems, appearing on this way, is the principal impossibility to generalize the theory of analytical functions of the complex variable for the quaternions. This impossibility is caused by the non-commutativity of quaternionic multiplication.

The poly-numbers structure does not contain the difficulties that appear in non-commutative or un-associative number algebras. Therefore we may expect that there is a possibility to construct the fractals based on poly-numbers, and such fractals could be much more interesting than the quaternionic ones. Turning to the H_n-numbers for example it is easy to see that it is impossible to construct interesting fractals using the usual for Julia sets dependencies. For example:

$$X_{(i+1)} = X_{(i)}^2 + C. \tag{51}$$

it is related with the very simple structure of H_n-numbers, H_3 in particular. In the special basis the analitical functions of H_n-variable break up to n functions of one variable. Therefore the iterative process may be turned to n independent one dimensional iterative process, which is not very interesting. But there is a great possibility to introduce some additional operations for the poly-numbers (one of them is the normal conjugation). These new operations may be used to build more complicated non-breaking iterative processes.

Thus, we can propose several simple non-trivial iterative processes for H_3:
$X_{i+1} = F(X_i)$:

1. $F(X) = \tilde{X} + C$,
2. $F(X) = [X, \tilde{X}] + C$,
3. $F(X) = [X, [X, 1]] + C$,
4. $F(X) = [X, [X, [X, 1]]] + C$,
5. $F(X) = X \cdot [X, 1] + C$,
6. $F(X) = [X, [X, C]] - 1$,
7. $F(X) = [X \cdot X, X] + C = [X, X] \cdot [X, 1] + C$,

where $C \in H_3$. The initial numbers for these iterative processes were taken on the planes which are perpendicular (in Euclid meaning) to the straight $1 \cdot t$. The parameter t indicates the point where the straight and the plane intersect. With $t = 0$ the plain contains the coordinate zero. It is interesting that with $C = 0$, $t = 0$ processes 2,3,4 gives the convergence area that looks like a round hexagon.

The scrutiny for the convergence of the process 1 gives some interesting in geometric aspect three-dimensional convergence areas. Appropriate results for process 7 are even more interesting.

Conclusion

The constructions proposed above, of course, may be generalized farther. So, we can examine the n-dimensional linear space with poly-linear symmetric form (3), divide the arguments manifold on two complexes and say that this form is a scalar product of these two complexes. This shall cause a further generalization of the notions introduced above.

It is undoubted that the normal conjugation has it's own algebraic meaning. We have proposed useful generalization algorithm for well-known from the Euclid and pseudo-Euclid spaces geometrical objects and values, such as scalar product, orthogonality, angles an so on.

The introduction of additional operations on the hypercomplex numbers turns them into something more than linear algebras. These operations allow us to obtain the geometries which have much more inner symmetries than the poly-numbers themselves contain. It would be appropriate to introduce the term "linear geometry", besides the usual "linear algebra". The new term contains the old one plus all possible independent poly-linear linear operations which natural follow from some constructions of linear algebra itself.

The construction of many-dimensional fractal sets is one of the perspective directions of applying the potential of such linear geometries.

Probably, we should underline once again, that the fractal sets, constructed by the mean of the introduced by authors specific $(n-1)$-nary operation, are the objects of poly-number, instead of arbitrary, space. This fact makes them perspective, unlike the quaternion-based fractals. It is well known that the quaternionic multiplication is not commutative. Therefore quaternions have poor mathematical perspectives. Thus, it is impossible to create a complete analytical functions theory. Since there no such problem with poly-numbers and taking into account the hypothetic possibility of the replacement of the Minkowsky space with one of the poly-spaces [6], the proposed approach seems to be very perspective.

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GENERALIZED ANALYTICAL FUNCTIONS AND THE CONGRUENCE OF THE GEODETIC

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Some properties of generalized-analytical functions of poly-number variable are being studied in this job. We can confront many spaces of affine connectedness with the $\{f^i; \Gamma^i_{kj}\}$ class of such functions. In each space the congruence of geodetic associated with the given class of general-analytic functions is defined. If the vector field f^i is tangent to one of the geodetic of congruence in each point of space there are certain restrictions for the generalized-analytical function itself.

Introduction

Very impressive success of the theory of complex variable and it's applications to physics makes us to search for a generalization of this theory for spaces of more than two dimensions. It is possible that the construction of poly-number variable [1] is one of such generalizations. We have to put some additional conditions to allow automatically apply such functions for theoretic physics models and some concrete physical questions.

Generalized analytical function (for further details see [1]) – is the pair $\{f^i; \gamma^i_k\}$:

$$\frac{\partial f^i}{\partial x^k} + \gamma^i_k = p^i_{kj} \dot{f}^j, \quad \text{или} \quad \tilde{\nabla}_k f^i = p^i_{kj} \dot{f}^j, \tag{1}$$

where f^i, $\dot{f}^i$ – single-covariant vector fields in the space $\{\mathbf{M_n}; \mathbf{P_n}\}$, $\mathbf{M_n}$ – n-dimensional elementary manifold admissive of inter-single-valued correspondence $\mathbf{M_n} \leftrightarrow \mathbf{P_n}$ on n-dimensional space of poly-numbers $\mathbf{P_n}$, and the objects γ^i_k while switching to another frame of reference transform as the objects $(\Gamma^i_{kj} f^i)$, where Γ^i_{kj} - objects of the affine connectivity. We postulate that one of the necessary properties of the space $\{\mathbf{M_n}; \mathbf{P_n}\}$ is that it's tangent space is isomorphous to the space of associative-commutative hypercomplex numbers (poly-numbers) $\mathbf{P_n}$. in any point $X \in \{\mathbf{M_n}; \mathbf{P_n}\}$. Due to the presence of the inter-single-valued transformation $\mathbf{M_n} \leftrightarrow \mathbf{P_n}$ we may introduce the special frames of references in the space $\{\mathbf{M_n}; \mathbf{P_n}\}$. In such frames of reference we define the rules of poly-number multiplication, which does not depend on the concerned point. If $\mathbf{P_n} \ni e_1, e_2, ..., e_n$ – a basis, than

$$e_i e_j = p^k_{ij} e_k. \tag{2}$$

Let ε^i – the coordinates of a unity breakdown than

$$\varepsilon^i p^k_{ij} = \delta^k_j. \tag{3}$$

Using this formulae and the formulae (1), we obtain an explicit stating for the generalized derivative

$$\dot{f}^i = \varepsilon^k \tilde{\nabla}_k f^i \tag{4}$$

and the Cauchy-Riemann correlations:

$$\tilde{\nabla}_k f^i - p^i_{kj} \varepsilon^m \tilde{\nabla}_m f^j = 0. \tag{5}$$

We may juxtapose a manifold of the affine connectivity $\mathbf{L_n}(\Gamma^i_{kj})$ to any generalized-analytical function $\{f^i;\gamma^i_k\}$. The objects of the affine connectivity Γ^i_{kj}, are a solutions of the equation set

$$\Gamma^i_{kj}f^j=\gamma^i_j. \tag{6}$$

Thus defined manifold of functions with a same object of connectivity forms a manifold (a function class), noted $\{f^i;\Gamma^i_{kj}\}$.

In the space of the affine connectivity always exist a parameter τ, such that the equation set of geodetic $x^i=x^i(\tau)$ takes [2] form

$$\frac{d^2x^i}{d\tau^2}+\Gamma^i_{kj}\frac{dx^k}{d\tau}\frac{dx^j}{d\tau}=0. \tag{7}$$

If we replace the connectivity object Γ^i_{kj} with a different one:

$$\tilde{\Gamma}^i_{kj}=\Gamma^i_{kj}+\frac{1}{2}(p_k\delta^i_j+p_j\delta^i_k)+S^i_{kj}, \tag{8}$$

where p_i – an arbitrary single-covariant field, and S^i_{kj} – an arbitrary tensor, which is antisymmetric by the down indexes, i.e. torsion tensor, than the geodetic remain the same. (see, for example, [2]).

Congruence of geodetic, appropriate for generalized-analytical function.

Let $\{f^i;\gamma^i_k\}$ – generalized-analytical function, and vector field f^i defines the congruence of geodetic with connection object (8), where the object Γ^i_{kj} is related with the concerned generalized-analytical function by the relation (6), moreover the tangent vector along the geodetic $x^i=x^i(\tau)$ is

$$\frac{dx^i}{d\tau}=f^i. \tag{9}$$

Than the differential equations (7) with Γ^i_{kj} replaced by $\tilde{\Gamma}^i_{kj}$ become relations that define generalized-analytical function

$$f^k\tilde{\nabla}_kf^i+(p_mf^m)f^i=0, \tag{10}$$

or

$$f^kp^i_{kj}\dot{f}^i+(p_mf^m)f^i=0. \tag{11}$$

Thus, to define geodetics congruence by the way given above (or, as we speak farther, to have X-property), the generalized-analytical function has to satisfy the relations (10), (11).

We call generalized-analytical functions with X-property the X-*functions.*

The equation set (11) is a set of linear equations for n unknowns $\dot{f}^i$ is consistent, since there certainly is one solution.

$$\dot{f}^i=-(p_mf^m)\varepsilon^i. \tag{12}$$

If the matrix

$$(a_{ij})=f^kp^i_{kj} \tag{13}$$

is non-degenerate in some area, than (12) is the only solution of system (11) in this area of the space $\{\mathbf{M_n}; \mathbf{P_n}\}$.

The n-th degree of "norm"in the poly-number $X \in P_n$ space may be expressed in terms of the form

$$\Omega(X) = det(x^k p^i_{kj}). \tag{14}$$

This form's value does not depend on basis:

$$\Omega(YX) = \Omega(Y)\Omega(X) \tag{15}$$

with any $X, Y \in P_n$; at last $\Omega(1) = 1$. Thus, we may define the n-th degree of "norm" by

$$|\,X|^n = \Omega(X) \tag{16}$$

or by

$$|\,X|^n = |\Omega(X)|. \tag{17}$$

On account of the said above we may expect that the solutions of the equation (10) will strongly depend on X-function equal zero or not.

Let us demonstrate that for arbitrary poly-numbers the analytic function

$$F(X) = \omega X + V_0 \tag{18}$$

(ω – an arbitrary real number, and V_0 – an arbitrary poly-number) is namely a function that define the congruence of the geodetics, i.e. an X-function.

If $\omega \neq 0$, than it may be written as

$$F(X) = \omega(X - X_0), \tag{19}$$

where X_0 – an arbitrary poly-number. Let us substitute (18) into (10) and, taking into account that for analytic functions $\gamma^i_k = 0$, we obtain

$$f^i[\omega + (p_m f^m)] = 0. \tag{20}$$

Since $p_m - m$ for arbitrary functions-components, we may always construct such m components, that $(p_m f^m) = -\omega$. Which was to be proved.

Let us find out a kind of curves, defined by the function (18). To do it, we have to find a general solution of the system of ordinary differential equations

$$\frac{dx^i}{d\tau} = \omega x^i + v^i_0. \tag{21}$$

It has the appearance of

$$x^i = v^i_0 \tau + a^i e^{\omega\tau}. \tag{22}$$

We imply by the congruence of curves in some area of n-dimensional space the $(n-1)$-parametric family of curves. At that one and only one curve passes through every point of this n-dimensional space.

There is $(2n+1)$ independent real parameters and the parameter along the curve in the general solution (22). Therefore parameters v^i_0, a^i, ω have to be expressible as $(n-1)$ independent parameter for equations (22) to define the congruence. And the region of variation of the parameter τ may be limited according to the values of these $(n-1)$ independent parameters. If we fix the direction of the parameter τ changing (for example – from lesser to bigger values), every curve gets a direction, i.e. it has a view of a current line or a "field line".

Despite of simplicity of appearance of the general solution (22), these formulas define a great variety of congruences of curves. And not all of them are straight, i.e. geodetic. Thus, the manifold of solutions of (10 includes the X-functions as a subset. This means that the fulfilment of (10) is necessary, but not enough for the generalized-analytical function to have the X-property.

In physics we often meet the condition $\nabla_i f^i = 0$. The law of conservation of charge and the 4-vector calibration of electro-magnetic field are expressed like that for example.

Let us calculate the same convolution product for the generalized-analytic function. We obtain

$$\tilde{\nabla}_i f^i = p^i_{ij} \dot{f}^j. \tag{23}$$

For X-function in case the condition (12) is satisfied we have

$$\tilde{\nabla}_i f^i = -(p_m f^m), \tag{24}$$

and for X-function (18), (19)

$$\tilde{\nabla}_i f^i = n\omega. \tag{25}$$

Examples of analytic X-function

Complex numbers

Let us take up the analytic function

$$F(z) = u(x, y) + iv(x, y) \tag{26}$$

of complex variable

$$z = x + iy, \quad i^2 = -1. \tag{27}$$

For first, let us write out the matrix(13)

$$(a_{ij}) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \tag{28}$$

and calculate it's determinant

$$det(a_{ij}) = u^2 + v^2. \tag{29}$$

Thus, for complex numbers the following formulae(16) is true

$$det(a_{ij}) = |F(z)|^2. \tag{30}$$

Let us solve the equation set (10). In this case it takes form

$$\begin{cases} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + (p_1 u + p_2 v)u = 0, \\ \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + (p_1 u + p_2 v)v = 0. \end{cases} \tag{31}$$

Using the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{32}$$

From this equation set we have two sets:

$$\begin{cases} (u^2+v^2)\left[\frac{\partial u}{\partial x}+(p_1u+p_2v)\right]=0, \\ (u^2+v^2)\frac{\partial u}{\partial y}=0, \end{cases} \tag{33}$$

and

$$\begin{cases} (u^2+v^2)\frac{\partial v}{\partial x}=0, \\ (u^2+v^2)\left[\frac{\partial u}{\partial y}+(p_1u+p_2v)\right]=0. \end{cases} \tag{34}$$

Let us examine these equation set in the area $u^2+v^2\neq 0$. In that case, reducing by this non-zero factor and writing the integrability conditions of the obtained equation sets, we have

$$\frac{\partial}{\partial x}(p_1u+p_2v)=\frac{\partial}{\partial y}(p_1u+p_2v)=0 \quad \Rightarrow \quad (p_1u+p_2v)=const, \tag{35}$$

and the only solution in this case

$$F(z)=\omega z+w_0, \tag{36}$$

where ω – an arbitrary real number, $w_0=u_0+iv_0$ – an arbitrary complex number.

Let us calculate a convolution $\nabla_i f^i$ of two X-functions (36), we get

$$\nabla_i f^i=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=2\omega, \tag{37}$$

which matches the formula (25).

So we have proven that all analytic X-functions of complex variable have the appearance of (36). There is no analytic X-function of complex variable (excluding a constant), for which $\nabla_i f^i\equiv 0$.

Hyperbolic numbers, H_2

Let us consider an analytic function

$$F(z)=u(x,y)+jv(x,y) \tag{38}$$

of hyperbolic variable

$$z=x+jy, \quad j^2=1. \tag{39}$$

Let's calculate the matrix (13)

$$(a_{ij})=\begin{pmatrix} u & v \\ v & u \end{pmatrix} \tag{40}$$

and it's determinant

$$det(a_{ij})=u^2-v^2. \tag{41}$$

Thus, if $v=\pm u$ the matrix (a_{ij}) is degenerate, and for hyperbolic numbers formulae (16) is true (16)

$$det(a_{ij})=|F(z)|^2, \tag{42}$$

if we take the square of norm in H_2 space as

$$|z|^2=x^2-y^2. \tag{43}$$

The relations (10) for hyperbolic numbers have the same appearance as for complex numbers, and the Cauchy-Riemann equations change a bit :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{44}$$

– therefore we only have to change the common factor in the equations (33), (34) to $(u^2 - v^2)$. Doing that, we obtain not a single, but three qualitative different solutions:

$$F_{(0)}(z) = \omega z + w_0, \tag{45}$$

where ω - an arbitrary real number and w_0 – an arbitrary hyperbolic number;

$$F_{(1)}(z) = f_{(1)}(x + y)(1 + j), \tag{46}$$

where $f_{(1)}(\xi)$ – an arbitrary one real number function;

$$F_{(2)}(z) = f_{(2)}(x - y)(1 - j), \tag{47}$$

where $f_{(2)}(\xi)$ – an arbitrary one real number function. In the ψ-basis:

$$\begin{gathered} \psi_{1,2} = \tfrac{1}{2}(1 \pm j), \quad \psi_1\psi_1 = \psi_1, \quad \psi_2\psi_2 = \psi_2, \psi_1\psi_2 = 0, \\ x + jy = (x + y)\psi_1 + (x - y)\psi_2 = \xi^1\psi_1 + \xi^2\psi_2 \end{gathered} \tag{48}$$

the two latter X-functions take the appearance of

$$F_{(1)}(z) = 2f_{(1)}(\xi^1)\psi_1, \quad F_{(2)}(z) = 2f_{(2)}(\xi^2)\psi_2, \tag{49}$$

at that $|F_{(1)}(z)| = 0, \quad |F_{(2)}(z)| = 0$.

So, the analytic X-functions of H_2 variable are more multifarious than appropriate functions of complex variable. It is related with the presence of the divisors of zero in H_2 algebra.

Let us calculate the scalar $\nabla_i f^i$ of three obtained X-functions:

$$\nabla_i f^i_{(0)} = 2\omega, \quad \nabla_i f^i_{(1)} = 2\dot{f}_{(1)}(x + y), \quad \nabla_i f^i_{(2)} = 2\dot{f}_{(2)}(x - y). \tag{50}$$

Note that there are no analytic X-functions of H_2 variable (excluding a constant) for which $\nabla_i f^i \ \equiv \ 0$.

Hyper-complex numbers H_4 These poly-numbers algebra is isomorphous to the algebra of real diagonal square matrices 4×4. It is the most easy to work with such numbers in ψ-basis: ψ_1, ψ_2, ψ_3, ψ_4;

$$\psi_i\psi_j = p^k_{ij}\psi_k, \quad p^k_{ij} = \begin{cases} 1, & \text{если} \quad i = j = k, \\ 0, & \text{in all other cases.} \end{cases} \tag{51}$$

An arbitrary analytic function of H_4-variable has an appearance of:

$$F(x) = \varphi^1(\xi^1)\psi_1 + \varphi^2(\xi^2)\psi_2 + \varphi^3(\xi^3)\psi_3 + \varphi^4(\xi^4)\psi_4, \tag{52}$$

where φ^i – arbitrary even functions of a real variable , and ξ^i – the coordinates of $X \in P_n$ in ψ-basis. The matrix (13) has the appearance of

$$(a_{ij}) = \begin{pmatrix} \varphi^1 & 0 & 0 & 0 \\ 0 & \varphi^2 & 0 & 0 \\ 0 & 0 & \varphi^3 & 0 \\ 0 & 0 & 0 & \varphi^4 \end{pmatrix} \tag{53}$$

and it's determinant equal

$$det(a_{ij}) = \varphi_1\varphi_2\varphi_3\varphi_4. \tag{54}$$

Thus, for hyper-complex numbers H_4 the formulae (16) is true:

$$|F|^4 = det(a_{ij}), \tag{55}$$

if the fourth degree of norm in H_4 space is

$$|\,X|^4 = \xi^1\xi^2\xi^3\xi^4. \tag{56}$$

The equation set (10) after substituting (52) into it is written like: (52)

$$\varphi^i \left[\frac{\partial\varphi^{i_-}}{\partial\xi^{i_-}} + p_m\varphi^m\right] = 0, \tag{57}$$

where $i \equiv i_-$ (no summation). As we noted above, the qualitative difference of the equation set (57) solutions is related with the presence of the divisors of zero in the poly-numbers system. Let us classify poly-numbers $X \neq\ 0$ in H_4 space in the following way:
А) X is not a divisor of zero;
Б) three coordinates $\xi^i, \xi^j, \xi^k, \quad i \neq j, \quad i \neq\ j, \quad j \neq\ k$ not equal zero, and the fourth coordinate equal zero;
В) only two coordinates ξ^i and ξ^j, $i \neq j$ differ from zero, and another two coordinate equal zero;
Г) only one coordinate ξ^i is not zero.

According to this classification we classify the solutions of the equation set (57):

$$\text{А)} \qquad F_{(0)}(X) = \omega X + W_0, \tag{58}$$

where ω – an arbitrary real number, a W_0 – an arbitrary poly-number;

$$\text{Б)} \qquad F_{(i,j,k)}(X) = \omega(\xi^i\psi_{i_-} + \xi^j\psi_{j_-} + \xi^k\psi_{k_-}) + \zeta_0^i\psi_{i_-} + \zeta_0^j\psi_{j_-} + \zeta_0^k\psi_{k_-}, \tag{59}$$

where ω, ζ_0^m – four arbitrary real numbers for each X-function of this kind;

$$\text{В)} \qquad F_{(i,j)}(X) = \omega(\xi^i\psi_{i_-} + \xi^j\psi_{j_-}) + \zeta_0^i\psi_{i_-} + \zeta_0^j\psi_{j_-}, \tag{60}$$

where ω, ζ_0^m – three arbitrary real numbers for each X-function of this kind;

$$\text{Г)} \qquad F_{(i)}(X) = \varphi^i(\xi^{i_-})\psi_{i_-}, \tag{61}$$

where $\varphi^i(\xi^{i_-})$ – an arbitrary flat function of a real variable for each X-function of this kind;

Let us calculate the scalar $\nabla_m \varphi^m$ of each obtained X-function.

$$\text{А)}\nabla_m \varphi^m = 4\omega, \quad \text{Б)}\nabla_m \varphi^m = 3\omega,$$
$$\text{В)}\nabla_m \varphi^m = 2\omega, \ \Gamma)\nabla_m \varphi^m = \dot{\varphi}^i(\xi^{i_-}). \tag{62}$$

Thus, there are no analytic X-functions of H_4 variable (excluding a constant) for which $\nabla_m \varphi^m \equiv 0$.

Non-degenerate X-functions

Let us call X-function *non-degenerate* if it is not a divisor of zero, i.e. $|F(X)| \neq 0$.

Than it follows from the above-stated that such generalized-analytic function has an appearance of

$$\{f^i; \gamma^i_k\} = \{f^i; -\frac{\partial f^i}{\partial x^k} + \delta^i_k a(x)\}, \tag{63}$$

where f^i – an arbitrary flat vector field, and $a(x)$ an arbitrary scalar field. Thus, there are non-degenerate X-functions for any poly-numbers, all of them have an appearance of (63), at that

$$\dot{f}^i = \varepsilon^i a(x), \qquad \tilde{\nabla}_i f^i = na(x). \tag{64}$$

Formally the non-constant non-degenerate X-functions with $\tilde{\nabla}_i f^i = 0$ do exist, but they are trivial, since the scalar field $a(x)$ at that identically equal zero. Mark that the derivative of the non-degenerate X-function in the basis generally has an appearance of $e_1 = 1, e_2, ..., e_n$

$$\dot{F}(X) = a(x) + 0e_2 + 0e_3 + ... + e_n. \tag{65}$$

Let us find out the conditions for the product of two non-degenerate X-functions $F_{(1)}(X), F_{(2)}(X)$ to be a non-degenerate X-function $F_{(3)}(X)$ too. Since $|F_{(1)}(X)F_{(2)}(X)| = |F_{(1)}(X)|\,|F_{(2)}(X)|$,, the function $F_{(3)}(X)$ is non-degenerate.

All we have to do is to check the fulfillment of the formulae (63) for it. From the article [1] we take the formulae for the poly-number product of two generalized-analytical functions:

$$\{f^i_{(1)}; \gamma^i_{(1)k}\}\{f^i_{(2)}; \gamma^i_{(2)k}\} = \{f^i_{(3)}; \gamma^i_{(3)k}\}, \tag{66}$$

where

$$\gamma^i_{(3)k} = p^i_{i_1 i_2}(f^{i_2}_{(2)}\gamma^{i_1}_{(1)k} + f^{i_1}_{(1)}\gamma^{i_2}_{(2)k}). \tag{67}$$

Let us demand all γ-objects in the formulae (67) to have the appearance, defined by the formulae (63). Than, after some transforms we obtain:

$$a_{(3)}\delta^i_k = a_{(1)}p^i_{kj}f^j_{(2)} + a_{(2)}p^i_{kj}f^j_{(1)}. \tag{68}$$

These are the wanted conditions for two functi

Conclusion

In this article have introduced the generalized-analytical functions of an arbitrary poly-number variable, which have been called the non-degenerate X-functions and they are the equivalent of function $F(z) = z$ of complex variable z. While these functions are not divisors of zero, they may define a congruence of geodetics in space $\{\mathbf{M_n};\mathbf{P_n}\}$. At that the derivative of such function is a poly-numeral unity multiplied by a scalar field. Formally the non-constant non-degenerate X-functions with $\tilde{\nabla}_i f^i = 0$ do exist, but they are trivial, since the scalar field $a(x)$ at that identically equal zero. Possible, namely the non-degenerate X-functions shall play the very same fundamental role as the complex variable z does in theory of analytical functions of complex variable, i.e. non-degenerate X-function $F(z) = z$.

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ON THE NORM OF BIQUATERNIONS AND OTHER ALGEBRAS WITH CENTRAL CONJUGATION

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The concept of central conjunction is introduced in this article. We apply it to algebras of biquaternions and bioctaves. With the given analysis method of the conjunctions permitted by algebra we derive some new results. Thus the alternative algebras with central conjunction are proven to have the multiplicative norm of second degree (that is in general not real). The consequence of this fact is that these algebras (biquaternions and bioctaves particulary) have the multiplicative real norm of degree higher than 2. This norm has several different but equivalent views. The quadrascalar and quadravector multiplications are introduced. Some results for algebras of biquaternions, diquaternions and bioctaves are given in terms of isotropic basis. The developed methods may be useful in the geometrical and physical usage of concerned algebras.

Introduction

Biquaternions is an prospective hyper-complex algebra, which is the natural language of the relativistic physics. Many people, once charmed by the quaternions, were subsequently impressed by the biquaternions. A good review of the known possibilities of biquaternion applications is given in [1]. While studying the biquaternions ten years ago, in 1994 year, the author have stated for himself some facts and formulaes. At that time he considered them too trivial to publish. But it turned out that some of these facts are still unknown, even to those, who work with biquaternions. As a result of this some questions are still wrong covered in the literature. Most people think that: a) biquaternions possess a real norm of the second degree and b) the norm of the biquaternions is not multiplicative. (i.e. the norm of the multiplication not equal the multiplication of the norms). One may meet this statement even in some encyclopedic reviews, [2]. Thus, this norm (which is mechanically written as a sum and a difference of squares) does not coincide with the biquaternion multiplication table and is really artificial for them. One rarely meets the correct statement that the biquaternions possess a complex norm of the second degree (and, consequently, a real norm of the fourth degree). And the author have not met a pointing that this norm is multiplicative. It is possible that such inattention for the norms of the degree higher than two is related with the fact that all the classic semisimple Lie groups are the groups of invariance of some quadratic form. Since the geometries are related with some groups (according to the Erlangen program of Klein), forms of the degree higher than two seem to be superfluous.

This clause is based on two ideas: 1) the consideration of the set of conjugations, specified in algebra, allows to prove a lot ofb facts with more simple and more general way, than with the help of direct algebraic calculations; 2) it is more correct to scrutinize the poly-norms, that naturally appear in algebra, instead of assigning a quadratic norm to them.

The norms and scalar product of the degree higher than two on the hyper-complex algebras were introduced by R. D. Schafer [5] – [7], who stated some fundamental facts in this area. Nowadays in Russia, the idea of applying the poly-norms ofb hyper-complex numbers in physics and geometry is actively popularized by D. G. Pavlov [9],[10].

For the better understanding of the question, we scrutinize theb biquaternions, diquaternions and bi-octaves at one time.

Algebras of biquaternions, diquaternions and bi-octaves

Quaternion is (see [2], for example) is a hyper-complex number

$$\mathbf{a} = a_0 + a_1\mathbf{q}_1 + a_2\mathbf{q}_2 + a_3\mathbf{q}_3, \tag{1}$$

where a_i – real numbers, and the orts $\mathbf{q}_i$ are multiplied according to the rule

$$1\mathbf{q}_k = \mathbf{q}_k 1 = \mathbf{q}_k, \qquad \mathbf{q}_j\mathbf{q}_k = -\delta_{jk}1 + \varepsilon_{jkn}\mathbf{q}_n, \tag{2}$$

where δ_{jk} и ε_{jkn} – Kroneker delta and Levy-Civita tensor.$(j, k, n = 1, 2, 3)$.

Biquaternions are the quaternions defined over the complex numbers field.

Diquaternions – are the quaternions defined over the binary numbers field. In view of the aforesaid we still may write down these numbers as (1), while a_i are the complex (dual) numbers. However, we shall use a more voluntary interpretation of biquaternions (diquaternions). Namely, they are an 8-dimensional algebra over the field of real numbers, consisting of two blocks:

$$\mathbf{a} = a + k \cdot \mathbf{i}_0, \tag{3}$$

or, in the unfolded view,

$$\mathbf{a} = a_0 + a_1\mathbf{q}_1 + a_2\mathbf{q}_2 + a_3\mathbf{q}_3 + k_0\mathbf{i}_0 + k_1\mathbf{i}_1 + k_2\mathbf{i}_2 + k_3\mathbf{i}_3, \tag{4}$$

where a, k – the quaternions with the real coefficients. $\mathbf{i}_0$ is an exterior unity, commuting with quaternions a, k; $\mathbf{i}_0^2 = -1$ for biquaternions and $+1$ for diquaternions,b unities $\mathbf{i}_j$ – is a result of an exterior (tensor) product $\mathbf{i}_0 \otimes \mathbf{q}_j$.(We use the same designation $\mathbf{i}_0$ for both algebras for a convenience, since they do not mix in this clause). From this it follows the rule of the multiplication for biquaterions (diquaternions) in a block and a tabular style:

$$(a + k \cdot \mathbf{i}_0)(b + l \cdot \mathbf{i}_0) = ab \mp kl + (al + kb) \cdot \mathbf{i}_0. \tag{5}$$

$\times$	1	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{i}_0$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
1	1	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{i}_0$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{q}_1$	$\mathbf{q}_1$	-1	$\mathbf{q}_3$	$-\mathbf{q}_2$	$\mathbf{i}_1$	$-\mathbf{i}_0$	$\mathbf{i}_3$	$-\mathbf{i}_2$
$\mathbf{q}_2$	$\mathbf{q}_2$	$-\mathbf{q}_3$	-1	$\mathbf{q}_1$	$\mathbf{i}_2$	$-\mathbf{i}_3$	$-\mathbf{i}_0$	$\mathbf{i}_1$
$\mathbf{q}_3$	$\mathbf{q}_3$	$\mathbf{q}_2$	$-\mathbf{q}_1$	-1	$\mathbf{i}_3$	$\mathbf{i}_2$	$-\mathbf{i}_1$	$-\mathbf{i}_0$
$\mathbf{i}_0$	$\mathbf{i}_0$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	∓ 1	$\mp\mathbf{q}_1$	$\mp\mathbf{q}_2$	$\mp\mathbf{q}_3$
$\mathbf{i}_1$	$\mathbf{i}_1$	$-\mathbf{i}_0$	$\mathbf{i}_3$	$-\mathbf{i}_2$	$\mp\mathbf{q}_1$	± 1	$\mp\mathbf{q}_3$	$\pm\mathbf{q}_2$
$\mathbf{i}_2$	$\mathbf{i}_2$	$-\mathbf{i}_3$	$-\mathbf{i}_0$	$\mathbf{i}_1$	$\mp\mathbf{q}_2$	$\pm\mathbf{q}_3$	± 1	$\mp\mathbf{q}_1$
$\mathbf{i}_3$	$\mathbf{i}_3$	$\mathbf{i}_2$	$-\mathbf{i}_1$	$-\mathbf{i}_0$	$\mp\mathbf{q}_3$	$\mp\mathbf{q}_2$	$\pm\mathbf{q}_1$	± 1

. (*Tab.* 1)

Duoquaternions are the quaternions defined over theb dual numbers field. They are not scrutinized in this clause.

Octonions, or *octaves,* (see. [3] and [4]) are the hyper-complex numbers like

$$\mathbf{a} = a_0 + a_1\mathbf{q}_1 + a_2\mathbf{q}_2 + a_3\mathbf{q}_3 + A_0\mathbf{e}_0 + A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3, \tag{6}$$

where the ort $\mathbf{q}_i, \mathbf{e}_j$ multiplication rule is defined on the grounds of the *duplication formula of Cayley-Dickson* for quaternions:

$$(a + A \cdot \mathbf{e}_0)(b + B \cdot \mathbf{e}_0) = ab - \bar{B}A + (Ba + A\bar{b}) \cdot \mathbf{e}_0, \tag{7}$$

(a, b, A, B – quaternions, $\bar{b}$ – the conjugated quaternion b, see farther about it). If we dismiss the difference between theb imaginary elements $\mathbf{q}_i$и $\mathbf{e}_j$ ($\mathbf{e}_j \equiv \mathbf{q}_{j+4}$), the multiplication rule may be expressed like this:

$$\mathbf{q}_i \cdot \mathbf{q}_j \ = \ -\delta_{ij} \cdot 1 + C_{ijk}\mathbf{q}_k, \qquad i, j, k = 1, 2, \ldots, 7, \tag{8}$$

where the totally antisymmetric octonion structured constants C_{ijk} equal

$$C_{123} = C_{145} = C_{176} = C_{246} = C_{257} = C_{347} = C_{365} = 1 \tag{9}$$

and turns to zero with other indices combinations.

Shall we give the octaves multiplication table for a convenience.

$\times$	1	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
1	1	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{q}_1$	$\mathbf{q}_1$	-1	$\mathbf{q}_3$	$-\mathbf{q}_2$	$\mathbf{e}_1$	$-\mathbf{e}_0$	$-\mathbf{e}_3$	$\mathbf{e}_2$
$\mathbf{q}_2$	$\mathbf{q}_2$	$-\mathbf{q}_3$	-1	$\mathbf{q}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_0$	$-\mathbf{e}_1$
$\mathbf{q}_3$	$\mathbf{q}_3$	$\mathbf{q}_2$	$-\mathbf{q}_1$	-1	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_0$
$\mathbf{e}_0$	$\mathbf{e}_0$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	-1	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$
$\mathbf{e}_1$	$\mathbf{e}_1$	$\mathbf{e}_0$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{q}_1$	-1	$-\mathbf{q}_3$	$\mathbf{q}_2$
$\mathbf{e}_2$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_0$	$-\mathbf{e}_1$	$-\mathbf{q}_2$	$\mathbf{q}_3$	-1	$-\mathbf{q}_1$
$\mathbf{e}_3$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_0$	$-\mathbf{q}_3$	$-\mathbf{q}_2$	$\mathbf{q}_1$	-1

. (*Tab.* 2)

Finaly, *bioctaves (bioconions)* are the octaves, defined over the complex numbers field. Their multiplication rule in a blockb style follows from the definition:

$$(a + A \cdot \mathbf{e}_0 + k \cdot \mathbf{i}_0 + K \cdot \mathbf{f}_0)(b + B \cdot \mathbf{e}_0 + l \cdot \mathbf{i}_0 + L \cdot \mathbf{f}_0) = ab - \bar{B}A - kl + \bar{L}K +$$
$$(Ba + A\bar{b} - Lk - K\bar{l}) \cdot \mathbf{e}_0 + (al - \bar{L}A + kb - \bar{B}K) \cdot \mathbf{i}_0 + (La + K\bar{b} + Bk + A\bar{l}) \cdot \mathbf{f}_0, \tag{10}$$

where $\mathbf{f}_0 \equiv \mathbf{i}_0 \cdot \mathbf{e}_0$. It may be written as ab symbolic table (also we give the complete form of the multiplication table for a convenience.)

$\times$	b	$B \cdot \mathbf{e}_0$	$l \cdot \mathbf{i}_0$	$L \cdot \mathbf{f}_0$
a	ab	$Ba \cdot \mathbf{e}_0$	$al \cdot \mathbf{i}_0$	$La \cdot \mathbf{f}_0$
$A \cdot \mathbf{e}_0$	$A\bar{b} \cdot \mathbf{e}_0$	$-\bar{B}A$	$A\bar{l} \cdot \mathbf{f}_0$	$-\bar{L}A \cdot \mathbf{i}_0$
$k \cdot \mathbf{i}_0$	$kb \cdot \mathbf{i}_0$	$Bk \cdot \mathbf{f}_0$	$-kl$	$-Lk \cdot \mathbf{e}_0$
$K \cdot \mathbf{f}_0$	$K\bar{b} \cdot \mathbf{f}_0$	$-\bar{B}K \cdot \mathbf{i}_0$	$-K\bar{l} \cdot \mathbf{e}_0$	$\bar{L}K$

and (*Tab.* 3)

$\times$	1	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{i}_0$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{f}_0$	$\mathbf{f}_1$	$\mathbf{f}_2$	$\mathbf{f}_3$
1	1	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{i}_0$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{f}_0$	$\mathbf{f}_1$	$\mathbf{f}_2$	$\mathbf{f}_3$
$\mathbf{q}_1$	$\mathbf{q}_1$	-1	$\mathbf{q}_3$	$-\mathbf{q}_2$	$\mathbf{e}_1$	$-\mathbf{e}_0$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{i}_1$	$-\mathbf{i}_0$	$\mathbf{i}_3$	$-\mathbf{i}_2$	$\mathbf{f}_1$	$-\mathbf{f}_0$	$-\mathbf{f}_3$	$\mathbf{f}_2$
$\mathbf{q}_2$	$\mathbf{q}_2$	$-\mathbf{q}_3$	-1	$\mathbf{q}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_0$	$-\mathbf{e}_1$	$\mathbf{i}_2$	$-\mathbf{i}_3$	$-\mathbf{i}_0$	$\mathbf{i}_1$	$\mathbf{f}_2$	$\mathbf{f}_3$	$-\mathbf{f}_0$	$-\mathbf{f}_1$
$\mathbf{q}_3$	$\mathbf{q}_3$	$\mathbf{q}_2$	$-\mathbf{q}_1$	-1	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{i}_3$	$\mathbf{i}_2$	$-\mathbf{i}_1$	$-\mathbf{i}_0$	$\mathbf{f}_3$	$-\mathbf{e}_2$	$\mathbf{f}_1$	$-\mathbf{f}_0$
$\mathbf{e}_0$	$\mathbf{e}_0$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	-1	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$	$\mathbf{f}_0$	$-\mathbf{f}_1$	$-\mathbf{f}_2$	$-\mathbf{f}_3$	$-\mathbf{i}_0$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{e}_1$	$\mathbf{e}_1$	$\mathbf{e}_0$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{q}_1$	-1	$-\mathbf{q}_3$	$\mathbf{q}_2$	$\mathbf{f}_1$	$\mathbf{f}_0$	$-\mathbf{f}_3$	$\mathbf{f}_2$	$-\mathbf{i}_1$	$-\mathbf{i}_0$	$-\mathbf{i}_3$	$\mathbf{i}_2$
$\mathbf{e}_2$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_0$	$-\mathbf{e}_1$	$-\mathbf{q}_2$	$\mathbf{q}_3$	-1	$-\mathbf{q}_1$	$\mathbf{f}_2$	$\mathbf{f}_3$	$\mathbf{f}_0$	$-\mathbf{f}_1$	$-\mathbf{i}_2$	$\mathbf{i}_3$	$-\mathbf{i}_0$	$-\mathbf{i}_1$
$\mathbf{e}_3$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_0$	$-\mathbf{q}_3$	$-\mathbf{q}_2$	$\mathbf{q}_1$	-1	$\mathbf{f}_3$	$-\mathbf{f}_2$	$\mathbf{f}_1$	$\mathbf{f}_0$	$-\mathbf{i}_3$	$-\mathbf{i}_2$	$\mathbf{i}_1$	$-\mathbf{i}_0$
$\mathbf{i}_0$	$\mathbf{i}_0$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{f}_0$	$\mathbf{f}_1$	$\mathbf{f}_2$	$\mathbf{f}_3$	-1	$-\mathbf{q}_1$	$-\mathbf{q}_2$	$-\mathbf{q}_3$	$-\mathbf{e}_0$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$
$\mathbf{i}_1$	$\mathbf{i}_1$	$-\mathbf{i}_0$	$\mathbf{i}_3$	$-\mathbf{i}_2$	$\mathbf{f}_1$	$-\mathbf{f}_0$	$-\mathbf{f}_3$	$\mathbf{f}_2$	$-\mathbf{q}_1$	1	$-\mathbf{q}_3$	$\mathbf{q}_2$	$-\mathbf{e}_1$	$\mathbf{e}_0$	$\mathbf{e}_3$	$-\mathbf{e}_2$
$\mathbf{i}_2$	$\mathbf{i}_2$	$-\mathbf{i}_3$	$-\mathbf{i}_0$	$\mathbf{i}_1$	$\mathbf{f}_2$	$\mathbf{f}_3$	$-\mathbf{f}_0$	$-\mathbf{f}_1$	$-\mathbf{q}_2$	$\mathbf{q}_3$	1	$-\mathbf{q}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$\mathbf{e}_0$	$\mathbf{e}_1$
$\mathbf{i}_3$	$\mathbf{i}_3$	$\mathbf{i}_2$	$-\mathbf{i}_1$	$-\mathbf{i}_0$	$\mathbf{f}_3$	$-\mathbf{f}_2$	$\mathbf{f}_1$	$-\mathbf{f}_0$	$-\mathbf{q}_3$	$-\mathbf{q}_2$	$\mathbf{q}_1$	1	$-\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$\mathbf{e}_0$
$\mathbf{f}_0$	$\mathbf{f}_0$	$-\mathbf{f}_1$	$-\mathbf{f}_2$	$-\mathbf{f}_3$	$-\mathbf{i}_0$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	1	$-\mathbf{q}_1$	$-\mathbf{q}_2$	$-\mathbf{q}_3$
$\mathbf{f}_1$	$\mathbf{f}_1$	$\mathbf{f}_0$	$-\mathbf{f}_3$	$\mathbf{f}_2$	$-\mathbf{i}_1$	$-\mathbf{i}_0$	$-\mathbf{i}_3$	$\mathbf{i}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{q}_1$	1	$\mathbf{q}_3$	$-\mathbf{q}_2$
$\mathbf{f}_2$	$\mathbf{f}_2$	$\mathbf{f}_3$	$\mathbf{f}_0$	$-\mathbf{f}_1$	$-\mathbf{i}_2$	$\mathbf{i}_3$	$-\mathbf{i}_0$	$-\mathbf{i}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{q}_2$	$-\mathbf{q}_3$	1	$\mathbf{q}_1$
$\mathbf{f}_3$	$\mathbf{f}_3$	$-\mathbf{f}_2$	$\mathbf{f}_1$	$\mathbf{f}_0$	$-\mathbf{i}_3$	$-\mathbf{i}_2$	$\mathbf{i}_1$	$-\mathbf{i}_0$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{q}_3$	$\mathbf{q}_2$	$-\mathbf{q}_1$	1

(*Tab.* 4)

This table includes as fragments the multiplication tables of biquaternions and octaves.

Central conjugation

Algebras of biquaternions, diquaternions and bioctaves possess some wonderful properties, which follow from the existence of a "very nice" conjugation in these algebras.

The accepted interpretation of conjugation is that they are linear involutional anti-automorphisms:

$$
\begin{aligned}
C(\lambda\mathbf{a}+\mu\mathbf{b}) &= \lambda C(\mathbf{a})+\mu C(\mathbf{b}) \qquad (\lambda,\mu-\text{real numbers, linearity}),\\
C(C(\mathbf{a})) &= \mathbf{a} \qquad (\text{involution}),\\
C(\mathbf{ab}) &= C(\mathbf{b})\cdot C(\mathbf{a}) \qquad (\text{anti-automorphisms}).
\end{aligned} \tag{11}
$$

Thanks to these properties, the expressions

$$\Re(\mathbf{a}) = 1/2(\mathbf{a}+C(\mathbf{a})) \qquad (\textit{real part } \mathbf{a}), \tag{12}$$

$$N_r = \mathbf{a}\cdot C(\mathbf{a}) \qquad (\textit{right 2-norm } \mathbf{a}), \tag{13}$$

$$N_l = C(\mathbf{a})\cdot \mathbf{a} \qquad (\textit{left 2-norm } \mathbf{a}), \tag{14}$$

do not change with conjugation (let us call them *invariant* or *real* for the given conjugation):

$$C(\Re(\mathbf{a})) = \Re(\mathbf{a}), \qquad C(N(\mathbf{a})) = N(\mathbf{a}). \tag{15}$$

At that

$$\Re(C(\mathbf{a})) = \Re(\mathbf{a}), \qquad \text{but, in general} \quad N(C(\mathbf{a})) \neq N(\mathbf{a}). \tag{16}$$

Note that the invariant expressions are not necessary real numbers, in spite of they are introduced by the same formulas, as for the classic quaternion and octave algebras. However, so written norm seem to be more correct than the one with a sum and a difference of squares, which has no algebraical ground.

Let us call $\mathbf{a} \cdot C(\mathbf{a})$ and $C(\mathbf{a}) \cdot \mathbf{a}$ the *natural 2-norms* of the hyper-complex algebra (right and left) to underline their compliance with the multiplication table.

Note that there are not so many "good" hyper-complex algebras with conjugation. The quantity of the "bad" ones (without conjugation) is much greater.

We may introduce the quaternionic conjugation $\bar{\mathbf{a}}$ according to the following rule:

$$\bar{\mathbf{a}} = a_0 - a_1\mathbf{q}_1 - a_2\mathbf{q}_2 - a_3\mathbf{q}_3. \tag{17}$$

This conjugation may be expressed in terms of the operations of the given algebra (let us call such conjugations *complied with the algebraic multiplication*):

$$\bar{\mathbf{a}} = -1/2(\mathbf{a} + \mathbf{q}_1\mathbf{a}\mathbf{q}_1 + \mathbf{q}_2\mathbf{a}\mathbf{q}_2 + \mathbf{q}_3\mathbf{a}\mathbf{q}_3), \tag{18}$$

and as a consequence of that:

$$\Re(a) = 1/2(\mathbf{a} + \bar{\mathbf{a}}) = 1/4(\mathbf{a} - \mathbf{q}_1\mathbf{a}\mathbf{q}_1 - \mathbf{q}_2\mathbf{a}\mathbf{q}_2 - \mathbf{q}_3\mathbf{a}\mathbf{q}_3). \tag{19}$$

It is easy to make sure that (18) is true, if we consider that the mapping $\mathbf{a} \to -\mathbf{q}_k\mathbf{a}\mathbf{q}$

Note that the complex numbers conjugation may not be expressed in terms of summation and multiplication due to the commutativity of the complex numbers algebra. Since that it has to be introduced by hand.

Taking in consideration that the bi- and diquaternions are the directly doubled quaternions, the appropriate *quaternionic conjugation* $\bar{\mathbf{a}}$ for them may be introduced according to the obvious rule:

$$\begin{gathered}
\bar{\mathbf{a}} = a_0 - a_1\mathbf{q}_1 - a_2\mathbf{q}_2 - a_3\mathbf{q}_3, \\
\text{(the writing with complex coefficients)}, \\
\bar{\mathbf{a}} = a_0 - a_1\mathbf{q}_1 - a_2\mathbf{q}_2 - a_3\mathbf{q}_3 + k_0\mathbf{i}_0 - k_1\mathbf{i}_1 - k_2\mathbf{i}_2 - k_3\mathbf{i}_3, \\
\text{(the complete writing with real coefficients)}, \\
\bar{\mathbf{a}} = \overline{a + k \cdot \mathbf{i}_0} = \bar{a} + \bar{k} \cdot \mathbf{i}_0 \\
\text{(the brief writing)}.
\end{gathered} \tag{20}$$

Again, this conjugation is complied with the multiplication. For biquaternions (the upper sign) and diquaternions (the bottom):

$$\bar{\mathbf{a}} = -1/4(\mathbf{a} + \mathbf{q}_1\mathbf{a}\mathbf{q}_1 + \mathbf{q}_2\mathbf{a}\mathbf{q}_2 + \mathbf{q}_3\mathbf{a}\mathbf{q}_3) \pm 1/4(\mathbf{i}_0\mathbf{a}\mathbf{i}_0 - \mathbf{i}_1\mathbf{a}\mathbf{i}_1 - \mathbf{i}_2\mathbf{a}\mathbf{i}_2 - \mathbf{i}_3\mathbf{a}\mathbf{i}_3), \tag{21}$$

(We consider that the mapping $\mathbf{a} \to -\mathbf{q}_k\mathbf{a}\mathbf{q}_k$ changes every component, excluding $\mathbf{a}_0$, $\mathbf{a}_k$, $\mathbf{a}_0$, $\mathbf{a}_k$; the mapping $\mathbf{a} \to -\mathbf{i}_k\mathbf{a}\mathbf{i}_k$ does the same for diquaternions and the opposite for biquaternions).

Thus, this conjugation is not enough for the scrutiny of the 4-norm and 4-scalar product int the biquaternion and diquaternions algebras. Therefore we have to introduce another one, *the dual quaternionic conjugation* according to the following rule: Введем поэтому второе, *дуальное кватернионное сопряжение* по следующему правилу (записи в той же последовательности):

$$\widetilde{\mathbf{a}} = a_0^* - a_1^*\mathbf{q}_1 - a_2^*\mathbf{q}_2 - a_3^*\mathbf{q}_3,$$
$$\widetilde{\mathbf{a}} = a_0 - a_1\mathbf{q}_1 - a_2\mathbf{q}_2 - a_3\mathbf{q}_3 - k_0\mathbf{i}_0 + k_1\mathbf{i}_1 + k_2\mathbf{i}_2 + k_3\mathbf{i}_3.$$
$$\widetilde{\mathbf{a}} = \widetilde{\alpha + k \cdot \mathbf{i}_0} = \bar{\alpha} - \bar{k} \cdot \mathbf{i}_0 \tag{22}$$

The conjugation $\widetilde{\mathbf{a}}$ (22) is an involutional automorphism too. Therefore it deserves the name of an conjugation. But it is not complied with the algebraic multiplication: the ort $\mathbf{i}_0$ commutes with the other orts and can it not be reflected by summation and multiplication.

Remark 1. Let us consider the combination of conjugations $\bar{\widetilde{\mathbf{a}}}$ (writing down in the same order):

$$\bar{\widetilde{\mathbf{a}}} = a_0^* + a_1^*\mathbf{q}_1 + a_2^*\mathbf{q}_2 + a_3^*\mathbf{q}_3,$$
$$\bar{\widetilde{\mathbf{a}}} = a_0 + a_1\mathbf{q}_1 + a_2\mathbf{q}_2 + a_3\mathbf{q}_3 - k_0\mathbf{i}_0 - k_1\mathbf{i}_1 - k_2\mathbf{i}_2 - k_3\mathbf{i}_3,$$
$$\bar{\widetilde{\mathbf{a}}} = \overline{\widetilde{a + k \cdot \mathbf{i}_0}} = a - k \cdot \mathbf{i}_0. \tag{23}$$

In other words, the transformation $\bar{\widetilde{\mathbf{a}}}$ is a *complex conjugation*, exterior for the quaternions a, k. Let us designate it with $\mathbf{a}^*$. It is an involution and an automorphism (rather than an anti-automorphism):

$$\mathbf{a}^* = \mathbf{a} \qquad \text{и} \qquad (\mathbf{ab})^* = \mathbf{a}^*\mathbf{b}^*. \tag{24}$$

As a consequence, the expression $\mathbf{aa}^*$ changes with the conjugation of it's kind (i.e. it is not real for it):

$$(\mathbf{aa}^*)^* = \mathbf{a}^*\mathbf{a} \neq \mathbf{aa}^*. \tag{25}$$

In connection with that, the combined transformation (the complex conjugation) $\mathbf{a}^*$ is not a conjugation at all of this algebra. Furthermore, the complex conjugation $\mathbf{a}^*$ changes the sign of the ort $\mathbf{i}_0$, that commutes with all other orts of the algebras $\mathbb{H}_c$. Therefore it is not complied with algebraic multiplication.

The most important advantage of the basic conjugation $\bar{\mathbf{a}}$ over the dual conjugation $\widetilde{\mathbf{a}}$ is that the hyper-complex numbers which are real (invariant) relative to $\bar{\mathbf{a}}$, contain only the orts 1 and $\mathbf{i_0}$, therefore they commute (like real numbers do) with any numbers of the algebra. This property of the quaternionic conjugation is the very source of several good qualities of the quaternions, biquaternions, diquaternions, octaves and bioctaves. It is important that not all algebras do possess such a good conjugation.

More rigorously, the real (invariant) relative to the basal conjugation elements $\mathbf{r}$:

1) are algebraically closed;
2) commute with each one of the algebra's elements $\mathbf{ar} = \mathbf{ra}$;
3) their multiplication by any element of the algebra $\mathbf{r} \cdot \mathbf{ab} = \mathbf{ra} \cdot \mathbf{b}$.

In other words they belong to the commutative and , at one time, associative centers of the algebra, i.e. to the algebra's center in it's ordinary sense. Thus, let us introduce

Definition. *The conjugation, which real (invariant) numbers belong to the center of the algebra, we name* ***the central conjugation****.*

The conjugations, which real numbers belong to the center of the algebra, we name the central. Next, the algebras with the central conjugation we designate $\mathbb{A}_c$. In fact, all the scrutinized algebras (including the infinite consequence of the Cayley-Dickson algebras) with all their complexifications (and the hyperbolic duplications) are central.

Remark 2. *Any algebra with a natural norm of the 2nd degree* (i.e. the norm of the kind $\mathbf{a}\bar{\mathbf{a}}$, complying with the algebra's multiplication table), *) is algebra with the central conjugation* (the center is reduced to real numbers at that). That is why the results good for an algebra with the central conjugation, is good for an algebra with the square norm too.

Remark 3. Due to the fact that the commutative center of the algebra $\mathbb{A}_c$ is, in general, a subset of the associative center, it is true that

$$\mathbf{r}_i\mathbf{q}_n \cdot \mathbf{r}_j\mathbf{q}_m = \mathbf{r}_i\mathbf{r}_j \cdot \mathbf{q}_n\mathbf{q}_m.$$

This means that the algebra with the central conjugation $\mathbb{A}_c$ represent by itself an exterior (tensor) multiplication of it's center $\mathbb{Z}$ by some sub-algebra $\mathbb{A}_0$. Or, in other words, it is an algebra $\mathbb{A}_0$ over it's center $\mathbb{Z}$.

As a matter of principle, all argumentation of this article is valid even in case of more weak condition that the real (invariant) elements of the conjugation belong to the alternative center of an algebra, rather than the associative one. In this case the algebra $\mathbb{A}_c$ can not be reduced to an exterior multiplication of it's center $\mathbb{Z}$ by the sub-algebra $\mathbb{A}_0$. Still it is not clear enough whether such extension is informal indeed. Everywhere in this clause it is assumed that algebras possess a unity and they are given above a field of the characteristic 0.

Mono-associativity of algebras central with central conjunction

So then, for commutator $[\mathbf{a},\ \mathbf{b}]$ of the elements of algebras $\mathbb{A}_c$ with the central conjugation (including quaternions, biquaternions, diquaternions) it is true that:

$$[\mathbf{a}, \mathbf{r}] = 0, \quad \text{где} \quad \mathbf{r} = \bar{\mathbf{r}}, \tag{26}$$

($\bar{\mathbf{a}} \equiv C(a)$ – the algebra's besic conjugation $\mathbb{A}_c$) and in particular

$$[\mathbf{a}, \mathbf{b} + \bar{\mathbf{b}}] = 0, \qquad [\mathbf{a}, \mathbf{b}\bar{\mathbf{b}}] = 0 \tag{27}$$

from (27) it immediately follows

$$[\mathbf{a}, \mathbf{b}] = [\bar{\mathbf{b}}, \mathbf{a}], \tag{28}$$

since that

$$[\mathbf{a}, \bar{\mathbf{a}}] = [\mathbf{a}, \mathbf{a}] = 0, \qquad \text{that is} \qquad \mathbf{a}\bar{\mathbf{a}} = \bar{\mathbf{a}}\mathbf{a}. \tag{29}$$

And so the following is true:

Lemma 1. *In the algebras with central conjugation the right and the left 2-norms coincide.*

$$N_l(\mathbf{a}) \equiv \bar{\mathbf{a}}\mathbf{a} = N_r(\mathbf{a}) \equiv \mathbf{a}\bar{\mathbf{a}}. \tag{30}$$

Since that the following is true too:

$$\mathbf{a} \cdot \bar{\mathbf{a}}\mathbf{a} = \mathbf{a} \cdot \mathbf{r} = \mathbf{r} \cdot \mathbf{a} = \bar{\mathbf{a}}\mathbf{a} \cdot \mathbf{a} = \mathbf{a}\bar{\mathbf{a}} \cdot \mathbf{a} \qquad (\mathbf{r} \equiv \bar{\mathbf{a}}\mathbf{a} \in \text{algebra's center}), \tag{31}$$

or, briefly

$$\mathbf{a}\cdot\bar{\mathbf{a}}\mathbf{a} = \mathbf{a}\bar{\mathbf{a}}\cdot\mathbf{a}. \tag{32}$$

The straight consequence of the lemma 1 is the power associativity (mono-associativity) of algebras $\mathbb{A}_c$ – the associativity of the sub-algebra consisting of all possible degrees of one element ($\mathbf{a}^k\cdot\mathbf{a}^n = \mathbf{a}^{n+k}$). Indeed, as is known, the algebra is mono-associative, if two following identities are true:

$$\mathbf{a}\cdot\mathbf{a}\mathbf{a} = \mathbf{a}\mathbf{a}\cdot\mathbf{a} \qquad \text{и} \quad \mathbf{a}\mathbf{a}\cdot\mathbf{a}\mathbf{a} = (\mathbf{a}\mathbf{a}\cdot\mathbf{a})\mathbf{a}. \tag{33}$$

The first one of them immediately follows from (32), since any element $\mathbf{a}$ of the algebra $\mathbb{A}_c$ may be represented as a sum of the imaginary and real (invariant) relative to the basic conjugation parts $\mathbf{q}$ и $\mathbf{s}$:

$$\mathbf{a} = 1/2(\mathbf{a}-\bar{\mathbf{a}}) + 1/2(\mathbf{a}+\bar{\mathbf{a}}) = \mathbf{q}+\mathbf{s}; \quad C(\mathbf{q}+\mathbf{s}) = -\mathbf{q}+\mathbf{s}$$

and the multiplication of the real elements is associative ($\mathbf{a}\cdot\mathbf{s}\mathbf{a} = \mathbf{a}\mathbf{s}\cdot\mathbf{a}$). In the same way we may easy see that:

$$\mathbf{a}\bar{\mathbf{a}}\cdot\mathbf{a}\mathbf{a} = \mathbf{r}\cdot\mathbf{a}\mathbf{a} = \mathbf{r}\mathbf{a}\cdot\mathbf{a} = (\mathbf{a}\bar{\mathbf{a}}\cdot\mathbf{a})\mathbf{a},$$

The second property (33) follows from here since $\mathbf{s} = \Re(\mathbf{a})$ belongs to the associative center. Thus, we have just proved the

Theorem 1. *Each algebra with the central conjugation is mono-associative.*

And at the same time the

Theorem 1b. *Any algebra, possessing a complied with the algebra square norm, is mono-associative.*

In spite these results are simple (using the analysis of the acceptable conjugations) provable, they seem to be new.

Other associative properties of the algebras $\mathbb{A}_c$

To simplify the farther calculations shall we introduce the associator:

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\} \equiv (\mathbf{a}\mathbf{b})\mathbf{c} - \mathbf{a}(\mathbf{b}\mathbf{c}). \tag{34}$$

It is obviously linear by each element. In the associative algebras the associator identically equal zero. In the alternative ones it is anti-symmetrical by each argument.

$$\{\mathbf{a},\mathbf{b},\mathbf{b}\} = 0 \quad \Leftrightarrow \quad \{\mathbf{a},\mathbf{b},\mathbf{c}\} = -\{\mathbf{a},\mathbf{c},\mathbf{b}\} \qquad \text{(right alternativity)}, \tag{35}$$

$$\{\mathbf{a},\mathbf{a},\mathbf{b}\} = 0 \quad \Leftrightarrow \quad \{\mathbf{a},\mathbf{b},\mathbf{c}\} = -\{\mathbf{b},\mathbf{a},\mathbf{c}\} \qquad \text{(right alternativity)}. \tag{36}$$

As a consequence, the associator is cyclic:

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\} = \{\mathbf{c},\mathbf{a},\mathbf{b}\}. \tag{37}$$

The alternative algebras are obviously elastic:

$$\mathbf{a}\cdot\mathbf{b}\mathbf{a} = \mathbf{a}\mathbf{b}\cdot\mathbf{a}, \qquad \text{since in them} \tag{38}$$

$$\{\mathbf{a},\mathbf{b},\mathbf{a}\} = -\{\mathbf{a},\mathbf{a},\mathbf{b}\} = 0.$$

But the opposite is wrong: not all elastic algebras are associative. Note that the elasticity property is equivalent to:

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\} = -\{\mathbf{c},\mathbf{b},\mathbf{a}\}. \tag{39}$$

It is easy to demonstrate that any algebra obtained by the Cayley-Dickson duplication (7) from an elastic algebra is elastic too. Thus, the whole infinite chain of the Cayley-Dickson algebras, beginning from the real numbers, is elastic. As is easy to show and as is known, the following identities are true in every elastic algebra:

$$\{\mathbf{a},\mathbf{b},\mathbf{bc}\} = \{\mathbf{a},\mathbf{b},\mathbf{c}\}\mathbf{b}, \qquad \{\mathbf{a},\mathbf{b},\mathbf{cb}\} = \mathbf{b}\{\mathbf{a},\mathbf{b},\mathbf{c}\}, \tag{40}$$

From which the famous Moufang identities follow:

$$\begin{aligned} (\mathbf{ab}\cdot\mathbf{c})\mathbf{b} = \mathbf{a}(\mathbf{bcb}) \quad &- \text{ right},\\ (\mathbf{aba})\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{ac}) \quad &- \text{ left},\\ \mathbf{ab}\cdot\mathbf{ca} = \mathbf{a}(\mathbf{bc})\mathbf{a} \quad &- \text{ central}. \end{aligned} \tag{41}$$

By means of these correlations and their linearizations we may easily prove the Artin's theorem: in the alternative algebra any two elements generate an associative subalgebra.(During the linearization we make the identities linear by each variable. For this we change the repeating element $\mathbf{a}$ with, for example, $\mathbf{a}+\mathbf{d}$. After that we reduce the known identities with the repeating variables.)

We may demonstrate that for the central algebras the elasticity is equivalent to the important for physical applications property : *the jordanity*:

$$\mathbf{a}^2\mathbf{b}\cdot\mathbf{a} = \mathbf{a}^2\cdot\mathbf{ba} \qquad \text{или} \quad \{\mathbf{a}^2,\mathbf{b},\mathbf{a}\} = 0, \tag{42}$$

But, in general, the jordanity is stronger (narrower) than the elasticity.

For the algebras with central conjugation the alternativity is equivalent to the useful for calculations property (let us designate it *the conjugated alternativity*):

$$\mathbf{a}\cdot\bar{\mathbf{b}}\mathbf{b} = \mathbf{a}\bar{\mathbf{b}}\cdot\mathbf{b} \qquad \text{and at the same time} \quad \mathbf{a}\cdot\bar{\mathbf{a}}\mathbf{b} = \mathbf{a}\bar{\mathbf{a}}\cdot\mathbf{b}, \tag{43}$$

which means

$$\{\mathbf{a},\bar{\mathbf{b}},\mathbf{b}\} = 0 \qquad \text{и} \quad \{\mathbf{a},\bar{\mathbf{a}},\mathbf{b}\} = 0, \tag{44}$$

or in other words

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\} = -\{\mathbf{a},\bar{\mathbf{c}},\bar{\mathbf{b}}\} \qquad \text{и} \quad \{\mathbf{a},\mathbf{b},\mathbf{c}\} = -\{\bar{\mathbf{b}},\bar{\mathbf{a}},\mathbf{c}\}. \tag{45}$$

Note that the conjugated alternativity is in generally stronger than the alternativity: if the real elements belong to the commutative, but not associative center of the algebra, the conjugated alternativity means the alternativity, but not vice versa.

Let us sum up all the aforesaid in the form of the associative properties hierarchy (the upper levels mean the bottom ones, but not vice versa) associativity

associativity:	$\mathbf{a}\cdot\mathbf{bc} = \mathbf{ab}\cdot\mathbf{c}$		$\{\mathbf{a},\mathbf{b},\mathbf{c}\} = 0;$
alternativity:	$\mathbf{a}\cdot\mathbf{bb} = \mathbf{ab}\cdot\mathbf{b}$	$\{\mathbf{a},\mathbf{b},\mathbf{c}\} = -\{\mathbf{a},\mathbf{c},\mathbf{b}\}$	$\{\mathbf{a},\mathbf{b},\mathbf{b}\} = 0$
together with	$\mathbf{a}\cdot\mathbf{ab} = \mathbf{aa}\cdot\mathbf{b}$	$\{\mathbf{a},\mathbf{b},\mathbf{c}\} = -\{\mathbf{b},\mathbf{a},\mathbf{c}\}$	$\{\mathbf{a},\mathbf{a},\mathbf{b}\} = 0;$
jordanity:	$\mathbf{a}^2\cdot\mathbf{ba} = \mathbf{a}^2\mathbf{b}\cdot\mathbf{a}$		$\{\mathbf{a}^2,\mathbf{b},\mathbf{a}\} = 0;$
elastisity:	$\mathbf{a}\cdot\mathbf{ba} = \mathbf{ab}\cdot\mathbf{a}$	$\{\mathbf{a},\mathbf{b},\mathbf{c}\} = -\{\mathbf{c},\mathbf{b},\mathbf{a}\}$	$\{\mathbf{a},\mathbf{b},\mathbf{a}\} = 0;$
monoassociativity:	$\mathbf{a}^n\cdot\mathbf{a}^m = \mathbf{a}^{n+m}$	$\{\mathbf{a}^2,\mathbf{a},\mathbf{a}\} = 0$	$\{\mathbf{a},\mathbf{a},\mathbf{a}\} = 0;$

At last it is important for the farther that the associator of an arbitrary element of the algebra's center (in case of the $\mathbb{A}_c$ algebra - of a real element) with any element equal zero.

$$\{\mathbf{r}, \mathbf{a}, \mathbf{b}\} = 0. \tag{46}$$

The multiplicativity of the $\mathbb{A}_c$-algebra's 2-norm

The existence of the central conjugation (property (26)) in the $\mathbb{A}_c$ algebras together with the condition of their alternativity leads to the wonderful property of the multiplicativity (complex, binary etc.) of the 2-norm in these algebras. (The norm of the product equal the product of the norms.) Here is the line of reasoning (using the the property of the conjugated alternativity):

$$N_2(\mathbf{ab}) = \mathbf{ab} \cdot \bar{\mathbf{b}}\bar{\mathbf{a}} = \mathbf{a}(\mathbf{b}\bar{\mathbf{b}} \cdot \bar{\mathbf{a}}) = \mathbf{a}(\bar{\mathbf{a}} \cdot \mathbf{b}\bar{\mathbf{b}}) = \mathbf{a}\bar{\mathbf{a}} \cdot \mathbf{b}\bar{\mathbf{b}} = N_2(\mathbf{a})N_2(\mathbf{b}). \tag{47}$$

The proof is based on the property $\mathbf{ab} \cdot \bar{\mathbf{b}}\bar{\mathbf{a}} = \mathbf{a}(\mathbf{b}\bar{\mathbf{b}} \cdot \bar{\mathbf{a}})$. It is obviously fulfilled in the alternative (therefore conjugated alternative) algebras. Indeed,

$$\mathbf{ab} \cdot \bar{\mathbf{b}}\bar{\mathbf{a}} - \mathbf{a}(\mathbf{b}\bar{\mathbf{b}} \cdot \bar{\mathbf{a}}) = \mathbf{ab} \cdot \bar{\mathbf{b}}\bar{\mathbf{a}} - \mathbf{a}(\mathbf{b} \cdot \bar{\mathbf{b}}\bar{\mathbf{a}}) = \{\mathbf{a}, \mathbf{b}, \bar{\mathbf{b}}\bar{\mathbf{a}}\}.$$

Due to the conjugated alternativity, this expression equal

$$\{\mathbf{a}, \mathbf{b}, \bar{\mathbf{b}}\bar{\mathbf{a}}\} = -\{\mathbf{a}, \mathbf{ab}, \bar{\mathbf{b}}\} = (\mathbf{a} \cdot \mathbf{ab})\bar{\mathbf{b}} - \mathbf{a}(\mathbf{ab} \cdot \bar{\mathbf{b}}) =$$
$$= (\mathbf{aa} \cdot \mathbf{b})\bar{\mathbf{b}} - \mathbf{a}(\mathbf{a} \cdot \mathbf{b}\bar{\mathbf{b}}) = \mathbf{aa} \cdot \mathbf{b}\bar{\mathbf{b}} - \mathbf{a}(\mathbf{a} \cdot \mathbf{b}\bar{\mathbf{b}}) = \{\mathbf{a}, \mathbf{a}, \mathbf{b}\bar{\mathbf{b}}\} = 0.$$

And so, we have prooved

Theorem 2. *Each alternative algebra with central conjugation possess a multiplicative (in general not real) 2-norm* $N_2 = \mathbf{a}\bar{\mathbf{a}} = \bar{\mathbf{a}}\mathbf{a}$.

Since it is enough for algebra to be alternative, rather than associative, the octave (bi-, di-, bibi-, didi- etc.) algebras posses a multiplicative 2-norm (complex, binary etc.) a well as the quaternion (bi-, di-, bibi-, didi- etc.) algebras.

But what can we say in case of an arbitrary non-alternative algebra with central conjugation? It is easy to prove the following

Theorem 3. *In any algebra with central conjugation the 2-norm of the square of the element equal the square of the element's 2-norm :*

$$N_2(\mathbf{aa}) = N_2(\mathbf{a})N_2(\mathbf{a}). \tag{48}$$

Since so we have the

In any algebra with natural square norm the 2-norm of the square of the element equal the square of the element's 2-norm

Theorem 3b. *In any algebra with natural square norm the 2-norm of the square of the element equal the square of the element's 2-norm.*

Indeed, using the power associativity of the algebras $\mathbb{A}_c$:

$$N_2(\mathbf{aa}) = \mathbf{aa} \cdot \bar{\mathbf{a}}\bar{\mathbf{a}} = (\mathbf{a} \cdot \mathbf{a}\bar{\mathbf{a}})\bar{\mathbf{a}} = (\mathbf{a} \cdot \bar{\mathbf{a}}\mathbf{a})\mathbf{a} = (\mathbf{a}\bar{\mathbf{a}} \cdot \mathbf{a})\bar{\mathbf{a}} = \mathbf{a}\bar{\mathbf{a}} \cdot \mathbf{a}\bar{\mathbf{a}} = N_2(\mathbf{a})N_2(\mathbf{a}). \ \square$$

And again, in spite this result is so easy obtained with our conjugation analysis method, it seem to be new.

2-scalar and 2-vector product in $\mathbb{A}_c$

Let us obtain some auxiliary results for first.

Lemma 2. *For the conjugation of the associator it is true:*

$$\overline{\{\mathbf{a},\mathbf{b},\mathbf{c}\}} = -\{\bar{\mathbf{c}},\bar{\mathbf{b}},\bar{\mathbf{a}}\} \tag{49}$$

Indeed,

$$\overline{\{\mathbf{a},\mathbf{b},\mathbf{c}\}} = C((\mathbf{ab}\cdot\mathbf{c} - \mathbf{a}\cdot\mathbf{bc})) = \bar{\mathbf{c}}\cdot\bar{\mathbf{b}}\bar{\mathbf{a}} - \bar{\mathbf{c}}\bar{\mathbf{b}}\cdot\bar{\mathbf{a}} = -\{\bar{\mathbf{c}},\bar{\mathbf{b}},\bar{\mathbf{a}}\}.$$

Using lemma 2 it is easy to prove the following two theorems:

Theorem 4. *There are merely imaginary associators in the elastic algebras.*

Indeed, according to (46), the associator of elements $\mathbf{a},\mathbf{b},\mathbf{c}$ in $\mathbb{A}_c$ algebras may be reduced to the associator of merely imaginary elements $\mathbf{p},\mathbf{q},\mathbf{g}$. Since so,

$$C(\{\mathbf{a},\mathbf{b},\mathbf{c}\}) = C(\{\mathbf{p},\mathbf{q},\mathbf{g}\}) = -\{\bar{\mathbf{g}},\bar{\mathbf{q}},\bar{\mathbf{p}}\} = +\{\mathbf{g},\mathbf{q},\mathbf{p}\} = -\{\mathbf{p},\mathbf{q},\mathbf{g}\} = -\{\mathbf{a},\mathbf{b},\mathbf{c}\}. \quad \square$$

Theorem 5. *In each elastic algebra with central conjugation the multiplication of three elements under the badge of real part is associative.*:

$$\Re(\mathbf{ab}\cdot\mathbf{c}) = \Re(\mathbf{a}\cdot\mathbf{bc}). \tag{50}$$

(Note it is not correct for an arbitrary quantity of elements.)

Indeed, this equality means that

$$\Re(\mathbf{ab}\cdot\mathbf{c} - \mathbf{a}\cdot\mathbf{bc}) = 0, \qquad \text{i.e.} \quad \Re(\{\mathbf{a},\mathbf{b},\mathbf{c}\}) = 0,$$

Which is true for the elastic algebras due to the theorem 4. □

Next, applying (28) two times we have:

$$[\mathbf{a},\mathbf{b}] = [\bar{\mathbf{a}},\bar{\mathbf{b}}]. \tag{51}$$

This means that the commutator of two elements of $\mathbb{A}_c$ algebra is neatly imaginary relative to the central conjugation of this algebra:

$$\overline{[\mathbf{a},\mathbf{b}]} = \overline{(\mathbf{ab}-\mathbf{ba})} = \bar{\mathbf{b}}\bar{\mathbf{a}} - \bar{\mathbf{a}}\bar{\mathbf{b}} = -(\bar{\mathbf{a}}\bar{\mathbf{b}} + \bar{\mathbf{b}}\bar{\mathbf{a}}) = -[\bar{\mathbf{a}},\bar{\mathbf{b}}] = -[\mathbf{a},\mathbf{b}].$$

However, the anti-commutator $\{\mathbf{a},\mathbf{b}\} = \mathbf{ab}+\mathbf{ba}$ is not necessary real.

From (51) follows the

Theorem 6. *A cyclic permutation under the badge of real part in elastic algebras with central conjugation is possible.*

Indeed,

$$\mathbf{ab} - \mathbf{ba} = \bar{\mathbf{a}}\bar{\mathbf{b}} - \bar{\mathbf{b}}\bar{\mathbf{a}} \quad \Rightarrow \quad \mathbf{ab} + \bar{\mathbf{b}}\bar{\mathbf{a}} = \bar{\mathbf{a}}\bar{\mathbf{b}} + \mathbf{ba},$$

i.e.

$$\Re(\mathbf{ab}) = \Re(\bar{\mathbf{a}}\bar{\mathbf{b}}) = \Re(\overline{\mathbf{ba}}) = \Re(\mathbf{ba}), \tag{52}$$

and as a consequence, due to the theorem 5

$$\Re(\mathbf{a}\cdot\mathbf{bc}) = \Re(\mathbf{ab}\cdot\mathbf{c}) = \Re(\mathbf{c}\cdot\mathbf{ab}) = \Re(\mathbf{ca}\cdot\mathbf{b}) = \Re(\mathbf{b}\cdot\mathbf{ca}). \ \square \tag{53}$$

Let us introduce *the left and the right 2-scalar product* with the same formulas as in case of quaternions and octaves. It is significant, that such product may be a non-real number (because it does not posses the advantage of the algebraic compliance).

$$(\mathbf{a},\mathbf{b})_p \equiv \Re(\mathbf{a}\bar{\mathbf{b}}) = 1/2(\mathbf{a}\bar{\mathbf{b}} + \mathbf{b}\bar{\mathbf{a}})$$
$$(\mathbf{a},\mathbf{b})_l \equiv \Re(\bar{\mathbf{a}}\mathbf{b}) = 1/2(\bar{\mathbf{a}}\mathbf{b} + \bar{\mathbf{b}}\mathbf{a}). \tag{54}$$

Obviously,

$$(\mathbf{a},\mathbf{b}) = (\mathbf{b},\mathbf{a}) \qquad \text{и} \qquad (\bar{\mathbf{a}},\bar{\mathbf{b}})_p = (\mathbf{a},\mathbf{b})_l.$$

From (52) it follows

$$(\mathbf{a},\mathbf{b})_p = \Re(\mathbf{a}\bar{\mathbf{b}}) = \Re(\bar{\mathbf{a}}\mathbf{b}) = (\bar{\mathbf{a}},\bar{\mathbf{b}})_p = (\mathbf{a},\mathbf{b})_l.$$

In other words,*the left and the right 2-scalar products coincide for $\mathbb{A}_c$ algebras.*. Furthermore,

$$(\mathbf{a},\mathbf{b}) = (\bar{\mathbf{a}},\bar{\mathbf{b}}). \tag{55}$$

The 2-norm and the 2-scalar product are related with a simple formula:

$$N(\mathbf{a}+\mathbf{b}) = N(\mathbf{a}) + N(\mathbf{b}) + 2(\mathbf{a},\mathbf{b}), \tag{56}$$

and consequently,

$$(\mathbf{a},\mathbf{b}) = \frac{1}{2}\big(N(\mathbf{a}+\mathbf{b}) - N(\mathbf{a}) - N(\mathbf{b})\big), \tag{57}$$

and in particular,

$$(\mathbf{a},\mathbf{a}) = N(\mathbf{a}), \tag{58}$$

That is why the coincidence of the left and the right 2-norms and the coincidence of the left and the right 2-scalar products are directly related.

Lemma 3 The following is true for the alternative algebras with central conjugation.

$$(\mathbf{ab},\mathbf{cd}) + (\mathbf{ad},\mathbf{cb}) = 2(\mathbf{a},\mathbf{c})(\mathbf{b},\mathbf{d}), \tag{59}$$

$$(\mathbf{ab},\mathbf{cb}) = (\mathbf{a},\mathbf{c})(\mathbf{b},\mathbf{b}) = (\mathbf{a},\mathbf{c})N_2(\mathbf{b}). \tag{60}$$

These formulas are the more general expression of the property of multiplicativity of the $\mathbb{A}_c$ algebras norm. Since so they may be obtained by the linearization of the latter. (We replace $\mathbf{a}$ and $\mathbf{b}$ with $\mathbf{a}+\mathbf{c}$ and $\mathbf{b}+\mathbf{d}$ and reduce the identities of the multiplicativity of the norm by each variable in

$$N_2(\mathbf{ab}) = N_2(\mathbf{a})N_2(\mathbf{b}), \qquad \text{i.e.} \qquad (\mathbf{ab},\mathbf{ab}) = (\mathbf{a},\mathbf{a})(\mathbf{b},\mathbf{b}).$$

Now, as is customary, let us introduce *the left and the right vector products*

$$\langle\mathbf{a},\mathbf{b}\rangle_l \equiv \Im(\bar{\mathbf{a}}\mathbf{b}) = 1/2(\bar{\mathbf{a}}\mathbf{b} - \bar{\mathbf{b}}\mathbf{a})$$
$$\langle\mathbf{a},\mathbf{b}\rangle_p \equiv \Im(\mathbf{a}\bar{\mathbf{b}}) = 1/2(\mathbf{a}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{a}}) \tag{61}$$

Obviously:

$$\langle\mathbf{b},\mathbf{a}\rangle = -\langle\mathbf{a},\mathbf{b}\rangle, \qquad \langle\mathbf{a},\mathbf{a}\rangle = 0, \quad \langle\mathbf{a},\mathbf{b}\rangle_p = \langle\bar{\mathbf{a}},\bar{\mathbf{b}}\rangle_l. \tag{62}$$

Note, for vector products

$$\langle\mathbf{a},\mathbf{b}\rangle \neq \langle\bar{\mathbf{a}},\bar{\mathbf{b}}\rangle \qquad \text{or} \qquad \langle\mathbf{a},\mathbf{b}\rangle_p \neq \langle\mathbf{a},\mathbf{b}\rangle_l.$$

Instead, due to (46), in the elastic algebras with central conjugation:

$$\langle \mathbf{a}, \mathbf{b} \rangle^2 = \langle \bar{\mathbf{a}}, \bar{\mathbf{b}} \rangle^2, \qquad \text{i. e.} \quad \langle \mathbf{a}, \mathbf{b} \rangle_p^2 = \langle \mathbf{a}, \mathbf{b} \rangle_l^2, \tag{63}$$

thus, the following equality is wrong:

$$(\langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{c}, \mathbf{d} \rangle)^2 \neq (\langle \bar{\mathbf{a}}, \bar{\mathbf{b}} \rangle + \langle \bar{\mathbf{c}}, \bar{\mathbf{d}} \rangle)^2.$$

It is easy to show, that there is a formula, similar with (60) but for vector product. This is related with the multiplicativity of $\mathbb{A}_c$ alternative algebras.

$$\langle \mathbf{ca}, \mathbf{da} \rangle^2 = N_2^2(\mathbf{a}) \langle \mathbf{c}, \mathbf{d} \rangle^2 = \langle \mathbf{ac}, \mathbf{ad} \rangle^2. \tag{64}$$

We may strengthen this formula in the associative algebras $\mathbb{A}_c$ to:

$$\langle \mathbf{ca}, \mathbf{da} \rangle_p = N_2(\mathbf{a}) \langle \mathbf{c}, \mathbf{d} \rangle_p \qquad \text{и} \qquad \langle \mathbf{ac}, \mathbf{ad} \rangle_l = N_2(\mathbf{a}) \langle \mathbf{c}, \mathbf{d} \rangle_l. \tag{65}$$

In the elastic algebras$\mathbb{A}_c$:

$$\langle \mathbf{b}, \mathbf{a} \rangle_p \, \mathbf{a} = \mathbf{a} \langle \bar{\mathbf{a}}, \bar{\mathbf{b}} \rangle_p \qquad \text{и} \qquad \mathbf{a} \langle \mathbf{b}, \mathbf{a} \rangle_l = \langle \bar{\mathbf{a}}, \bar{\mathbf{b}} \rangle_l \, \mathbf{a}. \tag{66}$$

In the alternative algebras $\mathbb{A}_c$ the following identity is correct:

$$N_2(\mathbf{a}) N_2(\mathbf{b}) = (\mathbf{a}, \mathbf{b})^2 - \langle \mathbf{a}, \mathbf{b} \rangle^2. \tag{67}$$

Farther, if there is no index p or $l$pointed, we consider the vector product is right.

Geometric aspect of the associative properties: concerning the algebra directly related with the Minkowsky norm

We may easy demonstrate the importance of associative properties for geometry naturally generated by an algebra. The following example is simple, but vivid.

Let us ask ourselves, which hyper-complex algebra is appropriate for the Minkowsky metric:

$$s^2 = t^2 - x^2 - y^2 - z^2. \tag{68}$$

We may construct several fitting algebras, basing on the quaternion algebra. They belong to two classes, which properties some differ from each other. Let us consider two multiplication tables for example:

$\times$	1	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$
1	1	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$
$\mathbf{q}_1$	$\mathbf{q}_1$	1	$\mathbf{q}_3$	$-\mathbf{q}_2$
$\mathbf{q}_2$	$\mathbf{q}_2$	$-\mathbf{q}_3$	1	$\mathbf{q}_1$
$\mathbf{q}_3$	$\mathbf{q}_3$	$\mathbf{q}_2$	$-\mathbf{q}_1$	1

. (*Tab.* 5*a*)

$\times$	1	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$
1	1	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$
$\mathbf{q}_1$	$\mathbf{q}_1$	1	$-\mathbf{q}_3$	$-\mathbf{q}_2$
$\mathbf{q}_2$	$\mathbf{q}_2$	$\mathbf{q}_3$	1	$\mathbf{q}_1$
$\mathbf{q}_3$	$\mathbf{q}_3$	$\mathbf{q}_2$	$-\mathbf{q}_1$	1

. (*Tab.* 5*b*)

There is a conjugation in such algebras and it is given with the usual way (each ort is imaginary, excluding 1):

$$\bar{\mathbf{a}} = a_0 \cdot 1 - a_1 \mathbf{q}_1 - a_2 \mathbf{q}_2 - a_3 \mathbf{q}_3.$$

Since the real part of the argument is a real number, we have the algebras with central conjugation. The natural square norm is a real number too and does coincide with the Minkowsky's metric:

$$\mathbf{a}\bar{\mathbf{a}} = a_0^2 - a_1^2 - a_2^2 - a_3^2.$$

The concerned algebras are obviously non-associative and, furthermore, they are not alternative:

$$\mathbf{q}_3\mathbf{q}_3 \cdot \mathbf{q}_1 = \mathbf{q}_1, \qquad \text{but} \quad \mathbf{q}_3 \cdot \mathbf{q}_3\mathbf{q}_1 = \mathbf{q}_3(-\mathbf{q}_2) = -\mathbf{q}_1.$$

The second algebra is not even elastic (although, according the Theorem 1b it is mono-associative as an algebra with the natural square form). This fact is not as obvious as in case of the non-alternativity. The products of the orts will always satisfy the elasticity property (it is related with the fact that the square of an ort equal the real number ± 1). Using (46), we may reduce the elasticity to the elasticity of the multiplication of imaginary parts $\mathbf{p}, \mathbf{q}$ of the elements $\mathbf{a}, \mathbf{b}$:

$$\mathbf{q} \cdot \mathbf{pq} - \mathbf{qp} \cdot \mathbf{q} = 0 \qquad \text{or}$$

$$\mathbf{q} \cdot \mathbf{pq} + \overline{\mathbf{q} \cdot \mathbf{pq}} = 2\Re(\mathbf{q} \cdot \mathbf{pq}) = 0. \tag{69}$$

Thus, we have a useful elasticity criteria:

The algebra with central conjugation is elastic if the expression of imaginary elements $\mathbf{q} \cdot \mathbf{pq}$ *does not contain real members.*

And, as we may easy see, in the concerned algebra

$$\Re(\mathbf{q} \cdot \mathbf{pq}) = a_3(a_1b_2 - a_2b_1) \neq 0.$$

At last, as in all algebras with square norm, the norm of the square of an element equals the norm of the square:

$$N_2(\mathbf{aa}) = N_2(\mathbf{a})N_2(\mathbf{a}).$$

This identity may be checked directly. Since the norm

$$\mathbf{aa} = (a_0^2 + a_1^2 + a_2^2 + a_3^2) \cdot 1 + 2a_0a_1\mathbf{q}_1 + 2a_0a_2\mathbf{q}_2 + 2a_0a_3\mathbf{q}_3,$$

this identity may be written as

$$(a_0^2 - a_1^2 - a_2^2 - a_3^2)^2 = (a_0^2 + a_1^2 + a_2^2 + a_3^2)^2 - 4a_0a_1^2 - 4a_0a_2^2 - 4a_0a_3^2.$$

Now let's explore the firsthand geometric consequences of all these algebraic properties.

1)Since all these algebras are *non-associative*, the movement in them may not be given by means of inner automorphisms:

$$\mathbf{a}' = \mathbf{u} \cdot \mathbf{au}^{-1}, \tag{70}$$

Because the following, which is good for the associative algebras, does not work here:

$$\mathbf{a}' \cdot \mathbf{b}' = (\mathbf{u} \cdot \mathbf{au}^{-1}) \cdot (\mathbf{u} \cdot \mathbf{bu}^{-1}) = \mathbf{u} \cdot \mathbf{a} \cdot (\mathbf{u}^{-1} \cdot \mathbf{u}) \cdot \mathbf{bu}^{-1} = \mathbf{u} \cdot \mathbf{ab} \cdot \mathbf{u}^{-1} = (\mathbf{ab})'.$$

2) Since all these algebras are *non-alternative*, the movement in them may not be given by means of multiplication by the elements of a unit norm. The norm of the multiplication

in alternative algebras does not equal the multiplication of the norms, therefore the norm of the image does not equal the norm of the pre-image.

$$N_2(\mathbf{ea}) \neq N_2(\mathbf{e})N_2(\mathbf{a}) = N_2(\mathbf{a}).$$

Since the second algebra is *non-elastic*, we have the problem of defining the orthogonality of merely imaginary elements (which are supposed to be equivalent to the usual 3-d space vectors). Indeed, in the elastic algebras, according to (52) and (69)

3) Since the second algebra is *non-elastic*, we have the problem of defining the orthogonality of merely imaginary elements (which are supposed to be equivalent to the usual 3-d space vectors). Indeed, in the elastic algebras, according to (52) and (69)

$$\Re(\mathbf{qp}\cdot\mathbf{q}) = \Re(\mathbf{q}\cdot\mathbf{qp}) = 0.$$

Consequently,

$$0 = \Re(\mathbf{qp}\cdot\mathbf{q} + \mathbf{q}\cdot\mathbf{qp}) = \mathbf{qp}\cdot\mathbf{q} - \mathbf{q}\cdot\mathbf{pq} + \mathbf{q}\cdot\mathbf{qp} - \mathbf{pq}\cdot\mathbf{q} =$$
$$= \mathbf{q}(\mathbf{qp} - \mathbf{pq}) + (\mathbf{qp} - \mathbf{pq})\mathbf{q} = \mathbf{q} < \mathbf{p},\mathbf{q} > + < \mathbf{p},\mathbf{q} > \mathbf{q} = (\mathbf{q}, < \mathbf{q},\mathbf{p} >).$$

Thus, we have

Theorem 7. *In the elastic algebras with central conjugation the vector product of three arbitrary imaginary vectors is orthogonal to each one of them:*

$$(\mathbf{q}, < \mathbf{q},\mathbf{p} >) = 0. \qquad (71)$$

In the second algebra it is not so, therefore the notion of orthogonality is algebraically undefined.

All this shows that the demand to generate "the good geometries" is quite active constraint. The excessive moving away from the associativity leads to the superficial geometries.

2-norm, 2-scalar and 2-vector multiplication for the algebras of the biquaternions, diquaternions and bioctaves

2-scalar product for the biquaternions (upper sign) and di-quaternons (lower sign) is easy to calculate. For the orts it is

$$(\mathbf{q}_k, \mathbf{q}_s) = \delta_{ks} \quad (\mathbf{q}_k, \mathbf{i}_s) = \mathbf{i}_0\delta_{ks}, \quad (\mathbf{i}_k, \mathbf{i}_s) = \mp\delta_{ks}. \qquad (72)$$

In the component-wise and in the brief form the scalar product is:

$$(\mathbf{a},\mathbf{b}) = a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3 \mp (k_0l_0 + k_1l_1 + k_2l_2 + k_3l_3) +$$
$$+ (a_0l_0 + a_1l_1 + a_2l_2 + a_3l_3 + k_0b_0 + k_1b_1 + k_2b_2 + k_3b_3)\cdot\mathbf{i}_0, \qquad (73)$$

$$(\mathbf{a},\mathbf{b}) = (a,b) \mp (k,l) + \big((a,l) + (k,b)\big)\cdot\mathbf{i}_0. \qquad (74)$$

From here or basing on (20) it is easy to calculate the 2-norm $N_2(\mathbf{a}) = (\mathbf{a},\mathbf{a})$ of biquaternion and diquaternion algebras in the brief:

$$N_2(\mathbf{a}) = \mathbf{a}\bar{\mathbf{a}} = (a + k\cdot\mathbf{i}_0)(\bar{a} + \bar{k}\cdot\mathbf{i}_0) = a\bar{a} \mp k\bar{k} + (a\bar{k} + k\bar{a})\cdot\mathbf{i}_0, \quad \text{или}$$

$$N_2(\mathbf{a}) = N_2(a) \mp N_2(k) + 2(a,k)\cdot\mathbf{i}_0, \qquad (75)$$

and in the component-wise form:

$$N_2(\mathbf{a}) = a_0^2 + a_1^2 + a_2^2 + a_3^2 \ \mp (k_0^2 + k_1^2 + k_2^2 + k_3^2) \ + 2(a_0 k_0 + a_1 k_1 + a_2 k_2 + a_3 k_3) \cdot \mathbf{i}_0 \quad (76)$$

In the same way, the 2-scalar product for the bioctaves algebra is:

$$(\mathbf{q}_k, \mathbf{q}_s) = (\mathbf{e}_k, \mathbf{e}_s) = \delta_{ks} \quad (\mathbf{i}_k, \mathbf{i}_s) = (\mathbf{f}_k, \mathbf{f}_s) = \mp\delta_{ks}, \quad (\mathbf{q}_k, \mathbf{i}_s) = (\mathbf{e}_k, \mathbf{f}_s) = \mathbf{i}_0 \delta_{ks}. \quad (77)$$

The rest pairs of orts give 0. From here the 2-norm for the bioctave algebra in the brief::

$$\begin{aligned} N_2(\mathbf{a}) &= (a + A \cdot \mathbf{e}_0 + k \cdot \mathbf{i}_0 + K \cdot \mathbf{f}_0)(\bar{a} - A \cdot \mathbf{e}_0 + \bar{k} \cdot \mathbf{i}_0 - K \cdot \mathbf{f}_0) = \\ &= a\bar{a} + \bar{A}A - k\bar{k} - \bar{K}K + (a\bar{k} + k\bar{a} + \bar{K}A + \bar{A}K) \cdot \mathbf{i}_0 + 0 \cdot \mathbf{e}_0 + 0 \cdot \mathbf{f}_0 = \\ &= N_2(a) + N_2(A) - N_2(k) - N_2(K) + 2((a, k) + (A, K)) \cdot \mathbf{i}_0, \quad (78) \end{aligned}$$

and in the component-wise form:

$$\begin{aligned} N_2(\mathbf{a}) = a_0^2 + a_1^2 + a_2^2 + a_3^2 \ + A_0^2 + A_1^2 + A_2^2 + A_3^2 - k_0^2 - k_1^2 - k_2^2 - k_3^2 \ - K_0^2 - K_1^2 - K_2^2 - K_3^2 \\ + 2(a_0 k_0 + a_1 k_1 + a_2 k_2 + a_3 k_3 \ + A_0 K_0 + A_1 K_1 + A_2 K_2 + A_3 K_3) \cdot \mathbf{i}_0 \quad (79) \end{aligned}$$

Let us also give the 2-vector and for biquaternions and diquaternions in the brief form:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \langle a + \mathbf{i}_0 k, b + \mathbf{i}_0 l \rangle = \langle a, b \rangle \mp \langle k, l \rangle + (\langle a, l \rangle + \langle k, b \rangle)\mathbf{i}_0, \quad (80)$$

and in the component-wise form:

$$\begin{aligned} \langle \mathbf{a}, \mathbf{b} \rangle = \big(&-a_0 b_1 + a_1 b_0 - a_2 b_3 + a_3 b_2 \ \mp (-k_0 l_1 + k_1 l_0 - k_2 l_3 + k_3 l_2) \big) \cdot \mathbf{q}_1 + \\ \big(&-a_0 b_2 + a_1 b_3 + a_2 b_0 - a_3 b_1 \ \mp (-k_0 l_2 + k_1 l_3 + k_2 l_0 - k_3 l_1) \big) \cdot \mathbf{q}_2 + \\ \big(&-a_0 b_3 - a_1 b_2 + a_2 b_1 + a_3 b_0 \ \mp (-k_0 l_3 - k_1 l_2 + k_2 l_1 + k_3 l_0) \big) \cdot \mathbf{q}_3 + \\ &+ \big(-a_0 l_1 + a_1 l_0 - a_2 l_3 + a_3 l_2 \ - k_0 b_1 + k_1 b_0 - k_2 b_3 + k_3 b_2 \big) \cdot \mathbf{i}_1 + \\ &+ \big(-a_0 l_2 + a_1 l_3 + a_2 l_0 - a_3 l_1 \ - k_0 b_2 + k_1 b_3 + k_2 b_0 - k_3 b_1 \big) \cdot \mathbf{i}_2 + \\ &+ \big(-a_0 l_3 - a_1 l_2 + a_2 l_1 + a_3 l_0 \ - k_0 b_3 - k_1 b_2 + k_2 b_1 + k_3 b_0 \big) \cdot \mathbf{i}_3. \quad (81) \end{aligned}$$

Multi-norm and multi-scalar product for the $\mathbb{A}_c$ algebras

Which algebras do posses the multiplicative norm of the second degree? This question was thoroughly considered and solved in XIX-th century (see [3], [11], [12]).

According to the Gurvitz theorem, any algebra with unity and possessing a multiplicative positive-defined norm is isomorphous either to real or complex numbers, quaternions or octaves.

The generalized Frobenius theorem claims that any alternative algebra with division is isomorphous to one of the algebras from the same list

The generalized Frobenius theorem claims that any alternative algebra with division is isomorphous to one of the algebras from the same list. According to the Albert's theorem, the algebras with unity, which real elements are real numbers, and which posses the non-degenerated multiplicative square form, are the algebras of the complex and dual numbers, quaterinions and anti-quaternions, octaves and anti-octaves.

According to the Pontriagin's theorem, only the real and the complex numbers, quaternions and octaves are the associated locally compact alternative topological bodies.

Consequently, the spaces of biquaternions, diquaternions and bi-octaves are not simply connected.

Finally, as it follows from the Tsorn's theorem, the only simple alternative non-associative algebras are octaves, anti-octaves and bi-octaves.

In the 50-60ths year of XX century the problem of the multilicativity of more than 2 degree forms was stated and solved by R.D. Shafer [5] − [7] (with Kevin McCrimmon's supplements[8]).

The meaning of the forms of the n degree is following. Let us have the vector space V, possible infinite-dimensional, over the field F of the characteristic 0 or $p > n$ (real and complex numbers have the characteristic 0). Than the mapping $\mathbf{u} \to N(\mathbf{u})$ V to F is called *the form of the* n *degree* over V if

$$N(\lambda \mathbf{u}) = \lambda^n N(\mathbf{u}) \qquad \text{for any} \quad \lambda \in F,\ \mathbf{u} \in V.$$

The result of that time's researches is the following theorem.

May $\mathbb{U}$ – an algebra with unity, in general infinite-dimensional, over the field F of zero or $p > n$ characteristic. A non-degenerated form N permitting a composition over $\mathbb{U}$ do exist if and only if $\mathbb{U}$ is an alternative separable algebra $\mathbb{U} = \mathbb{U}_1 \oplus \ldots \oplus \mathbb{U}_r$, $\mathbb{U}_i$ – the simple algebras of the m_i degree, where

$$n = m_1 f_1 + \ldots + m_r f_r, \tag{82}$$

is satisfied for in case of the positive whole numbers $f_i (i = 1, \ldots, r)$. Furthermore, the form N over $\mathbb{U}$ is given with

$$N(\mathbf{u}) = [n_1(u_1)]^{f_1} \cdots [n_r(u_r)]^{f_r}, \tag{83}$$

, where $n_j(u_j)$ - a form over the simple algebra $\mathbb{U}_j$.

Sharfer's theorem. *May $\mathbb{U}$ – an algebra with unity, in general infinite-dimensional, over the field F of zero or $p > n$ characteristic. A non-degenerated form N permitting a composition over $\mathbb{U}$ do exist if and only if $\mathbb{U}$ is an alternative separable algebra $\mathbb{U} = \mathbb{U}_1 \oplus \ldots \oplus \mathbb{U}_r$, $\mathbb{U}_i$ – the simple algebras of the m_i degree, where*

$$n = m_1 f_1 + \ldots + m_r f_r, \tag{84}$$

is satisfied for in case of the positive whole numbers $f_i (i = 1, \ldots, r)$. Furthermore, the form N over $\mathbb{U}$ is given with

$$N(\mathbf{u}) = [n_1(u_1)]^{f_1} \cdots [n_r(u_r)]^{f_r}, \tag{85}$$

where $n_j(u_j)$ - a form over the simple algebra $\mathbb{U}_j$.

The notion of the *non-degenerated* norm of the n-degree is important in this theorem. Here is the meaning of that. To each n-norm we naturally connect the n-linear form of the *n-scalar product* of n hyper-complex numbers according to the Sharfer's formula:

The notion of the *non-degenerated* norm of the n-degree is important in this theorem. Here is the meaning of that. To each n-norm we naturally connect the n-linear form of the *n-scalar product* of n hyper-complex numbers according to the Sharfer's formula:

$$(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n) = \frac{1}{n!}\big[N(\mathbf{u}_1 + \ldots + \mathbf{u}_n) - \sum_{i=1}^{n} N(\mathbf{u}_1 + \ldots + \check{\mathbf{u}}_i + \ldots + \mathbf{u}_n)$$
$$+ \sum_{i<j} N(\mathbf{u}_1 + \ldots + \check{\mathbf{u}}_i + \ldots + \check{\mathbf{u}}_j + \ldots + \mathbf{u}_n) - \ldots + (-1)^{n-1} \sum N(\mathbf{u}_i)\big], \tag{86}$$

where $\breve{\mathbf{u}}_i$ means that $\mathbf{u}_i$ is left out. It is easy to see that

$$N_n(\mathbf{u}) = (\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u}), \qquad \text{since} \quad \sum_{k=0}^{n-1}(-1)^k C_n^k (n-k)^n = n!$$

It is obvious that the form $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$ has all properties, which may naturally be expected from the generalization of the scalar product. It is real, symmetric over each permutation of the involved vectors, linear over each one of them (and, in particular, it turns out to be zero, if one of the vectors is zero).

According to the Shafer's definition, the form of the n degree is called *non-degenerated* if from $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n) = 0$ for all $\mathbf{u}_2, \ldots, \mathbf{u}_n \in V$ follows $\mathbf{u}_1 = 0$.

The alternative algebras with central conjugation do satisfy Shafer theorem's condition (if their initial 2-norm is non-degenerated). Since so, the real non-degenerate multiplicative norm of the n degree exist in them. The Shafer's theorem does not provide us with the direct algorithm of constructing such norm in general case. But for the algebras with central conjugation this algorithm is clear enough, since we already know 2-norm, which is multiplicative, according to the foresaid. It is especially simple to construct n-norm if the concerned algebra may be obtained with the help of the chain of duplications (not necessary Cayley-Dickson's) from the real numbers, i. e. it is sequentially graduated and it has the 2^p dimensionality (almost all actually important hyper-complex algebras belong to this class). All we have to do is to get rid of the half of the algebra's center members using the conjugation set while duplicating the norm's degree. We use the following identity here: $(a^2 + b^2)(a^2 - b^2) = 0$ (a and b commute with each other, since they belong to the algebra's center). This process is to be repeated till there is only real number left. As a result, if the algebra's center consist of $r = 2^k$ elements, the norm of the algebra has the $n = 2r = 2^{k+1}$ degree.

The multiplicativity of the obtained n-norm follows from the 2-norm's multiplicativity. Let us prove it using the induction method. Indeed, it is true that

$$N_2(\mathbf{uv}) = N_2(\mathbf{u})N_2(\mathbf{v}),$$

may it be correct and for the $m = 2^k$ degree:

$$N_m(\mathbf{uv}) = N_m(\mathbf{u})N_m(\mathbf{v}).$$

But $N_m(\mathbf{u})$ in the $\mathbb{A}_c$ algebras belongs to the closed associative commutative algebra with unity, which is the sub-manifold of the algebra's center.

If this algebra (let us designate it +++) coincides with 1, the prove is complete. If no, we consider the algebra generated by 1 and some element r_1 of A_z. If we do not obtain A_z again, we add an element r_2 of A_z and consider the algebra constituted by all multiplications $1, r_1, r_2$ (with any real coefficients) an so on.

In the end we shall obtain the algebra A_z from the set $\{1, r_1, \ldots, r_z\}$, and from the set $\{1, r_1, \ldots, r_{z-1}\}$ we get the A_{z-1} – it's "half" sub-algebra. Each element of A_z may be written as

$$\mathbf{t} = \mathbf{s} + \mathbf{r}_z\mathbf{S},$$

where $\mathbf{s}, \mathbf{S}$ belongs to the sub-algebra A_{z-1}. Let us introduce the involution now:

$$C[\mathbf{s} + \mathbf{r}_z\mathbf{S}] = \mathbf{s} - \mathbf{r}_z\mathbf{S}, \qquad \text{и в частности} \quad C(\mathbf{r}_z) = -C(\mathbf{r}_z).$$

Than $C[\mathbf{r}_z^2] = \mathbf{r}_z^2$ and, considering the commutativity of A_z algebra, we obtain:

$$C[\mathbf{r}_1\mathbf{r}_2] = C[(\mathbf{s}_1 + \mathbf{r}_z\mathbf{S}_1)(\mathbf{s}_2 + \mathbf{r}_z\mathbf{S}_2)] = C[\mathbf{s}_1\mathbf{s}_2 + \mathbf{r}_z^2\mathbf{S}_1\mathbf{S}_2 + (\mathbf{s}_1\mathbf{S}_2 + \mathbf{s}_2\mathbf{S}_1)\mathbf{r}_z] = \\ \mathbf{s}_1\mathbf{s}_2 + \mathbf{r}_z^2\mathbf{S}_1\mathbf{S}_2 - (\mathbf{s}_1\mathbf{S}_2 + \mathbf{s}_2\mathbf{S}_1)\mathbf{r}_z = (\mathbf{s}_1 - \mathbf{r}_z\mathbf{S}_1)(\mathbf{s}_2 - \mathbf{r}_z\mathbf{S}_2) = C[\mathbf{r}_1]C[\mathbf{r}_2].$$

Now we may introduce the $2k$ degree norm according to the rule:

$$N_{2k}(\mathbf{u}) = N_k(\mathbf{u})C[N_k(\mathbf{u})]$$

and we may prove it's multiplicativity:

$$N_{2k}(\mathbf{uv}) = N_k(\mathbf{uv})C[N_k(\mathbf{uv})] = N_k(\mathbf{u})N_k(\mathbf{v})C[N_k(\mathbf{u})N_k(\mathbf{v})] = \\ N_k(\mathbf{u})N_k(\mathbf{v})C[N_k(\mathbf{u})]C[N_k(\mathbf{v})] = N_k(\mathbf{u})C[N_k(\mathbf{u})]N_k(\mathbf{v})C[N_k(\mathbf{v})] = N_{2k}(\mathbf{u})N_{2k}(\mathbf{v}).$$

so, the given method of the constructions of n-norm also proves the

Theorem 8. *Each alternative sequentially graduated algebra with the central conjugation and non-degenerated 2-norm possesses the non-degenerate multiplicative norm of n-th degree, which may be expressed with the help of the conjugations set via the initial 2-norm. At that the norm's dimensionality is twice higher then the dimensionality of the algebra's center (invariant relative to the basal conjugation) $n = 2r$.*

Since the n-norm is constructed on the ground of 2-norm, it automatically has many of 2-norm's properties. Do does the n-scalar product. Thus,

$$\begin{gathered} N_n(\bar{\mathbf{u}}) = N_n(\mathbf{u}), \\ (\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2, \ldots, \bar{\mathbf{u}}_n) = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n), \\ (\mathbf{vu}_1, \mathbf{vu}_2, \ldots, \mathbf{vu}_n) = N_n(\mathbf{v})(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n). \end{gathered} \tag{87}$$

4-norm for the biquaternions, diquaternions and bioctaves

It is especially easy to obtain a real norm of the biquaternions, diquaternions and bioctaves. It's of the fourth degree. The algorithm is obvious: 2-norm is a complex or double number; having multiplied it by the conjugated, we obtain a real number.

$$N_4(\mathbf{a}) = N_2(\mathbf{a})N_2(\mathbf{a})^* \tag{88}$$

Taking into consideration (24), we have:

$$N_4(\mathbf{a}) = \mathbf{a}\bar{\mathbf{a}}(\mathbf{a}\bar{\mathbf{a}})^* = \mathbf{a}\bar{\mathbf{a}}\mathbf{a}^*\bar{\mathbf{a}}^*. \tag{89}$$

In the block form for biquaternions and diquaternions:

$$N_4(a + k \cdot \mathbf{i}_0) = (N_2(a) \mp N_2(k))^2 \pm 4(a,k)^2, \tag{90}$$

and for the bioctaves:

$$N_4(a + A\cdot\mathbf{e}_0 + k\cdot\mathbf{i}_0 + K\cdot\mathbf{f}_0) = (N_2(a) + N_2(A) - N_2(k) - N_2(K))^2 + 4((a,k) + (A,K))^2. \tag{91}$$

In the component-wise form for biquaternions and diquaternions:

$$N_4(\mathbf{a}) = (a_0^2 + a_1^2 + a_2^2 + a_3^2 \ \mp (k_0^2 + k_1^2 + k_2^2 + k_3^2))^2 \ \pm 4(a_0k_0 + a_1k_1 + a_2k_2 + a_3k_3)^2, \tag{92}$$

And for bioctaves:

$$N_4(\mathbf{a}) = (a_0^2+a_1^2+a_2^2+a_3^2\ +A_0^2+A_1^2+A_2^2+A_3^2\ -k_0^2-k_1^2-k_2^2-k_3^2\ -K_0^2-K_1^2-K_2^2-K_3^2)^2 + 4(a_0k_0+a_1k_1+a_2k_2+a_3k_3\ +A_0K_0+A_1K_1+A_2K_2+A_3K_3)^2. \quad (93)$$

Obviously, all norms are non-negative. But they are not positive defined (and they can not be ones, due to the Frobenius theorem): $N_4(\mathbf{a}) = 0$ doesn't mean $\mathbf{a} = 0$. Indeed, in case of quaternions it is enough to take $a = A$ for the 4-norm to be zero:

$$N_4(a + a\cdot\mathbf{i}_0) = (N_2(a)+N_2(a))^2 - 4(a,a)^2 = 4N_2(a)^2 - 4N_2(a)^2 = 0.$$

For the biquaternions the situation is some more complex. To make the 4-norm we have to take A with the norm equal the norm of a, A has to be perpendicular to a also. For example: $a = 3\mathbf{i}_1 - 2\mathbf{i}_3,\ A = 2\mathbf{i}_0 + 3\mathbf{i}_2$.

The formula of the multiplicativity of the 4-norm of the bioctaves in the quaternionic form (using some simple, but some lengthy transforms, we may prove it directly) looks like:

$$\begin{gathered}\big(N_2(a)+N_2(A)-N_2(k)-N_2(K)\big)^2 + 4\big((a,k)+(A,K)\big)^2\cdot\\ \big(N_2(b)+N_2(B)-N_2(l)-N_2(L)\big)^2 + 4\big((b,l)+(B,L)\big)^2 =\\ \big(N_2(ab-\bar{B}A-kl+\bar{L}K)+N_2(Ba+A\bar{b}-Lk-K\bar{l})\\ -N_2(al-\bar{L}A+kb-\bar{B}K)-N_2(La+K\bar{b}+Bk+A\bar{l})\big)^2\\ +4\big((ab-\bar{B}A-kl+\bar{L}K,\ al-\bar{L}A+kb-\bar{B}K)\\ +(Ba+A\bar{b}-Lk-K\bar{l},\ La+K\bar{b}+Bk+A\bar{l})\big)^2\end{gathered} \quad (94)$$

We give here the identity of the multiplicativity of the 4-norm for bioctave algebra rather for the illustrative purpose.

$$\begin{gathered}\Big[(a_0^2+a_1^2+a_2^2+a_3^2\ +A_0^2+A_1^2+A_2^2+A_3^2\ -k_0^2-k_1^2-k_2^2-k_3^2\ -K_0^2-K_1^2-K_2^2-K_3^2)^2\\ +4(a_0k_0+a_1k_1+a_2k_2+a_3k_3\ +A_0K_0+A_1K_1+A_2K_2+A_3K_3)^2\Big]\cdot\\ \Big[(b_0^2+b_1^2+b_2^2+b_3^2\ +B_0^2+B_1^2+B_2^2+B_3^2\ -l_0^2-l_1^2-l_2^2-l_3^2\ -L_0^2-L_1^2-L_2^2-L_3^2)^2\\ +4(b_0l_0+b_1l_1+b_2l_2+b_3l_3\ +B_0L_0+B_1L_1+B_2L_2+B_3L_3)^2\Big] =\\ \Big[(a_0b_0-a_1b_1-a_2b_2-a_3b_3-A_0B_0-A_1B_1-A_2B_2-A_3B_3-k_0l_0+k_1l_1+k_2l_2+k_3l_3+K_0L_0+K_1L_1+K_2L_2+K_3L_3)^2\\ +(a_0b_1+a_1b_0+a_2b_3-a_3b_2+A_0B_1-A_1B_0-A_2B_3+A_3B_2-k_0l_1-k_1l_0-k_2l_3+k_3k_2-K_0L_1+K_1L_0+K_2L_3-K_3L_2)^2\\ +(a_0b_2-a_1b_3+a_2b_0+a_3b_1-k_0l_2+k_1l_3-k_2B_0-k_3B_1+A_0B_2+A_1B_3-A_2B_0-A_3B_1-K_0L_2-K_1L_3+K_2L_0+K_3L_1)^2\\ +(a_0b_3+a_1b_2-a_2b_1+a_3b_0-k_0l_3-k_1l_2+k_2l_1-k_3l_0+A_0B_3-A_1B_2+A_2B_1-A_3B_0-K_0L_3+K_1L_2-K_2L_1+K_3L_0)^2\\ +(a_0B_0-a_1B_1-a_2B_2-a_3B_3+A_0b_0+A_1b_1+A_2b_2+A_3b_3-k_0L_0+k_1L_1+k_2L_2+k_3L_3-K_0l_0-K_1l_1-K_2l_2-K_3l_3)^2\\ +(a_0B_1+a_1B_0-a_2B_3+a_3B_2-A_0b_1+A_1b_0-A_2b_3+A_3b_2-k_0L_1-k_1L_0+k_2L_3-k_3L_2+K_0l_1-K_1l_0+K_2l_3-K_3l_2)^2\\ +(a_0B_2+a_1B_3+a_2B_0-a_3B_1-A_0b_2+A_1b_3+A_2b_0-A_3b_1-k_0L_2-k_1L_3-k_2L_0+k_3L_1+K_0l_2-K_1l_3-K_2l_0+K_3l_1)^2\\ +(a_0B_3-a_1B_2+a_2B_1+a_3B_0-A_0b_3-A_1b_2+A_2b_1+A_3b_0-k_0L_3+k_1L_2-k_2L_1-k_3L_0+K_0l_3+K_1l_2-K_2l_1-K_3l_0)^2\\ -(a_0l_0-a_1l_1-a_2l_2-a_3l_3+k_0b_0-k_1b_1-k_2b_2-k_3b_3-A_0L_0-A_1L_1-A_2L_2-A_3L_3-K_0B_0-K_1B_1-K_2B_2-K_3B_3)^2\\ -(a_0l_1+a_1l_0+a_2l_3-a_3l_2+k_0b_1+k_1b_0+k_2b_3-k_3b_2+A_0L_1-A_1L_0-A_2L_3+A_3L_2+K_0B_1-K_1B_0-K_2B_3+K_3B_2)^2\\ -(a_0l_2-a_1l_3+a_2l_0+a_3l_1+k_0b_2-k_1b_3+k_2b_0+k_3b_1+A_0L_2+A_1L_3-A_2L_0-A_3L_1+K_0B_2+K_1B_3-K_2B_0-K_3B_1)^2\\ -(a_0l_3+a_1l_2-a_2l_1+a_3l_0+k_0b_3+k_1b_2-k_2b_1+k_3b_0+A_0L_3-A_1L_2+A_2L_1-A_3L_0+K_0B_3-K_1B_2+K_2B_1-K_3B_0)^2\\ -(a_0L_0-a_1L_1-a_2L_2-a_3L_3+A_0l_0+A_1l_1+A_2l_2+A_3l_3+k_0B_0-k_1B_1-k_2B_2-k_3B_3+K_0b_0+K_1b_1+K_2b_2+K_3b_3)^2\\ -(a_0L_1+a_1L_0-a_2L_3+a_3L_2-A_0l_1+A_1l_0-A_2l_3+A_3l_2+k_0B_1+k_1B_0-k_2B_3+k_3B_2-K_0b_1+K_1b_0-K_2b_3+K_3b_2)^2\\ -(a_0L_2+a_1L_3+a_2L_0-a_3L_1-A_0l_2+A_1l_3+A_2l_0-A_3l_1+k_0B_2+k_1B_3+k_2B_0-k_3B_1-K_0b_2+K_1b_3+K_2b_0-K_3b_1)^2\\ -(a_0L_3-a_1L_2+a_2L_1+a_3L_0-A_0l_3-A_1l_2+A_2l_1+A_3l_0+k_0B_3-k_1B_2+k_2B_1+k_3B_0-K_0b_3-K_1b_2+K_2b_1+K_3b_0)^2\Big]^2\end{gathered}$$

$$
\begin{aligned}
&+4\Big[(a_0b_0-a_1b_1-a_2b_2-a_3b_3\ -A_0B_0-A_1B_1-A_2B_2-A_3B_3-k_0l_0+k_1l_1+k_2l_2+k_3l_3\ +K_0L_0+K_1L_1+K_2L_2+K_3L_3)\cdot\\
&(a_0l_0-a_1l_1-a_2l_2-a_3l_3\ +k_0b_0-k_1b_1-k_2b_2-k_3b_3-A_0L_0-A_1L_1-A_2L_2-A_3L_3\ -K_0B_0-K_1B_1-K_2B_2-K_3B_3)\\
&+(a_0b_1+a_1b_0+a_2b_3-a_3b_2\ +A_0B_1-A_1B_0-A_2B_3+A_3B_2-k_0l_1-k_1l_0-k_2l_3+k_3k_2\ -K_0L_1+K_1L_0+K_2L_3-K_3L_2\ )\cdot\\
&(a_0l_1+a_1l_0+a_2l_3-a_3l_2\ +k_0b_1+k_1b_0+k_2b_3-k_3b_2+A_0L_1-A_1L_0-A_2L_3+A_3L_2\ +K_0B_1-K_1B_0-K_2B_3+K_3B_2)\\
&+(a_0b_2-a_1b_3+a_2b_0+a_3b_1\ -k_0l_2+k_1l_3-k_2B_0-k_3B_1+A_0B_2+A_1B_3-A_2B_0-A_3B_1\ -K_0L_2-K_1L_3+K_2L_0+K_3L_1)\cdot\\
&(a_0l_2-a_1l_3+a_2l_0+a_3l_1\ +k_0b_2-k_1b_3+k_2b_0+k_3b_1+A_0L_2+A_1L_3-A_2L_0-A_3L_1\ +K_0B_2+K_1B_3-K_2B_0-K_3B_1)\\
&+(a_0b_3+a_1b_2-a_2b_1+a_3b_0\ -k_0l_3-k_1l_2+k_2l_1-k_3l_0+A_0B_3-A_1B_2+A_2B_1-A_3B_0\ -K_0L_3+K_1L_2-K_2L_1+K_3L_0)\cdot\\
&(a_0l_3+a_1l_2-a_2l_1+a_3l_0\ +k_0b_3+k_1b_2-k_2b_1+k_3b_0+A_0L_3-A_1L_2+A_2L_1-A_3L_0\ +K_0B_3-K_1B_2+K_2B_1-K_3B_0)\\
&+(a_0B_0-a_1B_1-a_2B_2-a_3B_3+A_0b_0+A_1b_1+A_2b_2+A_3b_3-k_0L_0+k_1L_1+k_2L_2+k_3L_3-K_0l_0-K_1l_1-K_2l_2-K_3l_3)\cdot\\
&(a_0L_0-a_1L_1-a_2L_2-a_3L_3+A_0l_0+A_1l_1+A_2l_2+A_3l_3+k_0B_0-k_1B_1-k_2B_2-k_3B_3+K_0b_0+K_1b_1+K_2b_2+K_3b_3)\\
&+(a_0B_1+a_1B_0-a_2B_3+a_3B_2-A_0b_1+A_1b_0-A_2b_3+A_3b_2-k_0L_1-k_1L_0+k_2L_3-k_3L_2+K_0l_1-K_1l_0+K_2l_3-K_3l_2)\cdot\\
&(a_0L_1+a_1L_0-a_2L_3+a_3L_2-A_0l_1+A_1l_0-A_2l_3+A_3l_2+k_0B_1+k_1B_0-k_2B_3+k_3B_2-K_0b_1+K_1b_0-K_2b_3+K_3b_2)\\
&+(a_0B_2+a_1B_3+a_2B_0-a_3B_1-A_0b_2+A_1b_3+A_2b_0-A_3b_1-k_0L_2-k_1L_3-k_2L_0+k_3L_1+K_0l_2-K_1l_3-K_2l_0+K_3l_1)\cdot\\
&(a_0L_2+a_1L_3+a_2L_0-a_3L_1-A_0l_2+A_1l_3+A_2l_0-A_3l_1+k_0B_2+k_1B_3+k_2B_0-k_3B_1-K_0b_2+K_1b_3+K_2b_0-K_3b_1)\\
&+(a_0B_3-a_1B_2+a_2B_1+a_3B_0-A_0b_3-A_1b_2+A_2b_1+A_3b_0-k_0L_3+k_1L_2-k_2L_1-k_3L_0+K_0l_3+K_1l_2-K_2l_1-K_3l_0)\cdot\\
&(a_0L_3-a_1L_2+a_2L_1+a_3L_0-A_0l_3-A_1l_2+A_2l_1+A_3l_0+k_0B_3-k_1B_2+k_2B_1+k_3B_0-K_0b_3-K_1b_2+K_2b_1+K_3b_0)\Big]^2.
\end{aligned}
\tag{95}
$$

This elephant-like identity is a direct generalization of the famous eight squares identity. Furthermore, we assume this identity maximal and extraordinary. Since the bioctaves, according to the Tsorn's theorem, are the maximal simple alternative (non-associative) algebra, all the identities of the greater dimensionality are reducible to this one. Like the formulas of the squares sum, this identity reveals the existence of the bioctaves algebra, depite of the fact that it is random from the position of the real numbers.

The knowledge of the 4-norm allows us to introduce the *inverse element* for the bi(di)quaternions and bioctaves (and, after all, for any 4-norm algebra with central conjugation). Since

$$
N_4(\mathbf{a}) = N_2(\mathbf{a})N_2^*(\mathbf{a}) = \mathbf{a}\bar{\mathbf{a}}N_2^*(\mathbf{a}), \quad N_4(\bar{\mathbf{a}}) = \bar{\mathbf{a}}\mathbf{a}N_2^*(\bar{\mathbf{a}}), \quad N_4(\bar{\mathbf{a}}) = N_4(\mathbf{a}),
$$

then it is clear how to introduce the left and the right inverse element:

$$
\mathbf{a}_p^{-1} = \frac{\bar{\mathbf{a}}N_2^*(\mathbf{a})}{N_4(\mathbf{a})} = \frac{\bar{\mathbf{a}}\cdot\mathbf{a}^*\bar{\mathbf{a}}^*}{N_4(\mathbf{a})}, \quad \mathbf{a}_l^{-1} = \frac{N_2^*(\bar{\mathbf{a}})\bar{\mathbf{a}}}{N_4(\mathbf{a})} \equiv \mathbf{a}_p^{-1}. \tag{96}
$$

Since $N_2(\bar{\mathbf{a}}) = N_2(\mathbf{a})$, then *the left and the right inverse elements coincide in any algebra with central conjugation – with any degree of the norm, since the argumentation is simply generalizable.* the fact that (96) really give the inverse element:

$$
\mathbf{a}\mathbf{a}_p^{-1} = \mathbf{a}_l^{-1}\mathbf{a} = 1. \tag{97}
$$

follows from the associativity of the multiplication of real 2-norm and any element of the algebra. (the property (46)).

Thus, each element of algebra $\mathbb{A}_c$ with non-zero 4-norm has an inverse element.

Dual 4-norm for the biquaternons and diquaternons

The unexpected fact is that there is an absolutely different way to obtain the 4-norm for the biquaternons and diquaternons. In other words – to obtain a real number for an arbitrary element with the help of multiplications and conjugations.

If we take the dual conjugation $\widetilde{\mathbf{a}}$ (22) as a basal one instead of $\bar{\mathbf{a}}$, than there will be only invariant (real) members in the product $\mathbf{a}\widetilde{\mathbf{a}}$. These members are proportional to the orts $1, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$.

Since these ort anti-commutate with each other, we may multiply $\mathbf{a}\widetilde{\mathbf{a}}$ by it's complex conjugation $\mathbf{a}\widetilde{\mathbf{a}})^*$ (all members with orts $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ will change the sign) and obtain a real number.

So, we may introduce the 4-norm for bi(di)quaternions with the following way:

$$N_4^{\circledast}(\mathbf{a}) = \mathbf{a}\widetilde{\mathbf{a}}(\mathbf{a}\widetilde{\mathbf{a}})^*. \tag{98}$$

Since the dual conjugation is $\widetilde{\mathbf{a}} = \bar{a} - \mathbf{i}_0\bar{k}$ (22),

than the alternative 2-norm ($\mathbf{i}_0^2 = \mp 1$) for the bi(di)quaternions in the block form is

$$N_2^{\circledast}(a + k \cdot \mathbf{i}_0) = (a + k \cdot \mathbf{i}_0)(\bar{a} - \bar{k} \cdot \mathbf{i}_0) = N_2(a) \pm N_2(k) + 2\langle k, a\rangle \mathbf{i}_0. \tag{99}$$

Farther, $N_4^{\circledast}(\mathbf{a}) = N_2^{\circledast}(\mathbf{a})[N_2^{\circledast}(\mathbf{a})]^*$. But, since

$$(m + \mathbf{q} \cdot \mathbf{i}_0)(m - \mathbf{q} \cdot \mathbf{i}_0) = m^2 \pm \mathbf{q}^2$$

for a real m and neatly imaginary quaternion $\mathbf{q}$, than in the block form the second 4-norm for the bi(di)quaternions is

$$N_4^{\circledast}(a + k \cdot \mathbf{i}_0) = (N_2(a) \pm N_2(k))^2 \pm 4\langle k, a\rangle^2. \tag{100}$$

The alternative norm contains a vector product of the quaternions instead of the scalar product.

Since for the purely imaginary quaternion $\mathbf{q}^2 = -\sum_k \mathbf{q}_k^2$, than in the componentwise form we have (($\mathbf{i}_0^2 = -1, +1$)):

$$N_4^{\circledast}(\mathbf{a}) = \big(a_0^2 + a_1^2 + a_2^2 + a_3^2 \ \pm (k_0^2 + k_1^2 + k_2^2 + k_3^2)\big)^2 \ \mp 4\big(a_1k_0 - a_0k_1 + a_3k_2 - a_2k_3\big)^2$$
$$\mp 4\big(a_2k_0 - a_0k_2 + a_1k_3 - a_3k_1\big)^2 \mp 4\big(a_3k_0 - a_0k_3 + a_2k_1 - a_1k_2\big)^2. \tag{101}$$

Thedual 4-norm is absolutely dislike the first 4-norm. Nevertheless, shall we prove their equivalence. Considering (24) and that $\widetilde{\mathbf{a}} = \bar{\mathbf{a}}^*$, we obtain:

$$N_4^{\circledast}(\mathbf{a}) = \mathbf{a}\widetilde{\mathbf{a}}\mathbf{a}^*\widetilde{\mathbf{a}}^* = \mathbf{a}\widetilde{\mathbf{a}}\mathbf{a}^*\bar{\mathbf{a}} = \mathbf{a}(\bar{\mathbf{a}}\mathbf{a})^*\bar{\mathbf{a}}.$$

But $\bar{\mathbf{a}}\mathbf{a}$, with it's modification $(\bar{\mathbf{a}}\mathbf{a})^*$ belong to the algebra center. Consequently,

$$N_4^{\circledast}(\mathbf{a}) = \mathbf{a}(\bar{\mathbf{a}}\mathbf{a})^*\bar{\mathbf{a}} = \mathbf{a}\bar{\mathbf{a}}(\bar{\mathbf{a}}\mathbf{a})^* = N_4(\mathbf{a}).$$

Just as expected. This fact in the quaternion form for the biquaternions (for the diquaternions we have the same, but in inverse order):

$$(N_2(a) - N_2(k))^2 + 4(a, k)^2 = (N_2(a) + N_2(k))^2 + 4\langle k, a\rangle^2, \tag{102}$$

And, after some evident transformations we have the identity: (67)

$$N_2(a)N_2(k) = (a, k)^2 - \langle k, a\rangle^2.$$

In the real numbers form it looks pretty good too:

$$(a_0^2 + a_1^2 + a_2^2 + a_3^2)(A_0^2 + A_1^2 + A_2^2 + A_3^2) =$$
$$(a_0A_0 + a_1A_1 + a_2A_2 + a_3A_3)^2 + (a_1A_0 - a_0A_1 + a_3A_2 - a_2A_3)^2$$
$$+ (a_2A_0 - a_0A_2 + a_1A_3 - a_3A_1)^2 + (a_3A_0 - a_0A_3 + a_2A_1 - a_1A_2)^2. \tag{103}$$

4-scalar product for the biquaternions, diquaternions and bioctaves

In case of $n = 4$ the Shafer's formula +++ looks like this: (86):

$$(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = \frac{1}{24}\big[N_4(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}) \ - N_4(\mathbf{a}+\mathbf{b}+\mathbf{c}) - N_4(\mathbf{a}+\mathbf{b}+\mathbf{d}) - N_4(\mathbf{a}+\mathbf{c}+\mathbf{d}) - N_4(\mathbf{b}+\mathbf{c}+\mathbf{d}) \ + N_4(\mathbf{a}+\mathbf{b}) + N_4(\mathbf{a}+\mathbf{c}) + N_4(\mathbf{b}+\mathbf{c}) + N_4(\mathbf{a}+\mathbf{d}) + N_4(\mathbf{b}+\mathbf{d}) + N_4(\mathbf{c}+\mathbf{d}) \ - N_4(\mathbf{a}) - N_4(\mathbf{b}) - N_4(\mathbf{c}) - N_4(\mathbf{d})\big]. \quad (104)$$

Having considered (88),(56) after some calculations we obtain the formula of 4-scalar product in the $\mathbb{A}_c$ algebras with 4-norm:

$$(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = \frac{1}{6}\big[(\mathbf{a},\mathbf{b})(\mathbf{c},\mathbf{d})^* + (\mathbf{a},\mathbf{c})(\mathbf{b},\mathbf{d})^* + (\mathbf{a},\mathbf{d})(\mathbf{b},\mathbf{c})^* + (\mathbf{c},\mathbf{d})(\mathbf{a},\mathbf{b})^* + (\mathbf{b},\mathbf{d})(\mathbf{a},\mathbf{c})^* + (\mathbf{b},\mathbf{c})(\mathbf{a},\mathbf{d})^*\big], \qquad \text{or} \qquad (105)$$

$$(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = \frac{1}{6}\big[(\mathbf{a},\mathbf{b})(\mathbf{c}^*,\mathbf{d}^*) + (\mathbf{a},\mathbf{c})(\mathbf{b}^*,\mathbf{d}^*) + (\mathbf{a},\mathbf{d})(\mathbf{b}^*,\mathbf{c}^*) + (\mathbf{c},\mathbf{d})(\mathbf{a}^*,\mathbf{b}^*) + (\mathbf{b},\mathbf{d})(\mathbf{a}^*,\mathbf{c}^*) + (\mathbf{b},\mathbf{c})(\mathbf{a}^*,\mathbf{d}^*)\big]. \quad (106)$$

It is easy to get some useful consequences from this formula. Thus, to solve the geometrical questions the real 4-forms of two vectors are important. Basing on 4-form of 4-vectors we may introduce two such forms:

$$(\mathbf{a},\mathbf{a},\mathbf{b},\mathbf{b}) = \frac{1}{6}\big[N_2(\mathbf{a})N_2^*(\mathbf{b}) + 4(\mathbf{a},\mathbf{b})(\mathbf{a},\mathbf{b})^* + N_2(\mathbf{b})N_2^*(\mathbf{a})\big], \quad (107)$$

$$(\mathbf{a},\mathbf{a},\mathbf{a},\mathbf{b}) = \frac{1}{2}\big[N_2(\mathbf{a})(\mathbf{a},\mathbf{b})^* + (\mathbf{a},\mathbf{b})N_2^*(\mathbf{a})\big], \quad (108)$$

and it is usefull to remember the third one:

$$\{\mathbf{a},\mathbf{b}\} = (\mathbf{a},\mathbf{b})(\mathbf{a},\mathbf{b})^*. \quad (109)$$

We note the symmetric form too:

$$\mathbf{a}\circ\mathbf{b} = (\mathbf{a},\mathbf{a},\mathbf{a},\mathbf{b})+(\mathbf{a},\mathbf{b},\mathbf{b},\mathbf{b}) = \frac{1}{2}\big[(N_2(\mathbf{a})+N_2(\mathbf{b}))(\mathbf{a},\mathbf{b})^*+(\mathbf{a},\mathbf{b})(N_2^*(\mathbf{a})+N_2^*(\mathbf{b}))\big] \quad (110)$$

Using it, the formula of the 4-norm of the sum

$$N_4(\mathbf{a}+\mathbf{b}) = (\mathbf{a}+\mathbf{b},\mathbf{a}+\mathbf{b},\mathbf{a}+\mathbf{b},\mathbf{a}+\mathbf{b}) = = N_4(\mathbf{a}) + 4(\mathbf{a},\mathbf{a},\mathbf{a},\mathbf{b}) + 6(\mathbf{a},\mathbf{a},\mathbf{b},\mathbf{b}) + 4(\mathbf{a},\mathbf{b},\mathbf{b},\mathbf{b}) + N_4(\mathbf{b}) \quad (111)$$

looks like:

$$N_4(\mathbf{a}+\mathbf{b}) = N_4(\mathbf{a}) + N_4(\mathbf{b}) + N_2(\mathbf{a})N_2^*(\mathbf{b}) + N_2(\mathbf{b})N_2^*(\mathbf{a}) + 4\{\mathbf{a},\mathbf{b}\} + 4\mathbf{a}\circ\mathbf{b} \quad (112)$$

The properties of the forms $(\mathbf{a},\mathbf{a},\mathbf{b},\mathbf{b})$ and $(\mathbf{a},\mathbf{a},\mathbf{a},\mathbf{b})$ really differ from each other.

True, the symmetric form $(\mathbf{j}_p,\mathbf{j}_p,\mathbf{j}_q,\mathbf{j}_q)$ for the bioctaves (octaves, bi(di)quaternions) is

$$(\mathbf{j}_p,\mathbf{j}_p,\mathbf{j}_q,\mathbf{j}_q) = 1 \qquad \text{with} \quad p = q,$$
$$(\mathbf{j}_p,\mathbf{j}_p,\mathbf{j}_q,\mathbf{j}_q) = \pm 1/3 \qquad \text{with} \quad p \neq q. \quad (113)$$

Opposite, $\mathbf{j}_p \circ \mathbf{j}_q$ equals zero not only in case of $(\mathbf{j}_p, \mathbf{j}_q) = 0$, but in all cases if $\mathbf{j}_p$ и $\mathbf{j}_q$ are different:

$$(\mathbf{j}_p, \mathbf{j}_p, \mathbf{j}_p, \mathbf{j}_q) = 0 \qquad \text{if} \quad p \neq q, \tag{114}$$

and that's why

$$\mathbf{j}_p \circ \mathbf{j}_q = \delta_{pq}, \tag{115}$$

Thus,*the relation* $\mathbf{a} \circ \mathbf{b} = 0$ *may be assumed as the generalization of the orthogonality of vectors in case of the 4-norms*

It is usefull to remember while the calculations that

$$\mathbf{a}\mathbf{a}^* = \mathbf{a}^*\mathbf{a} \qquad \text{that's why} \quad \mathbf{a}\mathbf{b}^* + \mathbf{b}\mathbf{a}^* = \mathbf{b}^*\mathbf{a} + \mathbf{a}^*\mathbf{b}. \tag{116}$$

4-vector product for the biquaternions and bioctaves

It worths to go farther than Shafer and to generalize the vector product, not only the scalar. For the biquaternions and bioctaves and the similar ones we may propose the following 4-vector multiplication:

$$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle = \frac{1}{6}\big[\langle \mathbf{a}, \mathbf{b} \rangle \langle \mathbf{c}, \mathbf{d} \rangle^* + \langle \mathbf{a}, \mathbf{c} \rangle \langle \mathbf{d}, \mathbf{b} \rangle^* + \langle \mathbf{a}, \mathbf{d} \rangle \langle \mathbf{b}, \mathbf{c} \rangle^* + \langle \mathbf{c}, \mathbf{d} \rangle \langle \mathbf{a}, \mathbf{b} \rangle^* + \langle \mathbf{d}, \mathbf{b} \rangle \langle \mathbf{a}, \mathbf{c} \rangle^* + \langle \mathbf{b}, \mathbf{c} \rangle \langle \mathbf{a}, \mathbf{d} \rangle^*\big], \qquad \text{или} \tag{117}$$

$$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle = \frac{1}{6}\big[\langle \mathbf{a}, \mathbf{b} \rangle \langle \mathbf{c}^*, \mathbf{d}^* \rangle + \langle \mathbf{a}, \mathbf{c} \rangle \langle \mathbf{d}^*, \mathbf{b}^* \rangle + \langle \mathbf{a}, \mathbf{d} \rangle \langle \mathbf{b}^*, \mathbf{c}^* \rangle + \langle \mathbf{c}, \mathbf{d} \rangle \langle \mathbf{a}, \mathbf{b} \rangle^* + \langle \mathbf{d}, \mathbf{b} \rangle \langle \mathbf{a}, \mathbf{c} \rangle^* + \langle \mathbf{b}, \mathbf{c} \rangle \langle \mathbf{a}^*, \mathbf{d}^* \rangle\big]. \tag{118}$$

This 4-vector product is absolutely anti-symmetric by the permutations of any pair of vectors. As the 2-vector product, it is not a real number, but a hyper-complex vector. It is interesting that unlike 2-vector product (which is neatly imaginary), the 4-vector product is real relatively to the basal conjugation $\bar{\mathbf{r}}$ (therefore it contains only 1 and $\mathbf{i}_0$ orts). If the half of the norm's degree is an even number, than the n-vector product will be real. If $n/2$ is an odd number – imaginary (so the 2-vector product of the quaternions is imaginary).

If we formally create the 4-vector product of the quaternions (which are the algebra with the 2-norms, rather than 4-norms), we obtain a real number with an evident geometrical meaning. The 4-vector product equals the determinant of the matrix, composed from the coordinates of the vectors involved:

$$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \rangle = \begin{bmatrix} a_0 & b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}. \tag{119}$$

Thus, it equals the 4-volume of the parallelepiped, strained over four vectors, i.e. it coincides with the parallelepipedal product of the 4 order.

The norm and the scalar product of the quaternions in the isotropic basis

The structural properties of the algebra may be better understood in terms of the isotopic basis, which is constituted of the vectors with zero norm. As is known, the ideal in the semi-simple associative rings are generated by the idempotents. Let us take two isotropic idempotents ($e^2 = e$) $\mathbf{u}_0$ и $\mathbf{v}_0$ in the biquaternion algebra and obtain with their help an isotropic basis $\{\mathbf{u}_i, \mathbf{v}_k\}$:

$$\begin{array}{llll} \mathbf{u}_0 = 1 + \mathbf{i}_3, & \mathbf{u}_1 = \mathbf{q}_1(1 + \mathbf{i}_3), & \mathbf{u}_2 = \mathbf{q}_2(1 + \mathbf{i}_3), & \mathbf{u}_3 = \mathbf{q}_3(1 + \mathbf{i}_3), \\ \mathbf{v}_0 = 1 - \mathbf{i}_3, & \mathbf{v}_1 = \mathbf{q}_1(1 - \mathbf{i}_3), & \mathbf{v}_2 = \mathbf{q}_2(1 - \mathbf{i}_3), & \mathbf{v}_3 = \mathbf{q}_3(1 - \mathbf{i}_3), \end{array} \tag{120}$$

or, in the evident form:

$$\begin{array}{llll} \mathbf{u}_0 = 1 + \mathbf{i}_3, & \mathbf{u}_1 = \mathbf{q}_1 - \mathbf{i}_2, & \mathbf{u}_2 = \mathbf{q}_2 + \mathbf{i}_1, & \mathbf{u}_3 = \mathbf{q}_3 - \mathbf{i}_0, \\ \mathbf{v}_0 = 1 - \mathbf{i}_3, & \mathbf{v}_1 = \mathbf{q}_1 + \mathbf{i}_2, & \mathbf{v}_2 = \mathbf{q}_2 - \mathbf{i}_1, & \mathbf{v}_3 = \mathbf{q}_3 + \mathbf{i}_0. \end{array} \tag{121}$$

Here is the multiplication table of the biquaternion algebra in the isotropic basis:

$\times$	$\mathbf{u}_0$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{u}_0$	$\mathbf{u}_0$	0	0	$\mathbf{u}_3$	0	$\mathbf{v}_1$	$\mathbf{v}_2$	0
$\mathbf{u}_1$	$\mathbf{u}_1$	0	0	$-\mathbf{u}_2$	0	$-\mathbf{v}_0$	$\mathbf{v}_3$	0
$\mathbf{u}_2$	$\mathbf{u}_2$	0	0	$\mathbf{u}_1$	0	$-\mathbf{v}_3$	$-\mathbf{v}_0$	0
$\mathbf{u}_3$	$\mathbf{u}_1$	0	0	$-\mathbf{u}_0$	0	$\mathbf{v}_2$	$-\mathbf{v}_1$	0
$\mathbf{v}_0$	0	$\mathbf{u}_1$	$\mathbf{u}_2$	0	$\mathbf{v}_0$	0	0	$\mathbf{v}_3$
$\mathbf{v}_1$	0	$-\mathbf{u}_0$	$\mathbf{u}_3$	0	$\mathbf{v}_1$	0	0	$-\mathbf{v}_2$
$\mathbf{v}_2$	0	$-\mathbf{u}_3$	$-\mathbf{u}_0$	0	$\mathbf{v}_2$	0	0	$\mathbf{v}_1$
$\mathbf{v}_3$	0	$\mathbf{u}_2$	$-\mathbf{u}_1$	0	$\mathbf{v}_3$	0	0	$-\mathbf{v}_0$

. (*Tab.* 6)

We may see, that the set of vectors $\mathbf{u}_k$ и $\mathbf{v}_k$ constitute left ideals in the biquaternion algebra. (The left multiplication of any element by an arbitrary element of this set gives the element of this set). The ideals may not be two-suede due to the simplicity of the biquaternion algebra). The 2-scalar product table for isotropic basis:

$(\times)$	$\mathbf{u}_0$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{u}_0$	0	0	0	0	1	0	0	$\mathbf{i}_0$
$\mathbf{u}_1$	0	0	0	0	0	1	$-\mathbf{i}_0$	0
$\mathbf{u}_2$	0	0	0	0	0	$\mathbf{i}_0$	1	0
$\mathbf{u}_3$	0	0	0	0	$-\mathbf{i}_0$	0	0	1
$\mathbf{v}_0$	1	0	0	$-\mathbf{i}_0$	0	0	0	0
$\mathbf{v}_1$	0	1	$\mathbf{i}_0$	0	0	0	0	0
$\mathbf{v}_2$	0	$-\mathbf{i}_0$	1	0	0	0	0	0
$\mathbf{v}_3$	$\mathbf{i}_0$	0	0	1	0	0	0	0

. (*Tab.* 7)

From here we have the appearance of the 2-norm in the isotropic basis $\mathbf{u}_k, \mathbf{v}_k$. Let us write down $\mathbf{a}$ in the isotropic basis:

$$\begin{aligned} \mathbf{a} &= a_0 + a_1\mathbf{q}_1 + a_2\mathbf{q}_2 + a_3\mathbf{q}_3 + k_0\mathbf{i}_0 + k_1\mathbf{i}_1 + k_2\mathbf{i}_2 + k_3\mathbf{i}_3, \\ \mathbf{a} &= r_0\mathbf{u}_0 + r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + r_3\mathbf{u}_3 + s_0\mathbf{v}_0 + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + s_3\mathbf{v}_3, \end{aligned} \tag{122}$$

where the real numbers r_k, s_k:

$$\begin{aligned} r_0 &= 1/2(a_0 + k_3),\ r_1 = 1/2(a_1 - k_2),\ r_2 = 1/2(a_2 + k_1),\ r_3 = 1/2(a_3 - k_0),\\ s_0 &= 1/2(a_0 - k_3),\ s_1 = 1/2(a_1 + k_2),\ s_2 = 1/2(a_2 - k_1),\ s_3 = 1/2(a_3 + k_0). \end{aligned} \quad (123)$$

Than the two norm in the isotropic basis is

$$N_2(\mathbf{a}) = (\mathbf{a}, \mathbf{a}) = r_0 s_0 + r_1 s_1 + r_2 s_2 + r_3 s_3 + \mathbf{i}_0(r_0 s_3 - r_3 s_0 + r_2 s_1 - r_1 s_2) \quad (124)$$

From here we have the 4-norm in the isotropic basis:

$$N_4(\mathbf{a}) = (r_0 s_0 + r_1 s_1 + r_2 s_2 + r_3 s_3)^2 + (r_0 s_3 - r_1 s_2 + r_2 s_1 - r_3 s_0)^2. \quad (125)$$

The norm and the scalar product of the diquaternions if the isotropic basis

In the diquaternion case the situation with the isotropic basis is a bit more comlex, but the result is more simple.

As it follows from the table 5, the 2-norm on a square ground can not turn to zero. Thus, the situation changes on the ground of norm of the 4 degree, which may be turned to zero evidentially. Let us take two idempotents and pick the 4-isotropic basis like this:

$$\begin{aligned} &\mathbf{u}_0 = 1 + \mathbf{i}_0, \quad \mathbf{u}_1 = \mathbf{q}_1(1 + \mathbf{i}_0), \quad \mathbf{u}_2 = \mathbf{q}_2(1 + \mathbf{i}_0), \quad \mathbf{u}_3 = \mathbf{q}_3(1 + \mathbf{i}_0),\\ &\mathbf{v}_0 = 1 - \mathbf{i}_0, \quad \mathbf{v}_1 = \mathbf{q}_1(1 - \mathbf{i}_0), \quad \mathbf{v}_2 = \mathbf{q}_2(1 - \mathbf{i}_0), \quad \mathbf{v}_3 = \mathbf{q}_3(1 - \mathbf{i}_0), \end{aligned} \quad (126)$$

or in the evident form:

$$\begin{aligned} &\mathbf{u}_0 = 1 + \mathbf{i}_0, \quad \mathbf{u}_1 = \mathbf{q}_1 + \mathbf{i}_1, \quad \mathbf{u}_2 = \mathbf{q}_2 + \mathbf{i}_2, \quad \mathbf{u}_3 = \mathbf{q}_3 + \mathbf{i}_3,\\ &\mathbf{v}_0 = 1 - \mathbf{i}_0, \quad \mathbf{v}_1 = \mathbf{q}_1 - \mathbf{i}_1, \quad \mathbf{v}_2 = \mathbf{q}_2 - \mathbf{i}_2, \quad \mathbf{v}_3 = \mathbf{q}_3 - \mathbf{i}_3. \end{aligned} \quad (127)$$

Here is the multiplication table of the biquaternion algebra in the isotropic basis:

$\times$	$\mathbf{u}_0$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{u}_0$	$\mathbf{u}_0$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	0	0	0	0
$\mathbf{u}_1$	$\mathbf{u}_1$	$-\mathbf{u}_0$	$\mathbf{u}_3$	$-\mathbf{u}_2$	0	0	0	0
$\mathbf{u}_2$	$\mathbf{u}_2$	$-\mathbf{u}_3$	$-\mathbf{u}_0$	$\mathbf{u}_1$	0	0	0	0
$\mathbf{u}_3$	$\mathbf{u}_1$	$\mathbf{u}_2$	$-\mathbf{u}_1$	$-\mathbf{u}_0$	0	0	0	0
$\mathbf{v}_0$	0	0	0	0	$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	0	0	0	$\mathbf{v}_1$	$-\mathbf{v}_0$	$\mathbf{v}_3$	$-\mathbf{v}_2$
$\mathbf{v}_2$	0	0	0	0	$\mathbf{v}_2$	$-\mathbf{v}_3$	$-\mathbf{v}_0$	$\mathbf{v}_1$
$\mathbf{v}_3$	0	0	0	0	$\mathbf{v}_3$	$\mathbf{v}_2$	$-\mathbf{v}_1$	$-\mathbf{v}_0$

. (*Tab.* 8)

The vector sets $\mathbf{u}_k$ and $\mathbf{v}_k$ constitute the two-sided ideals in the diquaternion algebra, which may be expanded into their direct sum.

In other words the diquaternion algebra is not simple, but semi-simple. That is why the scalar product of the diquaternions in the isotropic basis is very simple:

$$(\mathbf{u}_i, \mathbf{u}_k) = \delta_{ik}\mathbf{u}_0, \quad (\mathbf{v}_i, \mathbf{v}_k) = \delta_{ik}\mathbf{v}_0, \quad (\mathbf{u}_i, \mathbf{v}_k) = 0, \quad (128)$$

Let us write down $\mathbf{a}$ in the isotropic basis:

$$\begin{aligned} \mathbf{a} &= a_0 + a_1\mathbf{q}_1 + a_2\mathbf{q}_2 + a_3\mathbf{q}_3 + k_0\mathbf{i}_0 + k_1\mathbf{i}_1 + k_2\mathbf{i}_2 + k_3\mathbf{i}_3, \\ \mathbf{a} &= r_0\mathbf{u}_0 + r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + r_3\mathbf{u}_3 + s_0\mathbf{v}_0 + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + s_3\mathbf{v}_3, \end{aligned} \tag{129}$$

where the real numbers r_k, s_k:

$$\begin{aligned} r_0 &= 1/2(a_0 + k_0),\ r_1 = 1/2(a_1 + k_1),\ r_2 = 1/2(a_2 + k_2),\ r_3 = 1/2(a_3 + k_3), \\ s_0 &= 1/2(a_0 - k_0),\ s_1 = 1/2(a_1 - k_1),\ s_2 = 1/2(a_2 - k_2),\ s_3 = 1/2(a_3 - k_3). \end{aligned} \tag{130}$$

shall we obtain the 2-norm in the isotropic basis:

$$N_2(\mathbf{a}) = (\mathbf{a}, \mathbf{a}) = (1 + \mathbf{i}_0)(r_0^2 + r_1^2 + r_2^2 + r_3^2) + (1 - \mathbf{i}_0)(s_0^2 + s_1^2 + s_2^2 + s_3^2) \tag{131}$$

Since for the double numbers $(1 + \mathbf{i}_0)(1 - \mathbf{i}_0) = 0$, than

$$\big(\mathbf{r}(1 + \mathbf{i}_0) + \mathbf{s}(1 - \mathbf{i}_0)\big)\big(\mathbf{r}(1 - \mathbf{i}_0) + \mathbf{s}(1 + \mathbf{i}_0)\big) = 2\mathbf{rs}.$$

Since so, the 4-norm breaks into the multiplication of two 2-norms:

$$N_4(\mathbf{a}) = N_2(\mathbf{a})N_2(\mathbf{a})^* = (r_0^2 + r_1^2 + r_2^2 + r_3^2)(s_0^2 + s_1^2 + s_2^2 + s_3^2). \tag{132}$$

Conclusions

The method of the algebra's scrutiny, based on the analysis of the permitted conjugations, allows to obtain not only already knownb results, but some interesting new. In particular, the working with the forms of higher than square degree becomes more simple.

Although, since already the 2-norm for the biquaternion and bioctave algebras is multiplicative, and since the 4-norm is unambiguously expressible via it, it is not clear, whether this movement from complex (double) 2-norm to real 4-norm does create some newb possibilities. The same is not clear in case of the 4-scalar and 4-vector products, since they are expressible via their 2-prototypes too. In any case it seems interesting to give this objects some geometrical or physical meaning.

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ON SOME DISTRIBUTIVE UNIVERSAL ALGEBRAS

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The examined type of sets, that are not rings, but in some sense are close to them. These sets are called 'Hyper-rings'. They consist of several additive groups, that intersect each over at the zero only. Yet, they are multiplicative groupoids (or groups, excluding the zero). The distributive laws are fulfilled.

Rings (and in particular the bodies and the fields) are the special case of the concerned sets. The given examples witness that such sets are highly disseminated. So, the idea that the real physical values may be "layed"in the ring is wrong, because they are subset of hyper-ring.

The real hyper-rings with unity can not be reduced to rings. Their additive groups are vector spaces, and they may be treated as a generalized Hyper-complex systems, in which we include the real binary (provided with summation and multiplication) distributive algebraic structures with neutral element, where the number of included vector spaces is more than one and finite.

The example of hyper-rings, suggesting that scrutiny of them is worth-able, are the second order matrices, that are mostly like unitary matrixes, but normalized not by unity. They are normalized by an arbitrary non-negative number. The complex numbers and the quaternions may be represented with such matrixes while they are the ones subspace.

§1. Hyper-body (hyper-field, hyper-ring) of the k-th order by additive groups

Universal algebras [1] are the objects of more general type than bodies, fields and rings. This article concerns the universal algebras class that includes bodies, fields and rings.

Let us name *hyper-body (accordingly ,hyper-ring)* of the k-th order by additive groups the manifold M, which possesses the following properties:

1) It constitute k a non-zero additive groups which intersect each other in zero only;

2) It constitute , except the zero, a multiplicative group or loop[1] (a multiplicative groupoid [q], accordingly, including the zero);

3) Among the multiplications of a_i, a_k from fixed additive groups A_i и A_k there are non-zero ones for any i and k.

4) If we multiply two elements of some two additive groups (that may coincide with each other), we obtain some element from a fixed additive group which depends on the efficients (and their queue, maybe) only. This means that the additive groups A_i и A_k are the elements of a groupoid:

$$A_i A_k = A_l$$

if i and k are fixed, l is fixed as well. Let us name this groupoid *the factor-groupoid* of the manifold M by additive groups.

5) the left and the right distributive laws are in operation.

We shall denominate the hyper-body of the k-th order by additive groups with the commutative multiplication *the hyper-field* of the k-th order by additive groups.

Let us call the hyper-rings, which additive groups are the n-dimensional vector spaces over the field P, and at that the n is same for every additive group, the *hyper-algebras of the k-th order n-th rank* over the field P if the following relations are true:

$$(\alpha a) \cdot b = a \cdot (\alpha b) = \alpha(a \cdot b),$$

where α is an element of the field P, and a, b- the elements of the hyper-ring.

The same hyper-algebras, which elements (except the zero) constitute a multiplicative group or a loop, we shall denominate *hyper-algebras of the k-th order n-th rank with division* over the field P

Thus, the usual bodies (fields, rings) are appear to be the hyper-bodies (hyper-fields,hyper-rings) of the first order by the whole manyfold M - the only additive group in this case.

There can be only one body(field) among the additive groups, because the unit may not belong to more than one additive group, according to item one. If the concerned manyfold is a hyper-body or a hyper-field, than the additive group with unit is a body or a field itself. Indeed, the product of the unit e on an element of this group is the same element and therefore belongs to this group. Then in that case, according to the item 4, the product of any two elements of this additive group belongs to the same additive group. Farther, there is an inverse element for every element a of the concerned additive group, and this inverse element belongs to the same additive group.Indeed, if the inverse element a^{-1} belonged to any other additive group, than, according to the item 4, $a^{-1}a$ would belong to the same additive group; but, in this case, $a^{-1}a$ could not be equal to e. Thus, the additive group, that includes the unit, is (excluding the zero)a multiplicative group or a loop.Consequently, it is a multiplicative group or a loop of a body or a field[1]. The concerned additive group (excluding the zero) in case of associative hyper-body (hyper-field, in particular) is a multiplicative sub-group of the whole multiplicative group and the normal divisor of the latter [1,3].

Indeed, for each element b of manyfold M (excluding the zero) the following correlation is true:

$$b^{-1}eb = e.$$

Consequently ,according to the item 4, if $h_1 \neq 0$ belongs to additive group with unit, than

$$b^{-1}h_1b = h_2,$$

where h_2 belongs to the same additive group too, quod erat demonstrandum.

Any additive group of an associative hyper-body (hyper-field) (excluding the zero) is adjacent class by the normal divisor.

Indeed, element $a \neq 0$ of some additive group satisfies the correlation $ae = a$; therefore, according to the item 4, $ah_1 = b$, where h_1 - any other element of the normal divisor H, and b is an element of the same additive group with a.

This means that the whole adjacent class, appropriate for a, belongs to the concerned additive group A_1.

Let us take an arbitrary element $a' \neq a,\ a' \in A_1$, so that $a' \neq 0$. The following correlation is true for a' and for a: $a'H \in A_1$. If h_i - an arbitrary element of H, than

$$a'h_i = aa^{-1}a'h_i.$$

but

$$a^{-1}a = e \in H.$$

[1] Additive group with unit of associative hyper-algebra with the division of the finite rank over field of real numbers is a associative algebra with the division of the finite rank over field of real numbers. And, according to the Frobenius theorem, it is isomorphous to real number, complex numbers, or quaternions [1].

Consequently, according to the item 4, $a^{-1}a' \in H$; let $a^{-1}a' = h_3$. Than $a'h_i = ah_3h_i = ah_4$, where $h_4 = h_3h_i \in H$, i. e. a' belongs to the same adjacent class with a, quod erat demonstrandum.

It worths reminding now that right adjacent classes by the normal divisor coincide with the left ones[1, 3]. Consequently, all additive groups of an associative hyper-body (excluding the zero) constitute a factor-group of it's multiplicative group [1,3] by additive group comprising the unit (excluding the zero).

Lets make a note of some other properties of hyper-rings (hyper-bodies, hyper-fields):

a)Lust like for rings it is proven [1] that the product (left or right) of the zero on any element is equal to zero.

b)We saw that the additive groups of the hyper-ring constitute a multiplicative groupoid - it's factor groupoid by this additive groups. It's easy to see that additive groups A_i are the only elements of the additive groups with the additive groups with the following addition law:

$$A_i + A_i = A_i,$$

so that A_i is the zero 0_i of such additive group, and it's opposite element. But not the common zero of all these additive groups.

In conclusion let us remark that hyper-rings (hyper-bodies, hyper-fields) are not, in general, rings (bodies, fields). But they may be the ones.

Hyper-rings (hyper-bodies, hyper-fields) by additive groups may be both of the finite and infinite order.

§2. Unnormalized to unity unitary and orthogonal matrices

Before to pass to examples, we shall consider the necessary for farther statement unnormalized to unity unitary and orthogonal matrices.

As is well known, the complex matrix of the $n-th$ order A refers to unitary if

$$AA^+ = A^+A = 1, \tag{1}$$

where A^+ – the Hermitian-conjugated matrix for A. Matrix A refers to orthogonal, if

$$A\widetilde{A} = \widetilde{A}A = 1, \tag{2}$$

where $\widetilde{A}$ – the transposed A (if A is real-valued, the unitary matrix is orthogonal[4]).

Let us concern a complex matrix A of the n-th order, which satisfies the following condition:

$$AA^+ = \lambda I, \tag{3}$$

where λ - some number, a I - identity. λ is real and non-negative. Indeed, for any index i

$$\lambda = (AA^+)_{ii} = \sum_k A_{ik}(A^+)_{ki} = \sum_k A_{ik}A^*_{ik} = \sum_k |A_{ik}|^2 \geq 0. \tag{4}$$

(equals sign takes place in case of $A = 0$).

Let us denote that if condition (3) is satisfied than

$$AA^+ = A^+A = \lambda I. \tag{5}$$

Indeed, with $\lambda \neq 0$

$$A(A^+/\lambda) = I, \tag{6}$$

i. e. $A^+/\lambda = A^{-1}$ – the matrix, inverse for A. But, as is well known, for inverse matrices

$$A^{-1}A = AA^{-1} = I, \qquad \text{i. e.} \quad (A^+/\lambda)A = A(A^+/\lambda) = I.$$

Consequently, (5) is true. For zero-matrix (5) is true too(with $\lambda = 0$).

Let us name the matrix, satisfying the condition (3) and, consequently, (4) (5), the unnormalized to unity unitary or *quasi-unitary.*

If A – a real matrix n-th order, than alongside with orthogonal matrices we may concern the matrices, satisfying the condition:

$$A\widetilde{A} = \lambda I. \tag{7}$$

Let us name such matrices unnormalized to unity orthogonal or *quasi-orthogonal.*

It is obvious λ – real number. It is non-negative. It follows from the fact that in the real area quasi-unitary matrixes are quasi-orthogonal. Just as for quasi-unitary matrices the following correlation may be proved:

$$\widetilde{A}A = A\widetilde{A} = \lambda I. \tag{8}$$

It easy to show that the real second-order and fourth-order matrices, which are isomorphous to the real and complex numbers accordingly, are quasi-orthogonal. And the complex matrix of the second order, that is isomorphous to quaternions, is quasi-unitary.

Let us show now that quasi-unitary matrices of the n-th order (excluding the zero) constitute a multiplicative group.

Indeed, the identity is quasi-unitary and represents the unity of the system. Farther, as we have seen, there is an inverse quasi-unitary matrix for every quasi-unitary matrix A (excluding the zero):

$$A^{-1} = A^+/\lambda.$$

Let's examine the product of AB of quasi-unitary A и B. We have:

$$AB \cdot (AB)^+ = ABB^+A^+ = \lambda_B(AA^+) = \lambda_B\lambda_A, \qquad \text{where} \quad \lambda_A = AA^+,\ \lambda_B = BB^+. \tag{9}$$

Consequently, the AB is a quasi-unitary matrix too. Thus, all the conditions that turn quasi-unitary matrices of the n-th order (excluding the zero) into the group by the multiplication, are satisfied. Let us name the product

$$AA^+ = \lambda_A = |A|^2 \tag{10}$$

the square of modulus of the quasi-unitary matrix, and

$$|A| = \sqrt{AA^+} \tag{11}$$

– it's *modulus* or *absolute value.*

As it follows from the formulae (10), for the determinant of A we have:

$$|det\,A|^2 = |A|^{2n},$$

from which

$$|det\,A| = |A|^n \tag{10'}$$

(n – matrix's order).

In the real area quasi-orthogonal matrices of the n-th order (excluding the zero) constitute a multiplicative group.

Modulus or absolute value of a quasi-orthogonal matrix is

$$|A| = \sqrt{A\widetilde{A}}. \tag{12}$$

Naturally, the formulae (10′) is true for real quasi-orthogonal matrices of the n-th order also.

According to (9) – (12), the modulus of the product of quasi-unitary (quasi-orthogonal) matrix is equal to the product of the modules of the efficients.

Let's designate $QU(n)$ the group (by the multiplication) of quasi-unitary matrices of the n-th order (excluding the zero). And, accordingly, $QO(n)$ – the group (by the multiplication) of quasi-orthogonal matrices of the n-th order (excluding the zero).

Let's designate $QU+(n)$ the group of quasi-unitary matrices of the n-th order with positive determinant. The same for quasi-orthogonal matrices – $QO^{+}(n)$.

Note that the totality of quasi-unitary and quasi-orthogonal matrixes of the n-th order may include the zero-matrix, which is not true for unitary and orthogonal matrixes.

§3. Examples of hyper-bodies, hyper-fields or hyper-rings

1) Quasi-orthogonal real matrixes of the second order, constituting (excluding the zero) multiplicative group $QO(2)$.

Let's consider a real matrix of the second order.

$$A = \begin{pmatrix} x & y \\ a_{21} & a_{22} \end{pmatrix}. \tag{13}$$

Transposing A, we get:

$$\widetilde{A} = \begin{pmatrix} x & a_{21} \\ y & a_{22} \end{pmatrix}. \tag{14}$$

We are interested in quasi-orthogonal matrixes. The quasi-orthogonality conditions (7) give:

$$(A\widetilde{A})_{11} = x^2 + y^2, \tag{15}$$

$$(A\widetilde{A})_{12} = xa_{21} + ya_{22} = 0, \tag{16}$$

$$(A\widetilde{A})_{21} = a_{21}x + a_{22}y = 0, \tag{16′}$$

$$(A\widetilde{A})_{22} = (A\widetilde{A})_{11} = a_{21}^2 + a_{22}^2 = x^2 + y^2. \tag{17}$$

Due to the coincidence of (16) with (16′), we have the equations set (16),(17) for unknown a_{21}, a_{22}:

$$xa_{21} + ya_{22} = 0, \tag{16}$$

$$a_{21}^2 + a_{22}^2 = x^2 + y^2. \tag{17}$$

Equation (16) gives:

$$a_{21} = -\frac{y}{x}\, a_{22}. \tag{16″}$$

Substituting (16″) в (17), we get:

$$\frac{(x^2+y^2)a_{22}^2}{x^2} = x^2+y^2. \tag{17'}$$

With $x \neq 0$ we have:

$$a_{22}^2 = x^2$$

from where:

$$a_{22} = \pm x. \tag{17''}$$

Substituting (17″) в (16″), we get:

$$a_{21} = \mp y. \tag{18}$$

Consequently, we have two solutions:

$$A_1 = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \tag{19}$$

$$A_2 = \begin{pmatrix} x' & y' \\ y' & -x' \end{pmatrix}. \tag{20}$$

With $x = 0, y \neq 0$ equations (16) и (17) give:

$$a_{22} = 0, \tag{21}$$

$$a_{21} = \mp y. \tag{22}$$

For matrix A we have:

$$A_1 = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}, \tag{19'}$$

$$A_2 = \begin{pmatrix} 0 & y' \\ y' & 0 \end{pmatrix}. \tag{20'}$$

Matrixes (19′) и (20′) are the particular case of the matrices A_1 and A_2 ((19) и (20)). At last, with $x = y = 0$ wi have, according to (17):

$$a_{21}^2 + a_{22}^2 = 0, \tag{23}$$

from where:

$$a_{21} = a_{22} = 0, \tag{24}$$

and we get the zero matrix.

$$A_1 = A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{25}$$

But the matrixes (25) are the particular case of the matrices A_1 and A_2 ((19) and (20))

Thus, quasi-orthogonal matrices of the second order, in general, may be reduced to matrices A_1 and A_2, which are defined by formulas (19) and (20).

Matrices A_1 and A_2 are defined by independent variables $x, y; x', y'$. As is known [2], the matrices A_1, defined by (19) are isomorphous to complex numbers, so that the matrix modulus is equal to the modulus of the complex number.

It easy to see, that matrices A_1 (complex numbers) constitute (excluding the zero) a multiplicative group $QO^{+}(2)$.

As is known, giving x and y various values and adding matrices of A_1 type (complex numbers), we get:

$$A_1|_{x_1,\,y_1} + A_1|_{x_2,\,y_2} = \begin{pmatrix} x_1 + x_2 & y_1 + y_2 \\ -y_1 - y_2 & x_1 + x_2 \end{pmatrix}, \tag{26}$$

i.e. we have quasi-orthogonal matrices (complex numbers) again. In exactly the same way an adding of of A_2 type matrices give the matrices of the same type. The sum $A_1 + A_2$ does not belong neither of the two types, i.e. it is not quasi-orthogonal matrix. The totalities of A_1 and A_2 matrices cross crossing each other nowhere except the zero matrix. Consequently, they are two additive groups, crossing each other at the zero only.

A product of two matrices of the A_1 type (complex numbers), as is known, is a matrix of the same type (i.e. complex numbers)

Farther we have:

$$A_3 = (A_2|_{x=x'_1,\,y=y'_1}) \cdot (A_2|_{x=x'_2,\,y=y'_2}) = \begin{pmatrix} x'_1 & y'_1 \\ y'_1 & -x'_1 \end{pmatrix} \begin{pmatrix} x'_2 & y'_2 \\ y'_2 & -x'_2 \end{pmatrix} =$$

$$= \begin{pmatrix} x'_1x'_2 + y'_1y'_2 & x'_1y'_2 - y'_1x'_2 \\ y'_1x'_2 - x'_1y'_2 & y'_1y'_2 + x'_1x'_2 \end{pmatrix}. \tag{27}$$

We can see that A_3 is a matrix of the A_1 type, i.e. complex number, though it depends on the efficients sequence.

Finally,

$$A_4 = A_1 \cdot A_2 = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \begin{pmatrix} x' & y' \\ y' & -x' \end{pmatrix} = \begin{pmatrix} xx' + yy' & xy' - yx' \\ -yx' + xy' & -yy' - xx' \end{pmatrix}, \tag{28}$$

$$A_5 = A_2 \cdot A_1 = \begin{pmatrix} x' & y' \\ y' & -x' \end{pmatrix} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} x'x - y'y & x'y + y'x \\ y'x + x'y & y'y - x'x \end{pmatrix}. \tag{29}$$

Thus, $A_1A_2 \neq A_2A_1$, but $A_4 = A_1A_2$ and $A_5 = A_2A_1$ are the matrices of the A_2 type. Thus we have made sure of the following:

a) Quasi-orthogonal real matrixes of the second order are not bodies, they are not rings even. They represent a hyper-body of the second order;

b) the matrices of A_1 typre (excluding the zero-matrix) represent the normal divisor of the multiplicative group, so that additive group A_2 (excluding the zero) is a adjacent class by the normal divisor A_1;

c) Due to the easy provable fact that the additive groups A_1 and A_2 – vector two-dimensional spaces over the real-numbers field, Quasi-orthogonal real matrixes of the second order represent a hyper-algebra of the second order of the second rank with the division by additive groups over the field of real numbers.

2) Quasi-unitary real matrixes of the second order, constituting (excluding the zero) multiplicative group $QU(2)$.

a) Now, let A – a complex matrix of the second order. As in the previous example, let us assume

$$A = \begin{pmatrix} x & y \\ a_{21} & a_{22} \end{pmatrix}. \tag{13}$$

Let us define the matrix elements a_{21} and a_{22} for quasi-unitary matrix. For A^+ we obtain:

$$A^+ = \begin{pmatrix} x^* & a_{21^*} \\ y^* & a_{22}^* \end{pmatrix}. \tag{30}$$

Let's sign out the quasi-unitarity condition:

$$(AA^+)_{11} = |x|^2 + |y|^2, \tag{31}$$

$$(AA^+)_{12} = xa_{21}^* + ya_{22}^* = 0, \tag{32}$$

$$(AA^+)_{21} = a_{21}x^* + a_{22}y^* = 0, \tag{33}$$

$$(AA^+)_{22} = |a_{21}|^2 + |a_{22}|^2 = |x|^2 + |y|^2. \tag{34}$$

By solving the set of equations (33), (34), we obtain:

$$a_{21} = -\frac{a_{22}y^*}{x^*}, \tag{35}$$

$$|a_{22}|^2 \frac{(|y|^2 + |x|^2)}{|x|^2} = |x|^2 + |y|^2. \tag{36}$$

For $x \neq 0$ formulae (36) gives:

$$|a_{22}| = |x|. \tag{37}$$

Preserving the commonality, we may write:

$$a_{22} = x^* e^{i\varphi}. \tag{38}$$

Substituting (38) into (35), we have:

$$a_{21} = -y^* e^{i\varphi}. \tag{39}$$

Consequently,

$$A = \begin{pmatrix} x & y \\ -y^* e^{i\varphi} & x^* e^{i\varphi} \end{pmatrix}. \tag{40}$$

For $x = 0$, $y \neq 0$ accordingly to (33), (34), we obtain:

$$a_{22} = 0, \tag{41}$$

$$|a_{21}| = |y|, \tag{42}$$

from where, preserving the commonality,

$$a_{21} = -y^* e^{i\varphi}. \tag{39$'$}$$

With $x = 0$, $y = 0$ we get, according to (34):

$$a_{21} = a_{22} = 0. \tag{43}$$

Thus, the general appearance of quasi-unitary matrix is given by formulae (40).

Let us name φ in the formulae (40) the angular parameter of quasi-unitary matrix of the second order. It easy to notice, that all the matrix with the same angular parameter φ constitute an additive group.

Let us consider two quasi-unitary matrices with angular parameters φ_1 and φ_2:

$$A_1 = \begin{pmatrix} x_1 & y_1 \\ -y_1^* e^{i\varphi_1} & x_1^* e^{i\varphi_1} \end{pmatrix}, \tag{44}$$

$$A_2 = \begin{pmatrix} x_2 & y_2 \\ -y_2^* e^{i\varphi_2} & x_2^* e^{i\varphi_2} \end{pmatrix}, \tag{45}$$

$$A_{12} = A_1 A_2 = \begin{pmatrix} x_{12} & y_{12} \\ -y_{12}^* e^{i\varphi_{12}} & x_{12}^* e^{i\varphi_{12}} \end{pmatrix}, \tag{46}$$

where

$$x_{12} = x_1 x_2 - y_1 y_2^* e^{i\varphi_2}, \tag{47}$$

$$y_{12} = x_1 y_2 + y_1 x_2^* e^{i\varphi_2}, \tag{48}$$

$$\varphi_{12} = \varphi_1 + \varphi_2 + 2k\pi \qquad (k \text{ – integer}). \tag{49}$$

The product A_{12}, as it should, is described by the formulae (46), so that it's angular parameter equals to the sum of the angular parameters of the efficients (to $2k\pi$).

It is easy to see from formulae (40) , that:

$$det\, A = |det\, A| e^{i\varphi}. \tag{40$'$}$$

Generally, for matrices of the n-th rank it is obvious that the equality (40′) is true, where φ is some angle. Let us name the φ in the formulae (40′) the angular parameter of the quasi-unitary matrix of the n-th order. It is obvious that and in the general case of quasi-unitary matrix of the n-th order the correlation (49) is true.

With $\varphi = 0$ quasi-unitary matrix of the n-th order (excluding the zero), constitute a multiplicative group $QSU(n)$ – the sub-group $QU(n)$.

Matrices

$$A_0 = \begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix} \tag{50}$$

with zero angular parameter, as it well known [4], isomorphous to quaternions. They represent (excluding the zero) a normal divisor of the whole multiplicative group, due it includes the identity. Thus, from the formulas (44)–(50) it follows that quasi-unitary matrices of the second order represent a hyper-body of the infinity-order. b) As a particular case let us examine the matrix of the such type:

$$A_{1,0} = A_1|_{\varphi=0} = \begin{pmatrix} x_1 & y_1 \\ -y_1^* & x_1^* \end{pmatrix}, \tag{44$'$}$$

$$A_{2,0} = A_2|_{\varphi=\pi} = \begin{pmatrix} x_2 & y_2 \\ y_2^* & -x_2^* \end{pmatrix}. \tag{45$'$}$$

The matrices $A_{1,0}$ are of the type A_0 (see.(50)) and, consequently, isomorphous to quaternions. The matrixes like $A_{1,0}$ and $A_{2,0}$ represent, according to the above, two additive groups. We have:

$$A_{10,10'} = A_{1,0} A'_{1,0} = \begin{pmatrix} x_1 & y_1 \\ -y_1^* & x_1^* \end{pmatrix} \begin{pmatrix} x'_1 & y'_1 \\ -y'^*_1 & x'^*_1 \end{pmatrix} = \begin{pmatrix} x_1 x'_1 - y_1 y'^*_1 & x_1 y'_1 + y_1 x'^*_1 \\ -x_1^* y'^*_1 - y_1^* x'_1 & x_1^* x'^*_1 - y_1^* y'_1 \end{pmatrix} \tag{51}$$

– matrices like $A_{1,0}$;

$$A_{20,20'} = A_{2,0}A'_{2,0} = \begin{pmatrix} x_2 & y_2 \\ y_2^* & -x_2^* \end{pmatrix} \begin{pmatrix} x'_2 & y'_2 \\ y'^*_2 & -x'^*_2 \end{pmatrix} = \begin{pmatrix} x_2x'_2 + y_2y'^*_2 & x_2y'_2 - x'^*_2y_2 \\ y_2^*x'_2 - x_2^*y'^*_2 & y_2^*y'_2 + x_2^*x'^*_2 \end{pmatrix} \tag{52}$$

– matrices like $A_{1,0}$;

$$A_{10,20} = A_{1,0}A_{2,0} = \begin{pmatrix} x_1x_2 + y_1y_2^* & x_1y_2 - y_1x_2^* \\ -y_1^*x_2 + x_1^*y_2^* & -y_1^*y_2 - x_1^*x_2^* \end{pmatrix} \tag{53}$$

– matrices like $A_{2,0}$;

$$A_{20,10} = A_{2,0}A_{1,0} = \begin{pmatrix} x_2x_1 - y_2y_1^* & x_2y_1 + y_2x_1^* \\ y_2^*x_1 + x_2^*y_1^* & y_2^*y_1 - x_1^*x_2^* \end{pmatrix} \tag{54}$$

– matrices like $A_{2,0}$; it easy to show, that the matrices, inverse for $A_{1,0}$ and $A_{2,0}$, are the matrices of the same type accordingly. Matrices of the $A_{1,0}$ and $A_{2,0}$ kind constitute, thus, hyper-body of the second order, at that matrices $A_{1,0}$ (excluding the zero) – a normal divisor of the multiplicative group.

3) Quasi-unitary matrices of the second order with quaternionic matrix elements and abrogated multiplication law

Let us examine a product of the matrices of the n-th order $A_{(1)}$ и $A_{(2)}$ with abrogated multiplication law:

$$D_{ik} = (A_{(1)} \circ A_{(2)})_{ik} = \sum_l A_{(1)il}P_{il,\,lk}A_{(2)lk}, \tag{55}$$

where $P_{il,\,lk}$ – operator,

$$P_{il,\,lk} = \begin{cases} 1 \text{ for } A_{(1)il} \text{ и } A_{(2)lk}, \text{ which do not permute,} \\ p \text{ for } A_{(1)il} \text{ и } A_{(2)lk}, \text{ which permute,} \end{cases} \tag{56}$$

i. e.

$$A_{(1)il}pA_{(2)lk} = A_{(2)lk}A_{(1)il}. \tag{57}$$

It is easy to show, that the concerned abrogation of the matrix multiplication law does not violate the distributive law. Also we obtain the conditions for the rule of conjugation of matrices to be true:

$$D^+ = (A_{(1)} \circ A_{(2)})^+ = A^+_{(2)} \circ A^+_{(1)}, \quad (D = A_{(1)} \circ A_{(2)}). \tag{a}$$

We have:

$$D_{ik} = (A_{(1)} \circ A_{(2)})_{ik} = \sum_l A_{(1)il}P_{il,lk}A_{(2)lk}, \quad D_{ki} = \sum_l A_{(1)kl}P_{kl,li}A_{(2)li}.$$

Inasmuch $(D^+)_{ik} = D^+_{ki}$, we get:

$$(D^+)_{ik} = ((A_{(1)} \circ A_{(2)})^+)_{ik} = \sum_l (A_{(2)li})^+P_{kl,li}(A_{(1)kl})^+. \tag{58'}$$

On the other hand,

$$((A^+_{(2)} \circ A_{(1)})^+)_{ik} = \sum_l (A^+_{(2)})_{il} P_{il,lk} (A^+_{(1)})_{lk} = \sum_l (A_{(2)li})^+ P_{il,lk} (A_{(1)kl})^+. \tag{58''}$$

Comparing the right sides (58′) and (58″), we obtain (a):

$$P_{il,lk} = P_{kl,li}. \tag{58}$$

Let us investigate matrices of the second order (40), but with x, y - quaternions. If we demand $x^+ e^{i\varphi}$ to be quaterions too, than $e^{i\varphi}$ has to be a real number, which is possible only with $\varphi = 0$ and $\varphi = \pi$. Thus, we consider that φ takes only two values, at that $e^{i\pi} = e^{-i\pi} = -1$. At the same time $e^{i\varphi}$ permuting with quaternions. For A^+_φ we have:

$$A^+_\varphi = \begin{pmatrix} x^+ & -ye^{-i\varphi} \\ y^+ & xe^{-i\varphi} \end{pmatrix}. \tag{59}$$

Farther,

$$A_\varphi A^+_\varphi = \begin{pmatrix} |x|^2 + |y|^2 & (-xy + yx)e^{-i\varphi} \\ (-y^+x^+ + x^+y^+)e^{i\varphi} & |y|^2 + |x|^2 \end{pmatrix}. \tag{60}$$

Due to non-commutative property of x and y, and x^+, y^+ non-diagonal elements of $A_\varphi A^+_\varphi$ do not turn into zero, so A_φ is not quasi-unitary with the usual multiplication law.

Let us introduce operators $P_{il,lk}$ into the multiplication law of matrices (40), having selected them thus matrix A_φ become quasi-unitary. Let's sign out coefficients $P_{il,lk}$ in a matrix form:

$$P = \begin{pmatrix} P_{11,11} & P_{12,21} & P_{11,12} & P_{12,22} \\ P_{21,11} & P_{22,21} & P_{21,12} & P_{22,22} \end{pmatrix}. \tag{61}$$

There are eight values of matrix P, at which the matrices A are quasi-unitary and constitute a quasi-unitary groupoid. For these variants matrices P

$$P_1 = \begin{pmatrix} 1 & 1 & 1 & p \\ 1 & p & p & p \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 1 & p & 1 \\ p & 1 & p & p \end{pmatrix}, \quad P_3 = \begin{pmatrix} p & 1 & 1 & p \\ 1 & p & p & 1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} p & 1 & p & 1 \\ p & 1 & p & 1 \end{pmatrix},$$
$$P_5 = \begin{pmatrix} 1 & p & 1 & p \\ 1 & p & 1 & p \end{pmatrix}, \quad P_6 = \begin{pmatrix} 1 & p & p & 1 \\ p & 1 & 1 & p \end{pmatrix}, \quad P_7 = \begin{pmatrix} p & p & 1 & p \\ 1 & p & 1 & 1 \end{pmatrix}, \quad P_8 = \begin{pmatrix} p & p & p & 1 \\ p & 1 & 1 & 1 \end{pmatrix}. \tag{62}$$

In all cases, as it follows from formulas, given in the supplement, the quasi-unitary matrices represent hyper-rings. These hyper-rings consist of two additive vector-spaces, represented by matrices $A\big|_{\varphi=0}$ and $A\big|_{\varphi=\pi}$. Also they are hyper-algebras of the second order of the eighth rank over the field of real numbers. It is easy to check the following condition is true:

$$(A_{\varphi 1} \circ A_{\varphi 2})^+ = A^+_{\varphi 2} A^+_{\varphi 1} \tag{63}$$

for each case. In the sixth case matrices $A\big|_{\varphi=0}$ represent the Kelly algebra[1] ($A\big|_{\varphi=0}$, are alternative [1], which is not true for matrices of the second order with quaternionic matrix elements and the multiplication law (55))

Matrices A_φ in this case represent the hyper-algebra of the second order of the eighth rank over the field of real numbers with division. We may show that not only for Kelly algebra, but for whole hyper-algebra of A_φ the following correlation is satisfied:

$$|A_{\varphi 1} \circ A_{\varphi 2}| = |A_{\varphi 1}| \cdot |A_{\varphi 1}|. \qquad (64)$$

In the rest seven case $|A_{\varphi 1} \circ A_{\varphi 2}| \neq |A_{\varphi 1}| \cdot |A_{\varphi 1}|$.

4) a. Quasi-orthogonal real matrices of the third order kind of

$$A_{00} = \begin{pmatrix} x & y & 0 \\ -y & x & 0 \\ 0 & 0 & \sqrt{x^2+y^2} \end{pmatrix} \ (a), \qquad A_{10} = \begin{pmatrix} x' & y' & 0 \\ y' & -x' & 0 \\ 0 & 0 & \sqrt{x'^2+y'^2} \end{pmatrix} \ (b), \qquad (65)$$

$$A_{01} = \begin{pmatrix} x'' & y'' & 0 \\ -y'' & x'' & 0 \\ 0 & 0 & -\sqrt{x''^2+y''^2} \end{pmatrix} \ (a), \quad A_{11} = \begin{pmatrix} x''' & y''' & 0 \\ y''' & -x''' & 0 \\ 0 & 0 & -\sqrt{x'''^2+y'''^2} \end{pmatrix} \ (b). \quad (66)$$

Matrices (65) – (66) themselves are quasi-orthogonal. But this property vanishes if we add this matrices to each other (even of the same type) using the usual rules. This Situation may be fixed if we change the summation rules.

To shorten the calculations, let us write down the matrices (65) – (66) in a compact form

$$A^l_{ik} = \begin{pmatrix} x_l & y_l & 0 \\ (-1)^{i+1}y_l & (-1)^i x_l & 0 \\ 0 & 0 & (-1)^k\sqrt{x_l^2+y_l^2} \end{pmatrix}, \quad (i,k=0,1). \qquad (67)$$

Now, shall we define the summ of two matrices of one type A^1_{ik} and A^2_{ik} in the following way:

$$A^0_{ik} = A^1_{ik}(+)A^2_{ik} = \begin{pmatrix} x_1+x_2 & y_1+y_2 & 0 \\ (-1)^{i+1}(y_1+y_2) & (-1)^i(x_1+x_2) & 0 \\ 0 & 0 & (-1)^k\sqrt{(x_1+x_2)^2+(y_1+y_2)^2} \end{pmatrix}. \qquad (68)$$

The summation law, defined by the formula (68) is associative. The result is an quasi-orthogonal matrix too. There are inverse matrices for (65) – (66) due to their quasi-orthogonality.

The product of the matrices kind of (65) – (66) or (67) is a matrix of the same type. We may write the following multiplication table.

–	A''_{00}	A''_{10}	A''_{01}	A''_{11}
A'_{00}	$\tilde{A}_{00}$	$\tilde{A}_{10}$	$\tilde{A}_{01}$	$\tilde{A}_{11}$
A'_{10}	$\tilde{A}_{10}'$	$\tilde{A}_{00}'$	$\tilde{A}_{11}'$	$\tilde{A}_{01}'$
A'_{01}	$\tilde{A}_{01}$	$\tilde{A}_{11}$	$\tilde{A}_{00}$	$\tilde{A}_{10}$
A'_{11}	$\tilde{A}_{11}'$	$\tilde{A}_{01}'$	$\tilde{A}_{10}'$	$\tilde{A}_{00}'$

Table № 1

(*Note.* First efficient is from the first column, the second – from the upper string. The product is in their crossing. For example: $A'_{10} \cdot A''_{01} = \tilde{A}_{11}$ etc.)

Consequently, the matrices kind of (65) – (66) or (67) (excluding the zero) constitute a group, matrices kind of (65a) (excluding the zero) – is a normal divisor of this group (since they include a identity);matrices kind of (65b) – (66b) (excluding the zero) – adjacent classes by the normal divisor. Together (without zero) they represent (see the table 1) the elements of a commutative factor-group – [1,3]. Table 1 gives a law for multiplication of a normal divisor kind of A_{00} and adjacent classes kind of A_{10}, A_{01} и A_{11}.

Matrices kind of (67) represent by themselves additive groups according to the summation rule (68) since the sum of two matrices is a matrix of the same type; there is a 0 (zero matrix) and an inverse matrix

$$-A^l_{ik}(x,y) = A^l_{ik}(-x,-y.)$$

Now let us show, that the distributive law acts for the matrices kind of (67) with (68). Indeed,

$$A^1_{ik} = \begin{pmatrix} x_1 & y_1 & 0 \\ (-1)^{i+1}y_1 & (-1)^i x_1 & 0 \\ 0 & 0 & (-1)^k\sqrt{x_1^2+y_1^2} \end{pmatrix}, \tag{69}$$

$$A^2_{ik} = \begin{pmatrix} x_2 & y_2 & 0 \\ (-1)^{i+1}y_2 & (-1)^i x_2 & 0 \\ 0 & 0 & (-1)^k\sqrt{x_2^2+y_2^2} \end{pmatrix}, \tag{70}$$

$$A^1_{ik}(+)A^2_{ik} = \begin{pmatrix} x_1+x_2 & y_1+y_2 & 0 \\ (-1)^{i+1}(y_1+y_2) & (-1)^i(x_1+x_2) & 0 \\ 0 & 0 & (-1)^k\sqrt{(x_1+x_2)^2+(y_1+y_2)^2} \end{pmatrix}. \tag{71}$$

$$A^3_{lm} = \begin{pmatrix} x_3 & y_3 & 0 \\ (-1)^{l+1}y_3 & (-1)^l x_3 & 0 \\ 0 & 0 & (-1)^m\sqrt{x_3^2+y_3^2} \end{pmatrix}, \tag{72}$$

$$\left(A^1_{ik}(+)A^2_{ik}\right)A^3_{lm} = \begin{pmatrix} (x_1+x_2)x_3+(y_1+y_2)y_3(-1)^{l+1} & (x_1+x_2)y_3+(y_1+y_2)x_3(-1)^l & 0 \\ (-1)^{i+1}(y_1+y_2)x_3+(-1)^{i+l+1}(x_1+x_2)y_3 & (-1)^{i+1}(y_1+y_2)y_3+(x_1+x_2)x_3(-1)^{i+l} & 0 \\ 0 & 0 & (-1)^{k+m}\sqrt{[(x_1+x_2)^2+(y_1+y_2)^2](x_3^2+y_3^2)} \end{pmatrix}. \tag{73}$$

Farther,

$$A^r_{ik}A^3_{lm} = \begin{pmatrix} (x_r x_3+y_r y_3(-1)^{l+1} & x_r y_3+y_r x_3(-1)^l & 0 \\ (-1)^{i+1}y_r x_3+(-1)^{i+l+1}y_3 x_r & (-1)^{i+1}y_3 y_r+(-1)^{i+l}x_r x_3 & 0 \\ 0 & 0 & (-1)^{k+m}\sqrt{(x_r^2+y_r^2)(x_3^2+y_3^2)} \end{pmatrix}. \tag{74}$$

$(r = 1, 2)$. After some simple calculations we obtain:

$$A^1_{ik}A^3_{lm}(+)A^2_{ik}A^3_{lm} = \begin{pmatrix} (x_1+x_2)x_3+(y_1+y_2)y_3(-1)^{l+1} & (x_1+x_2)y_3+(y_1+y_2)x_3(-1)^{l} & 0 \\ (x_1+x_2)y_3(-1)^{i+l+1}+(y_1+y_2)x_3(-1)^{i+1} & (x_1+x_2)x_3(-1)^{i+l}+(y_1+y_2)y_3(-1)^{i+1} & 0 \\ 0 & 0 & (-1)^{k+m}\sqrt{[(x_1+x_2)^2+(y_1+y_2)^2](x_3^2+y_3^2)} \end{pmatrix}. \quad (75)$$

From (73) и (75) it follows:

$$A^1_{ik}A^3_{lm}(+)A^2_{ik}A^3_{lm} = \big(A^1_{ik}(+)A^2_{ik}\big)A^3_{lm}, \quad (76)$$

i. e. the right distributive law. The left one may be proved in the same way.

We may see now, that quasi-orthogonal A_{ik} matrices with fixed represent additive groups, and their totality (excluding the zero) represents a multiplicative group.

At that A_{00} are isomorphous to complex numbers, if we use the sign $(+)$ while summating, not the sign $+$. The matrices A_{ik}, concerned separately with i, k fixed, represent two dimensional vector spaces over the field of real numbers. Thus, if we use (68) summation rule, the totality of the matrices of the third order A_{ik} represent a hyper-algebra of the fourth order of the second rank with the division over the field of real numbers.

b. It is easy to show with help of table 1, that each of the totalities $A_{00}, A_{10}; A_{00}, A_{01}; A_{00}, A_{11}$ represents hyper-sub-algebra of the second rank with division. (Tables 2,3,4 – the multiplication tables for these totalities, at that the factor-groups of multiplicative groups of each one of three hyper-sub-algebras are isomorphous.)

$-$	A''_{00}	A''_{10}
A'_{00}	$\tilde{A}_{00}$	$\tilde{A}_{10}$
A'_{10}	$\tilde{A}_{10}{}'$	$\tilde{A}_{00}{}'$

$-$	A''_{00}	A''_{01}
A'_{00}	$\tilde{A}_{00}$	$\tilde{A}_{01}$
A'_{01}	$\tilde{A}_{01}$	$\tilde{A}_{00}$

$-$	A''_{00}	A''_{11}
A'_{00}	$\tilde{A}_{00}$	$\tilde{A}_{11}$
A'_{11}	$\tilde{A}_{11}{}'$	$\tilde{A}_{00}{}'$

Tables 2 – 4

In view of the aforesaid we should remark the isomorphism of all finite groups of the given second order.[1]

If we assume i, k in A_{ik} are elements of a finite field Z_2 [2], that take values $0, 1$, at that $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, $1 + 1 = 0$ $(mod\, 2)$, than the aggregate i, k is a two dimensional vector over this field.

From tablers 1 and 2,3,4 we may see that the following condition is always true:

$$A_{i,k}A_{l,m} = A_{i+l,\,k+m}. \quad (77)$$

Consequently, if we constitute a direct sum of $A_{i,k}$

$$\widetilde{A} = \widetilde{A}_{00} \oplus \widetilde{A}_{10} \oplus \widetilde{A}_{01} \oplus \widetilde{A}_{11},$$

and perform multiplication of it's elements A according to distributive law and (77),we obtain a standard graduated algebra[5].

5) Hyper-algebra with division over the field of real numbers, corresponding to a ring of formal real power series.

Let us examine a ring of formal real power series M [1]:

$$M = \sum_{k=0}^{\infty} a_k x^k, \tag{78}$$

where a_k – the real numbers.

The totality of additive groups A_k with elements $a_k x^k$ represents a hyper-algebra of the infinite order with the division over the field of real numbers.

6) Hyper-algebra with division over the field of complex numbers, corresponding to a ring of formal complex Fourier power series kind of

$$F(x) = \sum_{k=-\infty}^{\infty} f_k e^{ikx}, \tag{79}$$

where f_k – complex numbers, k – whole numbers, x – real argument. The totality of additive groups F_k with elements $f_k e^{ikx}$ represents a hyper-algebra with division over the field of complex numbers of the first rank.

7) Real and imaginary axes on a standard group hyper-complex system of the n-th rank represents a hyper-algebra of the first orderwith division over a field of real numbers by these axes.

8) a) *Straight lines on a complex plane, that pass through the point of origin.*

Let us take up a totality of straight lines:

$$z = x + iy = \rho e^{i(\varphi + k\pi)}, \tag{80}$$

k – whole, $0 \leqslant \varphi \leqslant \pi$, $\rho = \sqrt{x^2 + y^2}$.

With even k we have a beam

$$z_1 = \rho e^{i\varphi}, \tag{81}$$

with odd – a beam

$$z_2 = -\rho e^{i\varphi}. \tag{80'}$$

Every straight line (φ is predetermined) represents an additive group. The real axe

$$z = \rho e^{ik\pi} \tag{82}$$

represents a field of real numbers, due it is (excluding the zero) a normal divisor for all straight lines that include the zero, excluding the zero.

Thus, we may see, that the complex plane is a hyper-algebra of the infinite order of the first rank over the field of real numbers by the straight lines, that pass through the point of origin. And, of course, the algebra of the second rank with division over the field of real numbers).

Real and imaginary axes of the complex plane. b) *Real and imaginary axes of the complex plane.*

These straight lines equations are:

$$z_1 = \rho_1 e^{ik_1\pi} - \tag{83}$$

real numbers

$$z_2 = \rho_2 e^{i(\pi/2 + k_2\pi)} - \tag{84}$$

an imaginary number (k_1 and k_2 – whole numbers).

Each one of these straight lines – an additive group.

Farther we have:

$$z_1 \cdot z_1' = \rho_1 \rho_1' e^{i(k_1+k_1')\pi} - \qquad (85)$$

real numbers,

$$z_2 \cdot z_2' = \rho_2 \rho_2' e^{i[\pi+(k_2+k_2')\pi]} = \rho_2 \rho_2' e^{i(k_2+k_2'+1)\pi} - \qquad (86)$$

real numbers,

$$z_1 \cdot z_2 = z_2 \cdot z_1 = \rho_1 \rho_2 e^{[\pi/2+i(k_1+k_2)\pi]} - \qquad (87)$$

imaginary numbers. Besides, an inverse numbers for real case are:

$$1/z_1 = 1/\rho_1 e^{-ik_1\pi} - \qquad (88)$$

real numbers; the inverse for imaginary numbers are

$$1/z_2 = 1/\rho_2 e^{i(-\pi/2-k_2\pi)} = 1/\rho_2 e^{i[\pi/2-(k_2+1)\pi]} - \qquad (89)$$

imaginary numbers.

Purely real and purely imaginary numbers (excluding the zero) represent a multiplicative group, and real numbers are a normal divisor of this group.

Thus, real and imaginary axes jointly represent a hyper-algebra of the second order of the first (by these axes) with division over the field of real numbers.

9) *Real and imaginary axes of quaternions* represent hyper-algebra of 4-th order of the 1-st rank with division over the field of real numbers.

Manifolds of nominate (measuring) real numbers, used in physics and geometry.

10) *Manifolds of nominate (measuring) real numbers, used in physics and geometry.*

Numbers without dimension are a particular case of such numbers. As is known we may summate only numbers with equal dimension (or no measuring at all). Numbers with equal measuring, thus, represent additive groups that intersect each other at zero only. And all real numbers (excluding the zero) represent a multiplicative group. Dimensionless real numbers (excluding the zero) represent a normal divisor of the multiplicative group. Consequently, dimension and dimensionless real numbers represent a hyper-algebra of the infinite order of the first rank with division over the field of dimensionless real numbers. Thus, physical magnitudes are "beyond of" rings.

11) The totality of dimension and dimensionless whole numbers represent a hyper-ring of the infinite order [1], due they may not always be divided one by each other.

three dimensional real vector space with the cross product as a product of two vectors.

12) Three dimensional real vector space with the cross product as a product of two vectors[1]. Vector spaces of polar and axial vectors represent additive groups of manifold M. Due these vectors, as usual, are not straightly summated, their sum is not a ring. The product of two polar vectors is an axial vector; The product of two axial vectors is also an axial vector. A product of an axial vector and a polar one (in any queue) is a polar vector. Consequently, the concerned manifold is a hyper-algebra of the second order of the third rank over the real numbers field. The space of axial vectors is a ring. Thus, the idea that cross product is a product in a ring is not complete.

Note that in all examples given above the factor-goupoids and factor-groups are commutative. Also note that the factor groups in examples 1,2b,7b are isomorphous as groups of the simple (second) order.

The list of hyper-bodies (hyper-fields, hyper-rings) could be continued. As is known, hyper-complex systems represent real algebras of the n-th rank with unity, $n > 2$ and finite. If we include the binary (with summation and multiplication) distributive algebraic structures with unity over the field of real numbers comprising more than one vector space in the hyper-complex systems than it is true that hyper-rings with unity are a generalized hyper-complex system if they are not reducible to rings.

Conclusion

The notions of hyper-ring, hyper-ring, hyper-field, hyper-algebra are direct generalizations of ring, ring, field, algebra, where two binary operations are introduced – the summation and multiplication, which are left- and right-distributive.

It important for physical applications to note the importance of that:

1) Physical values do not constitute the subset of a ring.

2) The cross-product is not, in general, a ring-multiplication in the manifold of polar and axial vectors.

The supplement

The product of quasi-unitary matrices of the second order $A_{\varphi 1} \cdot A_{\varphi 2}$ with quaternion matrix elements and the abrogated multiplication laws.

$$A_{\varphi_1} = \begin{pmatrix} x_1 & y_1 \\ -y_1^+ e^{i\varphi_1} & x_1^+ e^{i\varphi_1} \end{pmatrix}, \quad A_{\varphi_2} = \begin{pmatrix} x_2 & y_2 \\ -y_2^+ e^{i\varphi_2} & x_2^+ e^{i\varphi_2} \end{pmatrix}$$

$$1)\ P_1,\ A_{\varphi_1} \cdot A_{\varphi_2} = \begin{pmatrix} x_1x_2 - y_1y_2^+ e^{i\varphi_2} & x_1y_2 + x_2^+ y_1 e^{i\varphi_2} \\ -y_1^+ x_2 e^{i\varphi_1} - y_2^+ x_1^+ e^{i(\varphi_1+\varphi_2)} & -y_2 y_1^+ e^{i\varphi_1} + x_2^+ x_1^+ e^{i(\varphi_1+\varphi_2)} \end{pmatrix}.$$

$$2)\ P_2,\ A_{\varphi_1} \cdot A_{\varphi_2} = \begin{pmatrix} x_1x_2 - y_1y_2^+ e^{i\varphi_2} & y_2x_1 + y_1 x_2^+ e^{i\varphi_2} \\ -x_2 y_1^+ e^{i\varphi_1} - x_1^+ y_2^+ e^{i(\varphi_1+\varphi_2)} & -y_2 y_1^+ e^{i\varphi_1} + x_2^+ x_1^+ e^{i(\varphi_1+\varphi_2)} \end{pmatrix}.$$

$$3)\ P_3,\ A_{\varphi_1} \cdot A_{\varphi_2} = \begin{pmatrix} x_2x_1 - y_1y_2^+ e^{i\varphi_2} & x_1y_2 + x_2^+ y_1 e^{i\varphi_2} \\ -y_1^+ x_2 e^{i\varphi_1} - y_2^+ x_1^+ e^{i(\varphi_1+\varphi_2)} & -y_2 y_1^+ e^{i\varphi_1} + x_1^+ x_2^+ e^{i(\varphi_1+\varphi_2)} \end{pmatrix}.$$

$$4)\ P_4,\ A_{\varphi_1} \cdot A_{\varphi_2} = \begin{pmatrix} x_2x_1 - y_1y_2^+ e^{i\varphi_2} & y_2x_1 + y_1 x_2^+ e^{i\varphi_2} \\ -x_2 y_1^+ e^{i\varphi_1} - x_1^+ y_2^+ e^{i(\varphi_1+\varphi_2)} & -y_2 y_1^+ e^{i\varphi_1} + x_1^+ x_2^+ e^{i(\varphi_1+\varphi_2)} \end{pmatrix}.$$

$$5)\ P_5,\ A_{\varphi_1} \cdot A_{\varphi_2} = \begin{pmatrix} x_1x_2 - y_2^+ y_1 e^{i\varphi_2} & x_1y_2 + x_2^+ y_1 e^{i\varphi_2} \\ -y_1^+ x_2 e^{i\varphi_1} - y_2^+ x_1^+ e^{i(\varphi_1+\varphi_2)} & -y_1^+ y_2 e^{i\varphi_1} + x_2^+ x_1^+ e^{i(\varphi_1+\varphi_2)} \end{pmatrix}.$$

$$6)\ P_6,\ A_{\varphi_1} \cdot A_{\varphi_2} = \begin{pmatrix} x_1x_2 - y_2^+ y_1 e^{i\varphi_2} & y_2x_1 + y_1 x_2^+ e^{i\varphi_2} \\ -x_2 y_1^+ e^{i\varphi_1} - x_1^+ y_2^+ e^{i(\varphi_1+\varphi_2)} & -y_1^+ y_2 e^{i\varphi_1} + x_2^+ x_1^+ e^{i(\varphi_1+\varphi_2)} \end{pmatrix}.$$

$$7)\ P_7,\ A_{\varphi_1} \cdot A_{\varphi_2} = \begin{pmatrix} x_2x_1 - y_2^+ y_1 e^{i\varphi_2} & x_1y_2 + x_2^+ y_1 e^{i\varphi_2} \\ -y_1^+ x_2 e^{i\varphi_1} - y_2^+ x_1^+ e^{i(\varphi_1+\varphi_2)} & -y_1^+ y_2 e^{i\varphi_1} + x_1^+ x_2^+ e^{i(\varphi_1+\varphi_2)} \end{pmatrix}.$$

$$8)\ P_8,\ A_{\varphi_1} \cdot A_{\varphi_2} = \begin{pmatrix} x_2x_1 - y_2^+ y_1 e^{i\varphi_2} & y_2x_1 + y_1 x_2^+ e^{i\varphi_2} \\ -x_2 y_1^+ e^{i\varphi_1} - x_1^+ y_2^+ e^{i(\varphi_1+\varphi_2)} & -y_1^+ y_2 e^{i\varphi_1} + x_1^+ x_2^+ e^{i(\varphi_1+\varphi_2)} \end{pmatrix}.$$

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DEFORMATION PRINCIPLE AS FOUNDATION OF PHYSICAL GEOMETRY AND ITS APPLICATION TO SPACE-TIME GEOMETRY

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Physical geometry studies mutual disposition of geometrical objects and points in space, or space-time, which is described by the distance function d, or by the world function $\sigma = d^2/2$. One suggests a new general method of the physical geometry construction. The proper Euclidean geometry is described in terms of its world function $\sigma_{\rm E}$. Any physical geometry $\mathcal{G}$ is obtained from the Euclidean geometry as a result of replacement of the Euclidean world function $\sigma_{\rm E}$ by the world function σ of $\mathcal{G}$. This method is very simple and effective. It introduces a new geometric property: nondegeneracy of geometry. Using this method, one can construct deterministic space-time geometries with primordially stochastic motion of free particles and geometrized particle mass. Such a space-time geometry defined properly (with quantum constant as an attribute of geometry) allows one to explain quantum effects as a result of the statistical description of the stochastic particle motion (without a use of quantum principles).

Introduction

A geometry lies in the foundation of physics, and a true conception of geometry is very important for the consequent development of physics. It is common practice to think that all problems in foundations of geometry have been solved many years ago. It is valid, but this concerns the geometry considered to be a logical construction. Physicists are interested in the geometry considered as a science on mutual disposition of geometrical objects in the space or in the space-time. The two aspects of geometry are quite different, and one can speak about two different geometries, using for them two different terms. Geometry as a logical construction is a homogeneous geometry, where all points have the same properties. Well known mathematician Felix Klein [1] believed that only the homogeneous geometry deserves to be called a geometry. It is his opinion that the Riemannian geometry (in general, inhomogeneous geometry) should be qualified as a Riemannian geography, or a Riemannian topography. In other words, Felix Klein considered a geometry mainly as a logical construction. We shall refer to such a geometry as the mathematical geometry.

The geometry considered to be a science on mutual disposition of geometric objects will be referred to as a physical geometry, because the physicists are interested mainly in this aspect of a geometry. The physical geometries are inhomogeneous, in general, although they may be homogeneous also. On the one hand, the proper Euclidean geometry is a physical geometry. On the other hand, it is a logical construction, because it is homogeneous and can be constructed of simple elements (points, straights, planes, etc.). All elements of the Euclidean geometry have similar properties, which are described by axioms. Similarity of geometrical elements allows one to construct the mathematical (homogeneous) geometry by means of logical reasonings. The proper Euclidean geometry was constructed many years ago by Euclid. Consistency of this construction was investigated and proved in [2]. Such a construction is very complicated even in the case of the proper Euclidean geometry, because simple geometrical objects are used for construction of the more complicated ones, and one cannot construct a complicated geometrical object $\mathcal{O}$ without construction of the more simple constituents of this object.

Note that constructing his geometry, Euclid did not use coordinates for labeling of the space points. His description of the homogeneous geometry was coordinateless. It means that the coordinates are not a necessary attribute of the geometry. Coordinate system is a method of the geometry description, which may or may not be used. Application of coordinates and of other means of description poses the problem of separation of the geometry properties from the properties of the means of the description. Usually the separation of the geometry properties from the coordinate system properties is carried out as follows. The geometry is described in all possible coordinate systems. Transformations from one coordinate system to the another one form a group of transformation. Invariants of this transformation group are the same in all coordinate system, and hence, they describe properties of the geometry in question.

At this point we are to make a very important remark. *Any geometry is a totality of all geometric objects* $\mathcal{O}$ *and of all relations* $\mathcal{R}$ *between them.* Any geometric object $\mathcal{O}$ is a subset of points of some point set Ω, where the geometry is given. In the Riemannian geometry (and in other inhomogeneous geometries) the set Ω is supposed to be a n-dimensional manifold, whose points P are labelled by n coordinates $x = \{x^1, x^2, ...x^n\}$. This labelling (arithmetization of space) is considered to be a necessary attribute of the Riemannian geometry. Most geometers believe that the Riemannian geometry (and physical geometry), in general cannot be constructed without introduction of the manifold. In other words, they believe that the manifold is an attribute of the Riemannian geometry (and of any continuous geometry, in general). This belief is founded on the fact, that the Riemannian geometry is always constructed on some manifold. But this belief is a delusion. The fact, that we always construct the physical geometry on some manifold, does not mean that the physical geometry cannot be constructed without a reference to a manifold, or to a coordinate system. Of course, some labelling of the spatial points (coordinate system) is convenient, but this labelling has no relations to the construction of the geometry, and the physical geometry should be constructed without a reference to coordinate system. Application of the coordinate system imposes constraints on properties of the constructed physical geometry. For instance, if we use a continuous coordinate system (manifold) we can construct only continuous physical geometry. To construct a discrete physical geometry, the geometry construction is not to contain a reference to the coordinate system.

Here we present the method of the physical geometry construction, which does not contain a reference to the coordinate system and other means of description. It contains a reference only to the distance function d, which is a real characteristic of physical geometry.

If a geometry is inhomogeneous, and the straights located in different places have different properties, it is impossible to describe properties of straights by means of axioms, because there are no such axioms for the whole geometry. Mutual disposition of points in a physical (inhomogeneous) geometry, which is given on the set Ω of points P, is described by the distance function $d(P,Q)$

$$d: \qquad \Omega\times\Omega\rightarrow \mathbf{R}, \qquad d(P,P)=0, \qquad \forall P\in\Omega \tag{1}$$

where $\mathbf{R}$ denotes the set of all real numbers. The distance function d is the main characteristic of the physical geometry. Besides, the distance function d is an *unique* *characteristic* of any physical geometry. *The distance function d determines completely the physical geometry, and one does not need any additional information for determination of the physical geometry.* This statement is very important for construction of a physical geometry. It will be proved below. Any physical geometry $\mathcal{G}$ is constructed on the basis of the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ by means of a deformation, i.e. by a replacement of the Euclidean distance function d_{E} by the distance function of the geometry in question. For instance, constructing the Riemannian geometry, we replace the Euclidean infinitesimal distance $dS_{\mathrm{E}} = \sqrt{g_{\mathrm{E}ik}dx^i dx^k}$ by the Riemannian one $dS = \sqrt{g_{ik}dx^i dx^k}$. There is no method of the inhomogeneous physical geometry construction other, than the deformation of the Euclidean geometry (or some other homogeneous geometry) which is constructed as a mathematical geometry on the basis of its axiomatics and logic. Unfortunately, conventional method of the Riemannian geometry construction contains a reference to the coordinate system. But this reference can be eliminated, provided that we use finite distances d instead of infinitesimal distances dS.

For description of a physical geometry one uses the world function σ [3], which is connected with the distance function d by means of the relation $\sigma(P,Q) = \frac{1}{2}d^2(P,Q)$. The world function σ of the σ-space $V = \{\sigma, \Omega\}$ is defined by the relation

$$\sigma: \qquad \Omega \times \Omega \to \mathbf{R}, \qquad \sigma(P,P) = 0, \qquad \forall P \in \Omega \tag{2}$$

where $\mathbf{R}$ denotes the set of all real numbers. Application of the world function is more convenient in the relation that the world function is real, when the distance function d is imaginary and does not satisfy definition (1). It is important at the consideration of the space-time geometry as a physical geometry.

In general, a physical geometry cannot be constructed as a logical building, because any change of the world function should be accompanied by a change of axiomatics. This is practically aerial, because the set of possible physical geometries is continual. Does the world function contain full information which is necessary for construction of the physical geometry? It is a very important question. For instance, can one derive the space dimension from the world function in the case of Euclidean geometry? Slightly below we shall answer this question in the affirmative. Now we formulate the method of the physical geometry construction.

Let us imagine that the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ can be described completely in terms and only in terms of the Euclidean world function σ_{E}. Such a description is called σ-immanent. It means that any geometrical object $\mathcal{O}_{\mathrm{E}}$ and any relation $\mathcal{R}_{\mathrm{E}}$ between geometrical objects in $\mathcal{G}_{\mathrm{E}}$ can be described in terms of σ_{E} in the form $\mathcal{O}_{\mathrm{E}}(\sigma_{\mathrm{E}})$ and $\mathcal{R}_{\mathrm{E}}(\sigma_{\mathrm{E}})$. To obtain corresponding geometrical object $\mathcal{O}$ and corresponding relation $\mathcal{R}$ between the geometrical objects in other physical geometry $\mathcal{G}$, it is sufficient to replace the Euclidean world function σ_{E} by the world function σ of the physical geometry $\mathcal{G}$ in description of $\mathcal{O}_{\mathrm{E}}(\sigma_{\mathrm{E}})$ and $\mathcal{R}_{\mathrm{E}}(\sigma_{\mathrm{E}})$.

$$\mathcal{O}_{\mathrm{E}}(\sigma_{\mathrm{E}}) \to \mathcal{O}_{\mathrm{E}}(\sigma), \qquad \mathcal{R}_{\mathrm{E}}(\sigma_{\mathrm{E}}) \to \mathcal{R}_{\mathrm{E}}(\sigma)$$

Index 'E' shows that the geometric object is constructed on the basis of the Euclidean axiomatics. Thus, one can obtain another physical geometry $\mathcal{G}$ from the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ by a simple replacement of σ_{E} by σ. For such a construction one needs no axiomatics and no reasonings. One needs no means of descriptions (topological structures, continuity, coordinate system, manifold, dimension, etc.). In fact, one uses implicitly the axiomatics of the Euclidean geometry, which is deformed by the replacement $\sigma_{\mathrm{E}} \to \sigma$. This replacement may be interpreted as a deformation of the Euclidean space. Absence of a reference to the means of description is an advantage of the considered method of the geometry construction. Besides, there is no necessity to construct the whole geometry $\mathcal{G}$. We can construct and investigate only that part of the geometry $\mathcal{G}$ which we are interested in. Any physical geometry may be constructed as a result of a deformation of the Euclidean geometry.

The geometric object $\mathcal{O}$ is described by means of the skeleton-envelope method [4]. It means that any geometric object $\mathcal{O}$ is considered to be a set of intersections and joins of elementary geometric objects (EGO).

The finite set $\mathcal{P}^n \equiv \{P_0, P_1, ..., P_n\} \subset \Omega$ of parameters of the envelope function $f_{\mathcal{P}^n}$ is the skeleton of elementary geometric object (EGO) $\mathcal{E} \subset \Omega$. The set $\mathcal{E} \subset \Omega$ of points forming EGO is called the envelope of its skeleton $\mathcal{P}^n$. For continuous physical geometry the envelope $\mathcal{E}$ is usually a continual set of points. The envelope function $f_{\mathcal{P}^n}$, determining EGO is a function of the running point $R \in \Omega$ and of parameters $\mathcal{P}^n \subset \Omega$. The envelope function $f_{\mathcal{P}^n}$ is supposed to be an algebraic function of s arguments $w = \{w_1, w_2, ...w_s\}$, $s = (n+2)(n+1)/2$. Each of arguments $w_k = \sigma(Q_k, L_k)$ is a σ-function of two arguments $Q_k, L_k \in \{R, \mathcal{P}^n\}$, either belonging to skeleton $\mathcal{P}^n$, or coinciding with the running point R. Thus, any elementary geometric object $\mathcal{E}$ is determined by its skeleton and its envelope function.

For instance, the sphere $\mathcal{S}(P_0, P_1)$ with the center at the point P_0 is determined by the relation

$$\mathcal{S}(P_0, P_1) = \{R | f_{P_0 P_1}(R) = 0\}, \qquad f_{P_0 P_1}(R) = \sqrt{2\sigma(P_0, P_1)} - \sqrt{2\sigma(P_0, R)} \qquad (3)$$

where P_1 is a point belonging to the sphere. The elementary object $\mathcal{E}$ is determined in all physical geometries at once. In particular, it is determined in the proper Euclidean geometry, where we can obtain its meaning. We interpret the elementary geometrical object $\mathcal{E}$, using our knowledge of the proper Euclidean geometry. Thus, the proper Euclidean geometry is used as a sample geometry for interpretation of any physical geometry.

We do not try to repeat subscriptions of Euclid at construction of the geometry. We take the geometrical objects and relations between them, prepared in the framework of the Euclidean geometry and describe them in terms of the world function. Thereafter we deform them, replacing the Euclidean world function σ_{E} by the world function σ of the geometry in question. In practice the construction of the elementary geometry object is reduced to the representation of the corresponding Euclidean geometrical object in the σ-immanent form, i. e. in terms of the Euclidean world function. The last problem is the problem of the proper Euclidean geometry. The problem of representation of the geometrical object (or relation between objects) in the σ-immanent form is a real problem of the physical geometry construction.

It is very important, that such a construction does not use coordinates and other methods of description, because the application of the means of description imposes constraints on the constructed geometry. Any means of description is a structure St given on the basic Euclidean geometry with the world function σ_{E}. Replacement $\sigma_{\mathrm{E}} \to \sigma$ is sufficient for construction of unique physical geometry $\mathcal{G}_\sigma$. If we use an additional structure St for construction of physical geometry, we obtain, in general, other geometry $\mathcal{G}_{St}$, which coincide with $\mathcal{G}_\sigma$ not for all σ, but only for some of world functions σ. Thus, a use of additional means of description restricts the list of possible physical geometries. For instance, if we use the coordinate description at construction of the physical geometry, the obtained geometry appears to be continuous, because description by means of the coordinates is effective only for continuous geometries, where the number of coordinates coincides with the geometry dimension.

Constructing geometry $\mathcal{G}$ by means of a deformation we use essentially the fact that the proper Euclidean geometry $\mathcal{G}_{\rm E}$ is a mathematical geometry, which has been constructed on the basis of Euclidean axiomatics and logical reasonings.

We shall refer to the described method of the physical geometry construction as the deformation principle and interpret the deformation in the broad sense of the word. In particular, a deformation of the Euclidean space may transform an Euclidean surface into a point, and an Euclidean point into a surface. Such a deformation may remove some points of the Euclidean space, violating its continuity, or decreasing its dimension. Such a deformation may add supplemental points to the Euclidean space, increasing its dimension. In other words, the deformation principle is a very general method of the physical geometry construction.

The deformation principle as a method of the physical geometry construction contains two essential stages:

(i) Representation of geometrical objects $\mathcal{O}$ and relations $\mathcal{R}$ of the Euclidean geometry in the σ-immanent form, i.e. in terms and only in terms of the world function $\sigma_{\rm E}$.

(ii) Replacement of the Euclidean world function $\sigma_{\rm E}$ by the world function σ of the geometry in question.

A physical geometry, constructed by means of the only deformation principle (i.e. without a use of other methods of the geometry construction) is called T-geometry (tubular geometry) [5, 4, 6]. The T-geometry is the most general kind of the physical geometry.

Application of the deformation principle is restricted by two constraints.

1. Describing Euclidean geometric objects $\mathcal{O}(\sigma_{\rm E})$ and Euclidean relation $\mathcal{R}(\sigma_{\rm E})$ in terms of $\sigma_{\rm E}$, we are not to use special properties of Euclidean world function $\sigma_{\rm E}$. In particular, definitions of $\mathcal{O}(\sigma_{\rm E})$ and $\mathcal{R}(\sigma_{\rm E})$ are to have similar form in Euclidean geometries of different dimensions. They must not depend on the dimension of the Euclidean space.

2. The deformation principle is to be applied separately from other methods of the geometry construction. In particular, one may not use topological structures in construction of a physical geometry, because for effective application of the deformation principle the obtained physical geometry must be determined only by the world function (metric).

Description of the proper Euclidean space in terms of the world function

The crucial point of the T-geometry construction is the description of the proper Euclidean geometry in terms of the Euclidean world function $\sigma_{\rm E}$. We shall refer to this method of description as the σ-immanent description. Unfortunately, it was unknown for many years, although all physicists knew that the infinitesimal interval $dS = \sqrt{g_{ik}dx^i dx^k}$ is the unique essential characteristic of the space-time geometry, and changing this expression, we change the space-time geometry. From physical viewpoint the σ-immanent description is very reasonable, because it does not contain any extrinsic information. The σ-immanent description does not refer to the means of description (dimension, manifold, coordinate system). Absence of references to means of description is important in the relation, that there is no necessity to separate the information on the geometry in itself from the information on the means of description. The σ-immanent description contains only essential characteristic of geometry: its world function. At first the σ-immanent description was obtained in 1990 [5].

The first question concerning the σ-immanent description is as follows. Does the world function contain sufficient information for description of a physical geometry? The answer is affirmative, at least, in the case of the proper Euclidean geometry, and this answer is given by the prove of the following theorem.

Let σ-space $V=\{\sigma,\Omega\}$ be a set Ω of points P with the given world function σ

$$\sigma: \qquad \Omega\times\Omega\rightarrow\mathbf{R}, \qquad \sigma(P,P)=0, \qquad \forall P\in\Omega \tag{4}$$

where $\mathbf{R}$ denotes the set of all real numbers. Let the vector $\mathbf{P}_0\mathbf{P}_1=\{P_0,P_1\}$ be the ordered set of two points P_0, P_1, and its length $|\mathbf{P}_0\mathbf{P}_1|$ is defined by the relation $|\mathbf{P}_0\mathbf{P}_1|^2=2\sigma(P_0,P_1)$.

Theorem

The σ-space $V=\{\sigma,\Omega\}$ is the n-dimensional proper Euclidean space, if and only if the world function σ satisfies the following conditions, written in terms of the world function σ.

I. Condition of symmetry:

$$\sigma(P,Q)=\sigma(Q,P), \qquad \forall P,Q\in\Omega \tag{5}$$

II. Definition of the dimension:

$$\exists\mathcal{P}^n\equiv\{P_0,P_1,...P_n\}, \qquad F_n(\mathcal{P}^n)\neq0, \qquad F_k\left(\Omega^{k+1}\right)=0, \qquad k>n \tag{6}$$

where $F_n(\mathcal{P}^n)$ is the Gram's determinant

$$F_n(\mathcal{P}^n)=\det||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)||=\det||g_{ik}(\mathcal{P}^n)||, \qquad i,k=1,2,...n \tag{7}$$

The scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is defined by the relation

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)=\sigma(P_0,Q_1)+\sigma(P_1,Q_0)-\sigma(P_0,Q_0)-\sigma(P_1,Q_1) \tag{8}$$

Vectors $\mathbf{P}_0\mathbf{P}_i$, $i=1,2,...n$ are basic vectors of the rectilinear coordinate system K_n with the origin at the point P_0, and the metric tensors $g_{ik}(\mathcal{P}^n)$, $g^{ik}(\mathcal{P}^n)$, $i,k=1,2,...n$ in K_n are defined by the relations

$$\sum_{k=1}^{k=n}g^{ik}(\mathcal{P}^n)\,g_{lk}(\mathcal{P}^n)=\delta_l^i, \qquad g_{il}(\mathcal{P}^n)=(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_l), \qquad i,l=1,2,...n \tag{9}$$

III. Linear structure of the Euclidean space:

$$\sigma(P,Q)=\frac{1}{2}\sum_{i,k=1}^{i,k=n}g^{ik}(\mathcal{P}^n)\,(x_i(P)-x_i(Q))\,(x_k(P)-x_k(Q)), \qquad \forall P,Q\in\Omega \tag{10}$$

where coordinates $x_i(P)$, $i=1,2,...n$ of the point P are covariant coordinates of the vector $\mathbf{P}_0\mathbf{P}$, defined by the relation

$$x_i(P)=(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}), \qquad i=1,2,...n \tag{11}$$

IV: The metric tensor matrix $g_{lk}(\mathcal{P}^n)$ has only positive eigenvalues

$$g_k>0, \qquad k=1,2,...,n \tag{12}$$

V. The continuity condition: the system of equations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}) = y_i \in \mathbf{R}, \qquad i = 1, 2, ...n \tag{13}$$

considered to be equations for determination of the point P as a function of coordinates $y = \{y_i\}$, $i = 1, 2, ...n$ has always one and only one solution. Conditions II – V contain a reference to the dimension n of the Euclidean space.

As far as the σ-immanent description of the proper Euclidean geometry is possible, it is possible for any T-geometry, because any geometrical object $\mathcal{O}$ and any relation $\mathcal{R}$ in the physical geometry $\mathcal{G}$ is obtained from the corresponding geometrical object $\mathcal{O}_\mathrm{E}$ and from the corresponding relation $\mathcal{R}_\mathrm{E}$ in the proper Euclidean geometry $\mathcal{G}_\mathrm{E}$ by means of the replacement $\sigma_\mathrm{E} \to \sigma$ in description of $\mathcal{O}_\mathrm{E}$ an $\mathcal{R}_\mathrm{E}$. For such a replacement be possible, the description of $\mathcal{O}_\mathrm{E}$ and $\mathcal{R}_\mathrm{E}$ is not to refer to special properties of σ_E, described by conditions II – V. A formal indicator of the conditions II – V application is a reference to the dimension n, because any of conditions II – V contains a reference to the dimension n of the proper Euclidean space.

If nevertheless we use one of special properties II – V of the Euclidean space in the σ-immanent description of a geometrical object $\mathcal{O}$, or relation $\mathcal{R}$, we refer to the dimension n and, ultimately, to the coordinate system, which is only a means of description.

Let us show this in the example of the determination of the straight in the n-dimensional Euclidean space. The straight $\mathcal{T}_{P_0Q}$ in the proper Euclidean space is defined by two its points P_0 and Q $(P_0 \neq Q)$ as the set of points R

$$\mathcal{T}_{P_0Q} = \{R \mid \mathbf{P}_0\mathbf{Q}||\mathbf{P}_0\mathbf{R}\} \tag{14}$$

where condition $\mathbf{P}_0\mathbf{Q}||\mathbf{P}_0\mathbf{R}$ means that vectors $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ are collinear, i.e. the scalar product $(\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{R})$ of these two vectors satisfies the relation

$$\mathbf{P}_0\mathbf{Q}||\mathbf{P}_0\mathbf{R}: \qquad (\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{R})^2 = (\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{Q})\,(\mathbf{P}_0\mathbf{R}.\mathbf{P}_0\mathbf{R}) \tag{15}$$

where the scalar product is defined by the relation (8). Thus, the straight line $\mathcal{T}_{P_0Q}$ is defined σ-immanently, i. e. in terms of the world function σ. We shall use two different names (straight and tube) for the geometric object $\mathcal{T}_{P_0Q}$. We shall use the term "straight", when we want to stress that $\mathcal{T}_{P_0Q}$ is a result of deformation of the Euclidean straight. We shall use the term "tube", when we want to stress that $\mathcal{T}_{P_0Q}$ may be a many-dimensional surface.

In the Euclidean geometry one can use another definition of collinearity. Vectors $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ are collinear, if components of vectors $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ in some coordinate system are proportional. For instance, in the n-dimensional Euclidean space one can introduce rectangular coordinate system, choosing $n+1$ points $\mathcal{P}^n = \{P_0, P_1, ...P_n\}$ and forming n basic vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1, 2, ...n$. Then the collinearity condition can be written in the form of n equations

$$\mathbf{P}_0\mathbf{Q}||\mathbf{P}_0\mathbf{R}: \qquad (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{Q}) = a\,(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{R})\,, \qquad i = 1, 2, ...n, \tag{16}$$

where a is some real constant. Relations (16) are relations for covariant components of vectors $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ in the considered coordinate system with basic vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1, 2, ...n$. Let points $\mathcal{P}^n$ be chosen in such a way, that $(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}) \neq 0$. Then eliminating the parameter a from relations (16), we obtain $n-1$ independent relations, and the geometrical object

$$\mathcal{T}_{Q\mathcal{P}^n} = \{R \mid \mathbf{P}_0\mathbf{Q}||\mathbf{P}_0\mathbf{R}\} = \bigcap_{i=2}^{i=n} \mathcal{S}_i, \tag{17}$$

$$\mathcal{S}_i = \left\{R \left| \frac{(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{Q})}{(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q})} = \frac{(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{R})}{(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R})} \right.\right\}, \qquad i = 2, 3, ...n \tag{18}$$

defined according to (16), depends on $n+2$ points $Q,\mathcal{P}^n$. This geometrical object $\mathcal{T}_{Q\mathcal{P}^n}$ is defined σ-immanently. It is a complex, consisting of the straight line and the coordinate system, represented by $n+1$ points $\mathcal{P}^n = \{P_0, P_1, ...P_n\}$. In the Euclidean space the dependence on the choice of the coordinate system and on $n+1$ points $\mathcal{P}^n$ determining this system, is fictitious. The geometrical object $\mathcal{T}_{Q\mathcal{P}^n}$ depends only on two points P_0, Q and coincides with the straight line $\mathcal{T}_{P_0Q}$. But at deformations of the Euclidean space the geometrical objects $\mathcal{T}_{Q\mathcal{P}^n}$ and $\mathcal{T}_{P_0Q}$ are deformed differently. The points $P_1, P_2, ...P_n$ cease to be fictitious in definition of $\mathcal{T}_{Q\mathcal{P}^n}$, and geometrical objects $\mathcal{T}_{Q\mathcal{P}^n}$ and $\mathcal{T}_{P_0Q}$ become to be different geometric objects, in general. But being different, in general, they may coincide in some special cases.

What of the two geometrical objects in the deformed geometry should be interpreted as the straight line, passing through the points P_0 and Q in the geometry $\mathcal{G}$? Of course, it is $\mathcal{T}_{P_0Q}$, because its definition does not contain a reference to a coordinate system, whereas definition of $\mathcal{T}_{Q\mathcal{P}^n}$ depends on the choice of the coordinate system, represented by points $\mathcal{P}^n$. In general, definitions of geometric objects and relations between them are not to refer to the means of description.

But in the given case the geometrical object $\mathcal{T}_{P_0Q}$ is, in general, $(n-1)$-dimensional surface, whereas $\mathcal{T}_{Q\mathcal{P}^n}$ is an intersection of $(n-1)$ $(n-1)$-dimensional surfaces, i.e. $\mathcal{T}_{Q\mathcal{P}^n}$ is, in general, a one-dimensional curve. The one-dimensional curve $\mathcal{T}_{Q\mathcal{P}^n}$ corresponds better to our ideas on the straight line, than the $(n-1)$-dimensional surface $\mathcal{T}_{P_0Q}$. Nevertheless, in physical geometry $\mathcal{G}$ it is $\mathcal{T}_{P_0Q}$, that is an analog of the Euclidean straight line.

It is very difficult to overcome our conventional idea that the Euclidean straight line cannot be deformed into many-dimensional surface, and *this idea has been prevent for years from construction of T-geometries*. Practically one uses such physical geometries, where deformation of the Euclidean space transforms the Euclidean straight lines into one-dimensional lines. It means that one chooses such geometries, where geometrical objects $\mathcal{T}_{P_0Q}$ and $\mathcal{T}_{Q\mathcal{P}^n}$ coincide.

$$\mathcal{T}_{P_0Q} = \mathcal{T}_{Q\mathcal{P}^n} \tag{19}$$

Condition (19) of coincidence of the objects $\mathcal{T}_{P_0Q}$ and $\mathcal{T}_{Q\mathcal{P}^n}$, imposed on the T-geometry, restricts list of possible T-geometries.

Let us consider the metric geometry, given on the set Ω of points. The metric space $M = \{\rho, \Omega\}$ is given by the metric (distance) ρ.

$$\rho \ : \quad \Omega \times \Omega \to [0, \infty) \subset \mathbf{R} \tag{20}$$

$$\rho(P,P) = 0, \qquad \rho(P,Q) = \rho(Q,P), \qquad \forall P,Q \in \Omega \tag{21}$$

$$\rho(P,Q) \geq 0, \qquad \rho(P,Q) = 0, \quad \text{iff } P = Q, \qquad \forall P,Q \in \Omega \tag{22}$$

$$0 \leq \rho(P,R) + \rho(R,Q) - \rho(P,Q), \qquad \forall P,Q,R \in \Omega \tag{23}$$

where $\mathbf{R}$ denotes the set of all real numbers. At first sight the metric space is a special case of the σ-space (4), and the metric geometry is a special case of the T-geometry with additional constraints (22), (23) imposed on the world function $\sigma = \frac{1}{2}\rho^2$. However it is not so, because the metric geometry does not use the deformation principle. The fact, that the Euclidean geometry can be described σ-immanently, as well as the conditions (6) - (13), were not known until 1990. Additional (with respect to the σ-space) constraints (22), (23) are imposed to eliminate the situation, when the straight line is not a one-dimensional line. The fact is that, in the metric geometry the shortest (straight) line can be constructed only in the case, when it is one-dimensional.

Let us consider the set $\mathcal{EL}(P,Q,a)$ of points R

$$\mathcal{EL}(P,Q,a) = \{R | f_{P,Q,a}(R) = 0\}, \qquad f_{P,Q,a}(R) = \rho(P,R) + \rho(R,Q) - 2a \tag{24}$$

If the metric space coincides with the proper Euclidean space, this set of points is an ellipsoid with focuses at the points P, Q and the large semiaxis a. The relations $f_{P,Q,a}(R) > 0$, $f_{P,Q,a}(R) = 0$, $f_{P,Q,a}(R) < 0$ determine respectively external points, boundary points and internal points of the ellipsoid. If $\rho(P,Q) = 2a$, we obtain the degenerate ellipsoid, which coincides with the segment $\mathcal{T}_{[PQ]}$ of the straight line, passing through the points P, Q. In the proper Euclidean geometry, the degenerate ellipsoid is one-dimensional segment of the straight line, but it is not evident that it is one-dimensional in the case of arbitrary metric geometry. For such a degenerate ellipsoid be one-dimensional in the arbitrary metric space, it is necessary that any degenerate ellipsoid $\mathcal{EL}(P, Q, \rho(P,Q)/2)$ have no internal points. This constraint is written in the form

$$f_{P,Q,\rho(P,Q)/2}(R) = \rho(P,R) + \rho(R,Q) - \rho(P,Q) \geq 0 \tag{25}$$

Comparing relation (25) with (23), we see that the constraint (23) is introduced to make the straight (shortest) line to be one-dimensional (absence of internal points in the geometrical object determined by two points).

As far as the metric geometry does not use the deformation principle, it is a poor geometry, because in the framework of this geometry one cannot construct the scalar product of two vectors, define linear independence of vectors and construct such geometrical objects as planes. All these objects as well as other are constructed on the basis of the deformation of the proper Euclidean geometry.

Generalizing the metric geometry, Menger [7] and Blumenthal [8] removed the triangle axiom (23). They tried to construct the distance geometry, which would be a more general geometry, than the metric one. As far as they did not use the deformation principle, they could not determine the shortest (straight) line without a reference to the topological concept of the curve $\mathcal{L}$, defined as a continuous mapping

$$\mathcal{L}: \qquad [0,1] \to \Omega \tag{26}$$

which cannot be expressed only via the distance. As a result the distance geometry appeared to be not a pure metric geometry, what the T-geometry is.

Conditions of the deformation principle application

Riemannian geometries satisfy the condition (19). The Riemannian geometry is a kind of inhomogeneous physical geometry, and, hence, it uses the deformation principle. Constructing the Riemannian geometry, the infinitesimal Euclidean distance is deformed into the Riemannian distance. The deformation is chosen in such a way that any Euclidean straight line $\mathcal{T}_{\mathrm{E}P_0Q}$, passing through the point P_0, collinear to the vector $\mathbf{P}_0\mathbf{Q}$, transforms into the geodesic $\mathcal{T}_{P_0Q}$, passing through the point P_0, collinear to the vector $\mathbf{P}_0\mathbf{Q}$ in the Riemannian space.

Note that in T-geometries, satisfying the condition (19) for all points $Q, \mathcal{P}^n$, the straight line

$$\mathcal{T}_{Q_0;P_0Q} = \{R \mid \mathbf{P}_0\mathbf{Q} || \mathbf{Q}_0\mathbf{R}\} \tag{27}$$

passing through the point Q_0 collinear to the vector $\mathbf{P}_0\mathbf{Q}$, is not a one-dimensional line, in general. If the Riemannian geometries be T-geometries, they would contain non-one-dimensional geodesics (straight lines). But the Riemannian geometries are not T-geometries, because at their construction one uses not only the deformation principle, but some other methods, containing a reference to the means of description. In particular, in the Riemannian geometries the absolute parallelism is absent, and one cannot to define a straight line (27), because the relation $\mathbf{P}_0\mathbf{Q}||\mathbf{Q}_0\mathbf{R}$ is not defined, if points P_0 and Q_0 do not coincide. On one hand, a lack of absolute parallelism allows one to go around the problem of non-one-dimensional straight lines. On the other hand, it makes the Riemannian geometries to be inconsistent, because they cease to be T-geometries, which are consistent by the construction (see for details [9]).

The fact is that the application of *only deformation principle* is sufficient for construction of a physical geometry. Besides, such a construction is consistent, because the original Euclidean geometry is consistent and, deforming it, we do not use any reasonings. If we introduce additional structure (for instance, a topological structure) we obtain a fortified physical geometry, i.e. a physical geometry with additional structure on it. The physical geometry with additional structure on it is a more pithy construction, than the physical geometry simply. But it is valid only in the case, when we consider the additional structure as an addition to the physical geometry. If we use an additional structure in construction of the geometry, we identify the additional structure with one of structures of the physical geometry. If we demand that the additional structure to be a structure of physical geometry, we restrict an application of the deformation principle and reduce the list of possible physical geometries, because coincidence of the additional structure with some structure of a physical geometry is possible not for all physical geometries, but only for some of them.

Let, for instance, we use concept of a curve $\mathcal{L}$ (26) for construction of a physical geometry. The concept of curve $\mathcal{L}$, considered as a continuous mapping is a topological structure, which cannot be expressed only via the distance or via the world function. A use of the mapping (26) needs an introduction of topological space and, in particular, the concept of continuity. If we identify the topological curve (26) with the "metrical" curve, defined as a broken line

$$\mathcal{T}_{\text{br}} = \bigcup_i \mathcal{T}_{[P_i P_{i+1}]}, \qquad \mathcal{T}_{[P_i P_{i+1}]} = \left\{ R | \sqrt{2\sigma(P_i, P_{i+1})} - \sqrt{2\sigma(P_i, R)} - \sqrt{2\sigma(R, P_{i+1})} \right\} \tag{28}$$

consisting of the straight line segments $\mathcal{T}_{[P_i P_{i+1}]}$ between the points P_i, P_{i+1}, we truncate the list of possible geometries, because such an identification is possible only in some physical geometries. Identifying (26) and (28), we eliminate all discrete physical geometries and those continuous physical geometries, where the segment $\mathcal{T}_{[P_i P_{i+1}]}$ of straight line is a surface, but not a one-dimensional set of points. Thus, additional structures may lead to (i) a fortified physical geometry, (ii) a restricted physical geometry and (iii) a restricted fortified physical geometry. The result depends on the method of the additional structure application.

Note that some constraints (continuity, convexity, lack of absolute parallelism), imposed on physical geometries are a result of a disagreement of the applied means of the geometry construction. In the T-geometry, which uses only the deformation principle, there are no such restrictions. Besides, the T-geometry accepts some new property of a physical geometry, which is not accepted by conventional versions of physical geometry. This property, called the geometry nondegeneracy, follows directly from the application of arbitrary deformations to the proper Euclidean geometry.

The geometry is degenerate at the point P_0 in the direction of the vector $\mathbf{Q}_0\mathbf{Q}$, $|\mathbf{Q}_0\mathbf{Q}| \neq 0$, if the relations

$$\mathbf{Q}_0\mathbf{Q} \uparrow\uparrow \mathbf{P}_0\mathbf{R}: \qquad (\mathbf{Q}_0\mathbf{Q}.\mathbf{P}_0\mathbf{R}) = \sqrt{|\mathbf{Q}_0\mathbf{Q}| \cdot |\mathbf{P}_0\mathbf{R}|}, \qquad |\mathbf{P}_0\mathbf{R}| = a \neq 0 \tag{29}$$

considered as equations for determination of the point R, have not more, than one solution for any $a \neq 0$. Otherwise, the geometry is nondegenerate at the point P_0 in the direction of the vector $\mathbf{Q}_0\mathbf{Q}$. Note that the first equation (29) is the condition of the parallelism of vectors $\mathbf{Q}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$.

The proper Euclidean geometry is degenerate, i.e. it is degenerate at all points in directions of all vectors. Considering the Minkowski geometry, one should distinguish between the Minkowski T-geometry and Minkowski geometry. The two geometries are described by the same world function and differ in the definition of the parallelism. In the Minkowski T-geometry the parallelism of two vectors $\mathbf{Q_0Q}$ and $\mathbf{P_0R}$ is defined by the first equation (29). This definition is based on the deformation principle. In Minkowski geometry the parallelism is defined by the relation of the type of (16)

$$\mathbf{Q}_0\mathbf{Q}\uparrow\uparrow\mathbf{P}_0\mathbf{R}: \qquad (\mathbf{P}_0\mathbf{P}_i.\mathbf{Q}_0\mathbf{Q}) = a\,(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{R})\,, \qquad i=1,2,...n, \qquad a>0 \tag{30}$$

where points $\mathcal{P}^n = \{P_0, P_1, ...P_n\}$ determine a rectilinear coordinate system with basic vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1, 2, ..n$ in the n-dimensional Minkowski geometry (n-dimensional pseudo-Euclidean geometry of index 1). Dependence of the definition (30) on the points $(P_1, P_2, ...P_n)$ is fictitious, but dependence on the number $n+1$ of points $\mathcal{P}^n$ is essential. Thus, definition (30) depends on the method of the geometry description.

The Minkowski T-geometry is degenerate at all points in direction of all timelike vectors, and it is nondegenerate at all points in direction of all spacelike vectors. The Minkowski geometry is degenerate at all points in direction of all vectors. Conventionally one uses the Minkowski geometry, ignoring the nondegeneracy in spacelike directions.

Considering the proper Riemannian geometry, one should distinguish between the Riemannian T-geometry and the Riemannian geometry. The two geometries are described by the same world function. They differ in the definition of the parallelism. In the Riemannian T-geometry the parallelism of two vectors $\mathbf{Q_0Q}$ and $\mathbf{P_0R}$ is defined by the first equation (29). In the Riemannian geometry the parallelism of two vectors $\mathbf{Q_0Q}$ and $\mathbf{P_0R}$ is defined only in the case, when the points P_0 and Q_0 coincide. Parallelism of remote vectors $\mathbf{Q_0Q}$ and $\mathbf{P_0R}$ is not defined, in general. This fact is known as absence of absolute parallelism.

The proper Riemannian T-geometry is locally degenerate, i.e. it is degenerate at all points P_0 in direction of vectors $\mathbf{P}_0\mathbf{Q}$. In the general case, when $P_0 \neq Q_0$, the proper Riemannian T-geometry is nondegenerate, in general. The proper Riemannian geometry is degenerate, because it is degenerate locally, whereas the nonlocal degeneracy is not defined in the Riemannian geometry, because of the lack of absolute parallelism. Conventionally one uses the Riemannian geometry (not Riemannian T-geometry) and ignores the property of the nondegeneracy completely.

From the viewpoint of the conventional approach to the physical geometry the nondegeneracy is an undesirable property of a physical geometry, although from the logical viewpoint and from viewpoint of the deformation principle the nondegeneracy is an inherent property of a physical geometry. The nonlocal nondegeneracy is ejected from the proper Riemannian geometry by denial of existence of the remote vector parallelism. Nondegeneracy in the spacelike directions is ejected from the Minkowski geometry by means of the redefinition of the two vectors parallelism. To appreciate this, let us consider an example.

Simple example of nondegenerate space-time geometry

The T-geometry [4] is defined on the σ-space $V = \{\sigma, \Omega\}$, where Ω is an arbitrary set of points and the world function σ is defined by the relations

$$\sigma: \qquad \Omega\times\Omega\rightarrow\mathbf{R}, \qquad \sigma(P,Q)=\sigma(Q,P)\,, \qquad \sigma(P,P)=0, \qquad \forall P,Q\in\Omega \tag{31}$$

where $\mathbf{R}$ denotes the set of all real numbers. Geometrical objects (vector $\mathbf{PQ}$, scalar product of vectors $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$, collinearity of vectors $\mathbf{P}_0\mathbf{P}_1||\mathbf{Q}_0\mathbf{Q}_1$, segment of straight line $\mathcal{T}_{[P_0P_1]}$, etc.) are defined on the σ-space in the same way, as they are defined σ-immanently in the proper Euclidean space. Practically one uses the deformation principle, although it is not mentioned in all definitions.

Let us consider a simple example of the space-time geometry $\mathcal{G}_{\rm d}$, described by the T-geometry on 4-dimensional manifold $\mathcal{M}_{1+3}$. The world function $\sigma_{\rm d}$ is described by the relation

$$\sigma_{\rm d}=\sigma_{\rm M}+D\left(\sigma_{\rm M}\right)=\begin{cases}\sigma_{\rm M}+d & \text{if } \sigma_0<\sigma_{\rm M}\\ \left(1+\frac{d}{\sigma_0}\right)\sigma_{\rm M} & \text{if } 0\leq\sigma_{\rm M}\leq\sigma_0\\ \sigma_{\rm M} & \text{if } \sigma_{\rm M}<0\end{cases} \tag{32}$$

where $d\geq 0$ and $\sigma_0>0$ are some constants. The quantity $\sigma_{\rm M}$ is the world function in the Minkowski space-time geometry $\mathcal{G}_{\rm M}$. In the orthogonal rectilinear (inertial) coordinate system $x=(t,\mathbf{x})$ the world function $\sigma_{\rm M}$ has the form

$$\sigma_{\rm M}\left(x,x'\right)=\frac{1}{2}\left(c^2\left(t-t'\right)^2-\left(\mathbf{x}-\mathbf{x}'\right)^2\right) \tag{33}$$

where c is the speed of the light.

Let us compare the broken line (28) in Minkowski space-time geometry $\mathcal{G}_{\rm M}$ and in the distorted geometry $\mathcal{G}_{\rm d}$. We suppose that $\mathcal{T}_{\rm br}$ is timelike broken line, and all links $\mathcal{T}_{[P_iP_{i+1}]}$ of $\mathcal{T}_{\rm br}$ are timelike and have the same length

$$\left|\mathbf{P}_i\mathbf{P}_{i+1}\right|_{\rm d}=\sqrt{2\sigma_{\rm d}\left(P_i,P_{i+1}\right)}=\mu_{\rm d}>0,\qquad i=0,\pm1,\pm2,... \tag{34}$$

where indices "d" and "M" mean that the quantity is calculated by means of $\sigma_{\rm d}$ and $\sigma_{\rm M}$ respectively. Vector $\mathbf{P}_i\mathbf{P}_{i+1}$ is regarded as the momentum of the particle at the segment $\mathcal{T}_{[P_iP_{i+1}]}$, devided by the speed of the light c (we take for simplicity that $c=1$). The quantity $|\mathbf{P}_i\mathbf{P}_{i+1}|=\mu$ is interpreted as its (geometric) mass. It follows from definition (8) and relation (32), that for timelike vectors $\mathbf{P}_i\mathbf{P}_{i+1}$ with $\mu>\sqrt{2\sigma_0}$

$$\left|\mathbf{P}_i\mathbf{P}_{i+1}\right|_{\rm d}^2=\mu_{\rm d}^2=\mu_{\rm M}^2+2d,\qquad \mu_{\rm M}^2>2\sigma_0 \tag{35}$$

$$\left(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1}\right)_{\rm d}=\left(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1}\right)_{\rm M}+d \tag{36}$$

Calculation of the shape of the segment $\mathcal{T}_{[P_0P_1]}\left(\sigma_{\rm d}\right)$ in $\mathcal{G}_{\rm d}$ gives the relation

$$r^2(\tau)=\begin{cases}\tau^2\mu_{\rm d}^2\frac{\left(1-\frac{\tau d}{2(\sigma_0+d)}\right)^2}{\left(1-\frac{2d}{\mu_{\rm d}^2}\right)}-\frac{\tau^2\mu_{\rm d}^2\sigma_0}{(\sigma_0+d)}, & 0<\tau<\frac{\sqrt{2(\sigma_0+d)}}{\mu_{\rm d}}\\ \frac{3d}{2}+2d\left(\tau-1/2\right)^2\left(1-\frac{2d}{\mu_{\rm d}^2}\right)^{-1}, & \frac{\sqrt{2(\sigma_0+d)}}{\mu_{\rm d}}<\tau<1-\frac{\sqrt{2(\sigma_0+d)}}{\mu_{\rm d}}\\ \left(1-\tau\right)^2\mu_{\rm d}^2\left[\frac{\left(1-\frac{(1-\tau)d}{2(\sigma_0+d)}\right)^2}{\left(1-\frac{2d}{\mu_{\rm d}^2}\right)}-\frac{\sigma_0}{(\sigma_0+d)}\right], & 1-\frac{\sqrt{2(\sigma_0+d)}}{\mu_{\rm d}}<\tau<1\end{cases}, \tag{37}$$

where $r\left(\tau\right)$ is the spatial radius of the segment $\mathcal{T}_{[P_0P_1]}\left(\sigma_{\rm d}\right)$ in the coordinate system, where points P_0 and P_1 have coordinates $P_0=\{0,0,0,0\}$, $P_1=\{\mu_{\rm d},0,0,0\}$ and τ is a parameter along the segment $\mathcal{T}_{[P_0P_1]}\left(\sigma_{\rm d}\right)$ $(\tau\left(P_0\right)=0,\ \tau\left(P_1\right)=1)$. One can see from (37) that the characteristic value of the segment radius is $\sqrt{d}$.

Let the broken tube $\mathcal{T}_{\rm br}$ describe the "world line" of a free particle. It means by definition that any link $\mathbf{P}_{i-1}\mathbf{P}_i$ is parallel to the adjacent link $\mathbf{P}_i\mathbf{P}_{i+1}$

$$\mathbf{P}_{i-1}\mathbf{P}_i\uparrow\uparrow\mathbf{P}_i\mathbf{P}_{i+1}:\qquad \left(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1}\right)-\left|\mathbf{P}_{i-1}\mathbf{P}_i\right|\cdot\left|\mathbf{P}_i\mathbf{P}_{i+1}\right|=0 \tag{38}$$

Definition of parallelism is different in geometries $\mathcal{G}_{\mathrm{M}}$ and $\mathcal{G}_{\mathrm{d}}$. As a result links, which are parallel in the geometry $\mathcal{G}_{\mathrm{M}}$, are not parallel in $\mathcal{G}_{\mathrm{d}}$ and vice versa.

Let $\mathcal{T}_{\mathrm{br}}(\sigma_{\mathrm{M}})$ describe the world line of a free particle in the geometry $\mathcal{G}_{\mathrm{M}}$. The angle ϑ_{M} between the adjacent links in $\mathcal{G}_{\mathrm{M}}$ is defined by the relation

$$\cosh\vartheta_{\mathrm{M}} = \frac{(\mathbf{P}_{-1}\mathbf{P}_0.\mathbf{P}_0\mathbf{P}_1)_{\mathrm{M}}}{|\mathbf{P}_0\mathbf{P}_1|_{\mathrm{M}}\cdot|\mathbf{P}_{-1}\mathbf{P}_0|_{\mathrm{M}}} = 1 \tag{39}$$

The angle $\vartheta_{\mathrm{M}} = 0$, and the geometrical object $\mathcal{T}_{\mathrm{br}}(\sigma_{\mathrm{M}})$ is a timelike straight line on the manifold $\mathcal{M}_{1+3}$.

Let now $\mathcal{T}_{\mathrm{br}}(\sigma_{\mathrm{d}})$ describe the world line of a free particle in the geometry $\mathcal{G}_{\mathrm{d}}$. The angle ϑ_{d} between the adjacent links in $\mathcal{G}_{\mathrm{d}}$ is defined by the relation

$$\cosh\vartheta_{\mathrm{d}} = \frac{(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_{\mathrm{d}}}{|\mathbf{P}_i\mathbf{P}_{i+1}|_{\mathrm{d}}\cdot|\mathbf{P}_{i-1}\mathbf{P}_i|_{\mathrm{d}}} = 1 \tag{40}$$

The angle $\vartheta_{\mathrm{d}} = 0$ also. If we draw the broken tube $\mathcal{T}_{\mathrm{br}}(\sigma_{\mathrm{d}})$ on the manifold $\mathcal{M}_{1+3}$, using coordinates of basic points P_i and measure the angle ϑ_{dM} between the adjacent links in the Minkowski geometry $\mathcal{G}_{\mathrm{M}}$, we obtain for the angle ϑ_{dM} the following relation

$$\cosh\vartheta_{\mathrm{dM}} = \frac{(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_{\mathrm{M}}}{|\mathbf{P}_i\mathbf{P}_{i+1}|_{\mathrm{M}}\cdot|\mathbf{P}_{i-1}\mathbf{P}_i|_{\mathrm{M}}} = \frac{(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_{\mathrm{d}} - d}{|\mathbf{P}_i\mathbf{P}_{i+1}|^2_{\mathrm{d}} - 2d} \tag{41}$$

Substituting the value of $(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_{\mathrm{d}}$, taken from (40), we obtain for the case, when $d \ll \mu_{\mathrm{d}}^2$

$$\cosh\vartheta_{\mathrm{dM}} = \frac{\mu_{\mathrm{d}}^d - d}{\mu_{\mathrm{d}}^2 - 2d} \approx 1 + \frac{d}{\mu_{\mathrm{d}}^2}, \qquad d \ll \mu_{\mathrm{d}}^2 \tag{42}$$

Hence, $\vartheta_{\mathrm{dM}} \approx \sqrt{2d}/\mu_{\mathrm{d}}$. It means, that the adjacent link is located on the cone of angle $\sqrt{2d}/\mu_{\mathrm{d}}$, and the whole line $\mathcal{T}_{\mathrm{br}}(\sigma_{\mathrm{d}})$ has a random shape, because any link wobbles with the characteristic angle $\sqrt{2d}/\mu_{\mathrm{d}}$. The wobble angle depends on the space-time distortion d and on the particle mass μ_{d}. The wobble angle is small for the large mass of a particle. The random displacement of the segment end is of the order $\mu_{\mathrm{d}}\vartheta_{\mathrm{dM}} = \sqrt{2d}$, i.e. of the same order as the segment width. It is reasonable, because these two phenomena have the common source: the space-time distortion D.

One should note that the space-time geometry influences the stochasticity of particle motion nonlocally in the sense, that the form of the world function (32) for values of $\sigma_{\mathrm{M}} < \frac{1}{2}\mu_{\mathrm{d}}^2$ is unessential for the motion stochasticity of the particle of the mass μ_{d}.

Such a situation, when the world line of a free particle is stochastic in the deterministic geometry, and this stochasticity depends on the particle mass, seems to be rather exotic and incredible. But experiments show that the motion of real particles of small mass is stochastic indeed, and this stochasticity increases, when the particle mass decreases. From physical viewpoint a theoretical foundation of the stochasticity is desirable, and some researchers invent stochastic geometries, noncommutative geometries and other exotic geometrical constructions, to obtain the quantum stochasticity. But in the Riemannian space-time geometry the particle motion does not depend on the particle mass, and in the framework of the Riemannian space-time geometry it is difficult to explain the quantum stochasticity by the space-time geometry properties. Distorted geometry $\mathcal{G}_{\mathrm{d}}$ explains the stochasticity and its dependence on the particle mass freely. Besides, at proper choice of the distortion d the statistical description of stochastic $\mathcal{T}_{\mathrm{br}}$ leads to the quantum description (Schrödinger equation) [10]. It is sufficient to set

$$d = \frac{\hbar}{2bc}, \tag{43}$$

where $\hbar$ is the quantum constant, c is the speed of the light, and b is some universal constant, connecting the geometrical mass μ with the usual particle mass m by means of the relation

$$m = b\mu \tag{44}$$

In other words, the distorted space-time geometry (32) is closer to the real space-time geometry, than the Minkowski geometry $\mathcal{G}_{\mathrm{M}}$.

Statistical description of stochastic world tubes

Statistical description of world lines cannot be a probabilistic statistical description, because the number of world lines may be negative. Indeed, the density of world lines in the vicinity of the space-time point x is defined by the relation

$$dN = j^k dS_k \tag{45}$$

where dN is the flux of world lines through the spacelike 3-area dS_k. The 4-vector $j^k = j^k(x)$ describes the world-lines density in the vicinity of the point x. The quantity dN may be interpreted as the number of world lines in the vicinity of the point x. This number may be negative.

In the nonrelativistic case the relation (45) turns into the relation

$$dN = j^0 dS_0 = \rho dV \tag{46}$$

where the particle density $j^0 = \rho \geq 0$, and ρ may be a ground for introduction of the probability density. In the relativistic case one cannot introduce the probability density, because the world line density is described by the 4-vector j^k.

For statistical description of stochastic world lines we use the dynamical conception of statistical description (DCSD), which does not use the concept of the probability [11].

Let $\mathcal{S}_{\mathrm{st}}$ be stochastic particle, whose state X is described by variables $\left\{\mathbf{x}, \frac{d\mathbf{x}}{dt}\right\}$, where $\mathbf{x}$ is the particle position. Evolution of the particle state is stochastic, and there exist no dynamic equations for $\mathcal{S}_{\mathrm{st}}$. Evolution of the state of $\mathcal{S}_{\mathrm{st}}$ contains both regular and stochastic components. To separate the regular evolution components, we consider a set (statistical ensemble) $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ of many independent identical stochastic particles $\mathcal{S}_{\mathrm{st}}$. All stochastic particles $\mathcal{S}_{\mathrm{st}}$ start from the same initial state. It means that all $\mathcal{S}_{\mathrm{st}}$ are prepared in the same way. If the number N of $\mathcal{S}_{\mathrm{st}}$ is very large, the stochastic elements of evolution compensate each other, but regular ones are accumulated. In the limit $N \to \infty$ the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ turns into a dynamic system, whose state evolves according to some dynamic equations.

Let the statistical ensemble $\mathcal{E}_{\mathrm{d}}[\mathcal{S}_{\mathrm{d}}]$ of deterministic classical particles $\mathcal{S}_{\mathrm{d}}$ be described by the action $\mathcal{A}_{\mathcal{E}_{\mathrm{d}}[\mathcal{S}_{\mathrm{d}}(P)]}$, where P are parameters describing $\mathcal{S}_{\mathrm{d}}$ (for instance, mass, charge). Let under influence of some stochastic agent the deterministic particle $\mathcal{S}_{\mathrm{d}}$ turn into a stochastic particle $\mathcal{S}_{\mathrm{st}}$. The action $\mathcal{A}_{\mathcal{E}_{\mathrm{st}}[\mathcal{S}_{\mathrm{st}}]}$ for the statistical ensemble $\mathcal{E}_{\mathrm{st}}[\mathcal{S}_{\mathrm{st}}]$ is reduced to the action $\mathcal{A}_{\mathcal{S}_{\mathrm{red}}[\mathcal{S}_{\mathrm{d}}]} = \mathcal{A}_{\mathcal{E}_{\mathrm{st}}[\mathcal{S}_{\mathrm{st}}]}$ for some set $\mathcal{S}_{\mathrm{red}}[\mathcal{S}_{\mathrm{d}}]$ of identical interacting deterministic particles $\mathcal{S}_{\mathrm{d}}$. The action $\mathcal{A}_{\mathcal{S}_{\mathrm{red}}[\mathcal{S}_{\mathrm{d}}]}$ as a functional of $\mathcal{S}_{\mathrm{d}}$ has the form $\mathcal{A}_{\mathcal{E}_{\mathrm{d}}[\mathcal{S}_{\mathrm{d}}(P_{\mathrm{eff}})]}$, where parameters P_{eff} are parameters P of the deterministic particle $\mathcal{S}_{\mathrm{d}}$, averaged over the statistical ensemble, and this averaging describes interaction of particles $\mathcal{S}_{\mathrm{d}}$ in the set $\mathcal{S}_{\mathrm{red}}[\mathcal{S}_{\mathrm{d}}]$. It means that

$$\mathcal{A}_{\mathcal{E}_{\mathrm{st}}[\mathcal{S}_{\mathrm{st}}]} = \mathcal{A}_{\mathcal{S}_{\mathrm{red}}[\mathcal{S}_{\mathrm{d}}(P)]} = \mathcal{A}_{\mathcal{E}_{\mathrm{d}}[\mathcal{S}_{\mathrm{d}}(P_{\mathrm{eff}})]} \tag{47}$$

In other words, stochasticity of particles $\mathcal{S}_{\rm st}$ in the ensemble $\mathcal{E}_{\rm st}[\mathcal{S}_{\rm st}]$ is replaced by interaction of $\mathcal{S}_{\rm d}$ in $\mathcal{S}_{\rm red}[\mathcal{S}_{\rm d}]$, and this interaction is described by a change

$$P \to P_{\rm eff} \tag{48}$$

in the action $\mathcal{A}_{\mathcal{E}_{\rm d}[\mathcal{S}_{\rm d}(P)]}$.

The free particle has the unique parameter - its mass m, and the action $\mathcal{A}_{\mathcal{S}_{\rm d}}$ for the free deterministic particle has the form

$$\mathcal{S}_{\rm d}: \qquad \mathcal{A}_{\mathcal{S}_{\rm d}}[\mathbf{x}] = \int \frac{m}{2}\left(\frac{d\mathbf{x}}{dt}\right)^2 dt \tag{49}$$

where $\mathbf{x} = \mathbf{x}(t) = \{x^1(t), x^2(t), x^3(t)\}$, and the time t is the independent variable.

The action $\mathcal{A}_{\mathcal{E}_{\rm d}[\mathcal{S}_{\rm d}(P)]}$ for the pure statistical ensemble $\mathcal{E}_{\rm d}[\mathcal{S}_{\rm d}]$ of free deterministic particles $\mathcal{S}_{\rm d}$ has the form

$$\mathcal{E}_{\rm d}[\mathcal{S}_{\rm d}]: \qquad \mathcal{A}_{\mathcal{E}_{\rm d}[\mathcal{S}_{\rm d}]}[\mathbf{x}] = \int \frac{m}{2}\left(\frac{d\mathbf{x}}{dt}\right)^2 dt d\xi \tag{50}$$

where $\mathbf{x} = \mathbf{x}(t,\xi) = \{x^1(t,\xi), x^2(t,\xi), x^3(t,\xi)\}$. Independent variables $\xi = \{\xi_1, \xi_2, \xi_3\}$ label elements $\mathcal{S}_{\rm d}$ of the statistical ensemble $\mathcal{E}_{\rm d}[\mathcal{S}_{\rm d}]$. The variables ξ are known as Lagrangian coordinates. Statistical ensemble $\mathcal{E}_{\rm d}[\mathcal{S}_{\rm d}]$ is a continuous dynamic system, having infinite number of the freedom degrees, whereas the particle $\mathcal{S}_{\rm d}$ is the discrete dynamic system having six degrees of freedom.

If the particles are stochastic, the action $\mathcal{A}_{\mathcal{E}_{\rm st}[\mathcal{S}_{\rm st}]}$ for the pure statistical ensemble $\mathcal{E}_{\rm st}[\mathcal{S}_{\rm st}]$ of free quantum stochastic particles $\mathcal{S}_{\rm st}$ has the form

$$\mathcal{E}_{\rm st}[\mathcal{S}_{\rm st}]: \qquad \mathcal{A}_{\mathcal{E}_{\rm st}[\mathcal{S}_{\rm st}]}[\mathbf{x},\mathbf{u}] = \int \left\{\frac{m}{2}\left(\frac{d\mathbf{x}}{dt}\right)^2 + \frac{m}{2}\mathbf{u}^2 - \frac{\hbar}{2}\nabla\mathbf{u}\right\} dt d\xi \tag{51}$$

where $\mathbf{u} = \mathbf{u}(t,\mathbf{x})$ is a vector function of arguments $t,\mathbf{x}$ (not of t,ξ), and $\mathbf{x} = \mathbf{x}(t,\xi)$ is a vector function of independent variables t,ξ. The 3-vector $\mathbf{u}$ describes the mean value of the stochastic component of the particle motion, which is a function of the variables $t,\mathbf{x}$. The first term $\frac{m}{2}\left(\frac{d\mathbf{x}}{dt}\right)^2$ describes the energy of the regular component of the stochastic particle motion. The second term $m\mathbf{u}^2/2$ describes the energy of the random component of velocity. The components $\frac{d\mathbf{x}}{dt}$ and $\mathbf{u}$ of the total velocity are connected with different degrees of freedom, and their energies should be added in the expression for the Lagrange function density. The last term $-\hbar\nabla\mathbf{u}/2$ describes interaction between the regular component $\frac{d\mathbf{x}}{dt}$ and the random one $\mathbf{u}$. Note that $m\mathbf{u}^2/2$ is a function of $t,\mathbf{x}$. It influences on the regular component $\frac{d\mathbf{x}}{dt}$ as a potential energy $U(t,\mathbf{x},\nabla\mathbf{x}) = -m\mathbf{u}^2/2$, generated by the random component.

The dynamic system (51) is a statistical ensemble, because the Lagrange function density of the action (51) does not depend on ξ explicitly, and we can represent the action for the single system $\mathcal{S}_{\rm st}$

$$\mathcal{S}_{\rm st}: \qquad \mathcal{A}_{\mathcal{S}_{\rm st}}[\mathbf{x},\mathbf{u}] = \int \left\{\frac{m}{2}\left(\frac{d\mathbf{x}}{dt}\right)^2 + \frac{m}{2}\mathbf{u}^2 - \frac{\hbar}{2}\nabla\mathbf{u}\right\} dt \tag{52}$$

Unfortunately, the expression for the action (52) is only symbolic, because the differential operator $\nabla = \{\partial/\partial x^\alpha\}$, $\alpha = 1,2,3$ is defined in the continuous vicinity of the point $\mathbf{x}$, but not only for one point $\mathbf{x}$. The expression (52) ceases to be symbolic, only if $\hbar = 0$. In this case the last term, containing ∇ vanishes. Variation of (52) with respect to $\mathbf{u}$ gives $\mathbf{u}=0$, and the action (52) coincides with the action (50) for $\mathcal{S}_{\rm d}$. If $\hbar\neq 0$, the expression for the action (52) is not the well defined, and dynamic equations for $\mathcal{S}_{\rm st}$ are absent.

Dynamic equation for $\mathbf{u}$ is obtained from the action functional (51) by means of variation with respect to $\mathbf{u}$. If the quantum constant $\hbar=0$, it follows from the dynamic equation for $\mathbf{u}$, that $\mathbf{u}=0$, and the action (51) reduces to the form (50). In the general case $\hbar\neq 0$ we are to go to independent variables $\mathbf{x}$, because $\mathbf{u}$ is a function of $t,\mathbf{x}$. We obtain instead of (51)

$$\mathcal{E}_{\rm st}\left[\mathcal{S}_{\rm st}\right]: \qquad \mathcal{A}_{\mathcal{E}_{\rm st}\left[\mathcal{S}_{\rm st}\right]}\left[\mathbf{x},\mathbf{u}\right]=\int\left\{\frac{m}{2}\left(\frac{d\mathbf{x}}{dt}\right)^{2}+\frac{m}{2}\mathbf{u}^{2}-\frac{\hbar}{2}\nabla\mathbf{u}\right\}\rho dtd\mathbf{x} \tag{53}$$

$$\rho=\frac{\partial\left(\xi_{1},\xi_{2},\xi_{3}\right)}{\partial\left(x^{1},x^{2},x^{3}\right)}=\left(\frac{\partial\left(x^{1},x^{2},x^{3}\right)}{\partial\left(\xi_{1},\xi_{2},\xi_{3}\right)}\right)^{-1} \tag{54}$$

Variation of (54) with respect to $\mathbf{u}$ gives

$$\frac{\delta\mathcal{A}_{\mathcal{E}_{\rm st}\left[\mathcal{S}_{\rm st}\right]}}{\delta\mathbf{u}}=m\rho\mathbf{u}+\frac{\hbar}{2}\nabla\rho=0 \tag{55}$$

Resolving dynamic equation (55) with respect to $\mathbf{u}$ in the form

$$\mathbf{u}=-\frac{\hbar}{2m}\nabla\ln\rho \tag{56}$$

we can eliminate the mean stochastic velocity $\mathbf{u}$ from the action (53). We obtain instead of (53)

$$\mathcal{E}_{\rm st}\left[\mathcal{S}_{\rm st}\right]: \qquad \mathcal{A}_{\mathcal{E}_{\rm st}\left[\mathcal{S}_{\rm st}\right]}\left[\mathbf{x}\right]=\int\left\{\frac{m}{2}\left(\frac{d\mathbf{x}}{dt}\right)^{2}-U\left(\rho,\nabla\rho\right)\right\}\rho dtd\mathbf{x} \tag{57}$$

where

$$\rho U\left(\rho,\nabla\rho\right)=-\frac{\hbar^{2}}{8m}\frac{\left(\nabla\rho\right)^{2}}{\rho}-\frac{\hbar^{2}}{4m}\rho\nabla^{2}\ln\rho \tag{58}$$

and ρ is defined by (54). Eliminating divergence, we obtain instead of (58)

$$U\left(\rho,\nabla\rho\right)=\frac{\hbar^{2}}{8m}\frac{\left(\nabla\rho\right)^{2}}{\rho^{2}}+\frac{\hbar^{2}}{4m}\frac{1}{\rho}\nabla^{2}\rho \tag{59}$$

The last term in (59) does not give a contribution into dynamic equations, and it may be omitted. The action (57) turns into

$$\mathcal{E}_{\rm st}\left[\mathcal{S}_{\rm st}\right]: \qquad \mathcal{A}_{\mathcal{E}_{\rm st}\left[\mathcal{S}_{\rm st}\right]}\left[\xi\right]=\int\left\{\frac{m}{2}\left(\frac{d\mathbf{x}}{dt}\right)^{2}-\frac{\hbar^{2}}{8m}\frac{\left(\nabla\rho\right)^{2}}{\rho^{2}}\right\}\rho dtd\mathbf{x} \tag{60}$$

where variables $t,\mathbf{x}$ are independent variables, and variables ξ are considered to be dependent variables. The quantities ρ and $\frac{d\mathbf{x}}{dt}$ are functions of the dependent variables ξ derivatives with respect to t and $\mathbf{x}$

$$\rho=\frac{\partial\left(\xi_{1},\xi_{2},\xi_{3}\right)}{\partial\left(x^{1},x^{2},x^{3}\right)} \tag{61}$$

$$\frac{dx^{\alpha}}{dt}=\frac{\partial\left(x^{\alpha},\xi_{1},\xi_{2},\xi_{3}\right)}{\partial\left(t,\xi_{1},\xi_{2},\xi_{3}\right)}=\rho^{-1}\frac{\partial\left(x^{\alpha},\xi_{1},\xi_{2},\xi_{3}\right)}{\partial\left(t,x^{1},x^{2},x^{3}\right)}, \qquad \alpha=1,2,3 \tag{62}$$

Dynamic equations, generated by the action (60), are rather complicated. However, in terms of the wave function the action (60) takes a more simple form [12].

In terms of the two-component wave function ψ

$$\psi = \{\psi_1, \psi_2\}, \qquad \psi^* = \begin{Bmatrix} \psi_1^* \\ \psi_2^* \end{Bmatrix}, \qquad \rho \equiv \psi^*\psi \equiv \psi_1^*\psi_1 + \psi_2^*\psi_2, \tag{63}$$

the action (62) takes the form

$$\mathcal{E}_{\rm st}\left[\mathcal{S}_{\rm st}\right] : \qquad \mathcal{A}_{\mathcal{E}_{\rm st}[\mathcal{S}_{\rm st}]}\left[\psi, \psi^*\right] = \int \left\{ \frac{ib_0}{2}\left(\psi^*\partial_0\psi - \partial_0\psi^* \cdot \psi\right) - \frac{b_0^2}{2m}\nabla\psi^*\nabla\psi \right.$$
$$\left. + \frac{b_0^2}{8m}\sum_{\alpha=1}^{\alpha=3}(\nabla s_\alpha)^2\rho + \frac{b_0^2 - \hbar^2}{8\rho m}(\nabla\rho)^2 \right\} dt d\mathbf{x}, \tag{64}$$

where

$$s_\alpha \equiv \frac{\psi^*\sigma_\alpha\psi}{\rho}, \qquad \alpha = 1, 2, 3, \tag{65}$$

and σ_α are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{66}$$

Here the constant b_0 is an arbitrary constant. We transit from the action (60) to the action (64) by means of the change of variables, accompanied by the integration of dynamic equations and by the appearance of three arbitrary functions $\mathbf{g}(\xi) = \{g^1(\xi), g^2(\xi), g^3(\xi)\}$.

The change of variables, connecting dependent variables ξ and ψ, has the form (see Appendix A or [12])

$$\psi_\alpha = \sqrt{\rho}e^{i\varphi}u_\alpha(\xi), \qquad \psi_\alpha^* = \sqrt{\rho}e^{-i\varphi}u_\alpha^*(\xi), \qquad \alpha = 1, 2, \ldots, n, \tag{67}$$

$$\psi^*\psi \equiv \sum_{\alpha=1}^{n}\psi_\alpha^*\psi_\alpha, \tag{68}$$

where (*) means the complex conjugate. The quantities $u_\alpha(\xi)$, $\alpha = 1, 2, ...n$ are functions of only variables ξ, and satisfy the relations

$$-\frac{i}{2}\sum_{\alpha=1}^{n}\left(u_\alpha^*\frac{\partial u_\alpha}{\partial\xi_\beta} - \frac{\partial u_\alpha^*}{\partial\xi_\beta}u_\alpha\right) = g^\beta(\xi), \qquad \beta = 1, 2, 3, \qquad \sum_{\alpha=1}^{n}u_\alpha^*u_\alpha = 1. \tag{69}$$

Here φ is the new dependent variable, appearing from the fictitious temporal Lagrangian coordinate ξ_0, and b_0 is an arbitrary constant. The number n is such a natural number that equations (69) admit a solution. In general, n depends on the form of the arbitrary integration functions $\mathbf{g} = \{g^\beta(\xi)\}$, $\beta = 1, 2, 3$.

The meaning of the wave function ψ is not clear, and interpretation is produced on the basis of the action (53) or (60), where meaning of all quantities is quite clear. The action (53) describes the flow of some fluid with the density ρ, determined by the relation (54), and the flux density

$$\mathbf{j} = \rho\frac{d\mathbf{x}}{dt} \tag{70}$$

In terms of the wave function ψ these quantities have the form

$$\rho = \psi^* \psi, \qquad \mathbf{j} = -\frac{ib_0}{2m} \left(\psi^* \nabla \psi - \nabla \psi^* \cdot \psi \right) \tag{71}$$

The functions $\mathbf{g}$ determine vorticity of the fluid flow. If $\mathbf{g} = 0$, equations (69) have the solution $u_1 = 1$, $u_\alpha = 0$, $\alpha = 2, 3, ...n$. In this case the function ψ may have one component (other components vanish), and the fluid flow is irrotational. The function ψ has the form

$$\psi = \sqrt{\rho} e^{i\varphi}, \qquad \psi^* = \sqrt{\rho} e^{-i\varphi} \tag{72}$$

and the fluid velocity

$$\mathbf{v} = \frac{\mathbf{j}}{\rho} = \nabla \frac{b_0 \varphi}{m} \tag{73}$$

has the potential $b_0 \varphi / m$.

In the partial case of the irrotational fluid flow

$$s_\alpha \equiv \frac{\psi^* \sigma_\alpha \psi}{\rho} = \text{const}, \qquad \alpha = 1, 2, 3 \tag{74}$$

and the action (64) turns into the action

$$\mathcal{A}_{\mathcal{S}_\mathrm{q}} \left[\psi, \psi^* \right] = \int \left\{ \frac{ib_0}{2} \left(\psi^* \partial_0 \psi - \partial_0 \psi^* \cdot \psi \right) - \frac{b_0^2}{2m} \nabla \psi^* \nabla \psi + \frac{b_0^2 - \hbar^2}{8 \rho m} (\nabla \rho)^2 \right\} dt d\mathbf{x}, \tag{75}$$

If we choose the arbitrary constant b_0 in the form $b_0 = \hbar$, the action (75) turns into the action

$$\mathcal{A}_{\mathcal{S}_\mathrm{q}} \left[\psi, \psi^* \right] = \int \left\{ \frac{i\hbar}{2} \left(\psi^* \partial_0 \psi - \partial_0 \psi^* \cdot \psi \right) - \frac{\hbar^2}{2m} \nabla \psi^* \nabla \psi \right\} dt d\mathbf{x}, \tag{76}$$

having the Schrödinger equation

$$i\hbar \partial_0 \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi \tag{77}$$

as the dynamic equation. Expressions (71) for the density and the particle flux turn into the conventional expressions

$$\rho = \psi^* \psi, \qquad \mathbf{j} = -\frac{i\hbar}{2m} \left(\psi^* \nabla \psi - \nabla \psi^* \cdot \psi \right) \tag{78}$$

Interpretation of all quantities is obtained on the basis the fact, that the quantum description in terms of the Schrödinger equation is the special case of the statistical description in terms of the statistical ensemble (51).

Can we obtain the statistical ensemble (51) from the statistical ensemble (50) by means of the change $m \to m_{\mathrm{eff}}$? It is possible, if we represent the nonrelativistic action (50) as the nonrelativistic approximation

$$\mathcal{E}_\mathrm{d} \left[\mathcal{S}_\mathrm{d} \right] : \qquad \mathcal{A}_{\mathcal{E}_\mathrm{d}[\mathcal{S}_\mathrm{d}]} \left[\mathbf{x} \right] = \int \left\{ -mc^2 + \frac{m}{2} \left(\frac{d\mathbf{x}}{dt} \right)^2 \right\} dt d\xi \tag{79}$$

of the relativistic action

$$\mathcal{E}_\mathrm{d} \left[\mathcal{S}_\mathrm{d} \right] : \qquad \mathcal{A}_{\mathcal{E}_\mathrm{d}[\mathcal{S}_\mathrm{d}]} \left[\mathbf{x} \right] = -\int mc^2 \sqrt{1 - \frac{1}{c^2} \left(\frac{d\mathbf{x}}{dt} \right)^2} dt d\xi \tag{80}$$

The action (51) is obtained from the action (79) as a result of the change

$$m \to m_{\text{eff}} = m\left(1 - \frac{\mathbf{u}^2}{2c^2} + \frac{\hbar}{2mc^2}\nabla\mathbf{u}\right) \tag{81}$$

Practically, the change is produced only in the first term of the action (79), because the change in the second term gives additional term of the order of c^{-2}, which is small in the nonrelativistic approximation. Another version of the change in the action (79) has the form

$$m \to m_{\text{eff}} = m\left(1 - \frac{\hbar^2}{8m^2c^2}\frac{(\nabla\rho)^2}{\rho^2} - \frac{\hbar^2}{4m^2c^2}\frac{1}{\rho}\nabla^2\rho\right) \tag{82}$$

or

$$m \to m_{\text{eff}} = m\left(1 + \frac{\hbar^2}{8m^2c^2}\frac{(\nabla\rho)^2}{\rho^2}\right) \tag{83}$$

The relation (82) is obtained from (81) after substitution of (56). Producing the change (83) in the action (79), we obtain in the nonrelativistic approximation the action (60).

In the relativistic case instead of the change (81) we have

$$m^2 \to m_{\text{eff}}^2 = m^2\left(1 + u_l u^l + \lambda\partial_l u^l\right), \qquad \lambda = \frac{\hbar}{mc} \tag{84}$$

where the variables $u^k = u^k(x) = u^k(t, \mathbf{x})$, $k = 0, 1, 2, 3$ are new dependent variables, describing the mean value of the stochastic component of the particle 4-velocity. The change (84) in the action (80) for the statistical ensemble of free relativistic particles leads finally to the action [13]

$$\mathcal{A}[\psi, \psi^*] = \int\left\{\hbar^2\partial_k\psi^*\partial^k\psi - m^2c^2\rho - \frac{\hbar^2}{4}(\partial_l s_\alpha)\left(\partial^l s_\alpha\right)\rho\right\}d^4x \tag{85}$$

where ψ is the two-component wave function (67) - (69). The variables ρ, s_α are defined by the relation (71) and, besides, the constant $b_0 = \hbar$. The action (85) is the action for the statistical ensemble of free stochastic relativistic particles. In the case of irrotational flow, when the wave function ψ may be one-component, s_α =const, and the dynamic equation for the action (85) is the Klein-Gordon equation.

$$\hbar^2\partial_k\partial^k\psi + m^2c^2\psi = 0 \tag{86}$$

Determination of the effective mass

We are going to show, that the change (83) follows from the form of the world function (32). In reality in [10] the inverse problem has been solved. What is the geometry of the uniform space-time, if the statistical description of free nonrelativistic particles leads to the quantum description in terms of the Schrödinger equation? Having solved this problem, we obtained the world function (32). Now we show that the effective mass m_{eff} of the nonrelativistic particle is determined by the relation (83).

Mathematical formalism of theoretical physics is suited for application in the Minkowski space-time. Mathematical formalism for work in the distorted space-time V_{d} with the world function (32) is absent now. We are forced to work in the Minkowski space-time, using conventional technique and taking into account distortion of the space-time by means of some corrections.

Let introduce the notion of the adduced vector $\vec{p} = \vec{p}(a, P_0, P_1)$ as a totality of a real or imaginary number a and two points $\{P_0, P_1\}$

$$\vec{p} = \vec{p}(a, P_0, P_1) = a(\mathbf{P}_0\mathbf{P}_1) = a\mathbf{P}_0\mathbf{P}_1 \tag{87}$$

The number a is called the gauge of the adduced vector. The vector $\mathbf{P}_0\mathbf{P}_1$ is a partial case of the adduced vector $a\mathbf{P}_0\mathbf{P}_1$ with the gauge $a = 1$. The scalar product of two adduced vectors $a_1\mathbf{P}_0\mathbf{P}_1$ and $a_2\mathbf{Q}_0\mathbf{Q}_1$ is defined by the relation

$$(a_1\mathbf{P}_0\mathbf{P}_1.a_2\mathbf{Q}_0\mathbf{Q}_1) = a_1 a_2 (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) \tag{88}$$

We shall consider statistical ensemble of relativistic particles, described by the action (80) with the oriented mass m_{o}, defined by the relation

$$m_{\mathrm{o}} = b\mu_{\mathrm{o}}, \qquad \mu_{\mathrm{o}} = (\mathbf{P}_0\mathbf{P}_1.\vec{u}(R)) \tag{89}$$

where $\vec{u} = \vec{u}(R)$ is the unit adduced vector of the 4-velocity at the point $R \in \mathcal{T}_{[P_0P_1]}$, $\mathbf{P}_0\mathbf{P}_1$ is the momentum vector, divided by the speed of the light c. The quantity m_{o} is called the oriented mass because it depends on the mutual orientation of the momentum vector and of the 4-velocity. The oriented mass m_{o} has different sign for the particle and for the antiparticle.

The 4-velocity $\vec{u} = \vec{u}(R)$ is the unit adduced vector inside the segment $\mathcal{T}_{[P_0P_1]}$ in the space-time V_{d}

$$\vec{u}(R) = |\mathbf{P}_0\mathbf{R}|_{\mathrm{d}}^{-1}\mathbf{P}_0\mathbf{R} = (\mathbf{P}_0\mathbf{R}.\mathbf{P}_0\mathbf{R})_{\mathrm{d}}^{-1/2}\mathbf{P}_0\mathbf{R}, \qquad R \in \mathcal{T}_{[P_0P_1]} \tag{90}$$

$$(\vec{u}(R).\vec{u}(R))_{\mathrm{d}} = 1 \tag{91}$$

The particle mass, defined by the relation (89), is different in V_{d} and in V_{M}. As far as $R \in \mathcal{T}_{[P_0P_1]}$ and, hence, $\mathbf{P}_0\mathbf{R} \uparrow\uparrow_{\mathrm{d}} \mathbf{P}_0\mathbf{P}_1$.

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R})_{\mathrm{d}} = |\mathbf{P}_0\mathbf{P}_1|_{\mathrm{d}} \cdot |\mathbf{P}_0\mathbf{R}|_{\mathrm{d}}, \tag{92}$$

we obtain for m_{od}

$$\begin{aligned} m_{\mathrm{od}} &= b(\mathbf{P}_0\mathbf{P}_1.\vec{u}(R))_{\mathrm{d}} = b|\mathbf{P}_0\mathbf{R}|_{\mathrm{d}}^{-1}(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R})_{\mathrm{d}} \\ &= b|\mathbf{P}_0\mathbf{R}|_{\mathrm{d}}^{-1} \cdot |\mathbf{P}_0\mathbf{P}_1|_{\mathrm{d}} \cdot |\mathbf{P}_0\mathbf{R}|_{\mathrm{d}} = b|\mathbf{P}_0\mathbf{P}_1|_{\mathrm{d}} = b\mu_{\mathrm{d}} \end{aligned} \tag{93}$$

where b is the constant, defined by (44).

If the point R on the segment $\mathcal{T}_{[P_0P_1]}$ is not close to the ends P_0 and P_1, (i.e. $|\mathbf{P}_0\mathbf{R}|_{\mathrm{d}}^2 > 2\sigma_0$, $|\mathbf{P}_0\mathbf{P}_1|_{\mathrm{d}}^2 > 2\sigma_0$) and relation (36) is satisfied, we obtain for the oriented mass m_{oM} in V_{M}

$$\begin{aligned} m_{\mathrm{oM}} &= b(\mathbf{P}_0\mathbf{P}_1.\vec{u}(R))_{\mathrm{M}} = b|\mathbf{P}_0\mathbf{R}|_{\mathrm{d}}^{-1}(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R})_{\mathrm{M}} \\ &= b|\mathbf{P}_0\mathbf{R}|_{\mathrm{d}}^{-1}((\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R})_{\mathrm{d}} - 2d) \\ &= b|\mathbf{P}_0\mathbf{P}_1|_{\mathrm{d}} - 2db|\mathbf{P}_0\mathbf{R}|_{\mathrm{d}}^{-1} = b\mu_{\mathrm{d}} - \frac{2bd}{|\mathbf{P}_0\mathbf{R}|_{\mathrm{d}}} \end{aligned} \tag{94}$$

Thus, the particle mass m_{oM}, defined by the relation (89) and calculated in V_{M} depends on the point R on the surface of the segment $\mathcal{T}_{[P_0P_1]}$. We use in the action (80) some effective mass m_{eff}, calculated in accordance with (94) in the Minkowski space-time V_{M} by means of the relation

$$m_{\mathrm{eff}} = b(\mathbf{P}_0\mathbf{P}_1.\vec{u}_{\mathrm{eff}})_{\mathrm{M}} \tag{95}$$

where the adduced vector $\vec{u}_{\text{eff}}$ is the mean 4-velocity inside the segment $\mathcal{T}_{[P_0P_1]}$.

Let the point P be the center of the segment $\mathcal{T}_{[P_0P_1]}$, as it shown in figure 1. The points P' and P'' are centers of segments $\mathcal{T}_{[P_0'P_1']}$, $\mathcal{T}_{[P_0''P_1'']}$, of adjacent world tubes of the statistical ensemble. We consider nonrelativistic case, and the vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0'\mathbf{P}_1'$, $\mathbf{P}_0''\mathbf{P}_1''$ may be considered to be parallel in V_{M}. Let segments $\mathcal{T}_{[P_0P_1]}$, $\mathcal{T}_{[P_0'P_1']}$, $\mathcal{T}_{[P_0''P_1'']}$ be placed in such a way, that $P \in \mathcal{T}_{[P_0'P_1']}$, $P \in \mathcal{T}_{[P_0''P_1'']}$. The 4-velocity of the segment $\mathcal{T}_{[P_0'P_1']}$, determined by the vector $\mathbf{P}_0'\mathbf{P}$, and the 4-velocity of the segment $\mathcal{T}_{[P_0''P_1'']}$, determined by the vector $\mathbf{P}_0''\mathbf{P}$, make a contribution in the effective 4-velocity $\vec{u}_{\text{eff}}$ of the segment $\mathcal{T}_{[P_0P_1]}$. We suppose that the origin of the effective 4-velocity vector $\vec{u}_{\text{eff}}$ is placed at the point P. Let the spatial distance between the points P, P' and P, P'' be l. According to the relation (37) we obtain

$$l = \sqrt{-\left|\mathbf{PP}'\right|^2} = \sqrt{-\left|\mathbf{PP}''\right|^2} = r\left(0.5\right) = \sqrt{\frac{3d}{2}} = \sqrt{\frac{3\hbar}{4bc}} \tag{96}$$

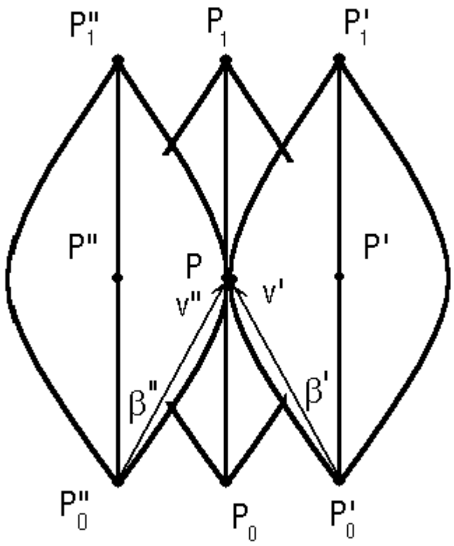

Рис. 1:

We choose the coordinate system with the origin at the point P and with the time axis directed along the vector $\mathbf{P}_0\mathbf{P}_1$. In the space-time V_{M} in this coordinate system we have covariant components of $\mathbf{P}_0\mathbf{P}_1$

$$\left(\mathbf{P}_0\mathbf{P}_1\right)_k = \left\{\mu_{\text{d}}c, 0, \right\} \tag{97}$$

The contravariant coordinates of the 4-velocity of the segment $\mathcal{T}_{[P_0'P_1']}$ $\mathbf{x}$ have the form

$$u^k = \left\{u^0, \mathbf{u}\right\} = \left\{\frac{1}{c\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}, -\frac{\mathbf{v}}{c\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\right\}, \qquad \mathbf{v} = -\frac{2\mathbf{x}}{\mu_{\text{d}}}c \tag{98}$$

where $\mathbf{x}$ are the spatial coordinates of the point P'. The effective 4-velocity at the point P is a sum of contributions of all segments $\mathcal{T}_{[P_0'P_1']}$

$$u_{\text{eff}}^0 = A\int \rho\left(\mathbf{x}\right)\delta\left(l^2-\mathbf{x}^2\right)u^0\left(\mathbf{x}\right)d\mathbf{x}, \tag{99}$$

$$l=\sqrt{\frac{3\hbar}{4bc}} \tag{100}$$

$$\mathbf{u}_{\text{eff}}=A\int\rho(\mathbf{x})\,\delta\left(l^2-\mathbf{x}^2\right)\mathbf{u}(\mathbf{x})\,d\mathbf{x}=-A\int\frac{\rho(\mathbf{x})\,\delta\left(l^2-\mathbf{x}^2\right)}{\sqrt{1-\left(\frac{2\mathbf{x}}{\mu_{\text{d}}}\right)^2}}\frac{2\mathbf{x}}{\mu_{\text{d}}}d\mathbf{x} \tag{101}$$

where the quantity ρ is the density of world lines in the statistical ensemble, and the quantity A is determined from the condition of normalization of $\vec{u}_{\text{eff}}(P)$

$$c^2\left(u_{\text{eff}}^0\right)^2-\mathbf{u}_{\text{eff}}^2=1 \tag{102}$$

Supposing that $\rho(\mathbf{x})$ changes slowly and expanding $\rho(\mathbf{x})$ in a series over $\mathbf{x}$, we obtain from (99), (101)

$$u_{\text{eff}}^0=A\int\frac{\rho}{c\sqrt{1-\left(\frac{2\mathbf{x}}{\mu_{\text{d}}}\right)^2}}\delta\left(l^2-\mathbf{x}^2\right)d\mathbf{x}=\frac{2\pi A\rho l}{c\sqrt{1-\left(\frac{2l}{\mu_{\text{d}}}\right)^2}} \tag{103}$$

$$\mathbf{u}_{\text{eff}}=-A\int\frac{(\mathbf{x}\nabla)\,\rho}{\sqrt{1-\left(\frac{2\mathbf{x}}{\mu_{\text{d}}}\right)^2}}\delta\left(l^2-\mathbf{x}^2\right)\frac{2\mathbf{x}}{\mu_{\text{d}}}cd\mathbf{x}=-\frac{4\pi Al^3\nabla\rho}{3\mu_{\text{d}}\sqrt{1-\left(\frac{2l}{\mu_{\text{d}}}\right)^2}} \tag{104}$$

where ρ is the value of the density at the point P. Substituting (103), (104) in (102) and using (100), we obtain

$$A=\frac{\sqrt{1-\left(\frac{2l}{\mu_{\text{d}}}\right)^2}}{2\pi l\rho\sqrt{1-\left(\frac{\hbar}{2m_{\text{d}}c}\nabla\ln\rho\right)^2}} \tag{105}$$

Substituting (105) in (103), we obtain

$$u_{\text{eff}}^0=\frac{1}{c\sqrt{1-\left(\frac{\hbar}{2m_{\text{d}}c}\nabla\ln\rho\right)^2}} \tag{106}$$

It follows from (95), (97) and (106)

$$m_{\text{eff}}=m_{\text{d}}cu_{\text{eff}}^0=m_{\text{d}}\frac{1}{\sqrt{1-\left(\frac{\hbar}{2m_{\text{d}}c}\nabla\ln\rho\right)^2}}=m_{\text{d}}\left(1+\frac{\hbar^2}{8m_{\text{d}}^2c^2}\left(\nabla\ln\rho\right)^2\right) \tag{107}$$

This result coincides with the relation (83).

We admit that there are another methods of calculation of the value of m_{eff}, which give another result. In this case we should choose another world function of the space-time V_{d}, which leads to the result (107), because we know that the effective mass, determined by the relation (107) agrees with the experimental data. We know about the distorted space-time geometry only that it generates stochastic motion of free particles. Information on its world function is obtained from the demand that the world function leads to the effective mass, which is determined by the relation (107).

Further development of the statistical description of geometrical stochasticity leads to a creation of the model conception of quantum phenomena (MCQP), which relates to the conventional quantum theory approximately in the same way as the statistical physics relates to the axiomatic thermodynamics. MCQP is the well defined relativistic conception with effective methods of investigation [14], whereas the conventional quantum theory is not well defined, because it uses incorrect space-time geometry, whose incorrectness is compensated by additional hypotheses (quantum principles). Besides, it has problems with application of the nonrelativistic quantum mechanics technique to the description of relativistic phenomena.

The geometry $\mathcal{G}_{\rm d}$ is a homogeneous geometry as well as the Minkowski geometry, because the world function $\sigma_{\rm d}$ is invariant with respect to all coordinate transformations, with respect to which the world function $\sigma_{\rm M}$ is invariant. In this connection the question arises, whether one could invent some axiomatics for $\mathcal{G}_{\rm d}$ and derive the geometry $\mathcal{G}_{\rm d}$ from this axiomatics by means of proper reasonings. Note that such an axiomatics is to depend on the parameter d, because the world function $\sigma_{\rm d}$ depends on this parameter. If $d = 0$, this axiomatics is to coincide with the axiomatics of the Minkowski geometry $\mathcal{G}_{\rm M}$. If $d \neq 0$, this axiomatics cannot coincide with the axiomatics of $\mathcal{G}_{\rm M}$, because some axioms of $\mathcal{G}_{\rm M}$ are not satisfied in this case. In general, the invention of axiomatics, depending on the parameter d and in the general case on the distortion function D, seems to be a very difficult problem. Besides, why invent the axiomatics? We had derived the axiomatics for the proper Euclidean geometry, when we constructed it before. There is no necessity to repeat this process any time, when we construct a new geometry. It is sufficient to apply the deformation principle to the constructed Euclidean geometry written σ-immanently. Application of the deformation principle to the Euclidean geometry is a very simple and general procedure, which is not restricted by continuity, convexity and other artificial constraints, generated by our preconceived approach to the physical geometry. (Bias of the approach is displayed in the antecedent supposition on the one-dimensionality of any straight line in any physical geometry, which reminds the statement of the ancient Egyptians that all rivers flow towards the North).

Thus, we have seen that the nondegeneracy of the physical geometry as well as non-one-dimensionality of the straight line are properties of the real physical geometries. The proper Euclidean geometry is a ground for all physical geometries. Although it is a degenerate geometry, it is beyond reason to deny an existence of nondegenerate physical geometries.

Thus, the deformation principle together with the σ-immanent description appears to be a very effective mathematical tool for construction of physical geometries.

1. The deformation principle uses results obtained at construction of the proper Euclidean geometry and does not add any additional supposition on properties of geometrical objects.

2. The deformation principle uses only the real characteristic of the physical geometry – its world function and does not use any additional means of description.

3. The deformation principle is very simple and allows one to investigate only that part of geometry which one is interested in.

4. Application of the deformation principle allows one to obtain the true space-time geometry, whose unexpected properties cannot be obtained at the conventional approach to physical geometry.

Transformation of the action for the statistical ensemble

To transform the action (60) to the description in terms of the wave function, we rewrite it in the form

$$\mathcal{E}_{\text{st}}\left[\mathcal{S}_{\text{st}}\right]: \qquad \mathcal{A}_{\mathcal{E}_{\text{st}}\left[\mathcal{S}_{\text{st}}\right]}\left[\mathbf{x}\right] = \int \left\{ \frac{m}{2}\left(\frac{d\mathbf{x}}{dt}\right)^2 - \frac{\hbar^2}{8m}\frac{\left(\nabla\rho\right)^2}{\rho^2}\right\} dt d\xi \tag{.108}$$

where

$$\rho = \frac{\partial\left(\xi_1,\xi_2,\xi_3\right)}{\partial\left(x^1,x^2,x^3\right)} = \left(\frac{\partial\left(x^1,x^2,x^3\right)}{\partial\left(\xi_1,\xi_2,\xi_3\right)}\right)^{-1} \tag{.109}$$

We introduce the independent variable ξ_0 instead of the variable $t = x^0$ and rewrite the action (.108) in the form

$$\mathcal{A}_{\mathcal{E}_{\text{st}}\left[\mathcal{S}_{\text{st}}\right]}\left[x\right] = \int \left\{ \frac{m\dot{x}^\alpha\dot{x}^\alpha}{2\dot{x}^0} - \frac{\hbar^2}{8m}\frac{\left(\nabla\rho\right)^2}{\rho^2}\right\} d^4\xi, \qquad \dot{x}^k \equiv \frac{\partial x^k}{\partial\xi_0} \tag{.110}$$

where $\xi = \{\xi_0, \boldsymbol{\xi}\} = \{\xi_k\}$, $k = 0,1,2,3$, $x = \left\{x^k\left(\xi\right)\right\}$, $k = 0,1,2,3$. Here and in what follows, a summation over repeated Greek indices is produced $(1-3)$.

Let us consider variables $\xi = \xi\left(x\right)$ in (.110) as dependent variables and variables x as independent variables. Let the Jacobian

$$J = \frac{\partial\left(\xi_0,\xi_1,\xi_2,\xi_3\right)}{\partial\left(x^0,x^1,x^2,x^3\right)} = \det\left|\left|\xi_{i,k}\right|\right|, \qquad \xi_{i,k} \equiv \partial_k\xi_i, \qquad i,k = 0,1,2,3 \tag{.111}$$

be considered to be a multilinear function of $\xi_{i,k}$. Then

$$d^4\xi = Jd^4x, \qquad \dot{x}^i \equiv \frac{\partial x^i}{\partial\xi_0} \equiv \frac{\partial\left(x^i,\xi_1,\xi_2,\xi_3\right)}{\partial\left(\xi_0,\xi_1,\xi_2,\xi_3\right)} = J^{-1}\frac{\partial J}{\partial\xi_{0,i}}, \qquad i = 0,1,2,3 \tag{.112}$$

After transformation to dependent variables ξ the action (.110) takes the form

$$\mathcal{A}_{\mathcal{E}_{\text{st}}\left[\mathcal{S}_{\text{st}}\right]}\left[\xi\right] = \int \left\{ \frac{m}{2}\frac{\partial J}{\partial\xi_{0,\alpha}}\frac{\partial J}{\partial\xi_{0,\alpha}}\left(\frac{\partial J}{\partial\xi_{0,0}}\right)^{-1} - \frac{\hbar^2}{8m}\frac{\left(\nabla\rho\right)^2}{\rho}\right\} d^4x, \tag{.113}$$

Here the dependent variable ξ_0 is fictitious

We introduce new variables

$$j^k = \frac{\partial J}{\partial\xi_{0,k}}, \qquad k = 0,1,2,3, \qquad \rho = j^0 \tag{.114}$$

by means of Lagrange multipliers p_k

$$\mathcal{A}_{\mathcal{E}_{\text{st}}\left[\mathcal{S}_{\text{st}}\right]}\left[\xi,j,p\right] = \int \left\{ \frac{m}{2}\frac{j^\alpha j^\alpha}{j^0} - \frac{\hbar^2}{8m}\frac{\left(\nabla\rho\right)^2}{\rho} + p_k\left(\frac{\partial J}{\partial\xi_{0,k}} - j^k\right)\right\} d^4x, \tag{.115}$$

Here and in what follows, a summation over repeated Latin indices is produced $(0-3)$.

Note that according to (.112), the relations (.114) can be written in the form

$$j^k = \left\{\frac{\partial J}{\partial\xi_{0,0}}, \frac{\partial J}{\partial\xi_{0,0}}\left(J^{-1}\frac{\partial J}{\partial\xi_{0,\alpha}}\right)\left(J^{-1}\frac{\partial J}{\partial\xi_{0,0}}\right)^{-1}\right\} = \left\{\rho, \rho\frac{dx^\alpha}{dt}\right\}, \qquad \rho \equiv \frac{\partial J}{\partial\xi_{0,0}} \tag{.116}$$

It is clear from (.116) that j^k is the 4-flux of particles, with $j^0 = \rho$ being its density.

Variation of (.115) with respect to ξ_i gives

$$\frac{\delta \mathcal{A}_{\mathcal{E}_{\rm st}[\mathcal{S}_{\rm st}]}}{\delta \xi_i} = -\partial_l \left(p_k \frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{i,l}} \right) = -\frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{i,l}} \partial_l p_k = 0, \qquad i = 0,1,2,3 \tag{.117}$$

Using identities

$$\frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{i,l}} \equiv J^{-1} \left(\frac{\partial J}{\partial \xi_{0,k}} \frac{\partial J}{\partial \xi_{i,l}} - \frac{\partial J}{\partial \xi_{0,l}} \frac{\partial J}{\partial \xi_{i,k}} \right) \tag{.118}$$

$$\frac{\partial J}{\partial \xi_{i,l}} \xi_{k,l} \equiv J \delta_k^i, \qquad \partial_l \frac{\partial^2 J}{\partial \xi_{0,k} \partial \xi_{i,l}} \equiv 0 \tag{.119}$$

one can test by direct substitution that the general solution of linear equations (.117) has the form

$$p_k = \frac{b_0}{2} \left(\partial_k \varphi + g^\alpha (\xi) \partial_k \xi_\alpha \right), \qquad k = 0,1,2,3 \tag{.120}$$

where $b_0 \neq 0$ is an arbitrary constant, $g^\alpha (\xi)$, $\alpha = 1,2,3$ are arbitrary functions of $\xi = \{\xi_1, \xi_2, \xi_3\}$, and φ is the dynamic variable ξ_0, which ceases to be fictitious. It is the conceptual integration, which allows one to introduce the wave function. Let us substitute (.120) in (.115). The term of the form $\partial_k \varphi \partial J / \partial \xi_{0,k}$ is reduced to Jacobian and does not contribute to dynamic equation. The terms of the form $\xi_{\alpha,k} \partial J / \partial \xi_{0,k}$ vanish due to identities (.119). We obtain

$$\mathcal{A}_{\mathcal{E}_{\rm st}[\mathcal{S}_{\rm st}]} [\varphi, \xi, j] = \int \left\{ \frac{m}{2} \frac{j^\alpha j^\alpha}{j^0} - j^k p_k - \frac{\hbar^2}{8m} \frac{(\nabla \rho)^2}{\rho} \right\} d^4 x, \qquad j^0 = \rho \tag{.121}$$

where quantities p_k are determined by the relations (.120)

Variation of the action (.121) with respect to j^k gives

$$p_0 = -\frac{m}{2} \frac{j^\alpha j^\alpha}{\rho^2} + \frac{\hbar^2}{8m} \left(\frac{(\nabla \rho)^2}{\rho^2} + 2 \nabla \frac{(\nabla \rho)}{\rho} \right) \tag{.122}$$

$$p_\beta = m \frac{j^\beta}{\rho}, \qquad \beta = 1,2,3 \tag{.123}$$

Now we eliminate the variables $\mathbf{j} = \{j^1, j^2, j^3\}$ from the action (.121), using relation (.123). We obtain

$$\mathcal{A}_{\mathcal{E}_{\rm st}[\mathcal{S}_{\rm st}]} [\rho, \varphi, \xi] = \int \left\{ -p_0 - \frac{p_\beta p_\beta}{2m} - \frac{\hbar^2}{8m} \frac{(\nabla \rho)^2}{\rho^2} \right\} \rho d^4 x, \tag{.124}$$

where p_k is determined by the relation (.120).

Now instead of dependent variables ρ, φ, ξ we introduce the n-component complex function ψ, defining it by relations (67) – (69)

The function ψ is constructed of the variable φ, the fluid density ρ and the Lagrangian coordinates ξ, considered as functions of $(t, \mathbf{x})$, as follows [12]. The n-component complex function $\psi = \{\psi_\alpha\}$, $\alpha = 1,2,\ldots,n$ is defined by the relations

$$\psi_\alpha = \sqrt{\rho} e^{i\varphi} u_\alpha(\xi), \qquad \psi_\alpha^* = \sqrt{\rho} e^{-i\varphi} u_\alpha^*(\xi), \qquad \alpha = 1,2,\ldots,n, \tag{.125}$$

$$\psi^* \psi \equiv \sum_{\alpha=1}^n \psi_\alpha^* \psi_\alpha, \tag{.126}$$

where (*) means the complex conjugate. The quantities $u_\alpha(\xi)$, $\alpha = 1, 2, \ldots, n$ are functions of only variables ξ, and satisfy the relations

$$-\frac{i}{2}\sum_{\alpha=1}^{n}\left(u_\alpha^*\frac{\partial u_\alpha}{\partial \xi_\beta} - \frac{\partial u_\alpha^*}{\partial \xi_\beta}u_\alpha\right) = g^\beta(\xi), \qquad \beta = 1,2,3, \qquad \sum_{\alpha=1}^{n} u_\alpha^* u_\alpha = 1. \tag{.127}$$

The number n is such a natural number that equations (.127) admit a solution. In general, n depends on the form of the arbitrary integration functions $\mathbf{g} = \{g^\beta(\xi)\}$, $\beta = 1, 2, 3$. The functions $\mathbf{g}$ determine vorticity of the fluid flow.

It is easy to verify that

$$\rho = \psi^*\psi, \qquad \rho p_0\left(\varphi, \xi\right) = -\frac{ib_0}{2}(\psi^*\partial_0\psi - \partial_0\psi^*\cdot\psi) \tag{.128}$$

$$\rho p_\alpha\left(\varphi, \xi\right) = -\frac{ib_0}{2}(\psi^*\partial_\alpha\psi - \partial_\alpha\psi^*\cdot\psi), \qquad \alpha = 1,2,3, \tag{.129}$$

The variational problem with the action (.124) appears to be equivalent to the variational problem with the action functional

$$\mathcal{A}_{\mathcal{E}_{\rm st}[\mathcal{S}_{\rm st}]}[\psi, \psi^*] = \int\left\{\frac{ib_0}{2}(\psi^*\partial_0\psi - \partial_0\psi^*\cdot\psi)\right.$$
$$\left. + \frac{b_0^2}{8m\rho}(\psi^*\nabla\psi - \nabla\psi^*\cdot\psi)^2 - \frac{\hbar^2}{8m}\frac{(\nabla\rho)^2}{\rho}\right\}{\rm d}^4x. \tag{.130}$$

We hope that in the case $n = 2$ equations (.127) have a solution for any functions $\mathbf{g}$, because in this case the number (four) of real components of ψ coincides with the number of hydrodynamic variables j^k $(k = 0, , 2, 3)$. (But this statement is not yet proved). For the two-component function ψ $(n = 2)$ the following identity takes place

$$(\nabla\rho)^2 - (\psi^*\nabla\psi - \nabla\psi^*\cdot\psi)^2 \equiv 4\rho\nabla\psi^*\nabla\psi - \rho^2\sum_{\alpha=1}^{\alpha=3}(\nabla s_\alpha)^2, \tag{.131}$$

$$\rho \equiv \psi^*\psi, \qquad s \equiv \frac{\psi^*\sigma\psi}{\rho}, \qquad \sigma = \{\sigma_\alpha\}, \qquad \alpha = 1,2,3, \tag{.132}$$

where σ_α are the Pauli matrices. In virtue of the identity (.131) the action (.130) reduces to the form

$$\mathcal{E}_{\rm st}\left[\mathcal{S}_{\rm st}\right]: \qquad \mathcal{A}_{\mathcal{E}_{\rm st}[\mathcal{S}_{\rm st}]}\left[\psi, \psi^*\right] = \int\left\{\frac{ib_0}{2}\left(\psi^*\partial_0\psi - \partial_0\psi^*\cdot\psi\right) - \frac{b_0^2}{2m}\nabla\psi^*\nabla\psi\right.$$
$$\left.+\frac{b_0^2}{8m}\sum_{\alpha=1}^{\alpha=3}(\nabla s_\alpha)^2\rho + \frac{b_0^2 - \hbar^2}{8\rho m}(\nabla\rho)^2\right\}d^4x, \tag{.133}$$

where $\mathbf{s}$ and ρ are defined by the relations (.132). Thus, we prove the relation (64).

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THE NILPOTENT VACUUM

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A fermionic state vector which is a nilpotent or square root of zero appears to be the most convenient packaging of the fundamental physical parameters space, time, mass and charge into a single unit. It also has the advantage of being a supersymmetric quantum field operator, which uniquely and simultaneously specifies both amplitude and phase for any fermionic state, and incorporates all the specific aspects required in BRST field quantization into a single package. The mathematical structure of the state vector immediately generates vacuum terms relevant to all four fundamental interactions, and explains the symmetry-breaking between them. By incorporating the vacuum aspects into our understanding of the fermion, we generate a 'string theory without strings'. The nilpotent vacuum operators suggest links with many well-known vacuum phenomena, including the Casimir effect and zero-point energy.

1. The nilpotent state vector

Фермионное состояние является наиболее эффективной формой объединения или упаковки в одну величину четырех фундаментальных параметров физики: времени, пространства, массы и заряда. В процессе упаковки соответствующие псевдоскалярные, векторные и скалярные единицы первых трех величин объединяются посредством применения одной из трех кватернионных единиц (quaternion charge units) по отдельности к каждой другой. Таким образом,

время	пространство	масса	заряд
i	$\mathbf{i}, \mathbf{j}, \mathbf{k}$	1	$\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$

строятся как новые псевдоскалярные, векторные и скалярные величины, энергия, импульс и масса покоя $(E, \mathbf{p}, m)$:

$i\boldsymbol{k}$	$\mathbf{i}\boldsymbol{i}\ \mathbf{j}\boldsymbol{i}\ \mathbf{k}\boldsymbol{i}$	$1\boldsymbol{j}$
E	$\mathbf{p}$	$m.$

В то же время, симметрия между кватернионными единицами нарушается для создания слабого, сильного и электрического зарядов (w, s, e) с соответствующими псевдоскалярными, векторными и скалярными характеристиками.

$$w \qquad s \qquad e$$

Отсюда следует, что составное дираковское состояние $(\pm\boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$ (где члены повсюду условно умножены на i) должно выражаться как через зарядовое состояние фермиона, так и через его энергетическое состояние. В действительности, процесс упаковки одновременно создает зарядовое пространство $(w - s - e)$, которое дуально фазовому пространству $(E - \mathbf{p} - m)$.

Составное фермионное или дираковское состояние $(\pm\boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$ является нильпотентным, или квадратным корнем из нуля, ввиду того что соотношение

$(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) = E^2 - p^2 - m^2 = 0$ – это просто стандартное релятивистское соотношение между энергией, импульсом и массой покоя. Здесь вектор импульса $\mathbf{p}$ рассматривается как многозначный, учитывая идею спина. В этом случае $\mathbf{pp} = \mathbf{p}.\mathbf{p} + i\mathbf{p} \times \mathbf{p} = (\sigma.\mathbf{p})(\sigma.\mathbf{p}) = pp = p^2$. Создание нильпотентного состояния как одного пакета требует одновременного порождения фундаментальных констант $\hbar, c$ и G, чтобы обеспечить подходящее масштабное соответствие членов. На деле, нильпотентная структура требует одновременного участия специальной относительности и квантования, а также их взаимной необходимости, – это единственный способ достижения данной цели. Члены E и $\mathbf{p}$ в состоянии могут представлять либо собственные значения, либо операторы, в зависимости от того, выбираем ли мы консервативное или неконсервативное представление состояния. Используя операторную версию состояния $(\pm \boldsymbol{k}\frac{\partial}{\partial t} \pm i\boldsymbol{i}\nabla + i\boldsymbol{j}m)$, мы можем выписать эквивалентную *сопряженную метрику* $(\pm \boldsymbol{k}t \pm i\boldsymbol{i}\mathbf{r} \pm i\boldsymbol{j}\tau)$, где τ – собственное время. В классическом пределе это превращается в определение специальной относительности в нильпотентной форме.

Нильпотент в операторной форме задает состояние в целом, и амплитуду и фазу, ввиду того, что фазовый член однозначно задается требованием нильпотентности собственного значения (или амплитуды). Это дополнительное ограничивающее условие возникает лишь в нильпотентной формулировке и приводит к тому, что для описания состояния не нужно ни уравнение Дирака, ни какое-либо другое. Это особенно важно в случае, когда операторы E и $\mathbf{p}$ заменяются ковариантными производными, или производными, содержащими полевые члены, ввиду того, что применяется тот же самый принцип. Решение уравнения Дирака в данном случае заменяется процессом нахождения фазового члена, который делает амплитуду дираковского состояния нильпотентной. В частном случае точечного заряда любого рода, со сферической симметрией как минимальным требованием, требование нильпотентности приводит к минимальному условию, эквивалентному обратному линейному (кулоновскому) потенциалу.

Благодаря включению дополнительных симметрий, возникших из-за нильпотентности, фермионное состояние становится автоматически вторично-квантованным, с встроенной суперсимметрией. Амплитуда и фаза задаются однозначно тем же оператором, и тем же способом являются квантованными. Знаки $\pm$ перед членами $\boldsymbol{k}E$ и $i\boldsymbol{i}\mathbf{p}$ представляют четыре одновременных "решения" для дираковского фермионного состояния: фермион / антифермион $(\pm \boldsymbol{k}E)$, спин вверх / спин вниз $(\pm i\boldsymbol{i}\mathbf{p})$, – и полное представление нильпотентного оператора является 4-компонентным вектором-строкой или столбцом, таким же образом, как и стандартный дираковский спинор. Отсюда мы можем определить антифермионное состояние, как состояние, имеющее форму $(\mp \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$. Фермион с перевернутым спином будет $(\pm \boldsymbol{k}E \mp i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$; бозон со спином 1: $(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)(\mp \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$ и бозон со спином 0: $(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)(\mp \boldsymbol{k}E \mp i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$. Конденсированное состояние Бозе-Эйнштейна, пара Купера, или их "бозонный" двуфермионный эквивалент примут форму $(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)(\pm \boldsymbol{k}E \mp i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$. В случае барионов, мы особым образом используем векторную структуру оператора $\mathbf{p}$ и строим смешанное (entangled) состояние, в котором первый ряд в спиноре имеет вид $(\boldsymbol{k}E \pm i\boldsymbol{i}p_x + i\boldsymbol{j}m)(\boldsymbol{k}E \pm i\boldsymbol{i}p_y + i\boldsymbol{j}m)(\boldsymbol{k}E \pm i\boldsymbol{i}p_z + i\boldsymbol{j}m)$. Барионное состояние, таким образом, имеет три компоненты (условно описываемые как "кварки"), представляющие шесть возможных "фаз" узнаваемой $SU(3)$ симметрии, в которых $\mathbf{p}$ есть соответственно $\pm i\mathbf{i}p_x, \pm i\mathbf{i}p_y, \pm i\mathbf{i}p_z$. Три компоненты составного бариона будут тогда иметь обычные свойства, соответствующие компонентам векторов, и могут быть разъединены не в большей мере, чем размерности пространства или импульса. Калибровочно инвариантный нелокальный "перенос" $\mathbf{p}$ между фазами будет происходить с постоянной скоростью, независимо от любой концепции физического разделения. Такая постоянная скорость изменения импульса эквивалентна постоянной силе или потенциалу, линейному по расстоянию.

Нильпотентное состояние естественным образом подчиняется принципу несовместности Паули, ввиду того, что $(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) = 0$, и нелокальность является другим автоматическим следствием. КЭД, КХД, КФД выводимы непосредственным образом, причем пропагаторы, определенные через нильпотентные состояния, устраняют инфракрасные расходимости [3]. Особое квантование поля становится ненужным, ввиду того что нильпотентные члены уже являются вторично проквантованными полевыми операторами. Они являются также точно суперсимметричными, с операторами $Q = (\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$ и $Q^+ = (\mp \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$, обращающими бозоны в фермионы, и фермионы в бозоны или бозоны в антифермионы соответственно. Точная суперсимметрия означает отождествление частиц с их собственными суперсимметричными партнерами, что предполагает вакуумную связь. Преобразования C, P, T могут быть представлены как:

$$
\begin{aligned}
&-\boldsymbol{j}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\boldsymbol{j} = (-\boldsymbol{k}E - i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m);\\
&\boldsymbol{i}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\boldsymbol{i} = (\boldsymbol{k}E - i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m);\\
&\boldsymbol{k}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\boldsymbol{k} = (-\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m),
\end{aligned}
$$

с CPT-инвариантностью, как легко выводимым следствием. Полуцелость спина фермионов может быть получена стандартным формальным путем, например:

$$
\begin{aligned}
[\hat{\sigma}, \mathcal{H}] &= [-\mathbf{1}, -\boldsymbol{j}(\mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3) + i\boldsymbol{k}m] = 2i\boldsymbol{j}\mathbf{1} \times \mathbf{p}\\
[\mathbf{L}, \mathcal{H}] &= -\boldsymbol{k}\boldsymbol{i}[\mathbf{r}, \mathbf{1}.\mathbf{p}] \times \mathbf{p} = -\boldsymbol{j}[\mathbf{r}, \mathbf{1}.\mathbf{p}] \times \mathbf{p} = -i\boldsymbol{j}\mathbf{1} \times \mathbf{p},
\end{aligned}
$$

так как $[\mathbf{r}, \mathbf{1}.\mathbf{p}]\psi = i\mathbf{1}\psi$. Тогда $[\mathbf{L} + \hat{\sigma}/2, \mathcal{H}] = 0$, что делает $\mathbf{L} + \hat{\sigma}/\mathbf{2}$ константой движения.

Однако, значение полуцелого спина в физическом смысле представляется таким, как это подразумевается естественной точной суперсимметрией: чисто фермионное состояние в некотором смысле может рассматриваться как неполное без своего вакуумного партнера.

2. Операторы дираковского вакуума

Четырехкомпонентный спинор, представляющий нильпотентное дираковское состояние, включает в себя четыре оператора рождения/уничтожения:

создание фермиона со спином вверх =
уничтожение антифермиона со спином вниз $(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$,
создание фермиона со спином вниз =
уничтожение антифермиона со спином вверх $(\boldsymbol{k}E - i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$,
создание антифермиона со спином вверх =
уничтожение фермиона со спином вниз $(-\boldsymbol{k}E - i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$,
создание антифермиона со спином вниз =
уничтожение фермиона со спином вверх $(-\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$.

Взяв любой из этих операторов, мы можем также указать вакуумные операторы, которые (предполагая подходящую нормализацию) оставляют состояние неизменным. Например, $(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$ остается неизменным после перемножения на $\boldsymbol{k}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$ любое число раз:

$$(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\boldsymbol{k}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\boldsymbol{k}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\dots$$

Однако, $\boldsymbol{k}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\mathbf{k}$ идентичен оператору рождения антифермиона $(-\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$, поэтому мы можем также записать это выражение с альтернативными членами, соответствующими рождению фермионов/антифермионов:

$$(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)(-\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)(\mathbf{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)(-\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\dots,$$

либо как процесс альтернативного рождения фермионов/бозонов через операторы суперсимметрии $QQ^{+}QQ^{+}\dots$ Состояние рождения антифермиона здесь действует как вакуумное "отражение" состояния рождения фермиона и обратно, в то время как настоящий фермион и его виртуальный двойник в комбинации создают суперсимметричного бозонного партнера, что идентично оригинальному фермионному рождению. Мы можем расширить данное рассуждение до утверждения, что действительное состояние рождения бозона, такое как $(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)(-\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$, будет порождать одновременно суперсимметричные виртуальные антифермионные и фермионные состояния как соответствующие вакуумные отражения компонент операторов рождения $(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$ и $(-\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$.

Выражение $\boldsymbol{k}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$, однако, не единственно для определения "вакуумного" состояния: $\boldsymbol{i}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$ и $\boldsymbol{j}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$ обладают теми же самыми свойствами, и также должны быть рассмотрены как вакуумные операторы. В случае $\boldsymbol{i}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$ вакуумное "отражение" требует изменения ориентации спина. В случае же $\boldsymbol{j}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$, фермион отражается как антифермион с дополнительным изменением спина. Каждый из этих трех случаев порождает суперсимметричные состояния бозонного типа, которые являются спин 1, спин 0 и конденсацией Бозе-Эйнштейна, для коэффициентов $\boldsymbol{k}$,$\boldsymbol{j}$ и $\boldsymbol{i}$ соответственно. Однако, не существует *дискретного* вакуумного эквивалента для $1(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$, потому что это исключается принципом Паули:

$$(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)1(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) = 0.$$

В то же время, два различных вакуумных состояния могут лишь комбинационным образом создать третье посредством физической части коэффициентов, то есть $E, \mathbf{p}$ или m.

Определение вакуумных состояний через коэффициенты $\boldsymbol{k}$,$\boldsymbol{j}$ и $\boldsymbol{i}$ приводит к новому пониманию четырех "решений", характеризующих дираковское состояние. Первый ряд спинора представляет фермионное/антифермионное состояние, в то время как остальные три ряда являются тремя дискретными отражениями вакуума. Три коэффициента могут также рассматриваться как последствия концепции дискретного (точечного) заряда.

$$\begin{array}{lll}
\boldsymbol{k}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) \quad \text{или} & i\boldsymbol{k}E(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) & \text{слабый вакуум,} \\
\boldsymbol{i}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) \quad \text{или} & \boldsymbol{i}\mathbf{p}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) & \text{сильный вакуум,} \\
\boldsymbol{j}(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) \quad \text{или} & \boldsymbol{j}m(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) & \text{электрический вакуум.}
\end{array}$$

Заряд в данной интерпретации является проявлением вакуума, и как в случае зарядов, три вакуума (vacua) совершенно независимы друг от друга, ничего не зная о существования остальных.

Вектор нильпотентного состояния включает в себя действительную и виртуальные компоненты, таким же образом, как он включает в себя массу и заряд, и zitterbewegung[1] может быть интерпретирован как переключение между ними. Это является причиной того, что векторы состояния суперсимметричны. Действительный фермион и множество его дуальных вакуумных образов в комбинации дают однозначное состояние бозонного спина, аналогичное консервативной физической системе, одновременно включающей в себя действие и противодействие третьего закона Ньютона или вириальное удвоение кинетической энергии в потенциальной энергии. По этой причине фермионные и антифермионные векторы состояний имеют идентичные компоненты, с единственным отличием – какое из состояний, с энергией $+E$ или $-E$, действительно реализуемо.

Рождение "действительного" фермиона отличается своим действительным коэффициентом (1) от "образных" вакуумных состояний, которые индуцируются слабыми, электрическими и сильными элементами, и описываются посредством кватернионных коэффициентов. Поэтому первый член дираковского 4-спинора имеет иной статус, чем остальные, как и временная координата имеет иной статус, чем пространственные координаты, в стандартном 4-векторе Минковского. В случае свободного фермиона (или бозона), этот статус особо важен. Вакуумные члены тогда не дают вклада в энергию частицы, учитывая, что ренормализация не является необходимой, как показывает нильпотентная версия квантовой электродинамики. При ренормализации уничтожается воздействие лишь "образных" члены, воздействие "действительного" члена остается неизменным.

3. БРСТ-квантование

Нильпотентный оператор Дирака, являющийся автоматически вторично-квантованным, уже включает в себя полное представление квантового поля. Тем не менее, более стандартные подходы к квантованию полей могут быть использованы для иллюстрации связи между операторами заряда и энергии, необходимой для построения нильпотентного формализма. Условно, взаимодействия совершаются посредством поглощения и испускания виртуальных бозонных квантов слабых, сильных и электрических полей. Процессы расширяются до бесконечности в вакууме, с бесконечной последовательностью "петлевых диаграмм" в фейнмановском формализме. Бесконечные процессы порождают расхождения, которые могут быть устранены посредством ренормализации, с переопределением значений массы и заряда в соответствии с мощностью взаимодействия. Однако, в нильпотентной формулировке свободный невзаимодействующий фермион не имеет определеного значения заряда, и бесконечная последовательность точно суперсимметричных бозонных или фермионных петель автоматически обращается в нуль [3]. "Ренормализация" предстает как просто механизм "пересчета" для фиксации значений при разных мощностях взаимодействий, которые при вычислении возмущений ограничены энергией обрезания, равной планковской массе.

Нильпотентные операторы специального вида используются и в стандартной квантовой теории поля, и было бы полезно представить связь между ними и членами формы $(\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$, рассмотренными как операторы энергии и одновременно заряда. Каноническое квантование электромагнитного поля использует кулоновскую калибровку, но это влечет нарушение лоренцевой инвариантности. Подход интеграла по путям позволяет нам использовать любую калибровку, и поэтому сохранять

[1] Zitterbewegung – букв. "пугливое, дрожащее движение" (нем.) – термин, введенный Шредингером для обозначения специфического движения микрочастиц.

инвариантность Лоренца, но проблема теперь состоит во введении нефизических или "фиктивных" полей духов Фаддеева-Попова. Версия, используемая в теории струн (БРСТ), устраняет духовые поля посредством объединения всей информации в единый оператор, примененный к лагранжиану. Существенно, что БРСТ оператор (δ_{BRST}) является нильпотентным. Этот оператор может быть использован для создания нетерового тока (J_μ), соответствующего сохраняющемуся нильпотентному БРСТ фермионному заряду (Q_{BRST}). Условие для определения физического состояния принимает вид:

$$Q_{BRST}|\psi\rangle = 0.$$

В дираковской нильпотентной формулировке оператор ($\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m$), который применяется только к физическим состояниям (на массовой оболочке), уже является вторично проквантованным и нильпотентным оператором формы δ_{BRST}. Он также является нильпотентным оператором *заряда,* формы Q_{BRST}, но расширенным для включения не только электромагнитных, но и слабых, сильных зарядов. В конечном итоге, он в форме собственных значений идентичен $|\psi\rangle$. Таким образом, три возможных понимания выражения ($\pm \boldsymbol{k}E \pm i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m$), применяются соответственно к: E и $\mathbf{p}$, интерпретируемым как дифференциальные операторы во времени и пространстве; $E, \mathbf{p}$ и m как к коэффициентам, определяющим природу зарядов, заданных посредством $\boldsymbol{k}$,$\boldsymbol{i}$ и $\boldsymbol{j}$; E и $\mathbf{p}$, интерпретируемым как собственные значения энергии и импульса. Поэтому нильпотентный оператор Дирака доставляет одновременно все характеристики, которые нужны для раздельных БРСТ-членов δ_{BRST}, Q_{BRST} и $|\psi\rangle$.

4. Слабый вакуум

Мы можем рассмотреть функцию заряда как "разбиение" непрерывного вакуума, которое мы никогда непосредственно не наблюдаем, в отличие от дискретного случая. Заряд становится видом вакуумного состояния, соответствующего квантово-полевой природе вектора состояния. Различные заряды соответствуют качественно разным вакуумным состояниям посредством их соотнесения с псевдоскалярными, векторными и скалярными коэффициентами. Три дискретные вакуумные структуры (discrete vacua) описывают только часть вакуума, которую воспринимает только соответствующий тип заряда.

Полный вакуум, который порождает зарядовое разбиение, есть выражение непрерывной или несчетной природы энергии-массы. Непрерывность неизбежно делает массу-энергию одномерными и однополярными, и, ввиду действительности, ограничивает ее одним математическим знаком, который чаще всего берется положительным. Мы можем интерпретировать это как следствие несимметричности основного состояния, или заполненного вакуума, который представляется негативной энергией или антифермионами. Физически это проявляет себя в поле Хиггса, которое нарушает симметрию зарядового сопряжения для слабых взаимодействий и дает массу покоя фермионам и слабым калибровочным бозонам.

Использование члена $\boldsymbol{k}E$ для слабого вакуума обеспечивает то, что мы для всего вакуума выразим непрерывность энергии-массы и одновременно необратимость времени. Никакое физическое состояние не может быть соотнесено с $-E$, хотя зарядово-сопряженное состояние $-i\boldsymbol{k}E$ может быть определено изменением знака оператора $i\boldsymbol{k}$. В принципе, это приводит к нарушению слабого зарядового сопряжения, что означает, что слабое взаимодействие безразлично к знаку слабого заряда, и может различать лишь фермионы и антифермионы. Для сохранения CPT-симметрии либо четность, либо симметрия обращения времени также должны быть нарушены.

Ввиду того, что оператор $\boldsymbol{k}$ меняет заряды фермиона на антифермион, слабый вакуум – единственный, который связан с уничтожением/рождением фермиона/антифермиона. Псевдоскалярный аспект означает, что вакуумное или зарядовое состояние, или потенциал, могут быть комплексными, что необходимо для нарушения CP. Псевдоскалярное представление также естественным образом предполагает биполярность, ввиду фундаментальной математической двойственности $\pm i$, и неразличимости знаков при нарушении слабого зарядового сопряжения. Это особое свойство слабого взаимодействия возникает как конечная причина различных фаз материи и фазовых переходов, когда неразличимость знака допускается для эффективного устранения слабой компоненты в фермион-фермионных комбинациях, и таким образом преодолевает аспекты принципа несовместности Паули.

Предполагая, что требование непрерывности энергии вакуума обеспечивает физическое преобладание материи над антиматерией, мы получаем, что вакуум должен обладать слабым дипольным моментом, проявляющимся как одностороннее вращение, представляемое как $1/2\hbar\omega$ мода колебания нулевого уровня энергии. В принципе, на это можно взглянуть как на причину появления лево-ориентированных фермионных спинов, где фермион создается одновременно со своим вакуумным отражением. Если слабый вакуум находится в непрерывном состоянии, или в состоянии, объявленном заполненным, посредством рождения слабых диполей, которые имеют дипольный момент или специфическую ориентацию, то мы можем также ожидать, что "флуктуации" в этом вакууме будут соответствовать рождению или уничтожению слабой дипольной фермион-антифермионной пары, каждый со спином 1/2, посредством гармонического осцилляторного механизма рождения-уничтожения. Флуктуации такого вида соответствуют силе Казимира или ван дер Ваальса с энергией нулевого уровня, в соответствии с потенциалом для флуктуации диполь-дипольного взаимодействия.

Наполненный слабый вакуум, необходимый для непрерывного состояния энергии, приводит к механизму Хиггса, посредством которого фермионы и слабо взаимодействующие бозоны приобретают массу. Чистый слабый заряд, может быть, в целом, лево-ориентированным, но мера противоположной ориентации появится, когда будут применены другие условия, которые при сохранении лоренц-инвариантности, эквивалентны появлению массы покоя. Типичное условие появляется, когда возникают не только слабые заряды. Заряд и инерциальная масса являются эффективно различными локализациями вакуума.

5. Сильный вакуум

Сильное взаимодействие, как мы его знаем, проявляется посредством нелокального глюонного моря, с переключением компонент импульса в членах как знака, так и направления, что включает шесть фаз. Это в точности то, что обеспечивается членом $\pm i\boldsymbol{i}\mathbf{p}$ в векторе состояния. Замечено, что барионная структура является, по существу, аффинной, распадаясь на компоненты глюонов и комбинации виртуальных барионов ad infinitum[2]. Это в точности то, что мы можем ожидать от аффинной природы оператора $\mathbf{p}$, компоненты которого могут быть разъединены (или зафиксированы) не более, чем соответственно размерности пространства. Векторная природа сильного оператора также означает, что сильный вакуум – единственный, который имеет определенные относительные фазы. В сильно взаимодействующих системах, фазы

[2] Ad infinitum – (лат.) до бесконечности.

связаны с наличием или отсутствием компонент электрического или слабого заряда. Там, где фазы, ассоциированные с этими компонентами, совпадают не остается возможности для различения фаз, а следовательно, нет сильного взаимодействия.

6. Электрический вакуум

Фермионные состояния относятся к состояниям со слабыми зарядами. Однако, существуют два типа фундаментальных фермионных состояний: кварк и лептон. Для кварков фазы **p** определены, и s заряды наличествуют; для лептонов они неопределенные, а s заряды отсутствуют. Как различные типы зарядов и вакуум существуют полностью независимо друг от друга, слабый заряд не должен отличаться в зависимости от присутствия или отсутствия сильного заряда. Таким образом, распределения слабых и электрических зарядов для кварков и фермионов должно следовать той же модели; поэтому, дробные электрические заряды, распределенные по кваркам, являются просто выражениями совершенной калибровочной инвариантности сильного взаимодействия, аналогично процессу создания дробного заряда в квантовом эффекте Холла, – и не являются собственными аспектами структуры кварка. В то же время, слабый заряд должен быть независим относительно присутствия или отсутствия e.

Фермионные состояния с электрическим зарядом и без него, стандартно описываются как $SU(2)_L$ состояния (вверх / вниз, нейтрино, электрон, и т. д.); они должны возникать как *явно* неразличимые относительно слабого взаимодействия. Обычно мы используем третью компоненту слабого изоспина (t_3), по аналогии с $SU(2)$ спина, как квантовое число для различия этих состояний. Для двух изоспиновых состояний, $t_3 = \pm 1/2$, но только для половины полного числа состояний (только для лево-ориентированных). Для свободных фермионов, квантовое число электрической силы принимает значение $Q = -1$, где представлен электрический заряд $(-e)$ (и взят, как обычно, с отрицательным знаком), снова для половины числа состояний (хотя и для другой половины). Если слабое и электрическое взаимодействия описаны некоторой калибровочной группой Великого Объединения, ортогональность и условия нормализации требуют смешанного соотношения, вводимого как $sin^2\theta_W$, определяемого посредством $Tr(t_3^2)/Tr(Q^2)$, и равного в данном случае 0.25.

Однако, соотношение не может быть применено только к свободным фермионам, если слабые взаимодействия независимы от присутствия или отсутствия сильного заряда. Таким образом, в точности та же пропорция смешения, с $sin^2\theta_W = 0.25$, должна существовать и для кварковых состояний, и по отдельности для каждой "цветовой" фазы, или направления импульса; так что слабое взаимодействие не может обнаруживаться посредством "цвета". Интерпретация "цвета" через фазы или направления импульса допускает мгновенное существование лишь одной кварковой фазы в трех. Таким образом мы получаем, что вариация заряда $0 \; 0 \; -e$ должна быть взята в противоположность как пустому фону, или "электрическому вакууму" $(0 \; 0 \; 0)$, так и заполненному фону $(e \; e \; e)$, так что два состояния слабого изоспина в трех цветах становятся:

$$\begin{array}{ccc} e & e & 0 \\ 0 & 0 & -e. \end{array}$$

Наиболее ясное проявление электрического вакуума должно, следовательно, входить в $SU(2)_L$ для слабого взаимодействия. Слабый вакуум, который полон и не может быть обратим, эффективно контрастирует с электрическим вакуумом, который может быть заполнен или опустошен, либо обращен для антифермионов. Однако, в то время как структуры $SU(3)$ и $U(1)$ прямо возникают из вектора $\mathbf{p}$ и скалярных m членов в дираковском состоянии, структура $SU(2)$ для слабого взаимодействия связана лишь с $SU(2)$-спин структурой, относящейся косвенным образом к псевдоскаляру E. Это происходит потому, что член E, входящий в уравнение, не представляет асимметрию физического E полным образом. $SU(2)$ для E является и $SU(2)$ для спиральности, и связано с $SU(2)_L$ для слабого изоспина лишь посредством матрицы (подобной CKM матрицы), включающей массу покоя. Это зависимость массы, связанная с заполненной природой вакуума в механизме Хиггса, который превращает $SU(2) \rightarrow SU(2)_L$.

Смешивание членов E и $\mathbf{p}$, или право- и лево-ориентированных компонент в нильпотентном векторе состояния, также эквивалентно смешиванию e и w зарядов, или электрических и слабых вакуумных структур, но это смешивание не может, по существу, влиять на слабое взаимодействие, которое не обладает право-ориентированными компонентами для фермионов. Таким образом, слабое взаимодействие должно быть одновременно лево-ориентированным для фермионных состояний и независимым относительно существования или отсутствия электрического заряда, который вводит право-ориентированный элемент.

7. Гравитационный вакуум

Три члена дираковского 4-спинора представляют три его дискретные вакуумные "отражения" фермиона; четвертый же (стандартно размещаемый в первом ряду), представляет само создание частицы. Ввиду того, что три вакуумные отражения порождаются членами, которые являются также операторами заряда, естественно заключить, что заряд является фундаментальным вакуумным генератором. В то же время, масса фермиона и соответствующая энергия вакуума могут быть рассмотрены как "порожденные" оператором "массы" (1). Так, мы можем рассмотреть гравитацию, силу, порожденную массой, как представляемой вакуумным оператором формы $1(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$. Однако, более вероятно, что гравитационный вакуум имеет форму $-1(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$, член, аннулирующий дираковское состояние.

Многие полагали, что гравитация является дискретной силой. Однако, она возникает из непрерывного вакуума и является единственным серьезным кандидатом на роль нелокальности для мгновенной квантовой корреляции дираковского состояния, и для источника бесконечного спектра нуль-энергии. Использование коэффициента 1 может быть взято как эквивалент утверждению о том, что гравитационный вакуум не может быть проквантован непосредственно. Одним из путей представления этого состоит в том, чтобы определить гравитационную энергию как отрицательную (ввиду силы отталкивания) и соотнести заполненный вакуум с отрицательным состоянием энергии, как предлагается в оригинальной теории позитрона Дирака, и как объясняется отсутствие антиматерии из основного состояния Вселенной. Тем не менее, существует дискретное вакуумное представление, связанное с массой. Это инерционная компонента, связанная с дискретной массой покоя, которая проявляет себя в структуре фермионного и бозонного заряда. В механизме Хиггса это знаменуется нелокальным конечным уровнем энергии для слабого вакуума. Инерционная компонента может быть рассмотрена как дискретная локальная реакция, заданная посредством $1(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ на непрерывную нелокальную гравитационную энергию, заданную посредством $-1(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)$. Можно сказать, что полная нуль-энергия "вселенной" появляется как комбинация положительного нильпотента (инерции, суммы зарядов) с негативным (гравитацией).

8. Сферически симметричные потенциалы, примененные к фермионному состоянию

Если мы определяем заряды как точечные источники, они, по определению, имеют сферическую симметрию. Введение сферической симметрии пространства, несомненно, является эквивалентным путем выражения сохранения углового момента, и мы можем показать, что три групповые симметрии, относящиеся к слабому, сильному и электрическому взаимодействиям, связаны с тремя независимыми аспектами сферической симметрии или сохранением углового момента. $SU(2)$ симметрия (слабая) означает независимость сохранения от направления вращения; $SU(3)$ симметрия (сильная) означает ее независимость от выбора осей; $U(1)$ симметрия (электрическая) означает ее независимость от длины радиус-вектора. На деле, дираковский нильпотент может быть выбран как выражение, содержащее всю необходимую информацию для получения полного углового момента, который описывает состояние. Три симметрии, и три раздельных закона сохранения углового момента вытекают из одного факта, что три части вектора фермионного состояния определяются соответственно псевдоскалярным, векторным и скалярным операторами.

Используя полярные координаты, мы можем записать вектор нильпотентного состояния фермиона,находящегося под воздействием потенциала $V(r)$ точечного источника в форме:

$$\Big(\boldsymbol{k}(E - V(r)) + \boldsymbol{i}(\frac{\partial}{\partial \mathbf{r}} + \frac{1}{r} \pm i\frac{j+1/2}{r}) + i\boldsymbol{j}m\Big),$$

где $j + 1/2$ – полный угловой момент. Целью теперь является найти фазовый член, к которому может быть применен данный оператор, чтобы сделать амплитуду нильпотентной, для всех типов потенциала $V(r)$. Фактически, мы уже знаем типы потенциалов, которые должны применяться для слабых, сильных и электрических сил, и также можем показать, что это именно те, которые допускают нильпотентные решения [1], [6].

Минимальным условием для сферической симметрии является обратный линейный (кулоновский) потенциал, $V = -A/r$, связанный с $\boldsymbol{k}E$. Это соотносится со скаляром $U(1)$ компоненты. В чистом случае, это соотносится с электрическим взаимодействием и дает нам характерное решение "водородного атома". Однако, кулоновский член является неотъемлемым аспектом как сильной, так и слабой силы, ввиду того, что каждый заряд с необходимостью имеет скалярную компоненту – эквивалент константы связи. Никакое нильпотентное решение не является возможным без нее, так как обратные линейные члены, ассоциированные с $\boldsymbol{i}$ кватернионным оператором, требуют присутствия члена того же самого типа, ассоциированного с $\boldsymbol{k}$ кватернионом. (Это является дополнительным требованием для $U(1)$ члена в слабом взаимодействии, и его связь с массой, которая эффективно меняет группу $SU(2)$ спина на $SU(2)_L$ слабого изоспина.)

В дополнении к кулоновскому члену, сильное взаимодействие требует, как мы видели, чтобы линейный потенциал $(-Br)$ допускал бы переключение между компонентами $\pm i\boldsymbol{i}p_x, \pm i\boldsymbol{i}p_y, \pm i\boldsymbol{i}p_z$ вектора импульса. Применение данного принципа к вектору состояния дает нам нильпотентные решения, имеющие фазовыми членами только члены вида $exp(\mp iEr \pm iqBr^2/2)r^{\pm iqA-1}$. Для малых r это приводит к асимптотической свободе, а при больших r – к инфракрасному заключению. "Активный" член в конфайнменте является векторной частью потенциала взаимодействия $(-Br)$, в то время как скалярная часть $(-A/r)$ остается "пассивной".

Для псевдоскалярного слабого взаимодействия необходим дипольный или мультипольный член $(-Cr^s)$, где $-3 \geq s$, в потенциале в дополнение к кулоновскому выражению. Однако, любая зависимость полиномиального типа r для потенциала, отличная от $s = \pm 1$, после комбинирования с обратным линейным членом, необходимым для сферической симметрии, дает решение гармонического осциллятора, безотносительно к специфике r зависимости. Т. о., состояние энергии принимает вид

$$E = \left(\frac{m}{j + 1/2} \right) (\pm iA + n'),$$

где n' – целое. Беря минимальное условие на A, фазовый член, необходимый для сферической симметрии, или случайное направление фермионного спина, как полуцелой величины $\pm 1/2i$, получаем решение:

$$E = \left(\frac{m}{j + 1/2} \right) (1/2 + n').$$

Естественно, слабое взаимодействие имеет в точности эти характеристики, создавая и уничтожая из вакуума пары фермион/антифермион спина 1/2, используя подходящие операторы создания или уничтожения, способом, присущим гармоническому осциллятору. Мы даже можем представить воздействие слабого дипольного момента как *причину* появления комбинаций фермион-антифермион из вакуумы или как обратный процесс взаимной аннигиляции. В дополнение, решение позволяет членам в выражении потенциала быть комплексными, что приводит к возможности нарушения CP, к комплексной константе связи или потенциалу, вводящих комплексность в лагранжиан, и, следовательно, в CKM матрицу.

Нильпотентная природа оператора Дирака на самом деле требует, чтобы один из трех зарядов или $E - \mathbf{p} - m$ членов был комплексным; таким образом, спин 1/2, или нильпотентное состояние, невозможен в принципе без введения комплексного аспекта. Комплексность члена, в этом смысле, приводит к рождению диполярного поля, алгебры, требующей одновременно положительных и отрицательных решений и никаких предпочтений. Другими словами, мы можем рассмотреть слабый заряд, как производящий не только фазовый член с двумя решениями, ни одно из которых не имеет привилегированного положения с математической точки зрения, но и механизм осцилляции между ними. Поэтому диполярная структура слабой силы существенным образом связана с комплексными аспектами слабого заряда. Комплексные числа не имеют привилегий в смысле знака, и слабый заряд имеет тенденцию вести себя независимо от знака: в то время как фермион и антифермион различимы, $+w$ и $-w$ не различимы. Комплексные уравнения с необходимостью имеют дуальные (комплексно-сопряженные) решения, и мы можем рассматривать слабый заряд, как несущий вместе с собой и заряд с другим знаком в качестве вакуумного образа. Создаваемый таким образом дипольный момент устанавливает ориентацию, которая обеспечивает киральность. Однако, физические соображения требуют преимущества материальных условий над заполненным слабым вакуумом, что обеспечивается соответствующим электромагнитным вакуумом, который либо является пустым, либо заполненным в одном из двух допустимых $SU(2)$ состояний.

9. Теория струн без струн

Модели суперструн и мембран позиционируются в физике как наиболее правдоподобные кандидаты на роль теорий великого объединения. Однако, существует распространенное мнение, что теория великого объединения не совпадет ни с одним из пяти известных классов теории струн, но предстанет как более фундаментальная, объединяющая теория, для которой эти известные классы окажутся модельно-зависимыми приближениями при дополнительных предположениях. Поэтому в идеальной теории струн или мембран должны исчезнуть модельно-зависимые предположения, фактически она должна стать струнной теорией без струн. Десять измерений пространства-времени, по-видимому, необходимы для построения квантовой полевой теории суперструн, в которой сокращаются все аномалии, в то время как расширение до одиннадцатого измерения необходимо для вложения в супермембраны всех классов теории струн. Нильпотентная дираковская теория удовлетворяет этим требованиям. Каждый нильпотент представляет 10 сохраняющихся величин и, поэтому, может быть построен в 10-ти мерном дуальном фазовом/зарядовом пространстве:

энергия	слабый заряд
3 компоненты импульса	3 компоненты сильного заряда
масса покоя	электрический заряд.

Это множество 10 "размерностей" соединяет в себе фундаментальную дуальность, включающую вакуум. Все частицы дуальны вакууму и существуют только в соотношении с ним (zitterbewegung становится динамическим проявлением этого), поэтому нам требуется десять частей информации одновременно для полного описания состояния частицы. Компоненты энергии и заряда появляются как взаимно исключающие заполнители вакуума или аспекты материи. В целом, одно множество из пяти компонент описывает частицу, а другое – дуальное вакуумное состояние, или одно множество представляет амплитуду, а другое – фазу. Однако, для задания состояния необходимы все десять компонент, а для преобразования из фазового пространства в "реальное" пространство мы можем просто использовать сопряженную метрику $\pm \boldsymbol{k}t \pm i\boldsymbol{i}\mathbf{r} + i\boldsymbol{j}\tau$). Существенным является то, что шесть "размерностей" (все, за исключением E и $\mathbf{p}$) являются фиксированными или компактифицированными, в точности в соответствии с требованиями теории струн. Также они ограничены симметриями, которые по своей природе являются сферическими, как, например $U(1)$-симметрия в теории Калуцы-Клейна, которая соответствует здесь частному случаю электрического заряда.

Одиннадцатое, или "мембранное" измерение является коммутативным гильбертовым пространством, связывающим все нильпотенты, которое существенным образом связано с гравитацией и мгновенными корреляциями. Тем не менее, мы должны осознать, что обе модели, 10-ти и 11-ти мерная, являются в действительности проявлениями более фундаментальной 3-мерности, заданной посредством трех кватернионных операторов $\boldsymbol{k},\boldsymbol{i},\boldsymbol{j}$. Квантовая нильпотентная структура всегда может быть задана 3-мерным представлением, через аффинную природу $\mathbf{p}$ или s оператора. Только одно направления спина и только одно состояние для цветного заряда является корректно-заданным. Данное понимание позволяет нам построить ренормализуемую теорию квантовой гравитации или квантовой гравитационной инерции.

Теории струн, по определению, являются также суперсимметричными. Это, конечно же, ненарушенная суперсимметрия дираковского нильпотента, которая позволяет задать состояния энергии и заряда одновременно. В принципе, ненарушенная суперсимметрия требует нулевой полной энергии вакуума, что мы и ожидаем, если связанная с материей полная энергия компенсируется отрицательной гравитационной энергией. Спонтанное нарушение симметрии в данной интерпретации вызвано не общим состоянием вакуума, а слабым дискретным вакуумом, который в силу механизма Дирака-Хиггса предпочитает состояния $+E$ состояниям $-E$ *дискретной* материи.

Существенно, что суперпространство, необходимое для суперсимметрии, постулирует четыре антисимметричные координаты как суперпартнеры пространства-времени; здесь они становятся массой и тремя зарядами. Вместе восемь координат составляют суперпространство, которое в данном формализме принимает характер нильпотентной алгебры Дирака.

10. Нуль-энергия и эффект Казимира

Одной из интерпретаций вакуума является "покой (неподвижность) вселенной", "реакционная" часть третьего закона Ньютона. Это показывает, как мы можем определить вакуум ссылаясь на "образный" заряд или "отражение" дискретного источника. Для дискретного слабого, сильного или электрического вакуума, это означает, что часть покоя вселенной распознается посредством подходящего заряда, и это является эффективным отрицанием данной компоненты. Однако, полный вакуум является *непрерывным* вакуумом, производимым посредством действительной (гравитационной) компоненты, и для любого заданного фермиона, он создает вектор состояния, эквивалентный $-1(\boldsymbol{k}E + i i\mathbf{p} + i\boldsymbol{j}m)$, с отрицательной энергией. Комбинация фермиона и полного вакуума порождает нуль-тотальность и нулевой вектор состояния. "Непрерывность" в данном контексте может означать лишь отсутствие дискретных уровней энергии, и именно это свойство приводит к возникновению бесконечной плотности виртуальной энергии и виртуальной энергии $1/2\hbar\omega$ для всех возможных мод колебаний, так называемого нулевого уровня энергии. Непрерывный вакуум поэтому составляется зеркальными образными состояниями всевозможных фермионных состояний, и именно такой непрерывный вакуум делает возможным нелокальную связь, предполагаемую принципом запрета Паули. Каждое возможное состояние дает виртуальную вакуумную энергию $1/2\hbar\omega$, как основное состояние гармонического осциллятора, которым, конечно, в точности и является. Для создания действительного фермионного состояния мы возбуждаем виртуальное вакуумное состояние $-1/2\hbar\omega$ до уровня $1/2\hbar\omega$, используя квант полной энергии $\hbar\omega$. Непрерывный вакуум, однако же, никогда не может быть подвергнут прямому точному наблюдению, ввиду своей непрерывности, и поэтому, понятие непрерывности с неизбежностью останется "потенциальным", или виртуальным.

Проявление непрерывности вакуума, которое мы наблюдаем, является широко известной силой Казимира отталкивания между незаряженными пластинками металла площади A, на малом расстоянии d:

$$F = \frac{\pi h c A}{480 d^4}.$$

Ввиду зависимости от $1/d^4$, эта сила проявляет себя вне области 1 μm как диполь-дипольное взаимодействие, в точности того же типа, что и сила Ван дер Ваальса сцепления молекул. Эта интерпретация предполагает нуль-флуктуации виртуальных фотонов в пространстве между пластинками или молекулами; но такой же результат возможно получить, используя нуль-флуктуации электронов в металлических поверхностях [7]. В данном случае это становится лондонским дисперсионным взаимодействием. Согласно другой картине (Хеллмана-Фейнмана), облака квантовых зарядов на двух пластинах, молекулах или других объектах по мере их приближения становятся деформированными в соответствии с изменением значения вероятности их распределения заряда. В этом случае, сила идентична причине химического соединения, вызванного классической электростатической силой [7].

Ввиду вышесказанного, сила Казимира является не отдельным феноменом, а аспектом классического электромагнитного взаимодействия. В то время как Петерсон и Метзгер [7] используют это в качестве средства устранения из вывода формулы таких неосмыслимых вещей как квантовые флуктуации, мы можем повернуть ход рассуждений таким образом, что обычная электромагнитная сила станет вакуумной проекцией. Обратно пропорциональная четвертому порядку сила Казимира между объектами, которые глобально электрически нейтральны, но локально образованы электростатическими диполями, должна предполагать обратный квадрат силы между отдельными заряженными частицами, из которых образованы эти объекты. И родственные эффекты конденсированной материи, такие как ядерные силы, должны описываться в тех же терминах, что и проявления казимировского типа фермионных или бозонных вакуумных флуктуаций, так же, как и взаимодействия между дискретными зарядами, определяемые вероятностными распределениями.

Если мы описываем силы, вызванные дискретными зарядами (электрическими, сильными, слабыми), как казимировского типа проявления вакуума, мы получаем прямую физическую интерпретацию для соответствующего использования квантовых операторов $\boldsymbol{j},\boldsymbol{i},\boldsymbol{k}$ как для этих трех зарядов, так и для операции соответствующего электрического, сильного и слабого вакуума (vacua) посредством $\boldsymbol{j}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m), \boldsymbol{i}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m), \boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$. Ввиду того, что операторы прикреплены соответственно к псевдоскаляру E, вектору $\mathbf{p}$ и скаляру m в векторе состояния, их вакуумы будут разными, и силы также будут вести себя различным образом. Однако, ключевым механизмом во всех казимировских вычислениях является то, что они предстают как результат выделения *дискретных* объектов из *непрерывной* среды, и имеют смысл лишь в контексте пар объектов. Создание пары дискретных объектов на некотором конечном расстоянии, порождает силу, поскольку создается пространств, защищенное от некоторых мод вакуумных колебаний вне этого пространства. В принципе, следовательно, все взаимодействия между дискретными заряженными объектами и даже величины констант связи могут быть рассмотрены как результат существования покоя вселенной как вакуумного состояния, в направлении принципа ренормализации и принципа Маха для параллельного случая инерционных масс.

В данной интерпретации, казимировский и связанные с ним эффекты становятся путем, на котором дискретный заряженный вакуум проявляет их в соотношении с непрерывным полным вакуумным фоном; они представляют разделение вакуума в соответствии с тремя типами зарядовых состояний. Число заполнения зарядовых состояний (то есть, имеют ли заряды единичные или нулевые значения) устанавливается на основе относительных фаз между компонентами вектора состояния. Это определяет тип частицы и возможные взаимодействия. Вакуум, однако же, становится механизмом, посредством которого данный процесс проявляется. Создание дискретных единиц с ненулевым числом заполнения порождает "искажения" вакуума, которые мы называем взаимодействиями, так же, как наличие дискретных источников порождает вакуумный отклик или искажение односвязного пространства, которое порождает эффект Ааронова-Бома и фазу Берри.

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DIVISION ALGEBRA, GENERALIZED SUPERSIMMETRIES AND OCTONIONIC M-THEORY

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This is the report of the talk given at the conference "Number, Time and Relativity", held at the Bauman University, Moscow, August 2004, concerning the recent research activity of the author and his collaborators about the inter-relation of the concepts of division algebras, representations of Clifford algebras, generalized supersymmetries with the introduction of an alternative description of the M-algebra in terms of the non-associative structure of the octonions.

Introduction.

The unification program aiming at a unified description of the known interactions as well as a consistent quantum formulation for gravity, nowadays mostly points towards higher-dimensional supersymmetric theories. At present the most promising, however still conjectural, candidate should live in eleven dimensions and goes under the name of M-theory [1]. The theoretical (and phenomenological) consistency requirements put on any possible candidate for unification necessarily lead to a systematic investigation of the properties of Clifford algebras and spinors in space-times of arbitrary dimension and signature. The generalized supersymmetries going beyond the standard HŁS scheme [2] admit the presence of bosonic abelian tensorial central charges associated with the dynamics of extended objects (branes). It is widely known since the work of [3] that supersymmetries are related to division algebras. Indeed, even for the generalized supersymmetries, classification schemes based on the associative division algebras ($\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$) are now available. For what concerns the remaining division algebra, the octonions, much less is known due to the complications arising from their non-associativity. Octonionic structures were, nevertheless, investigated in [4, 5] in application to superstring theory.

Octonions are not just a curiosity. They are the maximal division algebra. This fact alone already justifies that they should receive the same kind of attention paid to, let's say, the maximal supergravity. However, their importance is more than that, they are at the very heart of many exceptional structures in mathematics and can be held responsible for their existence. Among these exceptional structures we can cite the 5 exceptional Lie algebras and the exceptional Jordan algebras. Indeed, the G_2 Lie algebra is the automorphism group of the octonions, while F_4 is the automorphism group of the 3×3 octonionic-valued hermitian matrices realizing the exceptional $J_3(\mathbf{O})$ Jordan algebra. F_4 and the remaining exceptional Lie algebras E_6, E_7, E_8 are recovered from the so-called “magic square Tits' construction"which associates a Lie algebra to any given pair of division algebras, if at least one of these algebras coincides with the octonionic algebra [6].

It has been pointed out several times, [7, 8] that the exceptional Lie algebras fit well into the grand-unification scenario. Moreover, the E_8 Lie algebra enters, through the

$E_8 \times E_8$ tensor product, the anomaly-free heterotic string, while the G_2 holonomy of seven-dimensional manifolds is required, on phenomenological basis, to produce 4-dimensional $N = 1$ supersymmetric field theories by compactification of the eleven dimensions. This partial list of scattered pieces of evidence has brought to suggest, see e.g. [8], that for some deep reasons, Nature seems to prefer exceptional structures. In this context it deserves to be mentioned the special role of the exceptional Jordan algebra $J_3(\mathbf{O})$, not only associated to the unique consistent quantum mechanical system (in the Jordan framework, see [9]) based on a non-associative algebra, but also leading to a unique matrix Chern-Simon theory of Jordan type, see [10].

In this talk I will discuss the investigations presented in [11, 12] concerning the possibility of realizing general supersymmetries in terms of the non-associative division algebra of the octonions. In particular in [11] it was shown that the M algebra which supposedly underlines the M-theory comes in two (and only two, due to the absence of the complex and of the quaternionic structures) variants. Besides the standard realization of the M-algebra which involves real spinors and makes therefore use of the real structure, an alternative formulation, requiring the introduction of the octonionic structure, is also possible and can be exploited. This is made possible due to the existence of an octonionic description for the Clifford algebra defining the 11-dimensional Minkowskian spacetime and its related spinors. The features of this second variant, the octonionic M-superalgebra, are puzzling. While it is not at all surprising that it contains fewer bosonic generators, 52, w.r.t. the 528 of the standard M-algebra (this is after all expected, since the imposition of an extra structure always puts a constraint on a theory), what really came as an unexpected surprise is the fact that new conditions, not present in the standard M-theory, are now found. These conditions imply that the different brane-sectors are no longer independent. The octonionic 5-brane alone contains the whole set of degrees of freedom and is therefore equivalent to the octonionic $M1$ and $M2$ sectors. We can write this equivalence, symbolically, as $M5 \equiv M1 + M2$. This result is indeed very intriguing. It implies that quite non-trivial structures are found when investigating the octonionic construction of the M-theory. It is quite tempting to think that the exceptional structures that we mentioned before should be better understood from this octonionic variant of the M-algebra, rather than the standard real M-algebra.

The next passage consists in defining the closed algebraic structure which realizes the octonionic superconformal M-algebra. It turns out that the $OSp(1, 64)$ superconformal algebra of the real M-theory is replaced in the octonionic case by the $OSp(1, 8|\mathbf{O})$ superalgebra of supermatrices with octonionic-valued entries and total number of $7 + 232 = 239$ bosonic generators.

On Clifford algebras.

The classification of generalized supersymmetries requires the preliminary classification of Clifford algebras and spinors and of their association with division algebras.

To make this paper self-consistent, in this section we review the classification of the Clifford algebras associated to the $\mathbf{R}$,$\mathbf{C}$, $\mathbf{H}$ associative division algebras, following [13] and [14].

The most general irreducible *real* matrix representations of the Clifford algebra

$$\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2\eta^{\mu\nu}, \quad (.1)$$

with $\eta^{\mu\nu}$ being a diagonal matrix of (p, q) signature (i.e. p positive, $+1$, and q negative,

-1, diagonal entries)[3] can be classified according to the property of the most general S matrix commuting with all the Γ's ($[S, \Gamma^\mu] = 0$ for all μ). If the most general S is a multiple of the identity, we get the normal (**R**) case. Otherwise, S can be the sum of two matrices, the second one multiple of the square root of -1 (this is the almost complex, **C** case) or the linear combination of 4 matrices closing the quaternionic algebra (this is the **H** case). According to [13] the *real* irreducible representations are of **R**, **C**, **H** type, according to the following table, whose entries represent the values $p - q \ mod\ 8$

R	**C**	**H**
0, 2		4, 6
1	3, 7	5

(.2)

The real irreducible representation is always unique unless $p-q \ mod\ 8 = 1, 5$. In these signatures two inequivalent real representations are present, the second one recovered by flipping the sign of all Γ's ($\Gamma^\mu \mapsto -\Gamma^\mu$).

Let us denote as $C(p, q)$ the Clifford irreps corresponding to the (p, q) signatures. The normal (**R**), almost complex (**C**) and quaternionic (**H**) type of the corresponding Clifford irreps can also be understood as follows. While in the **R**-case the matrices realizing the irrep have necessarily real entries, in the **C**-case matrices with complex entries can be used, while in the **H**-case the matrices can be realized with quaternionic entries.

Let us discuss the simplest examples. The **C**-type $C(0, 1)$ Clifford algebra can be expressed either through the 2×2 matrix with real-valued entries $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or through the imaginary number i.

The **H**-type Clifford algebra $C(0, 3)$, on the other hand, can be realized as follows: *i)* with three 4×4 matrices with real entries, given by the tensor products $\tau_A \otimes \tau_1$, $\tau_A \otimes \tau_2$ and $\mathbf{1}_2 \otimes \tau_A$, where the matrices τ_A, τ_1 and τ_2 furnish a real irrep of $C(2, 1)$
$(\tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$,
ii) with three 2×2 complex-valued matrices given by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$,
iii) with the three imaginary quaternions e_i (see for more details the section **3**).

The formulas at the items *i)* and *ii)* provide the real and complex representations, respectively, for the imaginary quaternions. They can be straightforwardly extended to provide real and complex representations for the **H**-type Clifford algebras by substituting the quaternionic entries with the corresponding representations (the quaternionic identity 1 being replaced in the complex representation by the 2×2 identity matrix $\mathbf{1}_2$ and by the 4×4 identity matrix $\mathbf{1}_4$ in the real representation).

It is worth noticing that in the given signatures $p - q \ mod\ 8 = 0, 4, 6, 7$, without loss of generality, the Γ^μ matrices can be chosen block-antidiagonal (generalized Weyl-type matrices), i.e. of the form

$$\Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \tag{.3}$$

[3]Throughout this paper it will be understood that the positive eigenvalues are associated with space-like directions, the negative ones with time-like directions.

In these signatures it is therefore possible to introduce the Weyl-projected spinors, whose number of components is half of the size of the corresponding Γ-matrices[4].

A very convenient presentation of the irreducible representations of Clifford algebras with the help of an algorithm allowing to single out, in each arbitrary signature space-time, a representative (up to, at most, the sign flipping $\Gamma^\mu \leftrightarrow -\Gamma^\mu$) in each irreducible class of representations of Clifford's gamma matrices has been given in [14]. We recall and extend here the results presented in [14], making explicit the connection between the maximal-Clifford algebras in the table (.6) below and their division-algebra property.

The construction goes as follows. At first one proves that starting from a given D spacetime-dimensional representation of Clifford's Gamma matrices, one can recursively construct $D+2$ spacetime dimensional Clifford Gamma matrices with the help of two recursive algorithms. Indeed, it is a simple exercise to verify that if γ_i's denotes the d-dimensional Gamma matrices of a $D=p+q$ spacetime with (p,q) signature (namely, providing a representation for the $C(p,q)$ Clifford algebra) then $2d$-dimensional $D+2$ Gamma matrices (denoted as Γ_j) of a $D+2$ spacetime are produced according to either

$$\Gamma_j \equiv \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}$$

$$(p,q) \mapsto (p+1, q+1). \qquad (.4)$$

or

$$\Gamma_j \equiv \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix}$$

$$(p,q) \mapsto (q+2, p). \qquad (.5)$$

It is immediate to notice that the three matrices τ_A, τ_1, τ_2 introduced before and realizing the Clifford algebra $C(2,1)$ are obtained by applying either (.4) or (.5) to the number 1, i.e. the one-dimensional realization of $C(1,0)$.

All Clifford algebras of **R**-type are obtained by recursively applying the algorithms (.4) and (.5) to the Clifford algebra $C(1,0)$ ($\equiv 1$) and the Clifford algebras of the series $C(0,7+8m)$ (with m non-negative integer), which must be previously known. Similarly, all Clifford algebras of **H**-type are obtained by recursively applying the algorithms to the Clifford algebras $C(0,3+8m)$, while the **C**-type Clifford algebras are obtained by recursively applying the algorithms to the Clifford algebras $C(0,1+8m)$ and $C(0,5+8m)$. This is in accordance with the scheme illustrated in the table below, taken from [14]. We get

Table with the maximal Clifford algebras (up to $d=256$).

[4]This notion of Weyl spinors, which is convenient for our purposes, is different from the one usually adopted in connection with *complex*-valued Clifford algebras and has been introduced in [14].

	1	*	2	*	4	*	8	*	16	*	32	*	64	*	128	*	256	*
R	(1, 0)	⇒	(2, 1)	⇒	(3,2)	⇒	(4,3)	⇒	(5,4)	⇒	(6,5)	⇒	(7,6)	⇒	(8,7)	⇒	(9,8)	⇒
				↗	(1,2)	→	(2,3)	→	(3,4)	→	(4,5)	→	(5,6)	→	(6,7)	→	(7,8)	→
C			(0,1)															
				↘	(3,0)	→	(4,1)	→	(5,2)	→	(6,3)	→	(7,4)	→	(8,5)	→	(9,6)	→
						↗	(1,4)	→	(2,5)	→	(3,6)	→	(4,7)	→	(5,8)	→	(6,9)	→
H					(0,3)													
						↘	(5,0)	→	(6,1)	→	(7,2)	→	(8,3)	→	(9,4)	→	(10,5)	→
								↗	(1,6)	→	(2,7)	→	(3,8)	→	(4,9)	→	(5,10)	→
C							(0,5)											
								↘	(7,0)	→	(8,1)	→	(9,2)	→	(10,3)	→	(11,4)	→
								↗	(1,8)	→	(2,9)	→	(3,10)	→	(4,11)	→	(5,12)	→
R/O							(0,7)											
								↘	(9,0)	→	(10,1)	→	(11,2)	→	(12,3)	→	(13,4)	→
												↗	(1,10)	→	(2,11)	→	(3,12)	→
C											(0,9)							
												↘	(11,0)	→	(12,1)	→	(13,2)	→
														↗	(1,12)	→	(2,13)	→
H													(0,11)					
														↘	(13,0)	→	(14,1)	→
																↗	(1,14)	→
C															(0,13)			
																↘	(15,0)	→
																↗	(1,16)	→
R/O															(0,15)			
																↘	(17,0)	→

$$(.6)$$

Concerning the above table some remarks are in order. The columns are labeled by the matrix size d (in real components) of the maximal Clifford algebras. Their signature is denoted by the (p,q) pairs. Furthermore, the underlined Clifford algebras in the table can be named as “primitive maximal Clifford algebras". The remaining maximal Clifford algebras appearing in the table are the “maximal descendant Clifford algebras". They are obtained from the primitive maximal Clifford algebras by iteratively applying the two recursive algorithms (.4) and (.5). Moreover, any non-maximal Clifford algebra is obtained from a given maximal Clifford algebra by deleting a certain number of Gamma matrices (this point has been fully explained in [14] and will not be further elaborated here).

The maximal Clifford algebras generated by the $C(0,7+8m)$ series are associated to both the real ($\mathbf{R}$) and octonionic ($\mathbf{O}$) division algebras, since (.1), for the $(0,7+8m)$-signature, can be realized either associatively (in the normal, $\mathbf{R}$, case), or non-associatively through the octonions (see [14] and [16]).

The primitive maximal Clifford algebras $C(0,3)$ and $C(0,7)$ can be explicitly realized through, respectively, three 4×4 matrices (as already recalled) and seven 8×8 matrices given by

$$C(0,3) \equiv \begin{array}{l} \tau_A\otimes\tau_1, \\ \tau_A\otimes\tau_2, \\ \mathbf{1}_2\otimes\tau_A. \end{array} \qquad (.7)$$

and

$$C(0,7) \equiv \begin{array}{l} \tau_A\otimes\tau_1\otimes\mathbf{1}_2, \\ \tau_A\otimes\tau_2\otimes\mathbf{1}_2, \\ \mathbf{1}_2\otimes\tau_A\otimes\tau_1, \\ \mathbf{1}_2\otimes\tau_A\otimes\tau_2, \\ \tau_1\otimes\mathbf{1}_2\otimes\tau_A, \\ \tau_2\otimes\mathbf{1}_2\otimes\tau_A, \\ \tau_A\otimes\tau_A\otimes\tau_A. \end{array} \qquad (.8)$$

The complex primitive maximal Clifford algebras $C(0,1)$ and $C(0,5)$ can be obtained from $C(1,2)$ and $C(0,7)$, respectively, by deleting two gamma-matrices. From $C(0,7)$ we can, e.g., consider the last tensor-product column, eliminate the two terms containing τ_1 and τ_2 and replacing $\mathbf{1}_2\mapsto 1$, $\tau_A\mapsto i$, to get

$$C(0,5) \equiv \begin{array}{l} \tau_A\otimes\tau_1, \\ \tau_A\otimes\tau_2, \\ i\tau_1\otimes\mathbf{1}_2, \\ i\tau_2\otimes\mathbf{1}_2, \\ i\tau_A\otimes\tau_A. \end{array} \qquad (.9)$$

It is worth pointing out that the $C(0,1)$ and $C(0,5)$ series were correctly considered as “descendant"series in [14] due to the fact that they can be obtained from $C(1,2)$,

$C(0,7)$ after erasing extra-Gamma matrices. We find however convenient here to explicitly insert them in table (.6) and consider them as "primitive", since they admit a different division-algebra structure (they are almost complex, **C**) w.r.t. the normal (**R**)-type maximal Clifford algebras they are derived from.

The remaining primitive maximal Clifford algebras $C(0,x+8m)$, for positive integers $m = 1,2,\ldots$ and $x = 1,3,5,7$, can be recovered from the *mod* 8 properties of the Gamma-matrices. Let $\overline{\tau}_i$ be a realization of $C(0,x)$ for $x = 1,3,5,7$. By applying the (.4) algorithm to $C(0,7)$ we construct at first the 16×16 matrices realizing $C(1,8)$ (the matrix with positive signature is denoted as γ_9, ${\gamma_9}^2 = \mathbf{1}$, while the eight matrices with negative signatures are denoted as γ_j, $j = 1,2\ldots,8$, with ${\gamma_j}^2 = -\mathbf{1}$). We are now in the position [14] to explicitly construct the whole series of primitive maximal Clifford algebras $C(0,x+8n)$, through the formulas

$$C(0,x+8n) \;\equiv\; \begin{array}{ll} \overline{\tau}_i \otimes \gamma_9 \otimes \ldots & \ldots \quad \ldots \otimes \gamma_9, \\ \mathbf{1}_4 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots \quad \ldots \otimes \mathbf{1}_{16}, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots \quad \ldots \otimes \mathbf{1}_{16}, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_j \otimes \mathbf{1}_{16} \otimes \ldots & \ldots \quad \ldots \otimes \mathbf{1}_{16}, \\ \ldots & \ldots \quad \ldots, \\ \mathbf{1}_4 \otimes \gamma_9 \otimes \ldots & \ldots \quad \otimes\gamma_9 \otimes \gamma_j, \end{array} \tag{.10}$$

Please notice that the tensor product of the 16-dimensional representation is taken n times.

On division algebras.

In the previous section we furnished a simple algorithm to explicitly construct any given Clifford irrep of specified division-algebra type. It is convenient to review here the basic features of division algebras which will be needed in the following.

The four division algebra of real (**R**) and complex (**C**) numbers, quaternions (**H**) and octonions (**O**) possess respectively 0, 1, 3 and 7 imaginary elements e_i satisfying the relations

$$e_i \cdot e_j \;=\; -\delta_{ij} + C_{ijk}e_k, \tag{.11}$$

(i,j,k are restricted to take the value 1 in the complex case, $1,2,3$ in the quaternionic case and $1,2,\ldots,7$ in the octonionic case; furthermore, the sum over repeated indices is understood).

C_{ijk} are the totally antisymmetric division-algebra structure constants. The octonionic division algebra is the maximal, since quaternions, complex and real numbers can be obtained as its restriction. The totally antisymmetric octonionic structure constants can be expressed as

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \tag{.12}$$

(and vanishing otherwise).

The octonions are the only non-associative, however alternative (see [17]), division algebra.

Due to the antisymmetry of C_{ijk}, it is clear that we can realize (.1) by associating the $(0,3)$ and $(0,7)$ signatures to, respectively, the imaginary quaternions and the imaginary octonions.

For our later purposes it is of particular importance the notion of division-algebra principal conjugation. Any element X in the given division algebra can be expressed through the sum

$$X = x_0 + x_i e_i, \tag{.13}$$

where x_0 and x_i are real, the summation over repeated indices is understood and the positive integral i are restricted up to 1, 3 and 7 in the **C**, **H** and **O** cases respectively. The principal conjugate X^* of X is defined to be

$$X^* = x_0 - x_i e_i. \tag{.14}$$

It allows introducing the division-algebra norm through the product X^*X. The normed-one restrictions $X^*X = 1$ select the three parallelizable spheres S^1, S^3 and S^7 in association with **C**, **H** and **O** respectively.

Further comments on the division algebras and their relations with Clifford algebras can be found in [14] and [17].

On fundamental spinors.

In section **2** we discussed the properties of the Clifford irreps, presenting a method to explicitly construct them and mentioning their division-algebra structure. It is worth reminding that the division-algebra character of fundamental spinors does not necessarily (depending on the given space-time) coincide with the division-algebra type of the corresponding Clifford irreps.

Fundamental spinors carry a representation of the generalized Lorentz group with a minimal number of real components in association with the maximal, compatible, allowed division-algebra structure.

The following table, taken from the results in [18] and [13], see also [14], presents the comparison between division-algebra properties of Clifford irreps ($\mathbf{\Gamma}$) and fundamental spinors ($\mathbf{\Psi}$), in different space-times parametrized by $\rho = s - t \quad mod \quad 8$. We have

ρ	$\mathbf{\Gamma}$	$\mathbf{\Psi}$
0	**R**	**R**
1	**R**	**R**
2	**R**	**C**
3	**C**	**H**
4	**H**	**H**
5	**H**	**H**
6	**H**	**C**
7	**C**	**R**

(.15)

It is clear from the above table that, for $\rho = 2,3$, the fundamental spinors can accommodate a larger division-algebra structure than the corresponding Clifford irreps. Conversely, for $\rho = 6,7$, the Clifford irreps accommodate a larger division-algebra structure than the corresponding spinors. In several cases this mismatch of division-algebra structures plays an important role. For instance in [11] a method was introduced to construct superconformal algebras based on the minimal division algebra structure common to both Clifford irreps and fundamental spinors. This method can be straightforwardly modified to produce extended superconformal algebras based on the largest division-algebra structure. The price to be paid, in this case, would imply the introduction, for $\rho = 2, 3$, of reducible Clifford representations and, conversely, for $\rho = 6, 7$ of non-minimal spinors.

The reason behind the mismatch can be easily understood on the basis of the algorithmic construction of Section **2** and of table (.6). Indeed, all the maximal, descendant Clifford algebras appearing in table (.6) have all block-antidiagonal Gamma matrices with the exception of a single Gamma matrix given by $\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$. Therefore, all non-maximal Clifford algebras which are produced by erasing this extra Gamma matrix (a detailed discussion can be found in [14]) are of block-antidiagonal form. We recall now that the fundamental spinors carry a representation of the generalized Lorentz group whose generators are given by the commutators among Gamma matrices, $[\Gamma_i, \Gamma_j]$. For the non-maximal Clifford algebras under considerations these commutators are all in 2×2 block-diagonal forms, allowing to introduce a (generalized, in the sense specified in [14]) Weyl projection for fundamental spinors, with non-vanishing upper or lower components.

It is convenient to explicitly discuss the simplest Minkowskian cases where the mismatch appears (the general procedure can be straightforwardly read from table (.6)). In the ordinary $(3, 1)$ space-time the $(\mathbf{R})$ Clifford irrep is obtained as the non-maximal Clifford algebra $(3, 1) \subset (3, 2)$, obtained from the maximal $(\mathbf{R})$ $(3, 2)$ after erasing a time-like Gamma matrix. On the other hand, the fundamental complex spinors are obtained from the reducible Clifford representation $(3, 1) \subset (4, 1)$, obtained by erasing a space-like Gamma matrix from the $(\mathbf{C})$ Clifford irrep $(4, 1)$.

In the other Minkowskian cases we get

i) $(4, 1)$: $\mathbf{\Gamma}$ coincides with the maximal Clifford $(4, 1)$ $(\mathbf{C})$, while $\mathbf{\Psi}$ is constructed in terms of the reducible, non-maximal Clifford representation $(4, 1) \subset (6, 1)$ $(\mathbf{H})$,

ii) $(7, 1)$: $\mathbf{\Gamma}$ coincides with the non-maximal Clifford $(7, 1) \subset (7, 2)$ $(\mathbf{H})$, while $\mathbf{\Psi}$ is constructed in terms of the reducible, non-maximal Clifford representation $(7, 1) \subset (8, 1)$ $(\mathbf{C})$,

iii) $(8, 1)$: $\mathbf{\Gamma}$ coincides with the maximal Clifford $(8, 1)$ $(\mathbf{C})$, while $\mathbf{\Psi}$ is constructed in terms of the reducible, non-maximal Clifford representation $(8, 1) \subset (10, 1)$ $(\mathbf{R})$.

Generalized supersymmetries: the M and F algebra examples

Three matrices, denoted as A, B, C, have to be introduced in association with the three conjugations (hermitian, complex and transposition) acting on Gamma matrices [3]. Since only two of the above matrices are independent we choose here, following [14], to work with A and C. A plays the role of the time-like Γ^0 matrix in the Minkowskian space-time and is used to introduce barred spinors. C, on the other hand, is the charge conjugation matrix. Up to an overall sign, in a generic (s, t) space-time, A and C are given by the products of all the time-like and, respectively, all the symmetric (or antisymmetric) Gamma-matrices[5]. The properties of A and C immediately follow from their explicit construction, see [3] and [14].

[5]Depending on the given space-time (see [3] and [14]), there are at most two charge conjugations matrices, C_S, C_A, given by the product of all symmetric and all antisymmetric gamma matrices, respectively. In special space-time signatures they collapse into a single matrix C.

In a representation of the Clifford algebra realized by matrices with real entries, the conjugation acts as the identity, see (.14). In this case the space-like gamma matrices are symmetric, while the time-like gamma matrices are antisymmmetric, so that A can be identified with the charge conjugation matrix C_A.

For our purposes the importance of A and the charge conjugation matrix C lies on the fact that, in a D-dimensional space-time ($D = s + t$) spanned by $d \times d$ Gamma matrices, they allow to construct a basis for $d \times d$ (anti)hermitian and (anti)symmetric matrices, respectively. It is indeed easily proven that, in the real and the complex cases (the quaternionic case is different), the $\begin{pmatrix} D \\ k \end{pmatrix}$ antisymmetrized products of k Gamma matrices $A\Gamma^{[\mu_1 \cdots \mu_k]}$ are all hermitian or all antihermitian, depending on the value of $k \leq D$. Similarly, the antisymmetrized products $C\Gamma^{[\mu_1 \cdots \mu_k]}$ are all symmetric or all antisymmetric.

For what concerns the M-algebra, the 32-component real spinors of the $(10, 1)$-spacetime admit anticommutators $\{Q_a, Q_b\}$ which are 32×32 symmetric real matrices with, at most, $32 + \frac{32 \times 31}{2} = 528$ components. Expanding the r.h.s. in terms of the antisymmetrized product of Gamma matrices, we get that it can be saturated by the so-called M-algebra

$$\{Q_a, Q_b\} = (A\Gamma_\mu)_{ab} P^\mu + \left(A\Gamma_{[\mu\nu]}\right)_{ab} Z^{[\mu\nu]} + \left(A\Gamma_{[\mu_1 \ldots \mu_5]}\right)_{ab} Z^{[\mu_1 \ldots \mu_5]}. \qquad (.16)$$

Indeed, the $k = 1, 2, 5$ sectors of the r.h.s. furnish $11 + 55 + 462 = 528$ overall components. Besides the translations P^μ, in the r.h.s. the antisymmetric rank-2 and rank-5 abelian tensorial central charges, $Z^{[\mu\nu]}$ and $Z^{[\mu_1 \cdots \mu_5]}$ respectively, appear.

The (.16) saturated M-algebra admits a finite number of subalgebras which are consistent with the Lorentz properties of the Minkowskian eleven dimensions. There are 6 such subalgebras which are recovered by setting either one or two among the three sets of tensorial central charges P^μ, $Z^{[\mu\nu]}$, $Z^{[\mu_1 \cdots \mu_5]}$ identically equal to zero (a completely degenerate subalgebra is further obtained by setting the whole r.h.s. identically equal to zero).

The fact that the fundamental spinors in a $(10, 2)$-spacetime also admit 32 components is due to the existence of the Weyl projection. This implies that the saturated M-algebra admits a $(10, 2)$ space-time presentation, the so-called F-algebra, in terms of $(10, 2)$ Majorana-Weyl spinors $\tilde{Q}_{\tilde{a}}$, $\tilde{a} = 1, 2, \ldots, 32$.

In the case of Weyl projected spinors the r.h.s. has to be reconstructed with the help of a projection operator which selects the upper left block in a 2×2 block decomposition. Specifically, if $\mathcal{M}$ is a matrix decomposed in 2×2 blocks as $\mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 \\ \mathcal{M}_3 & \mathcal{M}_4 \end{pmatrix}$, we can define

$$P(\mathcal{M}) \equiv \mathcal{M}_1. \qquad (.17)$$

The saturated M-algebra (.16) can therefore be rewritten as

$$\left\{\tilde{Q}_{\tilde{a}}, \tilde{Q}_{\tilde{b}}\right\} = P\left(\tilde{A}\tilde{\Gamma}_{\tilde{\mu}\tilde{\nu}}\right)_{\tilde{a}\tilde{b}} \tilde{Z}^{[\tilde{\mu}\tilde{\nu}]} + P\left(\tilde{A}\tilde{\Gamma}_{[\tilde{\mu}_1 \ldots \tilde{\mu}_6]}\right)_{\tilde{a}\tilde{b}} \tilde{Z}^{[\tilde{\mu}_1 \ldots \tilde{\mu}_6]}, \qquad (.18)$$

where all tilde's are referred to the corresponding $(10, 2)$ quantities. The matrices in the r.h.s. are symmetric in the exchange $\tilde{a} \leftrightarrow \tilde{b}$. This time the rank-2 and selfdual rank-6 antisymmetric abelian tensorial central charges, $\tilde{Z}^{[\tilde{\mu}\tilde{\nu}]}$ and respectively $\tilde{Z}^{[\tilde{\mu}_1 \cdots \tilde{\mu}_6]}$, appear. Their total number of components is $66 + 462 = 528$, therefore proving the saturation of the r.h.s.. The saturated equation (.18) is named the F-algebra.

Real, complex and quaternionic generalized supersymmetries.

For real n-component spinors Q_a, the most general supersymmetry algebra is represented by

$$\{Q_a, Q_b\} = \mathcal{Z}_{ab}, \qquad (.19)$$

where the matrix $\mathcal{Z}$ appearing in the r.h.s. is the most general $n \times n$ symmetric matrix with total number of $\frac{n(n+1)}{2}$ components. For any given space-time we can easily compute its associated decomposition of $\mathcal{Z}$ in terms of the antisymmetrized products of k-Gamma matrices, namely

$$\mathcal{Z}_{ab} = \sum_k (A\Gamma_{[\mu_1 \ldots \mu_k]})_{ab} Z^{[\mu_1 \ldots \mu_k]}, \qquad (.20)$$

where the values k entering the sum in the r.h.s. are restricted by the symmetry requirement for the $a \leftrightarrow b$ exchange and are specific for the given spacetime. The coefficients $Z^{[\mu_1 \ldots \mu_k]}$ are the rank-k abelian tensorial central charges.

When the fundamental spinors are complex or quaternionic they can be organized in complex (for the **C** and **H** cases) and quaternionic (for the **H** case) multiplets, whose entries are respectively complex numbers or quaternions.

The real generalized supersymmetry algebra (.19) can now be replaced by the most general complex or quaternionic supersymmetry algebras, given by the anticommutators among the fundamental spinors Q_a and their conjugate ${Q^*}_{\dot{a}}$ (where the conjugation refers to the principal conjugation in the given division algebra, see (.14)). We have in this case

$$\{Q_a, Q_b\} = \mathcal{Z}_{ab} \quad , \quad \{{Q^*}_{\dot{a}}, {Q^*}_{\dot{b}}\} = {\mathcal{Z}^*}_{\dot{a}\dot{b}}, \qquad (.21)$$

together with

$$\{Q_a, {Q^*}_{\dot{b}}\} = \mathcal{W}_{a\dot{b}}, \qquad (.22)$$

where the matrix $\mathcal{Z}_{ab}$ (${\mathcal{Z}^*}_{\dot{a}\dot{b}}$ is its conjugate and does not contain new degrees of freedom) is symmetric, while $\mathcal{W}_{a\dot{b}}$ is hermitian.

The maximal number of allowed components in the r.h.s. is given, for complex fundamental spinors with n complex components, by

ia) $n(n+1)$ (real) bosonic components entering the symmetric $n \times n$ complex matrix $\mathcal{Z}_{ab}$ plus

iia) n^2 (real) bosonic components entering the hermitian $n \times n$ complex matrix $\mathcal{W}_{a\dot{b}}$.

Similarly, the maximal number of allowed components in the r.h.s. for quaternionic fundamental spinors with n quaternionic components is given by

ib) $2n(n+1)$ (real) bosonic components entering the symmetric $n \times n$ quaternionic matrix $\mathcal{Z}_{ab}$ plus

iib) $2n^2 - n$ (real) bosonic components entering the hermitian $n \times n$ quaternionic matrix $\mathcal{W}_{a\dot{b}}$.

The previous numbers do not necessarily mean that the corresponding generalized supersymmetry is indeed saturated. This is in particular true in the quaternionic case, see [15].

Any real generalized supersymmetry admitting a complex structure can be re-expressed in a complex formalism with n-component complex spinors and total number of $n(2n+1)$ (real) bosonic components split into $n(n+1)$ components entering the symmetric matrix $\mathcal{Z}$ and n^2 components entering the hermitian matrix $\mathcal{W}$. The situation is different in the quaternionic case. The quaternionic structure requires a restriction on the total number of bosonic generators. n-component quaternionic spinors can be described as $4n$-component real spinors. However, the r.h.s. of a quaternionic (.21) and (.22) superalgebra admits at most $4n^2+n$ bosonic components, instead of $8n^2+2n$ of the most general supersymmetric real algebra. The Lorentz-covariance further restricts the number of bosonic generators in a quaternionic supersymmetry algebra.

We conclude this section mentioning the two big classes of subalgebras, respecting the Lorentz-covariance, that can be obtained from (.21) and (.22) in both the complex and quaternionic cases. They are obtained by setting identically equal to zero either $\mathcal{Z}$ or $\mathcal{W}$, namely

I) $\mathcal{Z}_{ab} \equiv {\mathcal{Z}^*}_{\dot{a}\dot{b}} \equiv 0$, so that the only bosonic degrees of freedom enter the hermitian matrix $\mathcal{W}_{a\dot{b}}$ or, conversely,

II) $\mathcal{W}_{a\dot{b}} \equiv 0$, so that the only bosonic degrees of freedom enter $\mathcal{Z}_{ab}$ and its conjugate matrix ${\mathcal{Z}^*}_{\dot{a}\dot{b}}$.

Accordingly, in the following we will refer to the (complex or quaternionic) generalized supersymmetries satisfying the *I)* constraint as "hermitian"(or "type I") generalized supersymmetries, while the (complex or quaternionic) generalized supersymmetries satisfying the *II)* constraint will be referred to as "holomorphic"(or "type II") generalized supersymmetries.

Generalized supersymmetries and the octonionic M-superalgebra

As already recalled, in the $D = 11$ Minkowskian spacetime, where the M-theory should be found, the spinors are real and have 32 components. Since the most general symmetric 32×32 matrix admits 528 components, one can easily prove that the most general supersymmetry algebra in $D = 11$ can be presented as

$$\{Q_a, Q_b\} = (C\Gamma_\mu)_{ab}P^\mu + (C\Gamma_{[\mu\nu]})_{ab}Z^{[\mu\nu]} + (C\Gamma_{[\mu_1\ldots\mu_5]})_{ab}Z^{[\mu_1\ldots\mu_5]} \quad (.23)$$

(where C is the charge conjugation matrix), while $Z^{[\mu\nu]}$ and $Z^{[\mu_1\ldots\mu_5]}$ are totally antisymmetric tensorial central charges, of rank 2 and 5 respectively, which correspond to extended objects [21, 22], the p-branes. Please notice that the total number of 528 is obtained in the r.h.s as the sum of the three distinct sectors, i.e.

$$528 \;=\; 11 + 66 + 462. \quad (.24)$$

The algebra (.16) is called the M-algebra. It provides the generalization of the ordinary supersymmetry algebra, recovered by setting $Z^{[\mu\nu]} \equiv Z^{[\mu_1\ldots\mu_5]} \equiv 0$.

The octonionic M-superalgebra is introduced by assuming an octonionic structure for the spinors which, in the $D = 11$ Minkowskian spacetime, are octonionic-valued 4-component vectors. The algebra replacing (.16) is given by

$$\{Q_a, Q_b\} = \{{Q^*}_a, {Q^*}_b\} = 0, \qquad \{Q_a, {Q^*}_b\} = Z_{ab}, \quad (.25)$$

where $*$ denotes the principal conjugation in the octonionic division algebra and, as a result, the bosonic abelian algebra on the r.h.s. is constrained to be hermitian

$$Z_{ab} \;=\; {Z_{ba}}^*, \quad (.26)$$

leaving only 52 independent components.

The Z_{ab} matrix can be represented either as the $11 + 41$ bosonic generators entering

$$Z_{ab} = P^\mu(C\Gamma_\mu)_{ab} + Z_{\mathbf{O}}^{\mu\nu}(C\Gamma_{\mu\nu})_{ab}, \quad (.27)$$

or as the 52 bosonic generators entering

$$Z_{ab} = Z_{\mathbf{O}}^{[\mu_1 \ldots \mu_5]} (C\Gamma_{\mu_1 \ldots \mu_5})_{ab} \,. \qquad (.28)$$

Due to the non-associativity of the octonions, unlike the real case, the sectors individuated by (.27) and (.28) are not independent. Furthermore, as we have already seen for $k = 2$, in the antisymmetric products of k octonionic-valued matrices, a certain number of them are redundant (for $k = 2$, due to the G_2 automorphisms, 14 such products have to be erased). In the general case [14] a table can be produced expressing the number of independent components in D odd-dimensional spacetime octonionic realizations of Clifford algebras, by taking into account that out of the D Gamma matrices, 7 of them are octonionic-valued, while the remaining $D - 7$ are purely real. We get the following table, with the columns labeled by k, the number of antisymmetrized Gamma matrices and the rows by D (up to $D = 13$)

$D \setminus k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
7	1	7	7	1	1	7	7	1						
9	1	9	22	22	10	10	22	22	9	1				
11	1	11	41	75	76	52	52	76	75	41	11	1		
13	1	13	64	168	267	279	232	232	279	267	168	64	13	1

(.29)

For what concerns the octonionic equivalence of the different sectors, it can be symbolically expressed, in different odd space-time dimensions, according to the table

$D = 7$	$M0 \equiv M3$
$D = 9$	$M0 + M1 \equiv M4$
$D = 11$	$M1 + M2 \equiv M5$
$D = 13$	$M2 + M3 \equiv M6$
$D = 15$	$M3 + M4 \equiv M0 + M7$

(.30)

In $D = 11$ dimensions the relation between $M1 + M2$ and $M5$ can be made explicit as follows. The 11 vectorial indices μ are split into the 4 real indices, labeled by $a, b, c, \ldots$ and the 7 octonionic indices labeled by $i, j, k, \ldots$. The 52 independent components are recovered from $52 = 4 + 2 \times 7 + 6 + 28$, according to

4	$M1_a$	$M5_{[aijkl]} \equiv M5_a$
7	$M1_i$, $M2_{[ij]} \equiv M2_i$	$M5_{[abcdi]} \equiv M5_i, M5_{[ijklm]} \equiv \widetilde{M}5_i$
6	$M2_{[ab]}$	$M5_{[abijk]} \equiv M5_{[ab]}$
$4 \times 7 = 28$	$M2_{[ai]}$	$M5_{[abcij]} \equiv M5_{[ai]}$

(.31)

The octonionic superconformal M-algebra

The conformal algebra of the octonionic M-theory can be introduced [12] adapting to the eleven dimensions the procedure discussed in [5] for the 10 dimensional case. It requires the identification of the conformal algebra of the octonionic $D = 11$ M-algebra with the generalized Lorentz algebra in the $(11, 2)$-dimensional space-time. In such a space-time the octonionic Clifford's Gamma-matrices are 8-dimensional. The basis of the hermitian generators is given by the 64 antisymmetric two-tensors $C\Gamma_{[\mu_1\mu_2]}\mathcal{Z}^{\mu_1\mu_2}$ and the 168 antisymmetric three tensors $C\Gamma_{[\mu_1\mu_2\mu_3]}\mathcal{Z}^{\mu_1\mu_2\mu_3}$ (or, equivalently, by the 232 antisymmetric six-tensors $C\Gamma_{[\mu_1\ldots\mu_6]}\mathcal{Z}^{\mu_1\cdots\mu_6}$). This is already an indication that the total number of generators in the conformal algebra is 232. We will show that this is the case.

According to [5] the conformal algebra can be introduced as the algebra of transformations leaving invariant the inner product of Dirac's spinors. In $(11, 2)$ this is given by $\psi^\dagger C\eta$, where the matrix C, the analogous of the Γ^0, given by the product of the two space-like Clifford's Gamma matrices, is real-valued and totally antisymmetric. Therefore, the conformal transformations are realized by the octonionic-valued 8-dimensional matrices $\mathcal{M}$ leaving C invariant, i.e. satisfying

$$\mathcal{M}^\dagger C + C\mathcal{M} \;=\; 0. \tag{.32}$$

This allows identifying the (quasi)-group of conformal transformations with the (quasi-)group of symplectic transformations. Indeed, under a simple change of variables, C can be recast in the form

$$\Omega \;=\; \begin{pmatrix} 0 & \mathbf{1}_4 \\ -\mathbf{1}_4 & 0 \end{pmatrix}. \tag{.33}$$

The most general octonionic-valued matrix leaving invariant Ω can be expressed through

$$\mathbf{M} \;=\; \begin{pmatrix} D & B \\ C & -D^\dagger \end{pmatrix}, \tag{.34}$$

where the 4×4 octonionic matrices B, C are hermitian

$$B = B^\dagger, \qquad\qquad C = C^\dagger. \tag{.35}$$

It is easily seen that the total number of independent components in (.34) is precisely 232, as we expected from the previous considerations.

It is worth noticing that the set of matrices $\mathbf{M}$ of (.34) type forms a closed algebraic structure under the usual matrix commutation. Indeed $[\mathbf{M}, \mathbf{M}] \subset \mathbf{M}$ endows the structure of $Sp(8|\mathbf{O})$ to $\mathbf{M}$. For what concerns the supersymmetric extension of the superconformal algebra, we have to accommodate the 64 real components (or 8 octonionic) spinors of $(11, 2)$ into a supermatrix enlarging $Sp(8|\mathbf{O})$. This can be achieved as follows. The two 4-column octonionic spinors α and β can be accommodated into a supermatrix of the form

$$\left(\begin{array}{c|cc} 0 & -\beta^\dagger & \alpha^\dagger \\ \hline \alpha & 0 & 0 \\ \beta & 0 & 0 \end{array}\right). \tag{.36}$$

Under anticommutation, the lower bosonic diagonal block reduces to $Sp(8|\mathbf{O})$. On the other hand, extra seven generators, associated to the 1-dimensional antihermitian matrix A

$$A^\dagger \;=\; -A, \tag{.37}$$

i.e. representing the seven imaginary octonions, are obtained in the upper bosonic diagonal block. Therefore, the generic bosonic element is of the form

$$\left(\begin{array}{c|cc} A & 0 & 0 \\ \hline 0 & D & B \\ 0 & C & -D^{\dagger} \end{array}\right), \qquad (.38)$$

with A, B and C satisfying (.37) and (.35).

The closed superalgebraic structure, with (.36) as generic fermionic element and (.38) as generic bosonic element, will be denoted as $OSp(1,8|\mathbf{O})$. It is the superconformal algebra of the M-theory and admits a total number of 239 bosonic generators.

Conclusions.

We have seen that, contrary to what is commonly believed, an alternative formulation for the M superalgebra and the M superconformal algebra can be consistently introduced in association with the non-associative maximal division algebra of the octonions. It presents peculiar features, like the non-independence of the different octonionic brane sectors, which is a reflection of the higher-rank antisymmetric octonionic tensorial identities discussed in section **5**. The existence of this second variant of the M algebra is puzzling. It could be ultimately related with the arising of exceptional structures (exceptional Lie and Jordan algebras) in the "Theory Of Everything"[19].

Since imaginary octonions admits a geometrical description in terms of the seven sphere S^7, it could be speculated that the higher-dimensional octonionic descriptions, e.g. of the eleven dimensions, corresponds to a particular compactification of the eleven-dimensional M theory down to $AdS_4 \times S^7$. This compactification corresponds to a natural solution for the 11 dimensional supergravity, see [20].

The octonionic superconformal algebra $OSp(1,8|\mathbf{O})$ has been explicitly derived. It corresponds to a supersymmetric extension of a bosonic conformal algebra which is mathematically interesting since it corresponds to a closed algebraic structure which goes beyond the standard notion of conformal algebra of a given Jordan algebra, see [12].

Besides this aspect, the notion of hermitian (complex and quaternionic) and holomorphic (complex and quaternionic) supersymmetries, as consistently division-algebra constrained generalized supersymmetries, has been presented.

Physical implications of these mathematical structures are quite obvious. The classification of generalized supersymmetries allow to understand the web of interrelated dualities of different classes of theories which can be either analitically continued (let's say, to the Euclidean) or recovered through dimensional reduction.

As an example, we can cite that the analytic continuation of the M algebra was proven in [23] to correspond to an eleven-dimensional complex holomorphic supersymmetry. It was further shown in [15] that the same algebra also admits a 12-dimensional Euclidean presentation in terms of Weyl-projected spinors. These two examples of Euclidean supersymmetries can find application in the functional integral formulation of higher-dimensional supersymmetric models.

There is an interesting class of models which nicely fits in the framework here described and is currently under intense investigation. It is the class of superparticle models, introduced at first in [24] and later studied in [25], whose bosonic coordinates correspond to tensorial central charges. It was shown in [26] that a 4-dimensional theory of this kind leads to a tower of massless higher spin states, concretely implementing a

Fronsdal's proposal [27] of introducing bosonic tensorial coordinates to describe massless higher spin theories (admitting helicity states greater than two). This is an active area of investigation, the main motivation beingthe investigation the tensionless limit of superstring theory, corresponding to a tower of higher helicity massless particles (see e.g. [28]).

In a somehow "orthogonal"direction, a class of theories which can be investigated in the present framework is the class of supersymmetric extensions of Chern-Simon supergravities in higher dimensions, requiring as a basic ingredient a Lie superalgebra admitting a Casimir of appropriate order, see e.g. [29].

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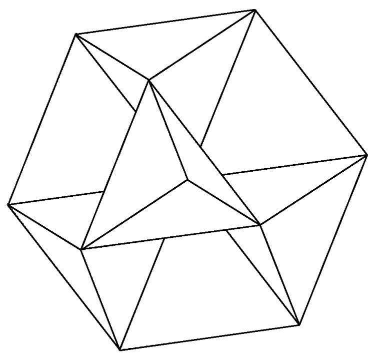

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