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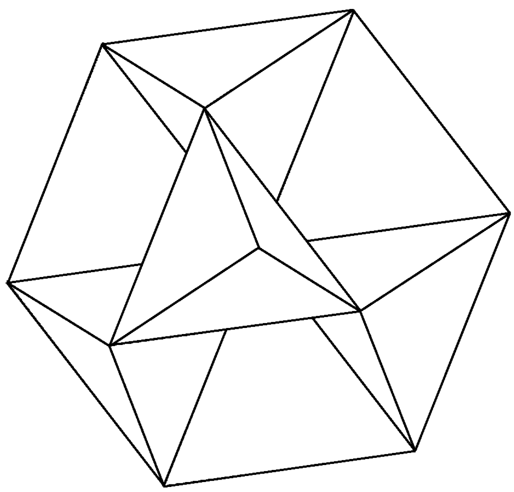

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From Editorial Board

NUMBER, GEOMETRY AND NATURE

Number is one of the most fundamental concepts not only in mathematics, but in general natural science as well. It may be primary even in comparison with such global categories as time, space, substance, matter, and field. That is why editing the first issue of the journal "Hypercomlex numbers in geometry and physics" the editorial board sincerely hopes that articles not only on numbers in general, but primarily the works that reveal their organic connection with the real world will find here their true scope.

The concept of number in its most general meaning unifies not only common numbers that all of us know from school, but also such objects as the quaternion, the octave, the matrices, etc. Without denying the importance of numbers of all types, let us well emphasize the class chain that has the following shape: natural → integer → rational → real → complex. At the same time our aim is to found the possibility of extending the given above classification to numbers of high dimensionality, including those that obey commutative-associative multiplication.

At first sight this plan seems to be absolutely unproductive, for in algebra there exists the Frobenius theorem that claims that multy-component numbers, as being structures subjected to arithmetic properties, end with the complex numbers. At the same time a special stress is laid on the fact that in the according algebras there are no the so-called divisors of zero. Of course, if we take into consideration the real and complex numbers, treating them as the standard, the zero divisor seems to be redundant. Nevertheless, from the point of view of physics and the pseudo-Euclidean geometry closely connected thereto, the zero divisor is one of the most natural objects, for the world lines of the light rays are related to it. The fact that the pseudo-Euclidean planes may be juxtaposed with the algebra of the commutative associative double numbers which have the zero divisors, may serve as the best proof of it. Habitual claims, that the double numbers are too primitive and cannot act as a real competitor to the complex, do not seem to be well-founded, as it would mean in terms of geometry that the Euclidean spaces are more important than the pseudo-Euclidean spaces. Long ago geometricians came to an agreement that both types of space have right to exist; therefore, that is why it is impossible to divide the double numbers as well as the complex numbers proper into the valuable ones and not rather. In our opinion the next conclusion is obvious: in the classification of the value number structures the double numbers should be placed close to the complex ones. If we do treat the double numbers as the fundamentals, then we will not have any argument to keep on ignoring the zero divisors, which means that it is quite possible to create number systems of a larger number of dimensions, and this does not contradict the Frobenius theorem.

The complex quaternions (they are also called biquaternions) are a nice example of such structures. Various interesting works published in the first issue of this journal are devoted to the exploration of the associative complex numbers and not to the ones that are commutative by multiplication. The hope of a success of this trend is based on the fact that the Poincare group, that plays an important role in modern physics, is a subgroup of the full group of continuous symmetries of the eight-dimensional real space of the biquaternions. On the other hand if we accept the fact that the divisor is independent we can build hypercomplex systems, that have commutative-associative multiplication, what has its additional advantages. It is suggested that we should pick them out in a new group of Poly-Numbers to emphasize the special status of such structures.

Lately much attention has not been paid to the exploration of Poly-Numbers, for their structure was commonly considered to be trivial. In a way this is true, but if we put in the first place not algebra but geometry then the multitude increases significantly. It is explained by the fact that spaces (that can be related to Poly-Numbers) as a rule are Finslerian spaces, where some non-linear reflections stand out from linear transformations.

No matter what will be the result of the generalization of the idea of the number, the existence of the Finslerian geometries is the reality, which means that we can explore physics in other or alternative ways. Why not try to change the geometrical basis of physics, and hope that the very geometric basis would be closer to non-quadratic structures, instead of searching hypercomplex structures corresponding to the classical Minkowskian space or to its quadratic modifications. Expecting rather a beautiful and effective confirmation of a close connection between mathematics and physics, we can assume a supposition, that the new geometry must be connected with the most common number structures as the basis of our a little bit risky plan. Here should emerge the Poly-Numbers that on one hand, as it is mentioned above, are quite trivial, but on the other hand are the elements of rather substantial geometries. Even if our expectation will not be fulfilled with Poly Numbers, there is still a vast number of alternatives, and taking into consideration the fundamental nature of the posed problem it is difficult to foresee which of the ways will turn out to be more productive.

GENERALIZATION OF SCALAR PRODUCT AXIOMS

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The concept of scalar product is vital in studying basic properties of either Euclidean or pseudo-Euclidean spaces. A generalizing of a special sub-class of Finslerian spaces, that we will call the polylinear, is presented in the work. The idea of scalar polyproduct and of related fundamental metric polyform has been introduced axiomatically. The definition of different metric parameters such as the vector length and the angle between vectors are founded on the idea. The concept of orthogonallity is also generalized. Some peculiarities of the geometry of the four-dimensional linear Finslerian space related to the algebra of commutative-associative hypercomplex numbers, that are called Quadranumerical, are proved in the concrete polyform.

1. The scalar product of the Euclidean spaces

For the last two thousand years that have past since the appearance of the famous "Beginnings" mathematics have tried a number of methods of describing the Euclidean spaces. The axiom systems by Euclid and Gilbert are the best well-known ones. But taking into consideration the modern attitude, the system of axioms that uses the ideas of the real number, the linear space, and the scalar product [1] is considered to be the most convenient. At the same time a few know that the latter case owes its appearance in geometry to a discovery of the non-commutative algebra of four-component hyper-complex numbers discovered in 1843 by William Hamilton, he called it the algebra of quaternions [2]. The discovery was preceded by several years of attempts to find three-component numbers, the triplets, that could be confronted to the vectors of the common space the same way as the complex numbers are confronted to the vectors of the Euclidean Plane. The solution was found when Hamilton rejected the commutative multiplication and in place of the triplets limited himself to the four-component numbers.

By definition a quaternion is a hyper-complex number, that can be presented as a linear combination:

$$X = x_0 + i \cdot x_1 + j \cdot x_2 + k \cdot x_3,$$

where x_i are real numbers, and i, j, k are pair-wisely different imaginary units, so that $i^2 = j^2 = k^2 = -1$ and $ij + ji = jk + kj = ki + ik = 0$. These rules including the rule of multiplication on the common real unit, sometimes are set into the so called table of multiplication of hypercomplex numbers, that in the case of quaternions looks the following way:

$$\begin{array}{c|cccc} & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & -1 & k & -j \\ j & j & -k & -1 & i \\ k & k & j & -i & -1 \end{array} .$$

Hamilton suggested that in the quaternion we should distinguish the scalar part x_0 from the vector part $\mathbf{V}_x = \mathbf{i} \cdot x_1 + \mathbf{j} \cdot x_2 + \mathbf{k} \cdot x_3$. In this case, as it is easy to check, the product of 2 vector quaternions is a common quaternion:

$$\mathbf{V}_x \mathbf{V}_y = (-x_1 y_1 - x_2 y_2 - x_3 y_3) + [\mathbf{i}(x_2 y_3 - x_3 y_2) + \mathbf{j}(x_3 y_1 - x_1 y_3) + \mathbf{k}(x_1 y_2 - x_2 y_1)],$$

whose scalar part has a symmetric bilinear form, and the vector part looks like a conventional vector multiplication. As a matter of fact, the term of scalar and vector product appeared right from here, and for the first time were introduced by Hamilton.

The first explorers of the quaternions were looking at them mainly as at an opportunity of using algebraic methods while operating with points and vectors of common space, though it is more natural to correspond these hyper-complex numbers with the four-dimensional space. Hamilton himself knew about this, he thought that this circumstance once would be used to describe the time. In this case quaternions would become a natural instrument not only in geometry, but also in physics.

Unfortunately, nowadays only some specialists know quaternions. It is explained by the fact that the idea of scalar product that originates from the quaternion algebra was very convenient and soon became an independent geometrical category, and practically stamped the hyper-complex numbers that had given birth to it. There began a debate among physics and mathematicians between the adherents of the quaternion algebra and of the arising vector calculus. As is well-known, the vector approach won, this fact to a certain extent owes to objective difficulties of quaternion diffusion into algebra and the function of the complex variable, that is conditioned to the peculiarities of non-commutative multiplication.

The scalar product that is connected with the quaternion can be applied only to the three-dimensional vectors. But if we separate the idea of scalar product from concrete numbers and generalize it to the field of arbitrary dimensionality, the advantages of the concept (the opportunity to define the length of vectors and angles between them mathematically) will still be preserved. For this we should postulate a symmetrical bilinear form of two vectors $(\mathbf{A}, \mathbf{B}) = \alpha_{ij} a_i b_j$ in the affine m-dimensional space. Reciprocally corresponding quadratic form $(\mathbf{A}, \mathbf{A})$ must be not negative. Then by definition we accept that the affine map that maps the vector $\mathbf{A}$ onto $\mathbf{A}'$ is congruent if it leaves the form invariant:

$$(\mathbf{A}, \mathbf{A}) = (\mathbf{A}', \mathbf{A}').$$

Two figures that can be mapped one onto another by a congruent reflection are congruent. By this fact the idea of congruence is defined in the axiomatic construction of the Euclidean geometry. For a congruent map takes place not only invariance of the quadratic form but also the invariance o the bilinear form:

$$(\mathbf{A}, \mathbf{B}) = (\mathbf{A}', \mathbf{B}').$$

For the vectors $\mathbf{A}$ and $\mathbf{A}'$ are congruent if and only if:

$$(\mathbf{A}, \mathbf{A}) = (\mathbf{A}', \mathbf{A}'),$$

it is possible to introduce the $(\mathbf{A}, \mathbf{A})$ as a numerical characteristic of the vector $\mathbf{A}$. But still it is more traditional to use the value of the positive square root of $(\mathbf{A}, \mathbf{A})$, that by definition is called the length of the vector $\mathbf{A}$ and usually is defined as

$$|\mathbf{A}| = (\mathbf{A}, \mathbf{A})^{1/2}.$$

Such definition lets us introduce the definition of the unit vector. Its relationship with common vectors is revealed in the following relation:

$$\mathbf{a} = \mathbf{A}/|\mathbf{A}|.$$

If $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{a}'$ and $\mathbf{b}'$, are two pairs of unit length vectors, then the figure, built by the first two vectors, is congruent to the figure, constructed by the two latter ones, only when the equality

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}')$$

is held true. The angle is considered to be the representative of congruency in the Euclidean spaces. But the mere numerical characteristic is related not to bilinear form of unit vectors, but to transcendental function of its inverse cosine

$$\phi = \arccos(\mathbf{a}, \mathbf{b}).$$

This definition of the angle is equivalent to the statement that the length of the arc on the unit sphere between the ends of the vectors $\mathbf{a}$ and $\mathbf{b}$ is the angle. Such complication of the numerical angle measure is compensated by the obtained property of additivity. When composing two angles laying on the same plane their value is summed up.

The property of perpendicularity of directions is a particular consequence of the idea of the angle. The perpendicular condition of two vectors consists in equality to 0 of the value of their bilinear form. The particular status of the perpendicular directions is accounted for many reasons, for example, for example by the simplification of the form of the quadratic metric function, presented in the basis all vectors of which are reciprocally perpendicular.

Two-dimensional case stands out among all the Euclidean spaces with quadratic metric function. This peculiarity is reflected in the Liouville theorem, that proves that in the three- or more-dimensional Euclidean (or pseudo-Euclidean) spaces the conformal transformations are limited to inversions, dilations, translations and rotations [3]. In other words, there are essentially more transformations that are related to conformal in the two-dimensional case. Mathematically this fact is reflected in the vast majority of analytical functions of the complex variable. To each of them a certain conformal reflection of the Euclidean plane is related.

2. The scalar product of the pseudo-Euclidean spaces

It is well-known that if a symmetrical bilinear form postulated over the affine space creates an alternating-sign quadratic form, then the geometry assigned by it becomes being of not Euclidean but Pseudo-Euclidean type [4]. We can unify both types of geometries by surrendering the claim about the positivity of the quadratic form. This unified system, in particular, can be presented with the following set:

(a): every 2 vectors $\mathbf{A}$ and $\mathbf{B}$ of the linear space are associated with a certain real number labeled by

$$k = (\mathbf{A}, \mathbf{B})$$

and called (as well as in the Euclidean case) the scalar product of these vectors;

(b) the scalar product is commutative regarding the permutation of vectors

$$(\mathbf{A}, \mathbf{B}) = (\mathbf{B}, \mathbf{A});$$

(c) the scalar product is distributive regarding the composition of vectors

$$(\mathbf{A}+\mathbf{C},\mathbf{B}) = (\mathbf{A},\mathbf{B}) + (\mathbf{C},\mathbf{B});$$

(d) the real multiplier can be isolated from the scalar product

$$(k\mathbf{A},\mathbf{B}) = k(\mathbf{A},\mathbf{B}).$$

The methods of defining the metric characteristics of pseudo-Euclidean spaces, which are the generalizing of the corresponding Euclidean parameters, do not change considerably, that enables us to save their names. So, transformations that leave the quadratic form moduli of all the vectors invariant are of congruent nature:

$$|(\mathbf{A},\mathbf{A})| = |(\mathbf{A}',\mathbf{A})'|.$$

The vector length is defined as a positive value of the square root of the moduli of the quadratic form:

$$|\mathbf{A}| = |(\mathbf{A},\mathbf{A})|^{1/2}.$$

But in this case there appear the so called isotropic and imaginary vectors. In the first case the length equals 0 even at non-zero components, and in the second case the quadratic form is negative. The angle between the two directions, as well as in the Euclidean case, is defined by congruence of the figure formed by two unit vectors, and by definition is treated as equal to the special function of their bilinear form:

$$\phi = \operatorname{arcch}(\mathbf{a},\mathbf{b}),$$

which ensures the additivity of the parameter under plane rotations. So, the angle equals the arc length between a pair of points on the unit sphere. But now, when calculating the angle, it is important to take into consideration the area in which the driving vector that is relative to the isotropic cone is lying, as the indicatrix stops being simply connected.

Also the perpendicular property of vectors is generalized in the pseudo-Euclidean spaces. In this case their scalar product must equal 0. It is customary to call such vectors orthogonal.

The pseudo-Euclidean spaces also admit the generalizing of the idea of a congruent reflection, which is defined as a transformation that saves the similarity of infinitesimal forms. Let us note that, as well as in the Euclidean case, the two-dimensional case, where conformal maps are wider than in higher dimensions, is distinguished in the pseudo-Euclidean space. Let us note another coincidence: The pseudo-Euclidean plane, as well as the Euclidean one, has an algebraic analogue called *double numbers* which differ from the complex by the fact that their square equals not -1, but +1. Such numbers along with the complex ones admit the idea of analytical functions where a correspondence of a conformal reflection of the pseudo-Euclidean plane [5] to each of them can be established. These peculiarities of two-dimensional spaces demonstrate the relationship between the geometries and commutative-associative algebras, for example, the algebras of complex and double numbers.

Apart from the pseudo-Euclidean case other approaches towards generalizing of the conception of the scalar product are known in geometry. The system of axioms for the so called unitary, where the metric function is set in the field of complex and not real numbers, and symplectic spaces where antisymmetric bilinear form [4, 6] is postulated in place of the symmetric, — are sequent to the scalar product.

Analyzing above examined examples of the usage of the concept of scalar product and its generalizing we can note that they are unified by connection with one or another bilinear form. But such form is just a special case of the polylinear form. Then there emerges a question whether it is possible to obtain a substantial geometry if we postulate the three-, four-, and so on up to polylinear symmetric form in place of the bilinear one?

3. The scalar polyproduct

Let us try to preserve all the axioms of the real number and m-dimensional affine spaces as the basis and add the following:

(a): to every of n vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots, \mathbf{Z}$ we will associate a real number denoted by

$$k = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots, \mathbf{Z}),$$

which we will call the scalar polyproduct;

(b): let us try to make it the way that the scalar polyproduct would be commutative with respect to permutation of any including vectors

$$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots, \mathbf{Z}) = (\mathbf{B}, \mathbf{A}, \mathbf{C}, \dots, \mathbf{Z}) = (\mathbf{C}, \mathbf{B}, \mathbf{A}, \dots, \mathbf{Z}) = \cdots = (\mathbf{Z}, \mathbf{C}, \mathbf{B}, \dots, \mathbf{A});$$

(c): distributive to their composing

$$(\mathbf{A}, \mathbf{B}, \mathbf{C} + \mathbf{E}, \dots, \mathbf{Z}) = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots, \mathbf{Z}) + (\mathbf{A}, \mathbf{B}, \mathbf{E}, \dots, \mathbf{Z});$$

(d): a real multiplier at any vector could be taken outside the scalar polyproduct:

$$(k\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots, \mathbf{Z}) = k(\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots, \mathbf{Z}).$$

These axioms just in a way differ from the corresponding axioms of the scalar product. Besides they can be unified into a concept of the symmetric polylinear form, and that is why we will call the space, endowed with one of the forms, *polylinear*. The above examined Euclidean and pseudo-Euclidean spaces, according to their primary definitions, are special cases of the polylinear spaces, in other words they comply to the above given axiom system when $n = 2$, that enables us to call them *bilinear*.

We will call the scalar polyproduct of the same vector, $\mathbf{A}, \mathbf{A}, \dots, \mathbf{A}$, by analogy with the quadratic form of the bilinear spaces, the *fundamental metric form* of the polylinear space, or simply *n-polyform* of the vector $\mathbf{A}$.

We will call the affine reflections of the polylinear space, that shift the vectors $\mathbf{A}$ into $\mathbf{A}'$, the *congruent* if they leave he moduli of the fundamental metric form invariant:

$$|(\mathbf{A}, \mathbf{A}, \mathbf{A}, \dots, \mathbf{A})| = |(\mathbf{A}', \mathbf{A}', \mathbf{A}', \dots, \mathbf{A}')|. \tag{1}$$

It is in our axiomatic construction of the polylinear space where the idea of congruence, and then of other metric notions, will be defined.

If there is a set of objects over which the axioms of the affine space are held true, we can choose any symmetric polylinear form in it and, therefore, the unambiguously connected n-polyform, and "assign" make the latter to be the fundamental metric form and on its basis define the conception of congruence as it has been done above. Then we a metrics gets introduced into the affine space with the help of the form, and it becomes a correct metric geometry. Such construction is not related neither to number of dimensions in the space nor to the specific number of dimensions in the fundamental form, nor with the type of the latter case.

It follows from the properties of the symmetry and from the linearity of the form $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z})$ where correlations, that are more general than (1), are held true for the congruent reflection of the polylinear space:

$$\begin{gathered}(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}, \mathbf{B}) = (\mathbf{A}', \mathbf{A}', \ldots, \mathbf{A}', \mathbf{B}'),\\ (\mathbf{A}, \mathbf{A}, \ldots, \mathbf{B}, \mathbf{B}) = (\mathbf{A}', \mathbf{A}', \ldots, \mathbf{B}', \mathbf{B}'),\\ \ldots\ldots\ldots\\ (\mathbf{A}, \mathbf{B}, \ldots, \mathbf{C}, \mathbf{Z}) = (\mathbf{A}', \mathbf{B}', \ldots, \mathbf{C}', \mathbf{Z}').\end{gathered}$$

In other words the congruent reflections of the polylinear spaces leave the polyforms invariant where the vectors are present in different combinations.

We will say that the two vectors of the polylinear space $\mathbf{A}$ and $\mathbf{A}'$ are congruent if the moduli of the corresponding n-polyforms are equal and are nonzero:

$$|(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}, \mathbf{A})| = |(\mathbf{A}', \mathbf{A}', \ldots, \mathbf{A}', \mathbf{A}')| \neq 0.$$

By definition it is possible to regard a n-polyform as a numerical parameter of the vector $\mathbf{A}$. But in place of this, as well as in the bilinear spaces, striving for additivity and unambiguity of the properties, we will use the positive root of the n-degree of the absolute value $(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}, \mathbf{A})$, calling it the vector length $\mathbf{A}$:

$$|\mathbf{A}| = |(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}, \mathbf{A})|^{1/n}.$$

Then the length of the sum of two codirected vectors equals the sum of their length. It is worth noting that this is not the only way of introducing the idea of length with additive properties, but in this approach the length is defined for the maximum number of directions coming from the affine space.

Now it becomes clear to which type of space we should relate the ones we try to construct with the help of the given above axioms or the scalar polyproduct. Firstly, these spaces are *Finslerian* [7,8] as their metric function is not limited by quadratic forms. Secondly they belong to the class known in the Finslerian geometry under the name of Minkowskian space [9], with which it is customary to associate the manifold where the indicatrices do not depend on the point. [The space of the Special theory of Relativity is a specific case of such spaces.] But the examined class of spaces is even smaller, as it is related to a strict idea of polylinear symmetric form. The latter case has a great significance as it becomes possible to introduce characteristics, that generalize such fundamental categories of geometry as the length, the angle, the orthogonality, the conformal reflection, etc. Let us conventionally call such spaces the *polylinear Finslerian spaces* (till the appearance of a more specific name let).

If $\mathbf{a}$ and $\mathbf{b}$, and also $\mathbf{a}'$ and $\mathbf{b}'$, are two pairs of unit vectors, then the figure, constructed with the first two vectors, will be congruent to the figure, constructed with the latter two, if a transformation mapping one figure onto the other there will be found. From the above examined properties of the polylinear forms it follows that such transformation can be found only if

$$\begin{gathered}(\mathbf{a}, \mathbf{a}, \ldots, \mathbf{b}) = (\mathbf{a}', \mathbf{a}', \ldots, \mathbf{b}'),\\ (\mathbf{a}, \mathbf{a}, \ldots, \mathbf{b}, \mathbf{b}) = (\mathbf{a}', \mathbf{a}', \ldots, \mathbf{b}', \mathbf{b}'),\\ \ldots\ldots\ldots\\ (\mathbf{a}, \mathbf{b}, \ldots, \mathbf{b}) = (\mathbf{a}', \mathbf{b}', \ldots, \mathbf{b}'). \qquad (2)\end{gathered}$$

This, in particular, entails that in the bilinear spaces the congruence of the pair of two unit vectors is related to the equality of only one form:

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}'), \tag{3}$$

which sets the idea of the angle as the parameter that characterizes the difference between two directions. The equality (3) along with the definition of the unit vector are tantamount to the axiom of the triangle congruence from the Hilbert system of axioms of the Euclidean space. Two triangles are congruent in the Euclidean space if the lengths of corresponding sides and angles between them are equal. One may can formulate analogous axioms also for the pseudo-Euclidean spaces. But it follows from the definition (2) that in the polylinear space with the dimension of the form of more than two the congruence of figures constructed of two unit vectors is defined by more than one circumstance. In the spaces with the three-linear form $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, the two forms must be equal to ensure that the figures would be congruent:

$$(\mathbf{a}, \mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{a}', \mathbf{b}'), \qquad (\mathbf{a}, \mathbf{b}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}', \mathbf{b}').$$

This seeming paradox has a very simple explanation. Usually speaking about a spatial figure, constructed on two vectors, it is thought as of a plain element held among sides, which are the driving vectors. But this is justified only in spaces with the bilinear form. In the spaces with the arbitrary polylinear form, the two vectors are now connected not with a plane but with a special cone-shaped surface, which configuration depends on the metric properties of the surrounding space. There can be more than one parameter, that defines the congruence of such fan-shaped figures, limited in the edges by unit vectors, that in particular is observed in spaces with three-linear symmetric form with two corresponding values.

On the basis of the above given brief analysis it becomes clear that polylinear spaces admit an introduction of analogous of the idea of the angle attributed to bilinear spaces. But we should take into account that the angle as the parameter in the bilinear spaces unifies simultaneously two properties: on the one hand, it serves as a characteristic of the difference between two directions, and on the other hand, is the parameter of one of types of congruent transformations called rotations. In the general case of the polylinear space each of the properties should be characterized by a proper value. It is meaningful to use the negative value of the n- polyform of the difference as the basis to getting the numerical parameter that would characterize the difference of directions of unit vectors: $\mathbf{a}$ and $\mathbf{b}$, to be more specific:

$$\begin{aligned}&(\mathbf{a}-\mathbf{b}, \mathbf{a}-\mathbf{b}, \ldots, \mathbf{a}-\mathbf{b}) = \\ &= (\mathbf{a}, \mathbf{a}, \ldots, \mathbf{a}) - C_n^1(\mathbf{a}, \mathbf{a}, \ldots, \mathbf{a}, \mathbf{b}) \pm \ldots (-1)^{n-1} C_n^{n-1}(\mathbf{a}, \mathbf{b}, \ldots, \mathbf{b}) + (-1)^n(\mathbf{b}, \mathbf{b}, \ldots, \mathbf{b}),\end{aligned}$$

where C_i^j are binomial coefficients. Consequently the scalar form of two unit vectors $\mathbf{a}$ and $\mathbf{b}$ reads

$$S(\mathbf{a}, \mathbf{b}) = -C_n^1(\mathbf{a}, \mathbf{a}, \ldots, \mathbf{a}, \mathbf{b}) \pm \ldots (-1)^{n-1} C_n^{n-1}(\mathbf{a}, \mathbf{b}, \ldots, \mathbf{b}) \tag{4}$$

or its function can play the role of a numerical parameter that defines the required property. Let us note that if the polylinear space is a two-bilinear one the expression (4) to the constant factor coincides with the definition of the common scalar product of two unit vectors. The value (4) can be called the *scalar product of two vectors* of the polylinear space. But may be it is even justified to divide the scalar product into items symmetrized in pairs:

$$S(\mathbf{a},\mathbf{b}) = C_n^1(-(\mathbf{a},\mathbf{a},\dots,\mathbf{a},\mathbf{b}) + (-1)^{n-1}(\mathbf{a},\mathbf{b},\mathbf{b}\dots,\mathbf{b}))$$
$$+ C_n^2((\mathbf{a},\mathbf{a},\dots,\mathbf{a},\mathbf{b},\mathbf{b}) + (-1)^{n-2}(\mathbf{a},\mathbf{a},\mathbf{b}\dots,\mathbf{b})) \pm \dots = S_1(\mathbf{a},\mathbf{b}) + S_2(\mathbf{a},\mathbf{b}) + \dots, \quad (5)$$

where every term $S_i(\mathbf{a},\mathbf{b})$ receives its proper value.

In the polylinear spaces there are pairs of vectors with definite ability of positional relationship similar to orthogonal vectors in the bilinear spaces. In the Finslerian space theory the corresponding idea is called the transversality. Let us call the vector $\mathbf{A}$ *transversal* to the vector $\mathbf{B}$, if $(\mathbf{A},\mathbf{A},\dots,\mathbf{A},\mathbf{B}) = 0$. It is seen here that the transversality is not commutative, that is, the vanishing $(\mathbf{A},\mathbf{A},\dots,\mathbf{A},\mathbf{B}) = 0$ does not entail $(\mathbf{B},\mathbf{B},\dots,\mathbf{B},\mathbf{A}) = 0$. But if we use the symmetrized forms (5), then the transversality, assigned by them, will have commutative properties. By definition, we will consider $\mathbf{A}$ and $\mathbf{B}$ *mutually transversal of the first degree*, when $S_1(\mathbf{A},\mathbf{B}) = \mathbf{0}$; and of the *second degree*, if $S_2(\mathbf{A},\mathbf{B}) = \mathbf{0}$, and so on up to $n/2$ or $(n-1)/2$ degree. Such differentiation of transversality demonstrates the ability of vectors of the linear Finslerian spaces to form pairs with a multitude of characteristic connection with the direction, – that generalizes the conception of orthogonality.

Apart from the quantities defined by the forms (4) it is meaningful to introduce one more "angle-like" characteristic in some polylinear spaces that have continuous congruent transformations like rotations. We will relate its value with the arc length in the unit sphere outlined by a ray simultaneously with a continuous one-parameter rotation. So generalized conception includes the property of the common angle - to be the additive measure that follows from the additivity of the length.

Not only pairs can be included into polyforms, but also three-, four-, etc., up to n different vectors. It is difficult to say to which quality consequences must lead this circumstance in the area of simple figures. Only one thing is clear: this property of polylinear spaces exists objectively that means that it should be as well taken into account.

There are such spaces among the polylinear ones where in one of the bases all the forms are nullified but for the ones that include only different vectors. For such spaces the fundamental metric forms take the following structure in the special basis:

$$(\mathbf{A},\mathbf{A},\dots,\mathbf{A}) = \pm a_1a_2\dots a_m \pm a_1a_2\dots a_{m-1}a_{m+1}$$
$$\pm\dots\pm a_2a_3\dots a_ma_{m+1}\pm\dots\pm a_{n-m}a_{n-m+1}\dots a_n. \quad (6)$$

Among these emerge the pseudo-Euclidean spaces labeled $(1, m-1)$, which play an important role in the modern theoretical physics. Though the classical quadratic form seems to be more convenient for the spaces, the second degree of the intervals in some of the isotropic bases looks like:

$$|\mathbf{A}|^2 = (\mathbf{A},\mathbf{A}) = a_1a_2 + a_1a_3 + a_1a_4 + \dots + a_{m-1}a_m = \sum_{k\neq l} a_ka_l.$$

For example, the square of interval of Minkowskian space $S^2 = (ct)^2 - x^2 - y^2 - z^2$ after the substitution

$$ct = \sqrt{3/8}(u+v+w+z), \quad x = \sqrt{1/8}(u-v+w-z),$$
$$y = \sqrt{1/8}(u+v-w-z), \quad z = \sqrt{1/8}(u-v-w+z)$$

(similar to (16)) gets an attractive symmetric form:

$$S^2 = uv + uw + uz + vw + vz + wz.$$

The expression (6) looks more concise in the cases with $n = m$, that is, when the dimension of the fundamental form coincides with the dimension of the space. In this case the nth degree of a vectors with respect to the corresponding basis takes on the form

$$|\mathbf{A}|^n = (\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}) = \pm a_1 a_2 \ldots a_n.$$

In these circumstances the specific role of the pseudo-Euclidean plane, where such correlations are held, is defined. It seems probable that there must exist a connection with associative-commutative algebras, that involves the appearance in the space of a large group of conformal reflections, only in spaces with $n = m$. At the same time the conformal reflections can be seen in a number of cases which follow from the works [10, 11] where the eight-dimensional biquaternions are examined, that, according to the above given axiom, have metric forms of the fourth degree which come outside the Liouville theorem. We can only hope that the property of some polylinear spaces has a vast group of conformal reflections which appears to be perspective in geometry as well as in physics.

On the other hand even superficial study of the properties of the polylinear spaces let us state that in some of them there are not only conformal, but also non-linear transformations that do not have analogies within common bilinear spaces. The presence of such transformations ensues merely from that the studied spaces require extension of the notion of orthogonality up to several respective members. As is well known, the nonlinear transformations that leave invariant ordinary orthogonality relates to conformal. In this connection it is natural to expect that the transformations retaining the transversality would occur preferable, too. This makes the existent polylinear spaces even more interesting.

4. Examples of polylinear spaces

There is a great number of polylinear spaces. The task to classify such spaces seems to be difficult even if we work with three-linear forms, not to mention the forms with a larger number of dimensions. But if we limit ourselves to the three-dimensional case, and if among symmetric three-dimensional spaces we examine those whose metric forms do not depend on permutation of vector components (it is suggested in the work [12], that examines a similar classification, to call them the *high-symmetric*) than we can single out 8 independent classes, where a fundamental canonical polyform can be related to each of them. The simplest look among all the forms has the following:

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1^3 + a_2^3 + a_3^3 = F_1;$$

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1^2 a_2 + a_1^2 a_3 + a_2^2 a_1 + a_2^2 a_3 + a_3^2 a_1 + a_3^2 a_2 = F_2;$$

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1 a_2 a_3 = F_3.$$

In the work [12] they are called *basic*. Any of the eight non-isomorphic high-symmetric tree-linear polyforms can be presented as a linear combination of the bases:

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}) = \omega_1 F_1 + \omega_2 F_2 + \omega_3 F_3.$$

But no matter how great the variety of spaces with three-linear symmetric form is, the space with the following form stands out with its concise symmetry:

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1 a_2 a_3.$$

As the result of its high involved symmetry we can confront the corresponding space with the algebra of commutative-associative numbers that is the sum of three real algebras. Let us call such hyper-complex system the *triple numbers* and label it as H_3. Mathematical, geometrical and may be physical structures related to the triple numbers are not trivial at all, that is proved in the works [13, 14] published in this issue. It will be noted that most three-linear polyforms cannot be juxtaposed by algebras in general [12].

In the four-dimensional polylinear spaces with $n = m$ the basic forms have the following shape:

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1^4 + a_2^4 + a_3^4 + a_4^4; \tag{7}$$

$$\begin{aligned}(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1^3(a_2 + a_3 + a_4) + a_2^3(a_1 + a_3 + a_4) \\ +a_3^3(a_1 + a_2 + a_4) + a_4^3(a_1 + a_2 + a_3);\end{aligned} \tag{8}$$

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1^2a_2^2 + a_1^2a_3^2 + a_1^2a_4^2 + a_2^2a_3^2 + a_2^2a_4^2 + a_3^2a_4^2; \tag{9}$$

$$\begin{aligned}(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1^2(a_2a_3 + a_2a_4 + a_3a_4) + a_2^2(a_1a_3 + a_1a_4 + a_3a_4)+ \\ a_3^2(a_1a_2 + a_1a_4 + a_2a_4) + a_4^2(a_1a_2 + a_1a_3 + a_2a_3);\end{aligned} \tag{10}$$

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A}) = a_1a_2a_3a_4, \tag{11}$$

and to each of them their particular, not isomorphic to others, geometries of the polylinear space.

As well as in the three-dimensional case the variety of four-dimensional polylinear spaces is not limited to these examples. It seems to be a very difficult task to present the full classification of corresponding geometries. Let us study at least one case before setting about its realization. For example, the geometry related to the most symmetric among the basic polyforms (7) – (11), and to be more specific (11). Its high symmetry again gives us an opportunity to confront the space defined by it to the algebra of commutative-associative hyper-complex numbers, that in order to be brief we will call the *Quadra numbers* labeled as H_4. Some of the properties of the space, related to the Quadra numbers are given in [15]. We can get the Quadra number algebra by adding the axiom of real numbers to the axiom of composing and multiplication of the following objects: $A = a_1 \cdot 1 + a_2 \cdot I + a_3 \cdot J + a_4 \cdot K$ and $B = b_1 \cdot 1 + b_2 \cdot I + b_3 \cdot J + b_4 \cdot K$, where a_i and b_i – real numbers called the components, and $1, I, J, K$ the basic units. We accepting by definition that the sum of the numbers A and B is called the number

$$C = (a_1 + b_1) \cdot 1 + (a_2 + b_2) \cdot I + (a_3 + b_3) \cdot J + (a_4 + b_4) \cdot K,$$

and their product – another number of the same class:

$$\begin{aligned}D = (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4) \cdot 1 + (a_1b_2 + a_2b_1 + a_3b_4 + a_4b_3) \cdot I+ \\ + (a_1b_3 + a_2b_4 + a_3b_1 + a_4b_2) \cdot J + (a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1) \cdot K,\end{aligned}$$

By the above given method we get the algebra of commutative-associative hyper-complex numbers the, where the multiplication table of basic units have the following look:

$$\begin{array}{c|cccc} & 1 & I & J & K \\ \hline 1 & 1 & I & J & K \\ I & I & 1 & K & J \\ J & J & K & 1 & I \\ K & K & J & I & 1 \end{array}.$$

It follows from the table that $I^2 = J^2 = K^2 = 1$, namely all its imaginary units are hyperbolic. We can get the same algebra another way: by applying for 2 times the agebra of the real number using two independent hyperbolical-imaginary units I and J the doubling operation. Let us denote the product of I and J as an independent object k, the number A from the corresponding multitude can be presented as a linear combination:

$$A = (a_1 + a_2 \cdot I) + (a_3 + a_4 \cdot I) \cdot J = a_1 + a_2 \cdot I + a_3 \cdot J + a_4 \cdot K,$$

where the symbol of the real unit 1, as it is accepted in the complex-numbers and quaternions, is omitted.

Let us call the numbers $\bar{A}, \hat{A}, \widetilde{A}$ *conjugate* to the number $A = a_1 + a_2 \cdot I + a_3 \cdot J + a_4 \cdot K$, if they look like:

$$\begin{aligned} \bar{A} &= a_1 - a_2 \cdot I + a_3 \cdot J - a_4 \cdot K, \\ \hat{A} &= a_1 + a_2 \cdot I - a_3 \cdot J - a_4 \cdot K, \\ \widetilde{A} &= a_1 - a_2 \cdot I - a_3 \cdot J + a_4 \cdot K. \end{aligned} \tag{12}$$

Notice that

$$\hat{\bar{\widetilde{A}}} = A. \tag{13}$$

The product of such fours, as it is easy to check by the direct substitution, are always real numbers

$$A\bar{A}\hat{A}\widetilde{A} = a_1^4 + a_2^4 + a_3^4 + a_4^4 - 2a_1^2a_2^2 - 2a_1^2a_3^2 - 2a_1^2a_4^2 - 2a_2^2a_3^2 - 2a_2^2a_4^2 - 2a_3^2a_4^2 + 8a_1a_2a_3a_4. \tag{14}$$

By analogy with the algebra of complex numbers we will relate the value to the fourth degree of the corresponding number modulus and denote it as $|A|^4$. The introduced conception has the common properties of the modulus:

$$|\lambda A| = |\lambda| \cdot |A|, \qquad |AB| = |A| \cdot |B|,$$

where λ is a real, and A, B are complex numbers. In the product the property of mutually conjugated to result in the real number let us introduce into the examined algebra the operation of division, interpreted as an action inverse to multiplication. So, let us understand the number

$$A^{-1} = \frac{\bar{A}\hat{A}\widetilde{A}}{|A|^4} \tag{15}$$

under the number A^{-1} which is inverse to A. Only the numbers whose module is non-zero have their inverse analogues. Such numbers do not have such analogs. The examined algebra is associated with the form (11). It can be proved by examining a shift from the basis $1, I, J, K$ to the basis $S_1, S_2, S_3, S_4,$, whose objects are connected with the initial correlation:

$$\begin{aligned} S_1 &= \frac{1}{4}(1 + I + J + K), \qquad & S_2 &= \frac{1}{4}(1 - I + J - K), \\ S_3 &= \frac{1}{4}(1 + I - J - K), \qquad & S_4 &= \frac{1}{4}(1 - I - J + K). \end{aligned} \tag{16}$$

These bases are the divisors of zero and are distinguished by the fact that their multiplication table is the most vivid one:

	S_1	S_2	S_3	S_4
S_1	S_1	0	0	0
S_2	0	S_2	0	0
S_3	0	0	S_3	0
S_4	0	0	0	S_4

.

We will call the divisor of zero with such properties the *principle*, and the bases formed of them – the *absolute*. The feedback of the units $1, I, J, K$ with the principle zero divisor of the algebra H_4 is evaluated the following way:

$$1 = S_1 + S_2 + S_3 + S_4, \qquad I = S_1 - S_2 + S_3 - S_4,$$

$$J = S_1 + S_2 - S_3 - S_4, \qquad K = S_1 - S_2 - S_3 + S_4.$$

It is easy not only to sum but also multiply and divide the numbers from H_4 written in the absolute basis. So, the product of two numbers A and B looks is following:

$$(AB) = (a'_1 b'_1)S_1 + (a'_2 b'_2)S_2 + (a'_3 b'_3)S_3 + (a'_4 b'_4)S_4,$$

and their fraction reads

$$\frac{A}{B} = \frac{a'_1}{b'_1}S_1 + \frac{a'_2}{b'_2}S_2 + \frac{a'_3}{b'_3}S_3 + \frac{a'_4}{b'_4}S_4.$$

(Henceforth the components with primes will relate to the absolute basis). the absolute basis reveals the structure of the quadrahypeboloic number algebra, which is isomorphic to the algebra of real diagonal matrices. The group of mutually conjugated written in the absolute basis looks like:

$$\begin{aligned}
A &= a'_1 S_1 + a'_2 S_2 + a'_3 S_3 + a'_4 S_4,\\
\bar{A} &= a'_2 S_1 + a'_1 S_2 + a'_4 S_3 + a'_3 S_4,\\
\hat{A} &= a'_3 S_1 + a'_4 S_2 + a'_1 S_3 + a'_2 S_4,\\
\widetilde{A} &= a'_4 S_1 + a'_3 S_2 + a'_2 S_3 + a'_1 S_4.
\end{aligned} \tag{17}$$

The modulus of the number A in such special basis looks like:

$$|A| = |a'_1 a'_2 a'_3 a'_4|^{1/4}, \tag{18}$$

that proves the correspondence of the algebra to geometry defined by the fundamental metric form (11). We can introduce the conception of function for the multitude of the Quadra numbers. The exponential function is one of the most interesting. Under it we will understand the following series:

$$e^X = 1 + X + \frac{X}{2!} + \dots,$$

where X is an arbitrary Quadra number. With the introduction of the exponential function we can examine along with the algebraic form of the number H_4 its exponential form. So, the number $A = a'_1 S_1 + a'_2 S_2 + a'_3 S_3 + a'_4 S_4$, where all the components of a'_i in the absolute basis are positive, corresponds to:

$$A = |A| e^{\alpha I + \beta J + \gamma K}, \tag{19}$$

where the positive value $|A|$ is its modulus. By analogy with the complex and double numbers we will call the real numbers α, β and γ, the *argument of the Quadra number A*. The connection of the arguments with the components a'_i in the absolute basis looks like:

$$\alpha = \frac{1}{4}\ln\frac{a'_1 a'_3}{a'_2 a'_4} = \frac{1}{4}(\ln a'_1 - \ln a'_2 + \ln a'_3 - \ln a'_4),$$

$$\beta = \frac{1}{4}\ln\frac{a_1'a_2'}{a_3'a_4'} = \frac{1}{4}(\ln a_1' + \ln a_2' - \ln a_3' - \ln a_4'),$$
$$\gamma = \frac{1}{4}\ln\frac{a_1'a_4'}{a_2'a_3'} = \frac{1}{4}(\ln a_1' - \ln a_2' - \ln a_3' + \ln a_4'),$$

where $\ln x$ is a logarithmic function of the real x. As the hyperboloic analog to the Euler formula works for every imaginary unit:

$$e^{\alpha I} = \cosh\alpha + I\sinh\alpha,$$

then the following expression for the exponent from an arbitrary Quadra number $X = \delta + \alpha I + \beta J + \gamma K$ is true:

$$e^X = (\cosh\delta + \sinh\delta)(\cosh\alpha + I\sinh\alpha)(\cosh\beta + J\sinh\beta)(\cosh\gamma + K\sinh\gamma), \tag{20}$$

where $\cosh x$ and $\sinh x$ are hyperbolic sinus and cosine. We can introduce an analogous function for the quadranumerical variable X as the following rows:

$$\cosh X = 1 + \frac{X^2}{2!} + \ldots, \qquad \sinh X = X + \frac{X^3}{3!} + \ldots.$$

We can connect the notion of the derivative with the function of the quadranumerical variable by the direction and analyticity the same way as the corresponding ideas are introduced into the algebra of double numbers [2]. The analyticity of the function from H_4 denotes the independence of its derivative from directions, [5] $dF = F'da$, and appears in simultaneous execution of 12 equations, which are analogs to the Cauchy-Riemann terms for the complex and double variables:

$$\frac{\partial U}{\partial a_1} = \frac{\partial V}{\partial a_2} = \frac{\partial W}{\partial a_3} = \frac{\partial Q}{\partial a_4}, \qquad \frac{\partial U}{\partial a_2} = \frac{\partial V}{\partial a_1} = \frac{\partial W}{\partial a_4} = \frac{\partial Q}{\partial a_3},$$
$$\frac{\partial U}{\partial a_3} = \frac{\partial V}{\partial a_4} = \frac{\partial W}{\partial a_1} = \frac{\partial Q}{\partial a_2}, \qquad \frac{\partial U}{\partial a_4} = \frac{\partial V}{\partial a_3} = \frac{\partial W}{\partial a_2} = \frac{\partial Q}{\partial a_1}, \tag{21}$$

where

$$F(A) = U(a_1,a_2,a_3,a_4) + V(a_1,a_2,a_3,a_4)I + W(a_1,a_2,a_3,a_4)J + Q(a_1,a_2,a_3,a_4)K$$

is an analytical function of a quadranumerical variable, and U, V, W, Q are hypercomplex-conjugated functions of four real arguments. In the algebra of quadranumbers there are 16 typical unit objects $e_1 - e_{16}$ that have in their basis, where the form (11) is written, the following components:

$$\begin{aligned}
e_1 &\leftrightarrow (1,1,1,1); & e_5 &\leftrightarrow (-1,-1,-1,-1);\\
e_2 &\leftrightarrow (1,-1,1,-1); & e_6 &\leftrightarrow (-1,1,-1,1);\\
e_3 &\leftrightarrow (1,1,-1,-1); & e_7 &\leftrightarrow (-1,-1,1,1);\\
e_4 &\leftrightarrow (1,-1,-1,1); & e_8 &\leftrightarrow (-1,1,1,-1);\\
e_9 &\leftrightarrow (1,-1,-1,-1); & e_{13} &\leftrightarrow (-1,1,1,1);\\
e_{10} &\leftrightarrow (1,1,-1,1); & e_{14} &\leftrightarrow (-1,-1,1,-1);\\
e_{11} &\leftrightarrow (1,-1,1,1); & e_{15} &\leftrightarrow (-1,1,-1,-1);\\
e_{12} &\leftrightarrow (1,1,1,-1); & e_{16} &\leftrightarrow (-1,-1,-1,1).
\end{aligned}$$

The vectors e_i that correspond to the numbers can be used to illustrate the presence in the Quadra space of two types of transverslity, that generalize the idea of orthogonal directions for the Finslerian space. This is true that the 2 symmetrized forms (5) enter the Quadra space. They look like:

$$S_1(\mathbf{a},\mathbf{b}) = (\mathbf{a},\mathbf{a},\mathbf{a},\mathbf{b}) + (\mathbf{a},\mathbf{b},\mathbf{b},\mathbf{b}) \tag{22}$$

and

$$S_2(\mathbf{a},\mathbf{b}) = (\mathbf{a},\mathbf{a},\mathbf{b},\mathbf{b}). \tag{23}$$

The equality to zero of any of them means the transversality of the corresponding directions. By direct substitutions of the components of vectors e_i in (22) and (23) we can make ourselves absolutely sure of the fact that every vector of the multitude faces 1, form mutually transversal pairs of the first order, and of the second with 8 of them. We can construct the basis that is an analog to the orthogonal from the four the first order transversal vectors. One of the specific cases of the basis is the above examined four-set $1, I, J, K$. It is impossible to construct basis from the second order transversal vectors as for each pair of the third and what is more fourth order do not have such correlation of directions.

5. Conclusion

The offered method of studying the examined class of Finslerian linear spaces, called polylinear, seems to be promising for it is based on the same principles as the scalar product. Let us note that the arising abilities let us move the focus of studies from the common vivid base to the soil of mathematical constructions. Thus the pseudo-Euclidean spaces demonstrate advantages of the analogous substitution. Not all geometrical effects are vivid in these spaces but the extension of the scalar product in its time was very useful.

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CHRONOMETRY OF THREE-DIMENSIONAL TIME

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The concept of the multi-dimensional time has tried not once to take its place in natural science, but every time under the pressure of some paradox was rejected. Meanwhile a philosophical question: why the space admits quite a number of dimensions and the time dos not, still preserves. In this work a new attempt has been made to resolve the matter, by switching from the traditional quadratic metrics to the Finslerian one, which may admit an arbitrary degree of the vector component that is included into the metric function. Though the offered method enables us to build continuums of time of any natural dimensionality, in order to demonstrate the specificity of the raised topic this study will focus on a simple (after rather trivial two-dimensional case) example of three temporial dimensions.

1. Introduction

The idea of space is accepted much easier and vividly than the idea of time. This circumstance is conditioned by the fact that the space is looked over all at one time, and above all in the three-dimensional shape, meanwhile we see just a side of the time and only in one dimension. This situation forced some scientists "to get rid" of the time, either limiting to fixed problems or driving the time into the condition of an extra space dimension. The first approach is related to Archimedes, the latter approach for the first time appeared in the works of Galiley, reached perfection in Lagrange's and in fact reigns nowadays, – though The Special Theory of Relativity practically confronted the category of time to space, denoting them absolutely different in their essence, having differences already on the geometrical level.

There grows the belief formulated for the first time by Synge [1] that Euclid put the natural science on the wrong track, as he took the space but not the time as the fundamental idea of the science. The lack of any adopted term for time studying according to Synge is the proof of such disregard. He suggested that we should use the word "chronometry" to define the branch of science that deals with the idea of time in the same wide meaning as geometry does with the idea of space. Though Synge is unlikely to mean the multi-dimensional time, his statement is applicable to this aspect of the problem.

2. Two-dimensional time

The essence of the multy-dimensional time, that serves as an alternative to the multy-dimensional space, can be illustrated by a paradoxical-seeming statement: practically all physicians know about the two-dimensional time, but by tradition go on looking at it in another way. We mean the pseudo-Euclidean plane. It is surprising that among all the Euclidean spaces only the two-dimensional is distinguished with its unique peculiarities, it is worth mentioning the following.

Firstly, the theorem of Louiville, that enumerates the types of possible conformal transformations, coming to translations, rotations, dilatations and inversions, is true for all the pseudo-Euclidean spaces with 3 or more dimensions. In the two-dimensional case the list of their conformal transformations is by far longer.

Secondly, there are several concepts of the total product of the plane vectors, and the majority of them have the inverse ones; meanwhile in other pseudo-Euclidean spaces only scalar product is introduced, as well as division is not defined at all.

Thirdly, isotropic vectors always divide the pseudo-Euclidean planes with the signature $(1, n-1)$ into 3 simply connected domains, with an exception of the plane, with 4 such domains.

Fourthly, it does not matter which of the two typical coordinates of the Euclidean space we will choose as the temporal and which as the spatial, as the result will change to permutation. Another case appears in planes with a bigger number of dimensions, where such symmetry collapses and to the time we can apply only change of the sign.

And finally, only the plane admits the accordance with the associative-commutative algebra, whose main objects are called the double numbers. Their algebra has all the characteristics of usual algebras of real and complex numbers, including the product commutativity, with an exception of presence of specific objects, called the divisiors of zero. Each divisor of zero has a counterpart such that their product is a divisor. Though the double numbers are trivial in comparison with the complex, even such algebras cannot be related with pseudo-Euclidean spaces with more than 2 dimensions.

But, thinking that the uniform order starts with 3 and more dimensions, scientists, due to some reasons, don't notice or at best attribute it to the reducible nature of the two-dimensional space. It is interesting to note that we face practically the same in the Euclidean case: the two-dimensional representatives stand separately out and are juxtaposed with the algebra of complex numbers.

We can make a supposition basing on only these two examples that because of some reasons the connection of some metric spaces with the commutative-associative algebra make them in a way distinguished and that is why the very algebras and the corresponding spaces deserve a special attention.

When we stated in the beginning that we there was no reason to treat the pseudo-Euclidean space as a special case of the multy-dimensional time, we based on the fact that in the space there is no objective reason for us to distinguish which of its directions can act as time and which not. Then we must admit that in such a space all non-isotropic directions are equal in rights. Their differentiation by physical meaning takes place only after subjectively choosing one quadrant as the field of future.

Note. The subjective choice is related mostly to the world line, an element of whose length is interpreted as the proper time of an observer, and the future region is defined as the consequence of the line direction.

Only after the given procedure the points of the facing quadrant automatically acquire the meaning of the past actions, and the points of the two side – become absolutely distant. But few things will change on the pseudo-Euclidean plane if we choose to use any other quadrant as the field of the future, as only all the others will trade places. With an exception of this inessential-seeming moment, any further construction in the pseudo-Euclidean plane does not differ from the construction in its usual interpretation as the time-space.

But a move to 3 and more dimensions leads to the fact that the difference between the pseudo-Euclidean space-time and the dimension-corresponding pure time becomes principal, and moreover if we think of the conceptual multy-dimensional time as of a possible geometrical alternative to the space of the Special Theory of Relativity, it is important to revise not only mathematical, but also philosophical attitudes towards the structure of physical reality.

3. Three-dimensional time

To make a move from the two-dimensional time model to the three-dimensional let us use the observation that in the case of the pseudo-Euclidean plane the corresponding geometry becomes related with the idea of the commutative-associative hyper-complex number, which are related to the commutative-associative hypercomlex algebras. William Hamilton is the pioneer of hyper-complex numbers; while speaking at one of the sittings of the Royal Irish Academy he stated that if there existed geometry - the pure mathematical space science, there must be the same pure time science, and such a science should be algebra [2]. It is paradoxical but he on the example of the quaternions, discovered by himself, disproved the multitude of principally different algebras. But let us take his statement, as a presentiment of the great mathematician, and by analogy with the algebra of binary numbers we will try to make the algebra of triple number, and try to correspond with them geometry, or using Synge's suggestion, the chronometry of the three-dimensional time.

The presence of the basis in binary numbers makes the expression for the second degree of the module to take an absolutely symmetrical form:

$$|\mathbf{X}|^2 = x'_1 x'_2, \tag{1}$$

It indirectly shows that there must be a basis for the numbers that admittedly can be an algebraic analog to the vectors of the three-dimensional time. In this basis the fourth degree of the module becomes connected with the next absolutely symmetrical form out of three components:

$$|\mathbf{X}|^3 = x'_1 x'_2 x'_3. \tag{2}$$

It is not difficult to make sure that the algebra of such numbers exists, it is commutative and associative, and is the direct sum of three real algebras that continues the tendency that started at the example of binary numbers, whose algebra becomes the direct sum of the two real. As is well known, the one-dimensional time can be compared with the real numbers themselves, that is another confirmation of the chosen algebraic way of searching for models of the multy-dimensional time.

The manifolds for which the differentials of the vector length are expressed by means of the types (1) − (2), are well known in geometry and are called the Finslerian spaces with the Berwald-Moore metric function [3]. Usually under the term Finslerian spaces we understand the manifold of the most common type with a null meaning of curvature and torsion. The concerned metric (2) is defines the linear space, that is why it is in near relation with Euclidean and pseudo-Euclidean spaces, though they do not look alike in everything.

Let us call the linear Finslerian spaces, whose metric function in one of the bases looks like:

$$F(x') = \left|\prod_{i=1}^{n} x'_i\right|^{1/n}, \tag{3}$$

the *n-dimensional time*. To have not only axiomatical but also physical right to use this name let us interpret every point of the spaces as an event, and every line as a world line of an inertial reference frame.

Notes. The concept of an event is introduced in this way that though having something common with the classical analogue introduced by Minkowski, still differs from the latter. This is related to the fact that the concept of event in the multy-dimensional time stops having a single meaning and becomes dependent on the reference frame. In other words the same point of the space should be interpreted as different events if the world lines are separated by isotropic hypersurfaces. The concepts of time and space are as if substituting with one another. There are cases 2^n of such domains in n-dimensional time, and every point may have the same number of interpretations. But there does not emerge polysemy if we examine only the reference frames where the world lines lie only in the light cone, and the concept of event practically does not differ from its classical analogue.

In such reference frame the interval of proper time between an arbitrary pair of the equals the length of the vector related to the event. It follows from the symmetry of the examined spaces that all their non-isotropic directions are absolutely equal in rights if we decide to relate, according to the given above thesis, the length of the vector to the proper time in the distinguished reference frame then its justified to call the spaces, this time not by definition, rather than because of physical reasons, the multy-dimensional time.

But still preserves the question: whether such verities have any connection with the real world? To approach the answer let us try to examine the properties and peculiarities of the three-dimensional time. We will start from examining its structure and isotropic subspaces.

4. Light pyramids

The form (2) nullifies in the points that correspond to the three distinguished planes, defined by the equalization:

$$x_1' = 0, \quad x_2' = 0, \quad x_3' = 0. \tag{4}$$

The vectors lying on the plane have the zero meaning of the modulus and in this meaning are isotropic. At the same time, lines, that simultaneously belong to 2 planes (as well as the point of intersection of all the 3) automatically become marked out. As there are only three lines, it is quite natural to try to connect the vectors with the special basis. This basis is unique up to permutation and the form (2) given above defines the value of an arbitrary number module and also the length of the vector, – all being of the simplest shape. Concerning the originality of such basis, we will give it a proper name of the Absolute basis.

In this respect the concerned space turns out to be arranged in an absolutely another way, than the usual Euclidean and pseudo-Euclidean spaces, where there are no preferred bases (with an exception of the pseudo-Euclidean plane), and that is why we usually try to turn the studying of analogous geometries into a non-coordinate form. The existence of special bases in the multy-dimensional time means that if some day a connection between corresponding varieties and the physical reality will be found then some frame of reference will play a clearly distinguished role.

The isotropic planes (4) can be thought about for example as they are presented on Fig. 1. As we can see on the picture the three-dimensional space is divided by isotropic planes into 8 equal camer-oktants, that are domains of simple connectedness in fact. At the same time every camera is separated from the 3 side ones by the two-dimensional isotropic planes, it borders upon isotropic rays with another 3 cameras and with the opposite one it contacts through only one point. By analogy we can characterize, only taking into consideration the dimension, the mentioned above the two-dimensional time, where all the space is divided by isotropic lines into 4 camera-octants. Every quadrant is separated from 2 adjoining ones by isotropic rays, and with the opposite borders through a point. At the same time the one-dimension time also obeys the rule, as we can look upon

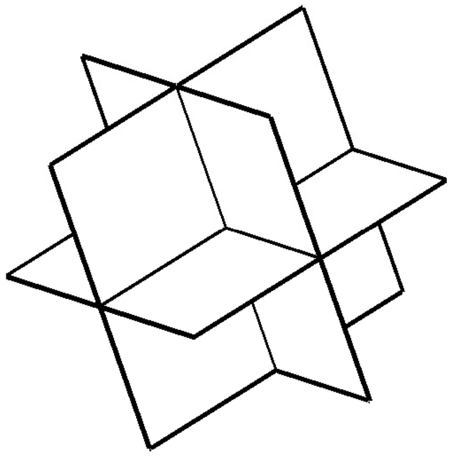

Figure 1: Isotropic planes of tree-dimensional time

the corresponding line as 2 opposite simply connected domains, divided by a special point, a zero that in a way can be considered to be an extreme singular case of the isotropic cone.

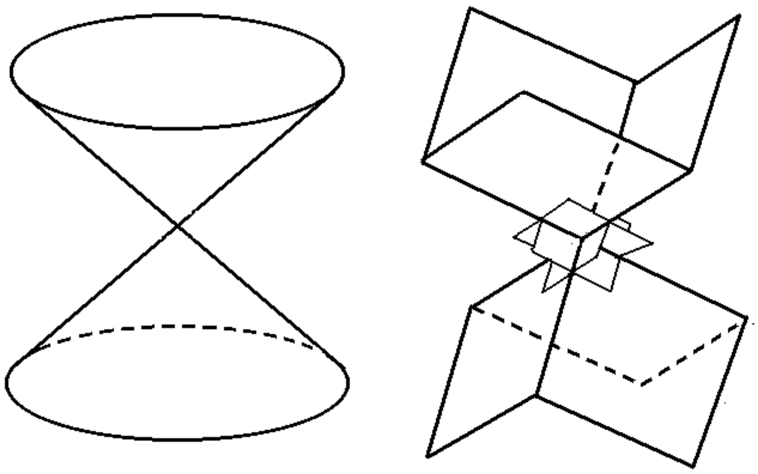

Figure 2: Light cones of tree-dimensional time (right) and tree-dimensional pseudo-Euclidian space (left)

If we choose 2 facing camera-octants from the 8 of the three-dimensional time and examine their united border we will get a figure depicted on Fig. 2. Such the sub-space looks like a light cone of the Euclidean space (depicted on the same picture to the left side) but for the fact that the first does not have a continuous axis symmetry. There are non-zero vectors in the inside of both facing octants, and the ends of the unit length vectors form 2 planes of a specific hyperboloid, which is the Finslerian analogue of the double-band hyperboloid of the pseudo-Euclidean space. Both figures are depicted on Fig. 3, the left corresponds to the three-dimensional time and represents only a quarter of the hyperboloid of space, which has 8 cavities, each for every simply connected area. The points of the figure satisfy the equalization: $|x'_1x'_2x'_3| = 1$, and its general form is represented on Fig. 4.

Among the unit vectors that are set against one and the same plane of such hyperboloid continuous transfers, exercised by the Abelian two-parameter group of linear

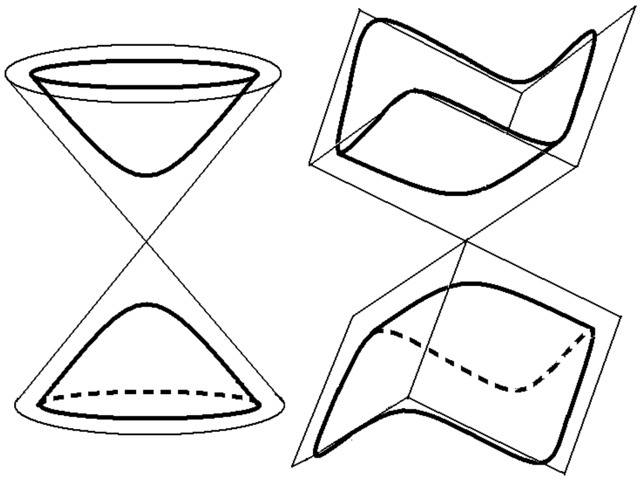

Figure 3: The fragments of unit hyperboloids

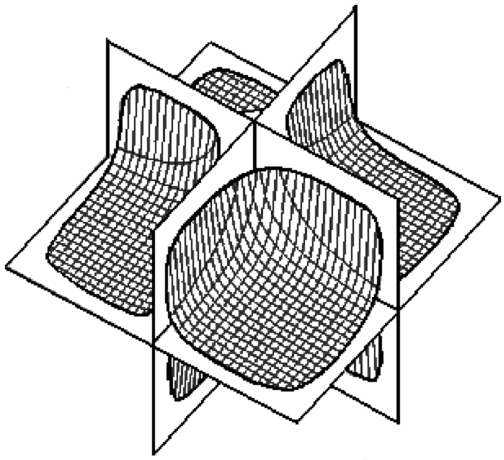

Figure 4: The eight-sheet hyperboloid of tree-dimensional time

transformations, is possible. The transformations can be displayed as a diagonal matrix:

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \tag{5}$$

with $a_1 a_2 a_3 = 1$. Transformations of the group are invariant to the interval of the three-dimensional time (2) and that is why it is its motion. In their character the motions are similar to the boosts of the corresponding pseudo-Euclidean space with the only difference that the points of the line stay static in the one-parameter turnings in space-time, and in the analogous case of the concerned space – only one single point. We will call transformations of the group the *hyperbolic turning of the three-dimensional time.*

Among motions of the space, apart from turnings, we can single out a three-parameter group of parallel shifts, that are a common idea in linear planes. There is no other continued transformation that would be invariant to the interval in the three-dimensional time.

The isotropic edges and unit hyperboloids of the distinguished group of facing octants whose ends are to end at infinity are depicted on Fig. 2 and Fig. 3, but due to

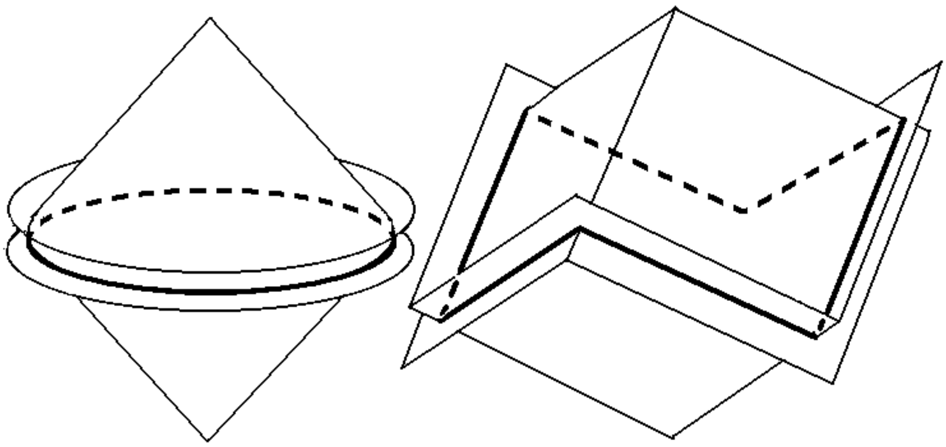

Figure 5: The two light cones couple intersection

the limited plane of the draft, their ends are cut short, but not at a plane, common for pseudo-Euclidean space, but in a more sophisticated way according to the following considerations. If we intersect the border of one of the octants with the border of the facing octant dislocated along their mutual axis we will get a rectilinear hexagon, and not a plane but the broken as it is demonstrated on Fig. 5. The volume that belongs to the interior of both octants is a common cube, and the mentioned above hexagon is composed of its edges that do not intersect the main axis.

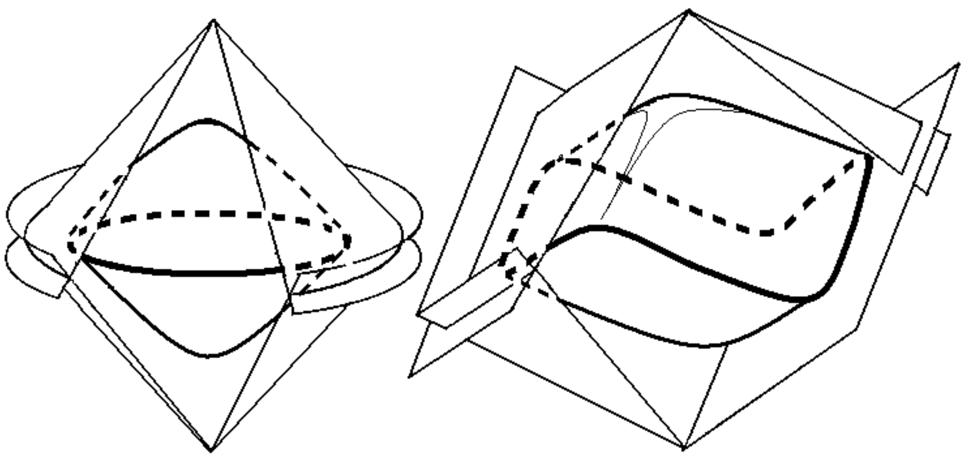

Figure 6: The two hyperboloids couple intersection with $0 < R < T$

Notes. We can say that in case of the n-dimensional time the figure that is the interception of two deposed towards each other facing cameras, consists of a half of $(n-2)$ edges of the formed by it hypercube, on top of all only edges that do not have common points with the main axis of symmetry participate in the formation.

If we construct two sets of concentric hyperboloids (per se they are Finslerian generalizing of spheres) inside the octants that form the cube with their centers in the opposite tops, the intersection of pairs with equal radius will result into a set of continuous closed graphs, whose form depends on the ratio of the corresponding to the curve radius of the hyperboloid R to half of the main diagonal of the cube T. When the radius of hyperboloids equal 0 they coincide with the isotropic edges of the octants, and their interception is a

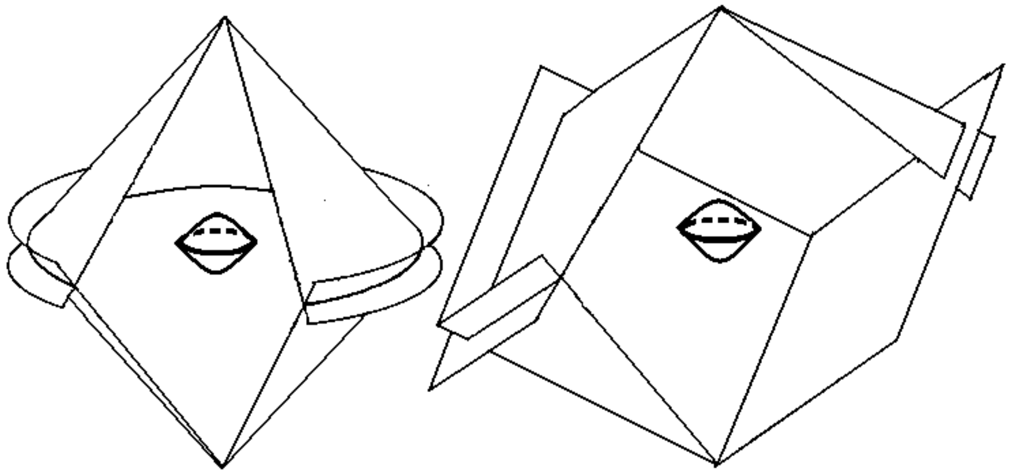

Figure 7: The two hyperboloids couple intersection with $R \approx T$

broken in space hexagon already examined on Fig. 5. When $0 < R < T$ the hyperboloids are intercepted on curves that look like the curve on Fig. 6. They are three-dimensional and have 6 round corners. While the value of the hyperboloid radius approaches to the value T the curves that are the result of their interception become more smooth and flattened out, and when $R \to T$ they turn into absolutely plane circumferences, though with infinitesimal radius Fig. 7.

In the three-dimensional pseudo-Euclidean space the analogous constructions lead to a group of concentric circumferences that lie in the same plane, you can see the circles on Fig. 5-7 to the right of them. The circumference that belongs to two light cones, that is corresponds to the interception of the pseudo-Euclidean sphere with $R = 0$ which in the Special Thery of Relativity is interpreted as a momentary position of the light front, that can be registered by the observer that is at the top of one of the cones, supposing that there is a flash at the top of the other. In general we should apply an analogous interpretation to the three-dimensional time case. So, the broken hexagon depicted on Fig. 5 can be interpreted as the multitude of points of the observer space, that is situated at the point T, with which it connects the momentary position of the light front, whose flash took place in $-T$. To make this situation true we must admit that the isotropic borders of the facing octants are analogues of the light cones of the past and future that corresponds in number of dimensions with the pseudo-Euclidean. This method looks rather natural and the only effort, in comparison with the common idea of the Special Theory of Relativity, we should make is to admit the borderness of the light cone. Taking into consideration that this borderness is executed in the space not available for the contemplation of the observer, the question whether is complies with the realities of our world turns out to be not so obvious.

Though we could save the name of light cones, usually used in the pseudo-Euclidean spaces in order not to emphasize peculiarities of geometry of the multy-dimensional time, for the isotropic borders of the simply-connected cameras, so let us call the corresponding figures the *light pyramids*, first of all singling out the pyramids of the past and future.

5. Planes of relative simultaneity

We should logically go further and accept an analogy not only between isotropic sub-spaces and the related to them light fronts but also we should put into correspondence with every common circle of two equal hyperboloids of the pseudo-Euclidean space an analogous curve, that is the interception of a pair of Finslerian spheres of the multy-dimensional time. There emerges quite a natural way to define the plane of the relative simultaneity of the three-dimensional time, as the same physical sense was played in the pseudo-Euclidean geometry by a plane represented with the above examined set of circles. Following the logic we should understand a multitude of points, equidistant in the meaning of the corresponding Finslerian metrics of two fixed points, under the simultaneous events of the multy-dimensional time. At the same time one of the fixed points coincides with the momentary position of the observer, and the second is the reflection of it with respect to the studies plenty of events.

The straight line that goes through the two points defines the inertial reference frame, but as it follows from the accepted definition of simultaneity now this property depends not only on the speed of the observer but also on his momentary position concerning the layer, to which he is going to give the equal time of performance. In the pseudo-Euclidean case (that has become practically classical) while defining the simultaneity meant only the relative speed of the relative speed of the reference frame, and the momentary position of the observer was not important. It is not so in the three-dimensional time and this circumstance seems to be one of the most important items, that differ the physical properties of the examined manifold from the common pseudo-Euclidean constructions.

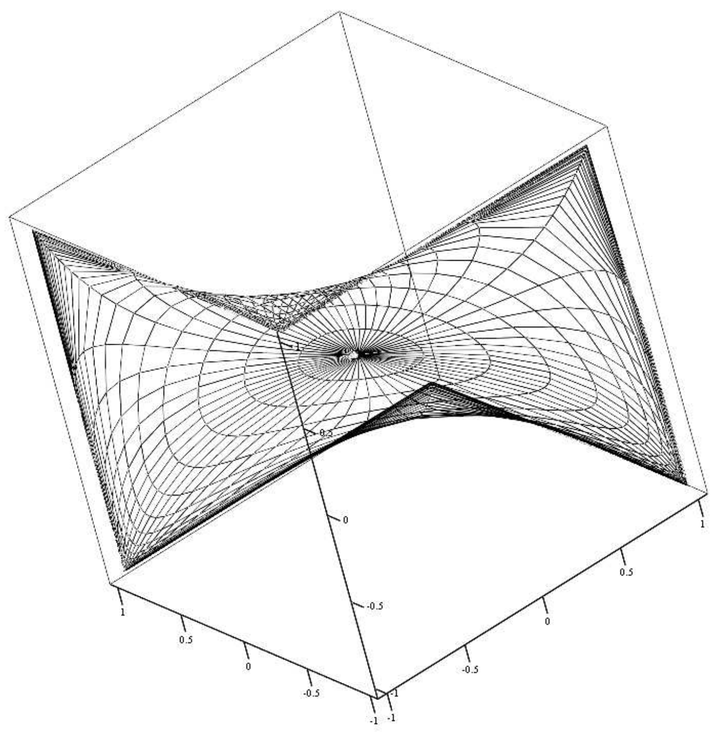

Figure 8: The simultaneous surface of three-dimensional time

It is convenient to describe the plane of simultaneity that corresponds to a fixed pair of points by an equalization that relates it coordinates to the coordinates of the initial affine space represented in the absolute basis. It is not difficult to get such equalization for an arbitrary pair of points, but it looks most vividly when momentary position of the observer is related to the point (T, T, T), and its reflexion has coordinates $(-T, -T, -T)$. In this case the equality of intervals leads to the equalization:

$$|(x'_1 + T)(x'_2 + T)(x'_3 + T)| = |(T - x'_1)(T - x'_2)(T - x'_3)|, \tag{6}$$

then after opening the brackets it leads to:

$$x'_1 x'_2 x'_3 + (x'_1 + x'_2 + x'_3)T^2 = 0. \tag{7}$$

The plane corresponding to the equalization is depicted on Fig. 8.

The curves examined on Fig. 5 and Fig. 7 mark points on the plane litarally equidistant from their geometrical center. Such curves in many ways are analogous to common concentric circles, though the related to it geometry does not coincides with the usual Euclidean.

On the other hand we can get a new group of curves, that corresponds to the multitude of radial lines of the Euclidean circle the canonic planes by intercepting the plane of simultaneity by canonic planes, called in the work [4] the *cones of rotation*, have tops in the point (T, T, T) and include the real axis. So, there is a net of curvilinear coordinates, that in the two-dimensional physical space play the same role as the polar scheme of coordinates does in the Euclidean plane.

Transformations that turn into themselves the plane of simultaneity so that the circles and radial curves at the same time map into the same curves and become in many ways analogous to spatial turns around the point of origin in the pseudo-Euclidean space, as the physical distance in either of the cases remain the same. But in the case of the three-dimensioal time these transformations are not linear, and on top of all do not leave invariant the three-dimensional intervals.

6. Physical distance and speed

It could seem that we have approached to the possibility of introduction into the three-dimensional time of two-dimensional physical distance and speed, it is enough to bring on the simultaneity plane in correspondence the set of circumferential and radial curves with the lines of the polar reference frame. But it is not like this. The fact is that the examined multitude does not admit the introduction as one-digit such physical notions as the distance and speed at least if the construction is based on the starting measurement of time intervals. What seems to be practically an obvious property of the pseudo-Euclidean spaces turns out to be not-compatible with the idea if the multy-dimensional time. This circumstance not only decr eases, but on the reverse increases the possibility of the multi-dimensional time to compete with the Minkowski space for being the geometrical basis of the real world. In fact, if we follow the idea of chronometry we should associate associate the time intervals, that are needed to send a desired signal and receive its reflection, with physical distance. But any attempt to unite this natural and vivid physical principle with the necessity of one-digitness comes upon obstacles. The idea of rejecting the one-digitness of the physical distance and speed seems to be a nice and far-reaching exit (cf. interpretations of quantum-mechanical uncertainty principle).

The above said does not mean that an entirely amorphous structure should replace the Euclidean geometry of the physical space. The analysis shows that our radical supposition touches upon not the quality, but only quantity aspect of the phenomenon. The distance and speed as independent physical categories are not completely excluded in the multy-dimensional time, but only change their status, getting the traits uncertainty on the initial geometric level. In particular the idea of equidistant in the physical meaning objects becomes dependent on which signals the observer, that defines this equidistance, uses as the reference. In its way the reference signals are defined by the principle of equality of proper times, where the hours pass in the corresponding inertial reference frames between sending, reflecting and receiving the signals. Taking into consideration that the time intervals are the only value that by definition are measurable in our Finslerian multitude, the task of distinguishing among the continuous specter of inclined world lines the ones would be characterized by the equality of intervals is quite possible. Let us note that we already used the method above, while defining the relatively simultaneous events. So, we can consider the signals to be etalons if their world lines start in one point, reach the plane of simultaneity and after refraction gather together and in another fixed point of the world line of the same observer. It is clear that all the intervals should be equal either before or after the refraction.

Such logic in constructing drives us to the fact that the physical space of the observer with its geometrical properties becomes in a way dependent on which set of reference signals define the geometry. So if the world lines of reference frames are practically parallel to the line of the observer, he starts to see a space, which in its characteristics practically coincides with the Euclidean. This is related to the fact that the ends of the vectors with the same value of the intervals in this cases lie (as it has been said above) on practically plane and ideal circle, and the latter while constructing the physical space plays the role of the Finslerian indicatrix. A common circle is the indicatrix of the two-dimensional Euclidean space. When tuning to the signals whose world lines are inclined more significantly, the ends of the corresponding vectors form this time not a circle, but a more sophisticated closed curve, which is not a plane one. At limit of the signals, whose speeds are interpreted as the light, this curve transforms into a broken hexagon, examined on Fig. 5. The geometry of the two-dimensional physical space is the Finslerian, and it is this geometry that differs greatly from the Euclidean, but in connection with the fact that the indicatrix even in this limit case is still closed and flattened out. The differences between the two geometries are not significant, in connection with which it is probably possible to mix them up, especially if the experimental cases are limited to low speeds.

So, if we suppose that our real world has a direct connection with the examined Finslerian geometry, the appearance of Euclidean and pseudo-Euclidean ideas in observer outlook should be a natural process of consistent approaches to a more exact description. On the other hand in our everyday life we use signals whose speed is by far lower than the light when we try to find the zones that manage the world. As the matter of fact we use the light only to identify the objects, and the distance is defined by other slower means – for example by a ruler. This circumstance leads us to the fact that when in special experiments really high-speed signals become of great importance, the geometry is considered to be defined before hand, and that is why even abnormal results will be treated anyhow, but only not in the direction of revising the obvious geometrical properties.

7. Conclusion

Among all the above listed properties and peculiarities of the three-dimensional time, as a representative of a very specific class (the non-linear) of Finslerian spaces, we should treat as the most important the one, thanks to which it is related to the most fundamental notion of mathematics – the number – which is the object of algebra, that has the most common arithmetical properties. We should emphasize ones more the fact that neither Euclidean nor pseudo-Euclidean spaces with three or more dimensions do not possess the analogous qualities. The quaternions and biquaternions used in similar situations are not genuine numbers, as there algebra has commutative multiplication, as the result of which the construction of a valuable theory that would generalize the theory of functions of the complex variable is not possible (or is extremely difficult). At the same time the given above examples demonstrate how common Euclidean and pseudo-Euclidean conceptions can come out of the idea of substitution of the pseudo-Euclidean metric to multy-temporal case – rather interesting and actual.

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FOUR-DIMENSIONAL TIME

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The generalized metric space, that can be called the flat four-dimensional time, is based on the Berwald-Moore's Finslerianview of metric function. This variety let us introduce physical notions: the event, the world lines, the reference frames, the multitude of relatively simultaneous events, the proper time, the three-dimensional distance, the speed, etc. It is demonstrated how from the point of the physical observer, associated with the world line, in absolutely symmetrical four-dimensional time the contraposition of the coordinate takes place, that defines its proper time, with the ones that appear as the result of the measurements made with the help of sample signals. When the signals correspond with lines, which are practically parallel to the world line of the observer, he starts to see the three-dimensional space which at the limit is the Euclidean space.

1. Introduction

For the last 100 years the idea, that the Pseudo-Euclidean metric with an alternating-sign quadratic dependence on the length of the vectors from the magnitude of its components lays in the basis of geometry, has taken root in physics. But still numerous and various attempts to connect all the known natural forces nature with the metric and make true the idea of the total geometrization of physics have failed. This drives to the idea that the reason lies not in the lack of scientists' creativity, but in the metrics itself, even better to say in the classical quadratic form, in place of which it is admittedly to use other dependences. Unfortunately, this attitude, the possibility of which indicated Riemann [1], was for the first time studied by Finsler [2], and up to nowadays used by hundreds of investigator [3], did not give eventual pictures. Though nowadays the work in this direction is continued, it considerably differs from many of them, as it is based on the idea of scalar poly-products, which is new for the Finslerian geometry, and metric form that is connected with one of the most fundamental notions in mathematics – the real number.

2. Multidimensional time .

The spaces that have unique correspondence with algebras, that are the sum of several real number algebras, stand out from Finslerian linear spaces. The metric functions do not depend on the point and in one of the bases look like:

$$F(x') = \left|\prod_{i=1}^{n} x'_i\right|^{1/n}, \tag{1}$$

where x'_i are the components of the vector and n is the number of dimensions. Such metric functions are well-known in the theory of Finslerian spaces and took the name of Berwald-Moore's function [3].

Geometries with such metrics in many ways are of the same type and the difference is related only to the dimension. The total equality of all non-isotropic directions is their main peculiarity. As any of such directions can be related to the proper time of the inertial reference frame, it is appropriate to call such spaces the *multi-dimensional time.*

Note. It seems that it is possible to relate a general line with an inertial reference frame in any linear space, where the element of the length is defined in every point. But in many spaces some reference frames do not admit the presence of isotropic connections with other lines that go in a parallel way with the given. For the viewer related to such reference frames, the existence of isotropic vectors, with which it is traditional to associate the light signals, becomes the origin to the idea of the physical distance and consequently the physical space.

The defined in this way spaces not always have the same shape as the one we got used to (in every day life and thanks to Euclid and Minkowski). At the same time we have to put a more general meaning than usually into the idea of physical space. On the other hand nothing prevent us from considering that in the sectors or dimensions, where isotropic connection is not set or have an extraordinary characteristics, that physical directions are undetectable, though representable from geometrical point of view. Consequently, it is quite logical to suppose the existence of some spaces, some parts of directions and even dimensions of which are not apparent from their physical side. From such point of view it would be interesting to analyze arbitrary linear spaces and in particular those, connected with quadratic forms and the Berwald-Moore's metrics treated over the field of complex numbers.

The chosen geometrical element of every n-dimensional time is its isotropic sub-space, that is a figure constructed from n-hyperplanes, that divide the multiformity into 2^n-equal simply connected cameras. Any of the cameras adjoins to the others, but for the facing, with which it borders in a point. The adjoining cameras can be classified according to the distinguished by the dimension of the frontier planes from 1 to $(n-1)$. All simply connected cameras are equal and have the shape of regular pyramids, n-hyperplanes of which start from the top and go to the infinity. We will call such pyramids, by analogy with isotropic cones of the Minkowski space, the light pyramids. Every *light pyramid* has n one-dimensional edges that can easily be connected with a special basis. In the basis the geometrical correlation of the multy-dimensional time appears in a vivid shape and, as such a basis is to permutation unique, it is quite natural to call it the *absolute*.

Any single vector that belongs to the inner area of a light pyramid can be continuously introduced into any other single vector that belongs to the same pyramid. The respective transformation form $n-1$-parametrical Abelian subgroup of movements, that leaves the initial metric function (1) invariant. The metrics of such transformations in the absolute basis is reduced to the diagonal shape:

$$\begin{pmatrix} a'_1 & 0 & \dots & 0 \\ 0 & a'_2 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & a'_n \end{pmatrix}, \tag{2}$$

where $\prod_{i=1}^{n} a'_i = 1$. The corresponding reflections can be classified as Hyperbolic turn (that in a way are analogous to the busts of the pseudo-Euclidean spaces) because such transformations leave on the place a point of convergence of the tops of all the pyramids and isotropic edges of the last at the same time turn into themselves. Among continuous movements of the multy-dimensional time along with hyperboloic turns there is also a n-parametrical subgroup of parallel transfers. The examined variety doesn't include any other continuous congruent transformations and that is why has less freedom than the spaces with quadratic types of metrics. The very circumstance made Helmholtz, Lee, Weyl prove a number of theorems that stated that the oneness of the quadratic metrics [4−6]. The main emphasis was made to maximum mobility in quadratic spaces. This according to them gave grounds to reject all other metric forms in the meaning of the basis of the real space-time. Let us note without rejecting the theorem accuracy that its approval is based on the examination of only the distinguished linear transformations, which means that it gives a chance to other theorems, where non-linear symmetries play the same role. In contrast to continuous congruent transformations the discrete group of symmetry of the multy-dimensional time excels the corresponding Euclidean- and pseudo-Euclidean spaces, but this is not enough to compete with the latter one. What really makes the multy-dimensional time the multy-dimensional time interesting is the presence of distinguished groups of non-linear transformations which are practically as fundamental as the groups of movements.

Such transformations save invariant not the intervals, but specific scalar forms of several vectors, that do not have direct analogous quadratic spaces, and that is why are not well-studied.

It is better to come to the understanding of such polyforms through the generalizing of the idea of the scalar product. It turns out that in a number of Finslerian linear spaces the poly-linear symmetry form of n vectors [7] (its special case is the classical bilinear form) can play the role of the scalar product. Let us call the poly-linear form the *scalar poly-product.* Founding on this generalizing we can enlarge with some Finslerian spaces such fundamental ideas of geometry as the length, the angle, the orthogonality, etc., the introduction of which is difficult due to some problems [8].

In the absolute basis the *scalar poly-product* of the multy-dimensional time looks like:

$$(\mathbf{A},\mathbf{B},\ldots,\mathbf{Z}) = \frac{1}{n!}\sum_{(i_1,i_2,\ldots,i_n)} a'_{i_1} b'_{i_2}\ldots z'_{i_n}, \quad \text{at} \quad i_j \neq i_k, \quad \text{if} \quad j \neq k. \tag{3}$$

It is not difficult to believe that with $\mathbf{A} = \mathbf{B} = \ldots = \mathbf{Z}$ the form (3) turns into the metric function (1). We can build the geometry of the linear time in an arbitrary natural scale using the poly-linear symmetrical form (3). But let us focus on this case if we base on common ideas about physical measurements and vivid typological detailedness of the four-dimensional space [9].

3. Four-dimensional time

According to (3) the scalar poly-product, that defines the four-dimensional time, in the absolute basis looks like:

$$(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) = \frac{1}{4!}\sum_{(i_1,i_2,i_3,i_4)} a'_{i_1} b'_{i_2} c'_{i_3} d'_{i_4}, \qquad \text{when} \quad i_j \neq i_k \text{ if } j \neq k, \tag{4}$$

it follows that the fourth degree of the vector length of such linear space is defined by the expression:

$$(\mathbf{X},\mathbf{X},\mathbf{X},\mathbf{X}) = |\mathbf{X}|^4 = x'_1 x'_2 x'_3 x'_4. \tag{5}$$

While turning to the basis analogous to the orthonormalized [7] (it is more visual than in the absolute case) the expression transforms into a more complicated but still symmetrical form:

$$|\mathbf{X}|^4 = x_1^4 + x_2^4 + x_3^4 + x_4^4 - 2(x_1^2x_2^2 + x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2) + 8x_1x_2x_3x_4. \tag{6}$$

In a number of cases it is more convenient to use the form picking out one of the coordinates, in particular x_1:

$$|\mathbf{X}|^4 = x_1^4 - 2(x_2^2 + x_3^2 + x_4^2)x_1^2 + 8(x_2x_3x_4)x_1 + (x_2^4 + x_3^4 + x_4^4 - 2x_2^2x_3^2 - 2x_2^2x_4^2 - 2x_3^2x_4^2). \quad (7)$$

The main arguments in favor of the chance of confronting the four-dimensional time to the real physical world is the presence of a group of continuous symmetries [10], that can be examined as an alternative to the linear group of spatial turning of the Minkowsky space. Not a scalar poly-product of the four-dimensional time (4) is an invariant to the transformations, but a specific form, that is defined by 2 vectors:

$$S(\mathbf{A}, \mathbf{B}) = \frac{(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{B})}{(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A})^{1/2}} + \frac{(\mathbf{A}, \mathbf{B}, \mathbf{B}, \mathbf{B})}{(\mathbf{B}, \mathbf{B}, \mathbf{B}, \mathbf{B})^{1/2}}. \quad (8)$$

Though the form $S(\mathbf{A}, \mathbf{B})$ is not an additive quantity of the vectors that belong to the interior of domain of a light pyramid, it complies with other very important characteristics of the common scalar product, to be more specific: the symmetry, the rule of multiplication by the vector, the sign distinctness and the triangle rule [10]. According to this there exists a principal opportunity in the four-dimensional time to introduce the idea of the three-dimensional distance, that corresponds to most of common conceptions of the physical quantity, but for the additivity. From philosophical point of view the last characteristic is very important. No, really, why should the rule of composition differ from the one of three-dimensional distances, as both values are relative? Such linearity appears only when we work with big distances, as well as the non-linearity of the rule of speed composing is essential only in the relativist field. At the same time an additional fundamental constant – the maximum possible magnitude of the physical system, or, in other words, the radius of the Universe, acts as the light speed in the three dimensional distance. For everyday distances we can still use the linear approximation, but in the space scale, in case of logical appliance of the multy-dimensional time conception, certain corrections should be made.

4. Plenty of relatively simultaneous events

We should first of all clarify the situation about a number of simultaneous events to give the definition of the four-dimensional time, three-dimensional speed and distance. Let us understand under it the total of points equidistant (of course in the meaning of the accepted Finslerian metrics (5)) from a pair of fixed events. In contrast to the Minkowskian space, where a multitude of points constitute hyperplanes, in the four-dimensional time the corresponding planes are non-linear [10]. Their form depends not only on the direction of the world line, that connects the fixed points, but also on the magnitude of the interval that separates them. This is the most fundamental difference from the space of the Special Theory of Relativity, as the idea of simultaneosity is defined now not only by the speed of the reference frame, but also by the interval of time that separates the instantaneous position of the observer and the examined spatial layer of events. So the relativism in the four-dimensional time touches upon not only the hyperbolic turns, with the help of which realizes the switch between one system to another, but also the transmission, that enables to change the reference point.

From philosophical point of view such generalization is quite logical, but in fact establishes a sort of relationship between the two subgroups of the total group of congruent symmetries. As an indirect affirmation of the made conclusion can serve the fact that in algebra transmissions lack the operation of composition, which are a part of the four-dimensional time, and hyperboloic turnings - multiplication, and mathematics do not question relationship between them. A natural way of introducing the idea of the physical distance in the four-dimensional time is offering a method that from conceptual point of view is analogous to the method of defining of the idea in the Minkowskian space. By definition under distance we can understand a value that equals (or is proportional) the tie intervals, that go along the world line of the observer, between sending some uniformly moving model signals to the world lines of the examined objects, and receiving the reflected signals. It leads to the fact that it is senseless to use the idea of distance towards single events in the four-dimensional time, and is productive concerning only chains of them, that are presented by certain lines. We can pay no attention to the fact in the Minkowskian space, as multitudes regarding simultaneous events are hyperplanes, as a result the distance defined for an arbitrary pair of parallel lines were still substantial and for a pair of points.

Not to overload the brief article with excessive community, but at the same time to be rather specific, we will give the result to which the described above algorithm drives only in one case - when the world line of the observer coincides with the real axis, it itself is situated at the point $(T, 0, 0, 0)$ and the necessary layer goes through the point $(0, 0, 0, 0)$ (Fig. 1) [Here and later on the appearing coordinates relate to the generalized orthogonal basis [7] that differs tremendously from the absolute].

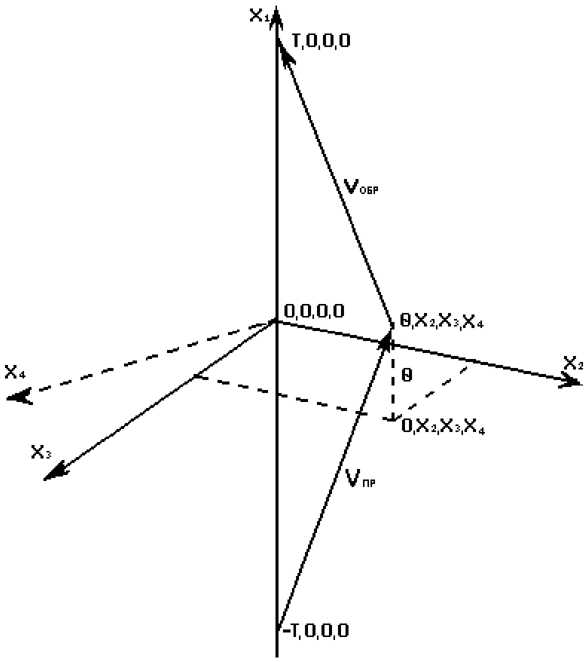

Figure 1: The world lines of direct and opposite signals with speed module

In this case the equalization, that relates the real coordinate θ of a point of the plane simultaneity to three other coordinates x_2, x_3 and x_4, follows from the rule of equality of the vector length that have the following components $(T + \theta, x_2, x_3, x_4)$ and $(T - \theta, -x_2, -x_3, -x_4)$. (Variable θ means deviation of concrete point from hyperplane $x_1 = 0$.) Using the expression for the magnitude of the interval (7) and at the same time concerning that for even degrees $(-x)^n = x^n$, we have:

$$(T+\theta)^4-2(x_2^2-x_3^2+x_4^2)(T+\theta)^2+8(x_2x_3x_4)(T+\theta)+(x_2^4+x_3^4+x_4^4-2x_2^2x_3^2-2x_2^2x_4^2-2x_3^2x_4^2) =$$
$$(T-\theta)^4-2(x_2^2+x_3^2+x_4^2)(T-\theta)^2+8(x_2x_3x_4)(T-\theta)+(x_2^4+x_3^4+x_4^4-2x_2^2x_3^2-2x_2^2x_4^2-2x_3^2x_4^2).$$

Opening the brackets and collecting terms we get:

$$T\theta^3 + (T^2 - x_2^2 - x_3^2 - x_4^2)T\theta + 2x_2x_3x_4T = 0. \tag{9}$$

introducing sizeless value $\eta = \theta/T, \chi_2 = x_2/T, \chi_3 = x_3/T, \chi_4 = x_4/T$ and taking into consideration that $T \neq 0$ we get a cubic equalization relatively to η:

$$\eta^3 + (1 - \chi_2^2 - \chi_3^2 - \chi_4^2)\eta + 2\chi_2\chi_3\chi_4 = 0. \tag{10}$$

Its real root characterizes the relative value of deflection of the simultaneity plane absciss from the coming through its center according to the hyperplane $x_1 = 0$. We will call such parameter the *coefficient of non-platitude.* When $\chi_2 \approx \chi_3 \approx \chi_4 \to 0$, η also stems to 0, we mean around the point $(0, 0, 0, 0)$ the plane of the simultaneity turns into the hyperplane $x_1 = 0$.

The plane of simultaneity has physical meaning only inside the light pyramide, that has the world line of the observer, in other case it would be necessary to admit the physical meaning of the superlight speed. Following the method of the Special Theory of Relativity, with every vector that start at $(-T, 0, 0, 0)$ and ends at the plane of simultaneity, or in other words at $(\eta T, x_2, x_3, x_4)$ it would be quite natural to connect the world line of the signal, that has a definite uniform speed. We will transform the signals of the vectors, if they have equal interval values, according to the value of the speed module: $|\mathbf{V}_{\text{dir}}|$. Logically the signal, that is confronted to the vector, connecting the points $(\eta T, x_2, x_3, x_4)$ and $(T, 0, 0, 0)$, has the value that is inverse to the speed $|\mathbf{V}_{\text{rev}}|$. On contrast to the Minkowskian space such vectors have components that differ not only in sign but also in value (Fig. 1), to be more specific: $\mathbf{V}_{\text{dir}} \leftrightarrow (\eta T + T, x_2, x_3, x_4)$ and $\mathbf{V}_{\text{rev}} \leftrightarrow (T - \eta T, -x_2, -x_3, -x_4)$. In the Minkowskian space the coefficient of the non-platitude η for every point of the plane of the simultaneity equals 0, as the result the components of the vectors that correspond to direct and inverse signal look like: $\mathbf{V}_{\text{dir}} \leftrightarrow (T, x_2, x_3, x_4)$ and $\mathbf{V}_{\text{rev}} \leftrightarrow (T, -x_2, -x_3, -x_4)$.

To give a definition of distance between the real axis and an arbitrary line parallel to it, which is totally defined by 3 fixed coordinates x_2, x_3, x_4, we should have a model signal, or even better to say vectors related to it, with the help of which it is possible to make intervals that would equal the distance of different directions. As well as in the space of the Special Theory of Relativity, in the four-dimensional time it is more convenient to relate such symbol signals to isotropic vectors, that at one end have the same beginning and from the other - they set against the plane of simultaneity. In the Minkowskian geometry a number of ends of such vectors represent an intersection of two light cones: the future with the top at point $(-T, 0, 0, 0)$ and the past whose top is deposed to $(T, 0, 0, 0)$. As is well known the result of such interception is a common sphere, that lies completely in the hyperplane $x_1 = 0$. This is typical only for spaces with a quadratic metric type. In any case in the fur-dimensional time an analogous figure that is the result of interception of two facing light pyramids, is not plane though consists of linear elements.

Tit is better to make sure of it using the three- and four-dimensional time [12] as the example, in particular looking at Fig. 2 where it is demonstrated the interception of two light pyramids. For comparison, an interception of two light cones of the three-dimensional pseudo-Euclidean space is demonstrated on the same picture. In the three-dimensional time the interior of domain, that belongs to either of the pyramids, is a common cube, one diagonal of which is a segment of the real axis $[-T, T]$. At the same time the interception of two light pyramids results in a figure, built from $(n - 2)$ edges of such cube, excluding the points $-T$ and T. In this case this is a hexagon $ABCDEF$ and it does not belong to the plane $x_1 = 0$, though compiles one of it rectilinear elements.

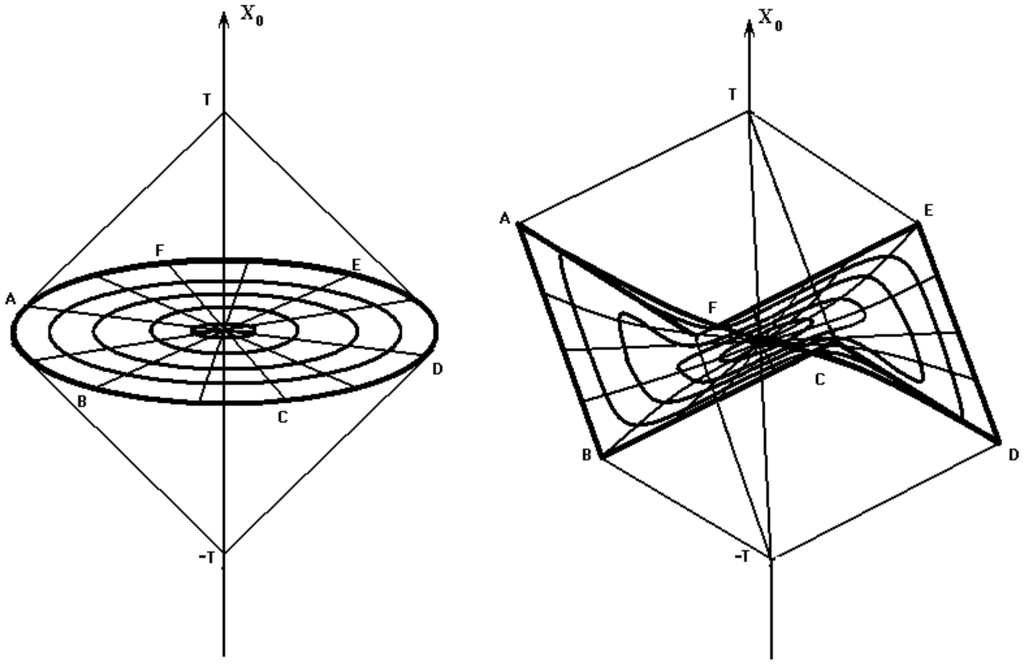

Figure 2: The simultaneous surface of three-dimensional time (right) and in three-dimensional pseudo-Euclidian space (left)

It is analogous in the four-dimensional time: the area that belongs to two facing light pyramids is a four-dimensional cube and the plane of the interception of their isotropic edges is built by 20 2-edges of the cube, that do not include the main diagonal $[-T, T]$.It is difficult to demonstrate this figure using a plane scheme that is why we will limit to the examined above a three-dimensional prototype. In the work [13] there was made an attempt to examine the corresponding dodecahedron (but it seems that the author has lost its principle four-dimensional character and depicted it as a common three-dimensional figure).

In the Minkowskian space the world lines that are parallel to the world line of the observer and touch the figure, which is the interception of two light cones, are accepted as equidistant points of the physical space of the observer, and the value proportional to the axis length of such double cone is referred as the distance. We can act in the analogous way in the four-dimensional time. In this case the parallel to the real axis lines, that come through the point of interception of the edges of two facing light pyramids, become equidistant from it, and in the role of the distant act the value that proportional to the main diagonal of the hypercube that is the result of such interception. In order to find the numerical value of it we should choose 2 real roots from the equalization:

$$x_1^4 - 2(x_2^2 + x_3^2 + x_4^2)x_1^2 + 8(x_2 x_3 x_4)x_1 + (x_2^4 + x_3^4 + x_4^4 - 2x_2^2 x_3^2 - 2x_2^2 x_4^2 - 2x_3^2 x_4^2) = 0, \quad (11)$$

which are nothing but the abscises of the interception point of the line, which is related to the coordinates x_2, x_3, x_4, and 4 isotropic hyperplanes. One of the roots $x_{1,1}$ corresponds to the point that belong to the pyramid of the past, another $x_{1,2}$ - to the future, as the other 2 redundant roots $x_{1,3}$ and $x_{1,4}$ belong to the edges of the plane of the side pyramids. In this case we can consider the distance to be half of the sum of the first 2 roots: $R_c = 1/2(x_{1,1} + x_{1,2})$, while the index "c" emphasizes that the value is defined by light signals.

The three-dimensional space that appears as the result of such procedure is the Finslerian and is characterized by its indicatrix whose role plays the described above

[13] dodecahedron. The space in its characteristics is quite close to the Euclidean, it comes from the convexity and two-dimensional restraint of its indicatrices, that does not differ greatly from the indicatrix of the Euclidean space, which is a common sphere. But the difference between the Euclidean sphere and the examined dodecahedron is rather principle to mix up their geometries. That is why there was made a conclusion in the work [13] that the idea that in the basis of the geometry of the real macro-world lies the four-dimensional time metrics. But still we think that while making the conclusion one very important circumstance, that when orientating in the real space the observer uses much slower signals rather than the light ones, was not taken into consideration. The light only helps, it is to identify the objects, as the comparison of their distances is realized by other slower means. The fact was not important in the Special Theory of Relativity as the indicatrix of the physical space did not depend on the speed of the signal. It is not like this in the multy-dimensional time. The more the relative speed of the probing signals differs from the light, the less the corresponding indicatrix distinguished from the hyper-plane, the more round become its angles and the more it looks like the three-dimensional sphere. At the limit when the relative speed of signals, with the help of which the physical space is examined, stems to 0, it stops being different from the Euclidean. So if we detect some static objects in the four-dimensional time with the help of the light, and define the distance with the help of other slower signals, so in this case we will come upon only the Euclidean geometry. Let us note that the very condition is complied in the vast majority of common for a man situations.

On the other hand it is not questioned that there is a principle opportunity to carry out an experiment in order to get to know which geometry better suits the real physical space – the Riemannian or the Finslerian. In this case it is important that the distance between fixed objects should be made by other light or slower signals. It is paradoxical but such experiments that do not accept double interpretation lack among the huge number of experimental materials. But the differences that should be traced are not large and that is why can be explained in different ways.

The above accepted conception of building the three-dimensional time explains why in absolutely equal in geometrical rights coordinates of the four-dimensional time the observer, associated with a world line, will register a significant difference between the coordinate that relate to his proper time and the other three. The answer lies in the topological difference between indicatrices of the geometrical and physical spaces. So if the first has the look of a specific 16-line hyperboloid, the second is a ring closed in two dimensions, its right form though depends on the used in measurements signals, is static from topological point of view.

5. Conclusions

Forms that save the scalar form (8), do not leave the intervals invariant, and tot ell the truth are not movements of the four-dimensional time. But as they turn the hyper-planes of the simultaneity (10) into themselves and do not change the three dimensional distances R_c they can act as common physical turns. There can emerge an explanation of the famous paradox - between the forward and rotatory movement. It is difficult to use the principle of relativity to the latter case, and the most famous attempt to examine it was made by Mach, who thought that the centrifugal forces owe their existence to the enormous mass of all the bodies in the Universe. According to Mach if we start turning the whole Universe a static small body will be affected by the centrifugal force that equals the force that emerge during the turning of the body itself. For many people it stays unclear the truth of the statement, and the question itself is still acute. In case we correspond to the real world in place of the Galileo or Pseudo-Euclidean metrics the geometry of the four-dimensional time the problem itself will not appear as the transformation that is responsible for the forward and rotatory movement, correspond to absolutely different continuous symmetries.

The analysis of the multiformity characteristics made in the work that claims to become an alternative to the Minkowski space is far from being finished. But the fact that we can give such condition for one of the most simple Finslerian metrics of the fourth degree that has nothing in common with the usual quadratic form, when it can stimulate not only classical but relative conceptions about the physical space, is worth paying attention to.

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FINSLEROID–SPACE SUPPLEMENTED BY ANGLE AND SCALAR PRODUCT

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The science of the past century has achieved great success on the basis of the geometrical quadratic concepts that were followed as logical and mathematical primaries. More profound ideas will imply using a more capacious class of geometries, for example the Finsler one which inscribes structures because the Finslerian indicatrices are no more isotropic in all directions. In the present work an attempt is made to resolve the respective difficulties of Finsler generalization by choosing the particular Finsleroid–type metric that implies one preferred direction, admitting the total axial symmetry around it. In this case, interesting constructive methods of introducing the concept of the angle and scalar product outside the frame of the Euclidean Geometry can conveniently be opened up.

"The Euclidean traditions are too strong to be rejected, and probably few generations of mathematicians are necessary to work off its influence." (Busemann [2], p. 8.)

Introduction

The quadratic method is the most convenient one to introduce the vector length. According to the method the length is defined by means of the square root of the quadratic form. For more than 20 centuries the Euclidean geometry and Euclidean rotations based on it have been served in theoretical constructions and predictions of results of experiments. The non-quadratic methods are developed in the Finsler geometry (see [1 – 6]).

Unfortunately, we must admit that much attention has not been paid in literature to studying the corresponding opportunities. By tradition the mathematical and theoretical physical concepts and equations are based on the method of introducing the vector length by the help of square root. And numerous interesting and deeply critical analysis (see, e.g., [7, 8]) of the geometrical structure of the space–time and methods of its generalization and comprehension usually go without even mentioning the existence of ideas and methods of the Finsler Geometry. In spite of high level of adequacy and accurate coincidence, it is still not clear how it is possible to express this degree of accuracy in numbers, for the Euclidean rotations do not possess a small parameter to evaluate.

In comparison with the common Euclidean metrics the Finslerian one introdudces the structure in metric geometry. While the unit surface of the Euclidean Geometry is a sphere that is isotropic in all directions, the introduction of geometrically preferred directions leads to generalizing the sphere and finally to generalizing the Euclidean Geometry. The corresponding, not isotropic, surface of the ends of the unit vectors (when issued from a fixed point) generates the Finsler metrics. Respective geometries can reflect the physical cases where the corresponding directional anisotropy is present. The Berwald–Moor metrics is totally anisotropic, for it supposes geometrically–emphasized directions whose number equals the number of manifold dimensions (accordingly, 3 in the three–dimensional case and 4 in the four–dimensional case). The Finsleroid–Geometry introduces only one preferred direction, supposing the total axial symmetry around it.

Actually, the task of generalizing the Euclidean metric function to the Finsler case seems to be too general and rather unclear to give a definite answer. But if we treat the problem from point of view of invariance and the possibility of introduction of the angle and scalar product, then we can endeavor to find constructive ways of defining the classes of Finsler spaces. As a result, there may emerge the methods of abandoning the Euclidean geometry.

Of course, no matter how motivated our desire to leave the borders of “quadratic conceptions” is, it is impossible to “overcome the square root completely”. The hierarchy of geometries takes its root in generalizing. It is clear that methods and ideas of the Euclidean geometry are present and work in the Riemann geometry. Many authors of works on the Finsler Geometry used “the associative Riemann Geometry”, introduced “the Riemann connection” or “the Finsler–Riemann connection”, introduced “associative relative Riemann geometry along the vector fields”, constructed “osculating Riemann space”, and “the Riemann development of the Finsler space along the curve”, etc. The mathematicians applied steadily the associative Euclidean geometry in the Minkowski spaces.

Any theory that abandons the concepts dictated by the quadratic form has the shape of a pyramid: going down to the basis of the “unique super-geometry” the researcher must enter the area of “the associative Finsler Geometry”, where in its turn appear different Riemann images, and then numerous Euclidean pictures.

The above facts are directly related to the Quadra-number geometry (developed recently in the work by Pavlov [9, 10]). In fact, it appears from examining the commutative hyper-numbers and relates the standard to them. By interpreting the component of the hyper-numbers as the component of the vector this metrics can be related to the type of “Berwald–Moor’s Finsler metric function”. Basing ourselves on the last case we can (and must) develop the theory of geometric correlations, including the introduction of the geodesic angle, perpendicularity,... – that do not coincide with analogous geometrical juxtaposition of the Riemann or Euclidean Geometry. Particularly, we cannot reject the latter one because we use graphic presentations and pictures, at least we have to simulate and construct them in the Euclidean space!

At the same time, this does not mean that the Finsler geometrical properties are prescribed uniquely to the Quadra–space. In fact, according to its own capacities, the poly–form theory makes it possible to introduce the corresponding angles and perpendicularity; in particular, such a generalization of the theory of “higher degree of metrics” was made in the works [9, 10].

Obviously, the Minkowski geometry has more invariants than the Euclidean one, and the Finsler aprroach – more than the Riemann one. In such context we should indicate that the Quadra–spaces have much more invariant objects, than the Finsler or Minkowski ones, and can offer a theory which is richer in geometrical concepts. In particular, this can be seen in the fact that the Euclidean geometry can be easily associated with the Quadra–space in many ways.

Philosophy and logic of associated problems. We can hardly overestimate the importance of Euclidean approaches and the fact that the Euclidean Quadratic geometry has already built up and keeps on building up the way of thinking and analysis of many scientists and researchers. For example, the Riemann geometry since its definition is based on the quadratic form (sometimes it used to be called “the geometry created by the quadratic form”), the theory of bundle spaces also applies the quadratic method (but it is more multifarious than the Riemann geometry), the Lagrangians in theories of physical fields are usually quadratic with respect to derivatives, the energy and impulse of the relativistic particle are connected by a quadratic form, etc. The Special and General Theory of Relativity are also based on the quadratic forms, but now possess pluses $\langle + \rangle$ as well as minuse $\langle - \rangle$ in the signature; the Lobachevski geometry is also related to the type.

Nowadays there are many books on geometry, where quite often different "models" of generalized geometries are presented and studied. In contrast to this the Geometry, and not "a model of geometry", is presented in Euclid's work.

Why the Euclidean geometry has lived through 2 millenniums? The reason is that the square root of a quadratic form is used to define length and vectors. We can come across this method everywhere: in practice, in mathematical and physical theories, and in experiments; it is also used nowadays. Logically it is the simplest way. But "the simplest" is not always "the most precise".

"The axiomaticians" during the last century have been analyzing the structure of the Euclidean geometry (remind Hilbert's famous work The Foundations of Geometry), and not the ways of constructive generalizing of the "quadraticity" of the Euclidean metrics.

It is quite easy to question any statement that declares about "high experimental accuracy" of the quadratic method of establishing the length. Has anyone and with what accuracy checked the Pythagorean theorem? Such check is hardly possible without the researcher using more general methods for comparison (profound research of the topic is out of the aim of the work, the readers may try to carry out their own analysis)

In fact, the Euclidivity of the geometry or its models is preserved till preserves the quadraticity of the definition of length. But we need something more than just courage to make the corresponding decisive step. This is a difficult task: we must find a good way to change the quadratic method of defining the length by a more general one and recast the equations of mathematical physics on the basis of the method in order to abandon the "Euclidivity". And this is a good task for the scientists of the new millennium. The conservative way of thinking as an obstacle in the way of geometrical progress can be effective only during a very short period of time.

The Length is the fundamental concept either in theoretical or applied science. We can compare it only with the concept of Number in its fundamentality. The development and application of the concept of the length have lead to creation of Geometry, and the concept of number – to Algebra.

Using the theory of the so–called Minkowski Space (it is also called Minkowski geometry) we can formulate quite a general and modern attitude. In the modern accurate mathematical language the Minkowski Space is often defined as the Finite–Dimensional Banach Space.

In the Minkowski spaces the length is introduced by the general definition that enables it to be defined by functions of a rather wide range of classes with minimum conditions on smoothness. The fibered manifold, where the fiberes are Minkowski spaces, are called the Finsler spaces.

During the last century many scientists have been studying the Minkowski geometry and Finsler geometry. More than 2000 works and a number of monographs have been published, but we should be very cautious while speaking about the achieved success. It is inevitable that we come across a large number of tensors in the Finsler geometry (that do not have non-trivial prototypes on the Riemann geometry), and it is not obvious that such numerical growth predetermines qualitative leap. By the latter case the Finsler geometry have spurned many mathematicians as it seems to be extremely difficult to study because of the great number of tensors (in comparison to this the Riemann geometry is quite economical: there is a metrical tensor, one set of coefficients of association and one curvature tensor).

But we should not be too pessimistic about the inconvenience of formalism. Especially nowadays, that there are few people that will be surprised by the multy–componentry of the objects neither in mathematics nor in theoretical physics. It is likely that the problems lie in another level, and to be more precise in the lack of definite key links. Here we can recall Busemann's remark, that the progress in motion should consist "not of the generalizing of the Riemann geometry, but of its results as well".

Should the development of the concept of "the Length" be in connection with the development of the concept of "the Number"?

If we turn to prehistory of the Euclidean geometry, to Pythagoras's activity, then we will learn about his tragedy when he learnt that the diagonal of the square is rationally not commensurable with the length of its sides. So, for Pythagoras it was a real catastrophe that the number did not correspond to the length. This "surprise" gave an impulse to development of the concept of the Number, and to be more precise, to creation of the theory of the irrational number. The developed correspondence between the Length and the Number made the basis of the Euclidean geometry and moreover of its axiomatics (for example suggested by Hilbert). In this regard the axiomatic of the Euclidean geometry developed by Hilbert was the culmination of the identity of the concept of arithmetical number and quadratic length, many geometric key concepts have been derived from arithmetical numerical properties.

The following move from the Euclidean geometry to the Riemann one does not add any new ideas to the dichotomy. The Riemann geometry is just "a fibration of the Euclidean geometries", so that in every level there works the Euclidean geometry and the common definition of length is used.

The Minkowski geometry abandons the definition of the length, but (though, as is well known, Minkowski started thinking about geometry while studying the theory of numbers) develops the problem without any connection with the concept of number. Pythagoras's tragedy does not matter any more! We can say the same about the modern Finsler spaces, that are just fibrations of the levels in the Minkowski space.

Such excursus into history enables us to show enough courage to state the following: *we should build the Finsler geometry in close connection with the development of the concept of the Number.*

We can hope that this idea will be principle for the successive development of the Finsler geometry in the present century. We should know on what level does the Finsler generalizing of the Length is needed to generalize the concept of the Number. The answer is not clear, though the reversed way of thinking is obvious: the non-Euclidean, not quadratic Finsler metric function should be the measure of the generalized number.

There emerges a very important question: where does poly-numbers are crucial in the Finsler geometry, so that you can do nothing without them? Pythagoras's tragedy is clear: the rational numbers are not enough to measure the length of the unit square diagonal. The origin of the transcendental numbers is also clear: the unit circumference diameter cannot be measured by the algebraic irrational number.

The anisotropy is presumed in generalizing the Euclidean geometry. In this connection the indicatrix becomes the key concept: it is the surface of ends of the unit vectors that issue at a fixed point. In the Euclidean geometry the sphere is the indicatrix. It symbolizes the isotropy of the space, equality of its properties in all directions. As the uniformity condition appears in the definition of the Minkowski space and the Finsler space, the indicatrix proves anisotrophy of any vectors (not necessarily the unit ones). The move from The Euclidean geometry to the Minkowski one symbolizes refusal from the total isotropy of the space and after the move the corresponding indicatrix cannot be a second-order surface any more.

From the point of view of anisotropy, The Berwald-Moor metrics is characterized by the presence of preferred directions, that in their number equal the number of dimensions of the space.

In the present work the necessary basic definitions and results of calculations of the associated values for the Finsleroid-geometry ($\mathcal{E}_g^{PD}$–geometry), that admit only one preferred direction are presented. Our previous investigation [5,6] showed that the study is promising. In fact, $\mathcal{E}_g^{PD}$-approach is applied to the development of new study of new field in the metric differential geometry and can be effective in Finsler and Minkowski geometries. The observation that the one-vector Finsler metric function associated with $\mathcal{E}_g^{PD}$-space quite naturally admits the promising two-vector generalizing , in this way generating the angle and the scalar product, is the key point of the article.

Attempts to introduce the angle and into the Finsler and Minkowski spaces always striked against the ambiguity:

"Therefore no particular angular measure can be entirely natural in Minkowski geometry. This is evidenced by the innumerable attempts to define such a measure, none of which found general acceptance". (Busemann [2], p. 279.)

"Unfortunately, there exists a number of distinct invariants in a Minkowskian space all of which reduce to the same classical euclidean invariant if the Minkowskian space degenerates into a euclidean space. Consequently, distinct definitions of the trigonometric functions and of angles have appeared in the literature concerning Minkowskian and Finsler spaces". (Rund [3], p. 26)

The fact that the attempts have never been unambiguous seems to be due to a lack of the proper tools. For the opinion was taken for granted that the angle ought to be defined or constructed in terms of the basic Finslerian metric tensor (and whence ought to be explicated from the initial Finslerian metric function). Let us doubt the opinion from the very beginning. Instead, we would like to raise alternatively the principle that the angle is a concomitant of the geodesics (and not of the metric function proper). The angle is determined by two vectors (instead of one vector in case of the length) and actually implies using a due extension of the Finslerian metric function to a two-vector metric function (to a scalar product). Below, the principle is applying to the Finsleroid space in a systematic way. The essence of the generalizing can be visualized in deformation of the Euclidean sphere (which is the indicatrix of the Euclidean space).

We devote the section 1 to geodesic equations. Remarkably, the equations admit a simple and clear solution. Then we can find the angle between two vectors. Usually it is expected that the angle measure should be additive (for the angles with the same vertex). The angle differs from the Euclidean angle in the quasi–Euclidean space only by the constant factor and consequently is additive. The cosine rule is held true when changing the Euclidean angle by the found angle. We get the corresponding scalar product.

Formally, the method of introducing the vector length with the help of the square root of a quadratic form lies in the basis of Euclidean conception. In the present work we use the concrete axial–symmetric generalizing of such method, basing ourselves on constructive ideas of the Finsler geometry. We introduce the corresponding Finsler metric function and in detail describe its basic properties and consequences. The generalizing is characterized by one non–dimentional parameter, that is denoted below as g.

Then the section 2 introduces designations, definitions and basic concepts of the space $\mathcal{E}_g^{PD}$. On this fact the supposition that the space includes one emphasized direction, that we will often call the Z axis, is based. The abbreviations FMF and FMT will be used to denote the Finsler metric function and Finsler metric tensor accordingly. The characteristic parameter g can take the value between -2 and 2; if $g = 0$ the space $\mathcal{E}_g^{PD}$ is driven to the common Euclidean space. After preliminary introduction of characteristic quadratic form B, that differs from the Euclidean sum of squares by the presence of the mixed term (see (2.22)), we define FMF K of the space $\mathcal{E}_g^{PD}$ with the help of the formula (2.30) – (2.33). The characteristic feature of the formula is the presence of the function "arctan". Then we calculate the tensorial values of the space. There is a phenomenon that simplifies the construction: the associated Cartan tensor that turns out to have a simple algebraic structure (2.66) – (2.67). In particular this unique phenomenon leads to the conclusion that the indicatrix of the space $\mathcal{E}_g^{PD}$ is a space of constant positive curvature. The curvature value depends on the parameter g according to the rule (2.73).

The section 3 introduces the concept of quasi–Euclidean reflection of the $\mathcal{E}_g^{PD}$–space. The concept turns out to be quite promising because the quasi–Euclidean space is simple in many aspects, so that the corresponding transformation simplifies different calculations. It is not flat, but is conformally flat. The section 4 gives idea about some interesting properties of the quasi–Euclidean metric tensor. Figures that illustrate the Finsleroids with different values of the parameter g are placed in the Appendix.

1. Derivation of geodesics and angle in associated quasi–euclidean space

For the space under study, the geodesics should be obtained as solutions to the equation

$$\frac{d^2R^p}{ds^2} + C_q{}^p{}_r(g;R)\frac{\partial R^q}{\partial ds}\frac{\partial R^r}{\partial ds} = 0 \tag{1}$$

which coefficients $C_p{}^q{}_r$ are given by the list placed at the end of Sec. 2. To avoid complications of calculations involved, it proves convenient to transfer the consideration in the quasi–euclidean approach (see Secs. 3 and 4). Accordingly, we put

$$\sqrt{g_{pq}(g;R)dR^pdR^q} = \sqrt{n_{pq}(g;t)dt^pdt^q} \tag{2}$$

and

$$R^p(s) = \mu^p(g;t^r(s)) \tag{3}$$

together with

$$\frac{dR^p(s)}{ds} = \mu^p_q(g;t^r(s))\frac{dt^q(s)}{ds}, \tag{4}$$

where $\mu^p(g;t^r)$ and $\mu^p_q(g;t^r)$ are the coefficients given, respectively by Eqs. (3.14) and (3.38) – (3.40). Let a curve C: $t^p = t^p(s)$ be given in the quasi-euclidean space, with *the arc-length parameter* s along the curve being defined by the help of the differential

$$ds = \sqrt{n_{pq}(g;t)dt^pdt^q}, \tag{5}$$

where $n_{pq}(g;t)$ is the associated quasi-euclidean metric tensor given by Eq. (3.49) in Part II. Respectively, *the tangent vectors*

$$u^p = \frac{dt^p}{ds} \tag{6}$$

to the curve are unit, in the sense that

$$n_{pq}(g;t)u^pu^q = 1. \tag{7}$$

Since $L_p = \partial S/\partial t^p$, we have

$$L_p u^p = \frac{dS}{ds}. \tag{8}$$

Here, $S^2(t) = n_{pq}(g;t)t^p t^q = r_{pq}t^p t^q$ (see Eq. (3.46)). Using Eq. (4.16) leads through well-known arguments to the following equation of geodesics in the quasi–euclidean space:

$$\frac{d^2\mathbf{t}}{ds^2} = \frac{1}{4}G^2\frac{\mathbf{t}}{S^2}H_{pq}u^p u^q, \tag{9}$$

where $H_{pq} = h^2(n_{pq} - L_p L_q)$ (see Eq. (4.4)) and $\mathbf{t} = \{t^p\}$. We obtain

$$\frac{d^2\mathbf{t}}{ds^2} = \frac{1}{4}g^2\frac{\mathbf{t}}{S^2}\left(1 - (\frac{dS}{ds})^2\right) = \frac{1}{4}g^2(a^2 - b^2)\frac{\mathbf{t}}{S^4} \tag{10}$$

and

$$\frac{d^2\mathbf{t}}{ds^2} = \frac{1}{4}g^2(a^2 - b^2)\frac{\mathbf{t}}{S^4} \tag{11}$$

with

$$S^2(s) = a^2 + 2bs + s^2, \tag{12}$$

where a and b are two constants of integration.

If we put

$$S(\Delta s) = \sqrt{a^2 + 2b\Delta s + (\Delta s)^2} \tag{13}$$

and

$$\mathbf{t}_1 = \mathbf{t}(0), \qquad \mathbf{t}_2 = \mathbf{t}(\Delta s), \tag{14}$$

then we get

$$a = \sqrt{(\mathbf{t}_1\mathbf{t}_1)} \tag{15}$$

and

$$S(\Delta s) = \sqrt{(\mathbf{t}_2\mathbf{t}_2)} \tag{16}$$

together with

$$(\mathbf{t}_1\mathbf{t}_2) = aS(\Delta s)\cos\Big[h\arctan\frac{\sqrt{a^2 - b^2}\,\Delta s}{a^2 + b\Delta s}\Big]. \tag{17}$$

Here, $\mathbf{t}_1$ and $\mathbf{t}_2$ are two vectors with the fixed origin O; they point to the beginning of the geodesic and to the end of the geodesic, respectively. The notation parenthesis couple (..) is used for the euclidean scalar product, so that $(\mathbf{t}_1\mathbf{t}_1) = r_{pq}t_1^p t_1^q$, $(\mathbf{t}_1\mathbf{t}_2) = r_{pq}t_1^p t_2^q$, and r_{pq} is a euclidean metric tensor; $r_{pq} = \delta_{pq}$ in case of orthogonal basis; δ stands for the Kronecker symbol. From (1.15)-(1.17) it directly follows that

$$\frac{\sqrt{a^2 - b^2}\,\Delta s}{a^2 + b\Delta s} = \tan\Big[\frac{1}{h}\arccos\frac{(\mathbf{t}_1\mathbf{t}_2)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\Big]. \tag{18}$$

The equality (1.18) suggests the idea to introduce

DEFINITION. *The $\mathcal{E}_g^{PD}$-associated angle* is given by

$$\alpha \stackrel{\text{def}}{=} \frac{1}{h}\arccos\frac{(\mathbf{t}_1\mathbf{t}_2)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}, \tag{19}$$

so that

$$\alpha = \frac{1}{h}\alpha_{euclidean}. \tag{20}$$

Such an angle is obviously *additive*:

$$\alpha(\mathbf{t}_1,\mathbf{t}_3)=\alpha(\mathbf{t}_1,\mathbf{t}_2)+\alpha(\mathbf{t}_2,\mathbf{t}_3). \tag{21}$$

Also,

$$\alpha(\mathbf{t},\mathbf{t})=0. \tag{22}$$

With the angle (1.19), we ought to propose

DEFINITION. Given two vectors $\mathbf{t}_1$ and $\mathbf{t}_2$, we say that the vectors are $\mathcal{E}_g^{PD}$-*perpendicular,* if

$$\cos\left(\alpha(\mathbf{t}_1,\mathbf{t}_2)\right)=0. \tag{23}$$

Since the vanishing (1.23) implies

$$\alpha_{quasi-euclidean}(\mathbf{t}_1,\mathbf{t}_2)=\frac{\pi}{2}, \tag{24}$$

in view of 1.20) we ought to conclude that

$$\alpha_{euclidean}(\mathbf{t}_1,\mathbf{t}_2)=\frac{\pi}{2}h\le\frac{\pi}{2}. \tag{25}$$

Therefore, vectors perpendicular in the quasi-euclidean sense proper look like acute vectors as observed from associated euclidean standpoint.

With the equality

$$(\sqrt{a^2-b^2}\,\Delta s)^2+(a^2+b\Delta s)^2\equiv a^2S^2(\Delta s), \tag{26}$$

we also establish the relations

$$\sqrt{a^2-b^2}\,\Delta s=aS(\Delta s)\sin\alpha \tag{27}$$

and

$$a^2+b\Delta s=aS(\Delta s)\cos\alpha. \tag{28}$$

They entail the equality

$$\frac{b}{\sqrt{a^2-b^2}}=\frac{S(\Delta s)\cos\alpha-a}{S(\Delta s)\sin\alpha} \tag{29}$$

from which the quantity b can be explicated.

Thus *each member of the involved set* $\{a,b,\Delta s,S(\Delta s)\}$ *can be explicitly expressed through the input vectors* $\mathbf{t}_1$ *and* $\mathbf{t}_2$. For many cases it is worth rewriting the equality (1.24) as

$$S^2(\Delta s)=(\Delta s)^2-a^2+2(a^2+b\Delta s). \tag{30}$$

Thus we have arrived at the following substantive items:

The $\mathcal{E}_g^{PD}$*-Case Cosine Theorem*

$$(\Delta s)^2=S^2(\Delta s)+a^2-2aS(\Delta s)\cos\alpha\,; \tag{31}$$

The $\mathcal{E}_g^{PD}$*-Case Two-Point Length*

$$(\Delta s)^2=(\mathbf{t}_1\mathbf{t}_1)+(\mathbf{t}_2\mathbf{t}_2)-2\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\cos\alpha\,; \tag{32}$$

The $\mathcal{E}_g^{PD}$*-Case Scalar Product*

$$< \mathbf{t}_1, \mathbf{t}_2 >= \sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\cos\alpha\,; \tag{33}$$

The $\mathcal{E}_g^{PD}$-Case Perpendicularity

$$< \mathbf{t}_1, \mathbf{t}_2 >= \sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}. \tag{34}$$

The identification

$$|\mathbf{t}_2 \ominus \mathbf{t}_1|^2 = (\Delta s)^2 \tag{35}$$

yields another lucid representation

$$|\mathbf{t}_2 \ominus \mathbf{t}_1|^2 = (\mathbf{t}_1\mathbf{t}_1) + (\mathbf{t}_2\mathbf{t}_2) - 2\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\cos\alpha\,. \tag{36}$$

The consideration can be completed by

THEOREM. *A general solution to the geodesic equation* (1.11) *can explicitly be found as follows:*

$$\mathbf{t}(s) =$$

$$= \frac{S(s)}{a}\,\frac{\sin\left[h\arctan\dfrac{\sqrt{a^2-b^2}\,(\Delta s - s)}{a^2+b\Delta s+(b+\Delta s)s}\right]}{\sin\left[h\arctan\dfrac{\sqrt{a^2-b^2}\,\Delta s}{a^2+b\Delta s}\right]}\,\mathbf{t}_1 + \frac{S(s)}{S(\Delta s)}\,\frac{\sin\left[h\arctan\dfrac{\sqrt{a^2-b^2}\,s}{a^2+bs}\right]}{\sin\left[h\arctan\dfrac{\sqrt{a^2-b^2}\,\Delta s}{a^2+b\Delta s}\right]}\,\mathbf{t}_2. \tag{37}$$

The euclidean limit proper is

$$\mathbf{t}(s)\Big|_{g=0} = \frac{(\Delta s - s)\mathbf{t}_1 + s\mathbf{t}_2}{\Delta s} = \mathbf{t}_1 + (\mathbf{t}_2 - \mathbf{t}_1)\frac{s}{\Delta s},$$

so that the geodesics become straight. From (1.35) the equality

$$(\mathbf{t}(s)\mathbf{t}(s)) = S^2(s) \tag{38}$$

follows, in agreement with (1.12). Since the general solution (1.35) is such that the right-hand side is spanned by two fixed vectors, $\mathbf{t}_1$ and $\mathbf{t}_2$, we are entitled concluding that *the geodesics under study are plane curves.*

2. Finsleroid-space $\mathcal{E}_g^{PD}$ of positive-definite type

Suppose we are given an N–dimensional vector space V_N. Denote by R the vectors constituting the space, so that $R \in V_N$. Any given vector R assigns a particular direction in V_N. Let us fix a member $R_{(N)} \in V_N$, introduce the straightline e_N oriented along the vector $R_{(N)}$, and use this e_N to serve as a R^N–coordinate axis in V_N. In this way we get the topological product

$$V_N = V_{N-1} \times e_N \tag{1}$$

together with the separation

$$R = \{\mathbf{R}, R^N\}, \qquad R^N \in e_N \quad \text{and} \quad \mathbf{R} \in V_{N-1}. \tag{2}$$

For convenience, we shall frequently use the notation

$$R^N = Z \tag{3}$$

and

$$R = \{\mathbf{R}, Z\}. \tag{4}$$

Also, we introduce a euclidean metric

$$q = q(\mathbf{R}) \tag{5}$$

over the $(N-1)$–dimensional vector space V_{N-1}.

With respect to an admissible coordinate basis $\{e_a\}$ in V_{N-1}, we obtain the coordinate representations

$$\mathbf{R} = \{R^a\} = \{R^1, \dots, R^{N-1}\} \tag{6}$$

and

$$R = \{R^p\} = \{R^a, R^N\} \equiv \{R^a, Z\}, \tag{7}$$

together with

$$q(\mathbf{R}) = \sqrt{r_{ab}R^aR^b}, \tag{8}$$

where r_{ab} are the components of a symmetric positive–definite tensor defined over V_{N-1}. The indices $(a, b, \dots)$ and $(p, q, \dots)$ will be specified over the ranges $(1, \dots, N-1)$ and $(1, \dots, N)$, respectively; vector indices are up, co–vector indices are down; repeated up–down indices are automatically summed; the notation δ^a_b will stand for the Kronecker symbol. The variables

$$w^a = R^a/Z, \quad w_a = r_{ab}w^b, \quad w = q/Z, \tag{9}$$

where

$$w \in (-\infty, \infty), \tag{10}$$

are convenient whenever $Z \neq 0$. Sometimes we shall mention the associated metric tensor

$$r_{pq} = \{r_{NN} = 1,\ r_{Na} = 0,\ r_{ab}\} \tag{11}$$

meaningful over the whole vector space V_N.

Given a parameter g subject to the inequality

$$-2 < g < 2, \tag{12}$$

we introduce the convenient notation

$$h = \sqrt{1 - \frac{1}{4}g^2}, \tag{13}$$

$$G = g/h, \tag{14}$$

$$g_+ = \frac{1}{2}g + h, \qquad g_- = \frac{1}{2}g - h, \tag{15}$$

$$g^+ = -\frac{1}{2}g + h, \qquad g^- = -\frac{1}{2}g - h, \tag{16}$$

so that

$$g_+ + g_- = g, \qquad g_+ - g_- = 2h, \tag{17}$$

$$g^+ + g^- = -g, \qquad g^+ - g^- = 2h, \tag{18}$$

$$(g_+)^2 + (g_-)^2 = 2, \tag{19}$$

$$(g^+)^2 + (g^-)^2 = 2, \tag{20}$$

and

$$g_+ \overset{g\to -g}{\Longleftrightarrow} -g_-, \qquad g^+ \overset{g\to -g}{\Longleftrightarrow} -g^-. \tag{21}$$

The characteristic quadratic form

$$B(g;R) = Z^2 + gqZ + q^2 \equiv \frac{1}{2}\Big[(Z+g_+q)^2 + (Z+g_-q)^2\Big] > 0 \tag{22}$$

is of the negative discriminant, namely

$$D_{\{B\}} = -4h^2 < 0, \tag{23}$$

because of Eqs. (2.12) and (2.13). Whenever $Z \neq 0$, it is also convenient to use the quadratic form

$$Q(g;w) \stackrel{\text{def}}{=} B/(Z)^2, \tag{24}$$

obtaining

$$Q(g;w) = 1 + gw + w^2 > 0, \tag{25}$$

together with the function

$$E(g;w) \stackrel{\text{def}}{=} 1 + \frac{1}{2}gw. \tag{26}$$

The identity

$$E^2 + h^2w^2 = Q \tag{27}$$

can readily be verified. In the limit $g \to 0$, the definition (2.22) degenerates to the quadratic form of the input metric tensor (2.11):

$$B\big|_{g=0} = r_{pq}R^pR^q. \tag{28}$$

Also

$$Q\big|_{g=0} = 1 + w^2. \tag{29}$$

In terms of this notation, we propose the FMF

$$K(g;R) = \sqrt{B(g;R)}\, J(g;R), \tag{30}$$

where

$$J(g;R) = \mathrm{e}^{\frac{1}{2}G\Phi(g;R)}, \tag{31}$$

$$\Phi(g;R) = \frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Big(\frac{q}{hZ} + \frac{G}{2}\Big), \qquad \text{if} \quad Z \geq 0, \tag{32}$$

$$\Phi(g;R) = -\frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Big(\frac{q}{hZ} + \frac{G}{2}\Big), \qquad \text{if} \quad Z \leq 0, \tag{33}$$

or in other convenient forms,

$$\Phi(g;R) = \frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Big(\frac{L(g;R)}{hZ}\Big), \qquad \text{if} \quad Z \geq 0, \tag{34}$$

$$\Phi(g;R) = -\frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Big(\frac{L(g;R)}{hZ}\Big), \qquad \text{if} \quad Z \leq 0, \tag{35}$$

where

$$L(g;R) = q + \frac{g}{2}Z, \tag{36}$$

and

$$\Phi(g;R) = \frac{\pi}{2} - \arctan\frac{hq}{A(g;R)}, \qquad \text{if} \quad Z \geq 0, \tag{37}$$

$$\Phi(g;R) = -\frac{\pi}{2} - \arctan\frac{hq}{A(g;R)}, \qquad \text{if} \quad Z \leq 0, \tag{38}$$

where

$$A(g;R) = Z + \frac{1}{2}gq. \tag{39}$$

This FMF has been normalized to show the handy properties

$$-\frac{\pi}{2} \leq \Phi \leq \frac{\pi}{2}, \tag{40}$$

$$\Phi = \frac{\pi}{2}, \quad \text{if} \quad q=0 \quad \text{and} \quad Z>0; \qquad \Phi = -\frac{\pi}{2}, \quad \text{if} \quad q=0 \quad \text{and} \quad Z<0. \tag{41}$$

We also have

$$\cot\Phi = \frac{hq}{A}, \qquad \Phi|_{Z=0} = \arctan\frac{G}{2}. \tag{42}$$

It is often convenient to use the indicator of sign ϵ_Z for the argument Z:

$$\epsilon_Z = 1, \quad \text{if} \quad Z>0; \qquad \epsilon_Z = -1, \quad \text{if} \quad Z<0; \tag{43}$$

Under these conditions, we call the considered space *the* $\mathcal{E}_g^{PD}$*–space*:

$$\mathcal{E}_g^{PD} = \{V_N = V_{N-1}\times e_N;\ R\in V_N;\ K(g;R);\ g\}. \tag{44}$$

The right–hand part of the definition (2.30) can be considered to be a function $\breve{K}$ of the arguments $\{g;q,Z\}$, such that

$$\breve{K}(g;q,Z) = K(g;R). \tag{45}$$

We observe that

$$\breve{K}(g;q,-Z) \neq \breve{K}(g;q,Z), \qquad \textit{unless} \quad g=0. \tag{46}$$

Instead, the function $\breve{K}$ shows the property of gZ*–parity*

$$\breve{K}(-g;q,-Z) = \breve{K}(g;q,Z). \tag{47}$$

The $(N-1)$–space reflection invariance holds true

$$K(g;R) \overset{R^a\leftrightarrow -R^a}{\Leftrightarrow} K(g;R). \tag{48}$$

It is frequently convenient to rewrite the representation (2.30) in the form

$$K(g;R) = |Z|V(g;w), \tag{49}$$

whenever $Z \neq 0$, with *the generating metric function*

$$V(g;w) = \sqrt{Q(g;w)}\, j(g;w). \tag{50}$$

We have

$$j(g;w) = J(g;1,w).$$

Using (2.25) and (2.31)–(2.35), we obtain

$$V' = wV/Q, \qquad V'' = V/Q^2, \tag{51}$$

$$(V^2/Q)' = -gV^2/Q^2, \qquad (V^2/Q^2)' = -2(g+w)V^2/Q^3, \tag{52}$$

$$j' = -\frac{1}{2}gj/Q, \tag{53}$$

and also

$$\frac{1}{2}(V^2)' = wV^2/Q, \qquad \frac{1}{2}(V^2)'' = (Q-gw)V^2/Q^2, \tag{54}$$

$$\frac{1}{4}(V^2)''' = -gV^2/Q^3, \tag{55}$$

together with

$$\Phi' = -h/Q, \tag{56}$$

where the prime ($'$) denotes the differentiation with respect to w.

Also,

$$(A(g;R))^2 + h^2q^2 = B(g;R) \tag{57a}$$

and

$$(L(g;R))^2 + h^2Z^2 = B(g;R). \tag{57b}$$

Sometimes it is convenient to use the function

$$E(g;w) \stackrel{\text{def}}{=} 1 + \frac{1}{2}gw. \tag{58}$$

The simple results for these derivatives reduce the task of computing the components of the associated FMT to an easy exercise, indeed:

$$R_p \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial K^2(g;R)}{\partial R^p}:$$

$$R_a = r_{ab}R^b\frac{K^2}{B}, \qquad R_N = (Z+gq)\frac{K^2}{B}; \tag{59}$$

$$g_{pq}(g;R) \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial^2 K^2(g;R)}{\partial R^p \partial R^q} = \frac{\partial R_p(g;R)}{\partial R^q}:$$

$$g_{NN}(g;R) = [(Z+gq)^2 + q^2]\frac{K^2}{B^2}, \qquad g_{Na}(g;R) = gqr_{ab}R^b\frac{K^2}{B^2}, \tag{60}$$

$$g_{ab}(g;R) = \frac{K^2}{B}r_{ab} - g\frac{r_{ad}R^d r_{be}R^e Z}{q}\frac{K^2}{B^2}. \tag{61}$$

The reciprocal tensor components are

$$g^{NN}(g;R) = (Z^2+q^2)\frac{1}{K^2}, \qquad g^{Na}(g;R) = -gqR^a\frac{1}{K^2}, \tag{62}$$

$$g^{ab}(g;R) = \frac{B}{K^2}r^{ab} + g(Z+gq)\frac{R^aR^b}{q}\frac{1}{K^2}. \tag{63}$$

The determinant of the FMT given by Eqs. (2.59) – (2.60) can readily be found in the form

$$\det(g_{pq}(g;R)) = [J(g;R)]^{2N}\det(r_{ab}) \tag{64}$$

which shows, on noting (2.31) – (2.33), that

$$\det(g_{pq}) > 0 \quad \textit{over all the definition range} \quad V_N \setminus 0. \tag{65}$$

The associated angular metric tensor

$$h_{pq} \stackrel{\text{def}}{=} g_{pq} - R_pR_q\frac{1}{K^2}$$

proves to be given by the components

$$h_{NN}(g;R) = q^2\frac{K^2}{B^2}, \qquad h_{Na}(g;R) = -Zr_{ab}R^b\frac{K^2}{B^2},$$

$$h_{ab}(g;R) = \frac{K^2}{B}r_{ab} - (gZ+q)\frac{r_{ad}R^dr_{be}R^e}{q}\frac{K^2}{B^2},$$

which entails

$$\det(h_{ab}) = \det(g_{pq})\frac{1}{V^2}.$$

The use of the components of the Cartan tensor (given explicitly in the end of the present section) leads, after rather tedious straightforward calculations, to the following simple and remarkable result.

PROPOSITION 1. *The Cartan tensor associated with the FMF* (2.30) *is of the following special algebraic form:*

$$C_{pqr} = \frac{1}{N}\left(h_{pq}C_r + h_{pr}C_q + h_{qr}C_p - \frac{1}{C_sC^s}C_pC_qC_r\right) \tag{66}$$

with

$$C_tC^t = \frac{N^2}{4K^2}g^2. \tag{67}$$

By the help of (2.65), elucidating the structure of the curvature tensor

$$S_{pqrs} \stackrel{\text{def}}{=} (C_{tqr}C_p{}^t{}_s - C_{tqs}C_p{}^t{}_r) \tag{68}$$

results in the simple representation

$$S_{pqrs} = -\frac{C_tC^t}{N^2}(h_{pr}h_{qs} - h_{ps}h_{qr}). \tag{69}$$

Inserting here (2.66), we are led to

PROPOSITION 2. *The curvature tensor of the space* $\mathcal{E}_g^{PD}$ *is of the special type*

$$S_{pqrs} = S^*(h_{pr}h_{qs} - h_{ps}h_{qr})/K^2 \tag{70}$$

with

$$S^* = -\frac{1}{4}g^2. \tag{71}$$

DEFINITION. FMF (2.30) introduces an $(N-1)$–dimensional indicatrix hypersurface according to the equation

$$K(g;R) = 1. \tag{72}$$

We call this particular hypersurface *the Finsleroid*, to be denoted as $\mathcal{F}_g^{PD}$.

Recalling the known formula $\mathcal{R} = 1 + S^*$ for the indicatrix curvature (see [4]), from (2.71) we conclude that

$$\mathcal{R}_{Finsleroid} = h^2 = 1 - \frac{1}{4}g^2, \qquad 0 < \mathcal{R}_{Finsleroid} \le 1. \tag{73}$$

Geometrically, the fact that the quantity (2.70) is independent of vectors R means that the indicatrix curvature is constant. Therefore, we have arrived at

PROPOSITION 3. *The Finsleroid $\mathcal{F}_g^{PD}$ is a constant-curvature space with the positive curvature value* (2.73).

Also, on comparing between the result (2.73) and Eqs. (2.22)–(2.23), we obtain

PROPOSITION 4. *The Finsleroid curvature relates to the discriminant* (2.23) *of the input characteristic quadratic form* (2.22) *simply as*

$$\mathcal{R}_{Finsleroid} = -\frac{1}{4}D_{\{B\}}. \tag{74}$$

Last, we write down the explicit components of the relevant Cartan tensor

$$C_{pqr} \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial g_{pq}}{\partial R^r} :$$

$$R^N C_{NNN} = gw^3V^2Q^{-3}, \qquad R^N C_{aNN} = -gww_aV^2Q^{-3},$$

$$R^N C_{abN} = \frac{1}{2}gwV^2Q^{-2}r_{ab} + \frac{1}{2}g(1-gw-w^2)w_aw_bw^{-1}V^2Q^{-3},$$

$$R^N C_{abc} = -\frac{1}{2}gV^2Q^{-2}w^{-1}(r_{ab}w_c + r_{ac}w_b + r_{bc}w_a) + gw_aw_bw_cw^{-3}\left(\frac{1}{2}Q + gw + w^2\right)V^2Q^{-3};$$

and

$$R^N C_N{}^N{}_N = gw^3/Q^2, \qquad R^N C_a{}^N{}_N = -gww_a/Q^2,$$

$$R^N C_N{}^a{}_N = -gw(1+gw)w^a/Q^2,$$

$$R^N C_a{}^N{}_b = \frac{1}{2}gwr_{ab}/Q + \frac{1}{2}g(1-gw-w^2)w_aw_b/wQ^2,$$

$$R^N C_N{}^a{}_b = \frac{1}{2}gw\delta^a_b/Q + \frac{1}{2}g(1+gw-w^2)w^aw_b/wQ^2,$$

$$R^N C_a{}^b{}_c = -\frac{1}{2}g\left(\delta^b_aw_c + \delta^b_cw_a + (1+gw)r_{ac}w^b\right)/wQ + \frac{1}{2}g(gwQ + Q + 2w^2)w_aw^bw_c/w^3Q^2.$$

The components have been calculated by the help of the formulae (2.50) – (2.53). The use of the contractions

$$R^N C_a{}^b{}_c r^{ac} = -g\frac{w^b}{w}\frac{1+gw}{Q}\left(\frac{N-2}{2} + \frac{1}{Q}\right)$$

and

$$R^N C_a{}^b{}_c w^a w^c = -g\frac{w}{Q^2}(1+gw)w^b$$

is handy in many calculations.

Also,

$$R^N C_N = \frac{N}{2}gwQ^{-1}, \qquad R^N C_a = -\frac{N}{2}g(w_a/w)Q^{-1},$$

$$R^N C^N = \frac{N}{2}gw/V^2, \qquad R^N C^a = -\frac{N}{2}gw^a(1+gw)/wV^2,$$

$$C^N = \frac{N}{2}gwR^N K^{-2}, \qquad C^a = -\frac{N}{2}gw^a(1+gw)w^{-1}R^N K^{-2},$$

$$C_p C^p = \frac{N^2}{4K^2}g^2.$$

3. Quasi–euclidean map of Finsleroid

It is possible to indicate the diffeomorphism

$$\mathcal{F}_g^{PD} \overset{i_g}{\Longrightarrow} \mathcal{S}^{PD} \tag{1}$$

of the Finsleroid $\mathcal{F}_g^{PD} \subset V_N$ to the unit sphere $\mathcal{S}^{PD} \subset V_N$:

$$\mathcal{S}^{PD} = \{R \in \mathcal{S}^{PD} : \ S(R) = 1\}, \tag{2}$$

where

$$S(R) = \sqrt{r_{pq}R^pR^q} \equiv \sqrt{(R^N)^2 + r_{ab}R^aR^b} \tag{3}$$

is the input euclidean metric function (see (2.11)).

The diffeomorphism (3.1) can always be extended to get the diffeomorphic map

$$V_N \overset{\sigma_g}{\Longrightarrow} V_N \tag{4}$$

of the whole vector space V_N by means of the homogeneity:

$$\sigma_g \cdot (bR) = b\sigma_g \cdot R, \quad b > 0. \tag{5}$$

To this end it is sufficient to take merely

$$\sigma_g \cdot R = ||R|| i_g \cdot \Bigl(\frac{R}{||R||}\Bigr), \tag{6}$$

where

$$||R|| = K(g; R). \tag{7}$$

Eqs. (3.1)–(3.7) entail

$$K(g; R) = S(\sigma_g \cdot R). \tag{8}$$

The identity (2.57) suggests to take the map

$$\bar{R} = \sigma_g \cdot R \tag{9}$$

by means of the components

$$\bar{R}^p = \sigma^p(g; R) \tag{10}$$

with

$$\sigma^a = R^a h J(g;R), \qquad \sigma^N = A(g;R)J(g;R), \tag{11}$$

where $J(g;R)$ and $A(g;R)$ are the functions (2.31) and (2.39). Indeed, inserting (3.11) in (3.3) and taking into account Eqs. (2.30) and (2.57), we get the identity

$$S(\bar{R}) = K(g;R) \tag{12}$$

which is tantamount to the implied relation (3.8).

PROPOSITION 5. The map given explicitly by Eqs. (3.9)–(3.11) assigns *the diffeomorphism between the Finsleroid and the unit sphere* according to Eqs. (3.1)–(3.8).

Therefore, we may also call the operation (3.1) *the quasi–euclidean map of Finsleroid.*

The inverse

$$R = \mu_g \cdot \bar{R}, \qquad \mu_g = (\sigma_g)^{-1}, \tag{13}$$

of the transformation (3.9) – (3.11) can be presented by the components

$$R^p = \mu^p(g;\bar{R}) \tag{14}$$

with

$$\mu^a = \bar{R}^a/hk(g;\bar{R}), \qquad \mu^N = I(g;\bar{R})/k(g;\bar{R}), \tag{15}$$

where

$$k(g;\bar{R}) \stackrel{\text{def}}{=} J(g;\mu(g;\bar{R})) \tag{16}$$

and

$$I(g;\bar{R}) = \bar{R}^N - \frac{1}{2}G\sqrt{r_{ab}\bar{R}^a\bar{R}^b}. \tag{17}$$

The identity

$$\mu^p(g;\sigma(g;R)) \equiv R^p \tag{18}$$

can readily be verified. Notice that

$$\frac{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}}{\bar{R}^N} = \frac{hq}{A(g;R)}, \qquad w^a = \frac{R^a}{R^N} = \frac{\bar{R}^a}{hI(g;\bar{R})}, \tag{19}$$

and

$$\sqrt{B}/Z = S/I, \qquad \sqrt{Q} = S/I. \tag{20}$$

The σ_g–image

$$\phi(g;\bar{R}) \stackrel{\text{def}}{=} \Phi(g;R)\big|_{R=\mu(g;\bar{R})} \tag{21}$$

of the function Φ described by Eqs. (2.31) – (2.42) is of a clear meaning of angle:

$$\phi(g;\bar{R}) = \arccos\frac{\bar{R}^N}{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}} = \begin{cases} \dfrac{\pi}{2} - \arctan\dfrac{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}}{\bar{R}^N}, & \text{if} \quad \bar{R}^N \ge 0; \\ -\dfrac{\pi}{2} - \arctan\dfrac{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}}{\bar{R}^N}, & \text{if} \quad \bar{R}^N \le 0; \end{cases} \tag{22}$$

which ranges over

$$-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}. \tag{23}$$

We have

$$\phi = \frac{\pi}{2}, \quad \text{if} \quad \bar{R}^a = 0 \quad \text{and} \quad \bar{R}^N > 0; \qquad \phi = -\frac{\pi}{2}, \quad \text{if} \quad \bar{R}^a = 0 \quad \text{and} \quad \bar{R}^N < 0, \tag{24}$$

and also

$$\phi|_{\bar{R}^N=0} = 0. \tag{25}$$

Comparing Eqs. (3.16) and (2.31) shows that

$$k = \mathrm{e}^{\frac{1}{2}G\phi}. \tag{26}$$

The right–hand parts in (3.11) are homogeneous functions of degree 1:

$$\sigma^p(g; bR) = b\sigma^p(g; R), \quad b > 0. \tag{27}$$

Therefore, the identity

$$\sigma^p_s(g; R)R^s = \bar{R}^p \tag{28}$$

should be valid for the derivatives

$$\sigma^q_p(g; R) \stackrel{\text{def}}{=} \frac{\partial\sigma^q(g; R)}{\partial R^p}. \tag{29}$$

The simple representations

$$\sigma^N_N(g; R) = \left(B + \frac{1}{2}gqA\right)\frac{J}{B}, \tag{30}$$

$$\sigma^N_a(g; R) = -\frac{g(ZA - B)}{2q}\frac{Jr_{ab}R^b}{B}, \tag{31}$$

$$\sigma^a_N(g; R) = \frac{1}{2}gq\frac{JR^a h}{B}, \tag{32}$$

$$\sigma^a_b(g; R) = \left(B\delta^a_b - \frac{gr_{bc}R^cR^aZ}{2q}\right)\frac{Jh}{B}, \tag{33}$$

and also the determinant value

$$\det(\sigma^q_p) = h^{N-1}J^N \tag{34}$$

are obtained. The relations

$$\sigma^a_bR^b = JhR^a(AZ+q^2)/B, \qquad r^{cd}\sigma^a_c\sigma^b_d = J^2h^2\left[r^{ab} - g(R^aR^bZ/qB) + \frac{1}{4}g^2(R^aR^bZ^2/B^2)\right]$$

are handy in many calculations involving the coefficients $\{\sigma^q_p\}$.

Henceforth, to simplify notation, we shall use the substitution

$$t^p = \bar{R}^p. \tag{35}$$

Again, we can note the homogeneity

$$\mu^p(g; bt) = b\mu^p(g; t), \quad b > 0, \tag{36}$$

for the functions (3.15), which entails the identity

$$\mu^p_s(g; t)t^s = R^p \tag{37}$$

for the derivatives

$$\mu^p_q(g;t) \stackrel{\text{def}}{=} \frac{\partial \mu^p(g;t)}{\partial t^q}. \tag{38}$$

We find

$$\mu^N_N = 1/k(g;t) - \frac{1}{2}g\frac{m(t)I(g;t)}{k(g;t)(S(t))^2}, \qquad \mu^N_a = \frac{1}{2}g\frac{r_{ac}t^cI^*(g;t)}{k(g;t)(S(t))^2}, \tag{39}$$

$$\mu^a_N = -\frac{1}{2}g\frac{m(t)\,t^a}{hk(g;t)(S(t))^2}, \qquad \mu^a_b = \frac{1}{hk(g;t)}\delta^a_b + \frac{1}{2}g\frac{t^Nt^ar_{bc}t^c}{m(t)\,hk(g;t)(S(t))^2}, \tag{40}$$

where

$$m(t) = \sqrt{r_{ab}t^at^b}, \tag{41}$$

$$I^*(g;t) = hm(t) - \frac{1}{2}gt^N, \tag{42}$$

and

$$S(t) = \sqrt{r_{rs}t^rt^s} \equiv \sqrt{(t^N)^2 + r_{ab}t^at^b}. \tag{43}$$

The relations

$$\frac{\partial(1/k(g;t))}{\partial t^N} = -\frac{1}{2}g\frac{m(t)}{hk(g;t)(S(t))^2}, \qquad \frac{\partial(1/k(g;t))}{\partial t^a} = \frac{1}{2}g\frac{t^Nr_{ab}t^b}{m(t)hk(g;t)(S(t))^2}$$

are obtained.

Also

$$R_p\mu^p_q = t_q, \qquad t_p\sigma^p_q = R_q. \tag{44}$$

The unit vectors

$$L^p \stackrel{\text{def}}{=} \frac{t^p}{S(t)}, \qquad L_p \stackrel{\text{def}}{=} r_{pq}L^q \tag{45}$$

fulfil the relations

$$L^q = l^p\sigma^q_p, \qquad l^p = \mu^p_qL^q, \qquad l_p = \sigma^q_pL_q, \qquad L_p = \mu^q_pl_q, \tag{46}$$

where $l^p = R^p/K(g;R)$ and $l_p = g_{pq}(g;R)l^q$ are the initial Finslerian unit vectors.

Now we use the explicit formulae (2.61) – (2.62) and (3.29) – (3.32) to find the transform

$$n^{rs}(g;t) \stackrel{\text{def}}{=} \sigma^r_p\sigma^s_qg^{pq} \tag{47}$$

of the FMT g_{pq} under the $\mathcal{F}^{PD}_g$–induced map (3.9) – (3.11), which results in

PROPOSITION 6. *One obtains the simple representation*

$$n^{rs} = h^2r^{rs} + \frac{1}{4}g^2L^rL^s. \tag{48}$$

The covariant version reads

$$n_{rs} = \frac{1}{h^2}r_{rs} - \frac{1}{4}G^2L_rL_s. \tag{49}$$

The determinant of this tensor is a constant:

$$\det(n_{rs}) = h^{2(1-N)}\det(r_{ab}). \tag{50}$$

Notice that

$$L^pL_p = 1, \quad n_{pq}L^q = L_p, \quad n^{pq}L_q = L^p, \quad n_{pq}L^pL^q = 1, \quad n_{pq}t^pt^q = (S(t))^2.$$

Eq. (5.47) obviously entails

$$g_{pq} = n_{rs}(g;t)\sigma^r_p\sigma^s_q. \tag{51}$$

4. Quasi-euclidean metric tensor

Let us introduce

DEFINITION. The metric tensor (3.48) – (3.49) is called *quasi-euclidean.*

DEFINITION. *The quasi-euclidean space*

$$\mathcal{Q}_N = \{V_N; n_{pq}(g;t); g\} \tag{1}$$

is an extension of the euclidean space $\{V_N; r_{pq}\}$ to the case $g \neq 0$.

The transformation (3.47) can be inverted to read

$$g_{pq} = \sigma^r_p\sigma^s_q n_{rs}. \tag{2}$$

For the angular metric tensor (see the formula going below Eq. (2.64)), from (3.46) and (4.2) we infer

$$h_{pq} = \sigma^r_p\sigma^s_q H_{rs}\frac{1}{h^2}, \tag{3}$$

where

$$H_{rs} \stackrel{\text{def}}{=} r_{rs} - L_r L_s \tag{4}$$

is the tensor showing the orthogonality property

$$L^r H_{rs} = 0. \tag{5}$$

One can readily find that

$$H_{rs} = h^2(n_{rs} - L_r L_s).$$

PROPOSITION 7. The quasi-euclidean metric tensor (3.48) – (3.49) *is conformal* to the euclidean metric tensor.

Indeed, if we consider the map

$$\bar{R}^p \to \tilde{R}: \qquad \tilde{R}^p = f(g;\bar{R})\bar{R}^p/h \tag{6}$$

with

$$f(g;\bar{R}) = a\left(g;\, \frac{1}{2}S^2(\bar{R})\right) \tag{7}$$

and use the coefficients

$$k^p_q \stackrel{\text{def}}{=} \frac{\partial \tilde{R}^p}{\partial \bar{R}^q} = (f\delta^p_q + a'\bar{R}^p\bar{R}_q)/h \tag{8}$$

to define the tensor

$$c^{pq}(g;\tilde{R}) \stackrel{\text{def}}{=} k^p_r k^q_s n^{rs}(g;\bar{R}), \tag{9}$$

we find that

$$c^{pq} = f^2 r^{pq} \tag{10}$$

whenever

$$f=\left[\frac{1}{2}S^2(\bar{R})\right]^{\gamma/2}, \tag{11}$$

where

$$\gamma=h-1\equiv\sqrt{1-\frac{g^2}{4}}-1 \tag{12}$$

is the parameter. The proof of Proposition 7 is complete.

Let us now use the obtained quasi-euclidean metric tensor $n_{pq}(g;t)$ to construct the associated *quasi-euclidean Christoffel symbols* $N_p{}^r{}_q(g;t)$. We find consecutively:

$$n_{pq,r}\stackrel{\text{def}}{=}\frac{\partial n_{pq}}{\partial t^r}=-\frac{1}{4}G^2(H_{pr}L_q+H_{qr}L_p)/S, \tag{13}$$

and

$$N_p{}^r{}_q=n^{rs}N_{psq}, \qquad N_{prq}=\frac{1}{2}(n_{pr,q}+n_{qr,p}-n_{pq,r}), \tag{14}$$

together with

$$N_{prq}(g;t)=-\frac{1}{4}G^2H_{pq}L_r/S, \tag{15}$$

which eventually yields

$$N_p{}^r{}_q(g;t)=-\frac{1}{4}G^2L^rH_{pq}/S. \tag{16}$$

Comparing the representation (4.16) with the identity (4.5) shows that

$$t^pN_p{}^r{}_q=0, \qquad N_p{}^s{}_s=0, \qquad N_t{}^s{}_rN_p{}^t{}_q=0. \tag{17}$$

Also,

$$\frac{\partial N_p{}^r{}_q}{\partial t^s}-\frac{\partial N_p{}^r{}_s}{\partial t^q}=-\frac{1}{4}G^2(H_{pq}H_s{}^r-H_{ps}H_q{}^r)/S^2. \tag{18}$$

Using the identities (4.17)-(4.18) in *the quasi-euclidean curvature tensor*:

$$R_p{}^r{}_{qs}(g;t)\stackrel{\text{def}}{=}\frac{\partial N_p{}^r{}_q}{\partial t^s}-\frac{\partial N_p{}^r{}_s}{\partial t^q}+N_p{}^w{}_qN_w{}^r{}_s-N_p{}^w{}_sN_w{}^r{}_q, \tag{19}$$

we arrive at the simple result:

$$R_{prqs}(g;t)=-\frac{1}{4}G^2(H_{pq}H_{rs}-H_{ps}H_{qr})/S^2. \tag{20}$$

This infers the identities

$$L^pR_{pqrs}=L^qR_{pqrs}=L^rR_{pqrs}=L^sR_{pqrs}=0. \tag{21}$$

Note. Because of the transformation rules (3.12) and (3.47), the representation (4.20) is tantamount to Eqs. (2.69)–(2.70). Therefore *we have got another rigorous proof of Proposition* 3, *and of Eq.* (2.71), *concerning the Finsleroid curvature.*

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Appendix

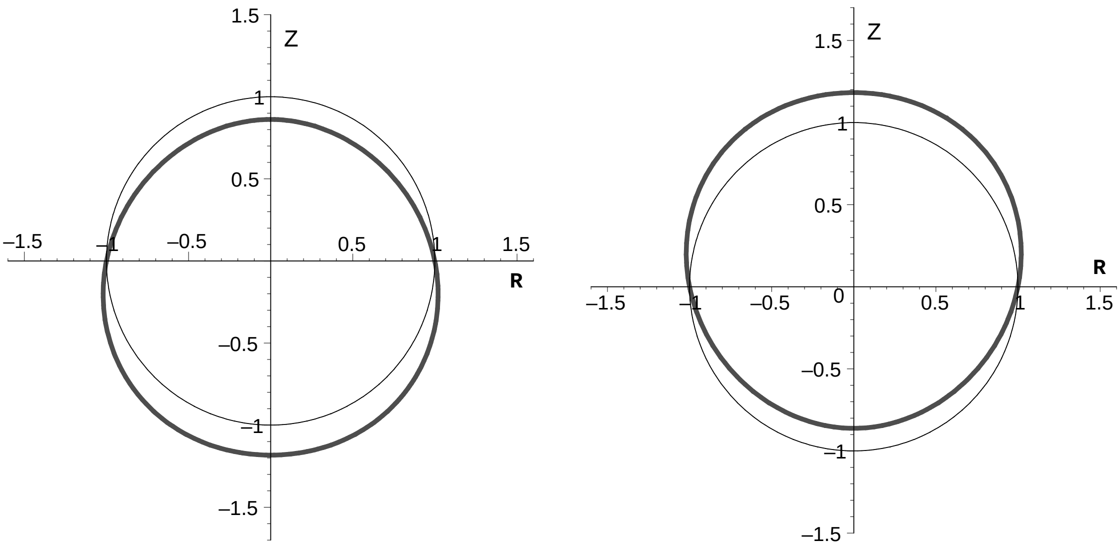

Figure 1: $g = 0.2$ and $g = -0.2$

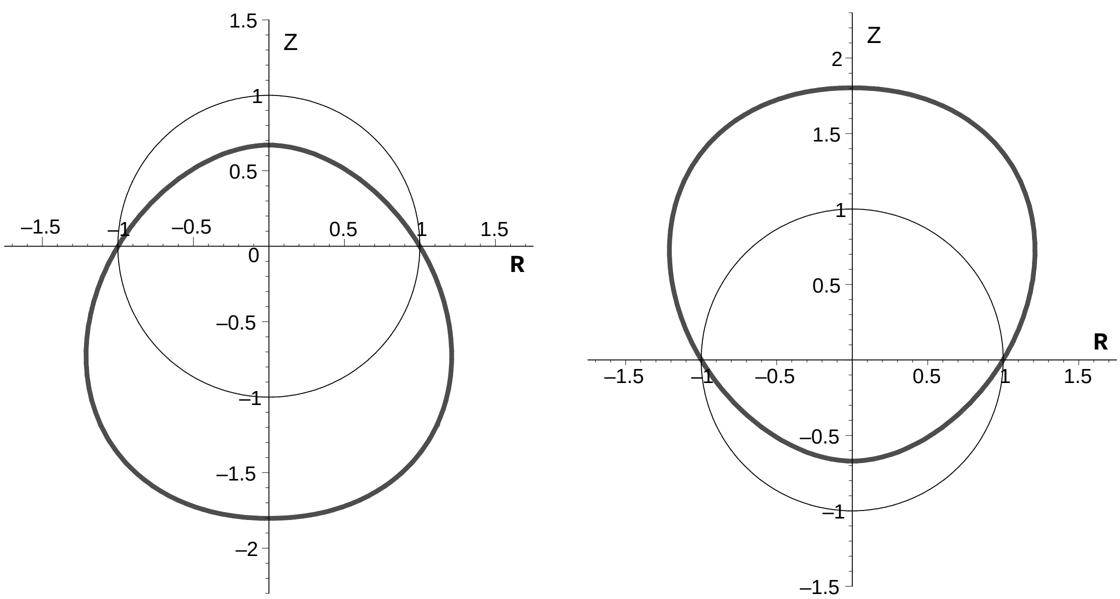

Figure 2: $g = 0.6$ and $g = -0.6$

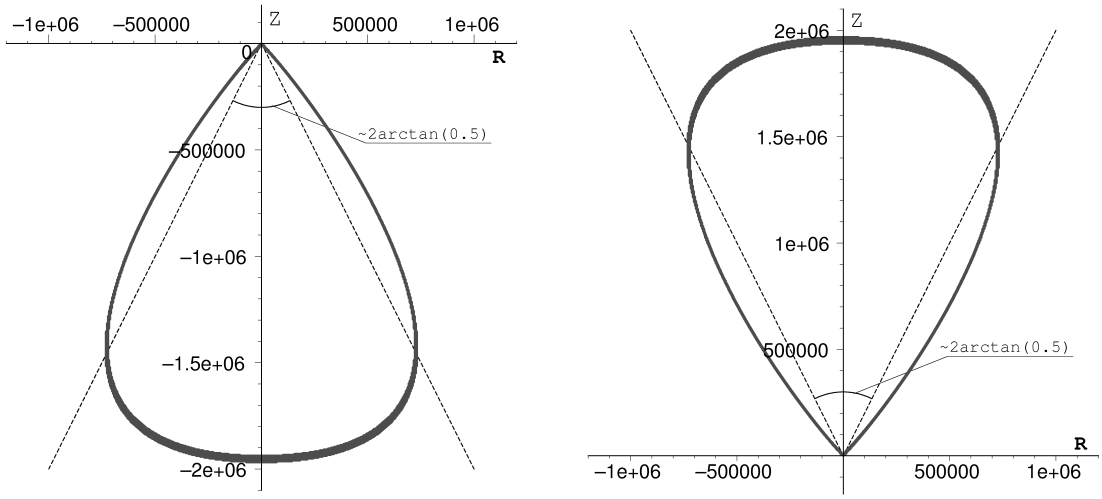

Figure 3: $g = 1.96$ and $g = -1.96$

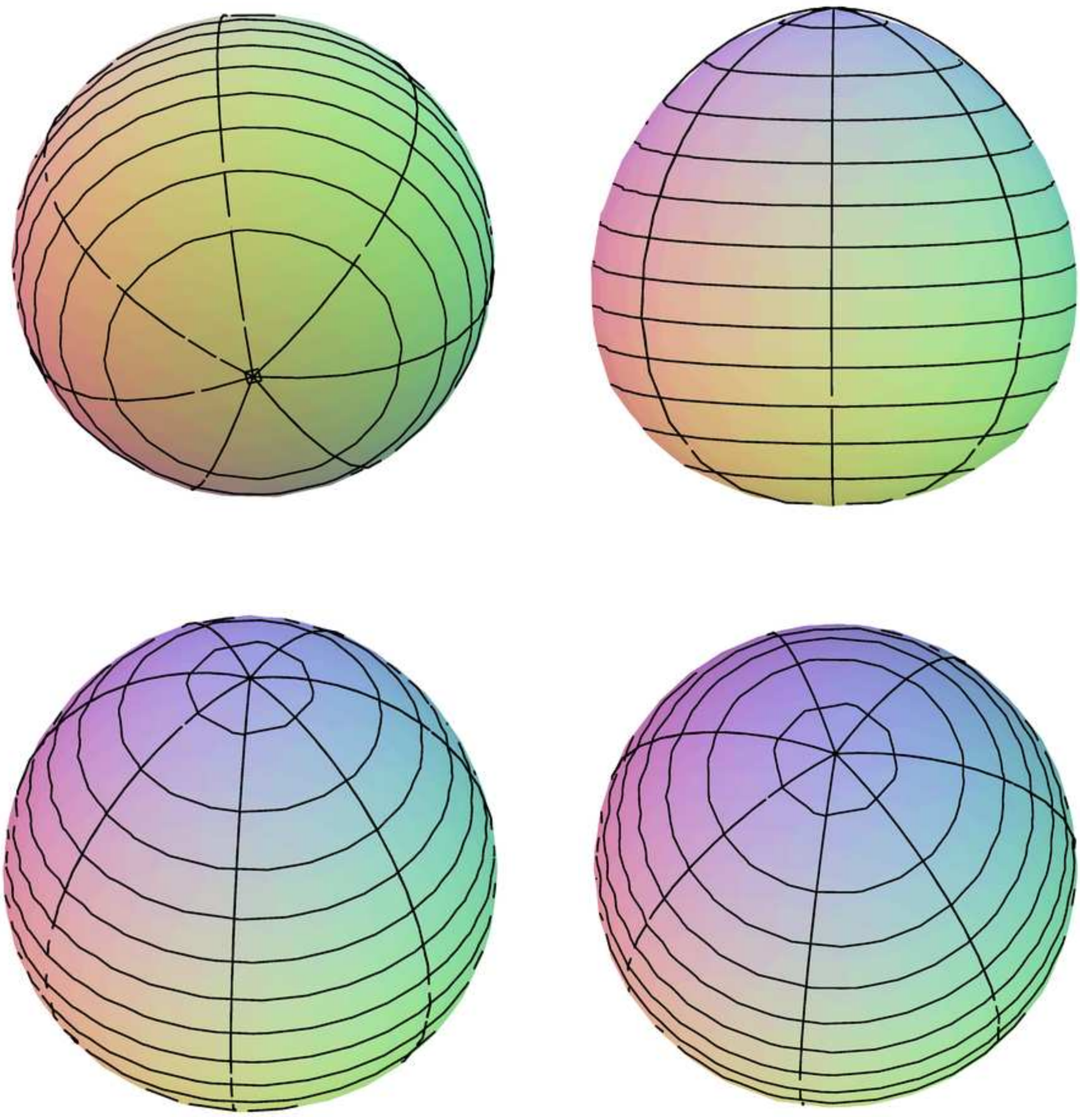

Figure 4: $3D$-images of Finsleroid; $g = 0.6$

PROPERTIES OF SPACES ASSOCIATED WITH COMMUTATIVE–ASSOCIATIVE H_3 AND H_4 ALGEBRAS

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In the first part of this work a real axis of the space associated with the H_3 algebra and the lines parallel to this axis are interpreted as the world lines of resting particles; surface of simultaneity is used for introduction of a distance between the real axis and a line parallel thereto. The coordinate system similar to a polar one can be introduced on this surface such that this allows us to reveal its simplest invariant transformations. In the second part of this paper the Lorentz transformations in form of special kind of rotations in the space associated with H_4 algebra are presented.

Introduction

The H_3 and H_4 algebras belong to the commutative–associative algebras of the H_n type which are of the simplest structure. These algebras are characterized by some preferred basis. The multiplication of numbers is realized in terms of this basis in a componentwise manner similarly to the addition in arbitrary algebras. On the other side, in H_n type algebras, which can be called hyperbolic, H_3 and H_4 algebras directly follow after the algebras of real (H_1) and double (H_2) numbers, which possessed important properties for their physical applications [6, 11]. We set forth an assumption of "inheriting" these properties by 3- and 4- dimensional algebras under consideration. As a motivation of this assumption we recall the relation between Berwald–Moor's metrics and H_4 algebra in Finsler generalization of the relativity theory [1]. From the point of view of possible applications, hyperbolic H_4 algebra is the most promising one because the $n = 4$ dimensional spaces have the topological preference [7]. However, H_3 algebra possesses one evident advantage. It is possible to use the computer visualization animation for figures, surfaces, and lines in the three dimensional metrical space associated with this algebra. Although it is not worth overestimating the analytical capacities of such applications, it gets a special visuality to geometric properties of this space. Therefore a sufficiently general approach to physical treatment of the hyperbolic space properties, offered in the first part of this paper, is represented for a space accounted with H_3 algebra. Its properties give the cube of norm as

$$|A|^3 = |a^1 a^2 a^3|,$$

where a^i are components of the vector in the preferred basis, combined from three numbers e_i, where $i = 1, 2, 3$, with properties $(e_i)^2 = e_i$, $e_i \cdot e_j = 0$ when $i \neq j$. Real numbers on a line can be shared in two classes: they are positive numbers, placed on the right side from zero, and the negative ones, placed on the left side from zero. Two isotropic lines in the double numbers algebra divide the pseudo-euclidian plane into 2^2 quadrants. Similarly to this the associated space is divided into 2^3 octants, and for all numbers appropriated to one octant points it is typical that the same sign combination of components is taken with respect to the preferred basis. The boundaries of the octants are three isotropic planes with equations $a^i = 0$, where $i = 1, 2, 3$. It will be noted also that since a hyperbolic algebras are algebras with a unity, defined by an expression

$$1 = e_1 + e_2 + \cdots + e_n,$$

two octants of the treated space can be preferably be selected. They are the octants, containing 1 and -1; they are characterized by numbers with all positive or all negative components, respectively.

Using considered algebras requires an availability of euclidian or pseudo-euclidian properties. In the order of algebras: the Dirac algebra [2], quaternions [3], biquaternions [5] – the existence of such properties provides a classical appearance of the norm of the number. However, there is a slight amount of such algebras, but amongst commutative-associative algebras only the double number algebra belongs to such class, in which a square of the norm of the numbers is given by

$$|A|^2 = |(a^1)^2 - (a^2)^2|$$

(see [4]). Chronogeometry method [8], [12] gives an other opportunity to establishing properties which are similar with the properties of euclidian or pseudo-euclidian spaces, in the spaces associated with the considered algebras; the first part of this paper is devoted to application of this method to H_3. Some more opportunity to establishing the sought properties appears on application of symmetric polyform associated with the algebra [9], which, for example, has the following form for H_3 algebra:

$$(A, B, C) = \frac{1}{3!}(a^1 b^2 c^3 + \cdots + a^3 b^2 c^1).$$

The second part of this paper is connected with such opportunity applied to H_4 algebra, where the form having appearance as pseudo-euclidian metric is determined by a polylinear form of four vectors.

1. A simultaneity surface in the commutative–associative algebras (as examplified by H_3)

1.1. *Axiomatics*

We shall treat the following statements, playing the role of axioms, as a principle to interpret physically the properties of the considered algebras class.

1. It is possible to connect an algebra number with some spatial-temporal event.

2. The real axis of the space, which direction is given by means of the unity of the algebras, is treated as a temporal axis, while the norm of the number is interpreted as an observer's time interval whose world line coincides with the vector corresponded to this number.

3. The increase of a relative velocity of particle or signal results in increasing an inclination of tangent line to the particles world line in the given point to the observer world line, and resting material points have world lines which are parallel to the observer line.

4. Light signals, which have a maximal velocity, are connected with isotropic hypersurfaces of the algebra; and it is supposed that the velocity of the light signals does not depend on their propagation direction. According to these statements two selected octants with 1 and -1, which are referred to above, are the analogs of the cone of the

future and the past Minkowski space in the space associated with H_3 algebra, respectively. Contrary to the Minkowski space in the considered space a domain outside these cones also possesses isotropic directions, because consists of six side cones. In this paper we restrict our attention to the most common particular case, when the observer world line coincides with the real axis.

1.2. *Exponential form of the H_3 algebra number representation with respect to the basis* $(1, j, k)$

Any number in the selected basis is represented as:

$$A = a^1 \cdot e_1 + a^2 \cdot e_2 + a^3 \cdot e_3.$$

For an exponential function in terms of this basis the following formula takes place:

$$\exp(a^1 \cdot e_1 + a^2 \cdot e_2 + a^3 \cdot e_3) = \exp(a^1) \cdot e_1 + \exp(a^2) \cdot e_2 + \exp(a^3) \cdot e_3. \quad (1)$$

Since in the considered algebra we get $|A|^3 = |a^1 a^2 a^3|$, any number with $a^i > 0$ is represented as

$$A = |A| \cdot \exp(b^1 e_1 + b^2 e_2 + b^3 e_3)$$

with a restriction

$$b_1 + b_2 + b_3 = 0, \quad (2)$$

which implies the identity:

$$|\exp(b^1 e_1 + b^2 e_2 + b^3 e_3)| = 1.$$

The other basis of the algebra is composed from vectors:

$$\begin{cases} 1 = e_1 + e_2 + e_3 \\ j = \sin\varphi_0 \cdot e_1 + \sin(\varphi_0 + 2\pi/3) \cdot e_2 + \sin(\varphi_0 + 4\pi/3) \cdot e_3 \\ k = \cos\varphi_0 \cdot e_1 + \cos(\varphi_0 + 2\pi/3) \cdot e_2 + \cos(\varphi_0 + 4\pi/3) \cdot e_3 \end{cases} \quad (3)$$

The vectors appearing in this basis are mutually orthogonal (in the usual euclidian sense), while an arbitrary parameter φ_0 can be treated in a certain sense as the angle of a simultaneous rotation of a pair of vectors j, k around the real axis. If t, x, y – are coordinates of the number in a new basis, then according to the transformation rules of coordinates of the number we have a system in the other basis:

$$\begin{cases} a^1 = t + \sin\varphi_0 \cdot x + \cos\varphi_0 \cdot y \\ a^2 = t + \sin(\varphi_0 + 2\pi/3) \cdot x + \cos(\varphi_0 + 2\pi/3) \cdot y \\ a^3 = t + \sin(\varphi_0 + 4\pi/3) \cdot x + \cos(\varphi_0 + 4\pi/3) \cdot y \end{cases} \quad (4)$$

from which it follows that $t = (a^1 + a^2 + a^3)/3$. Therefore by (2) the number representable in a exponential form in the basis (1,j,k) is given by

$$A = |A| \cdot e^{\alpha \cdot j + \beta \cdot k}.$$

If we modify this exponential representation, introducing an definition $\rho = \sqrt{\alpha^2 + \beta^2}$, we obtain

$$A = |A| \cdot e^{\rho(\cos\varphi \cdot j + \sin\varphi \cdot k)}. \quad (5)$$

Thus, in agreement with (5), the number at this representation is given by three parameters: the norm of the number $|A|$, the "radial coordinate" ρ, and the "angle coordinate" φ. Making use of (1) and (3), formula (5) takes simple and elegant form in components:

$$\begin{cases} a^1 = |A| \cdot \exp(\rho \sin[\varphi_0 + \varphi]) \\ a^2 = |A| \cdot \exp(\rho \sin[\varphi_0 + 2\pi/3 + \varphi]) \\ a^3 = |A| \cdot \exp(\rho \sin[\varphi_0 + 4\pi/3 + \varphi]) \end{cases}$$

1.3. *Method of setting the distance between the real axis and the parallel line.*

For determination of the distance between the world lines of resting particles, one of which lying on the real axis, we use the chronogeometry method. Consider the exchange of signals with the constant velocity $\nu \leq c$; for simplicity we shall arrange point-events of signal transmission and the reception of the reverse signal on the real axis symmetrically with respect to zero time moment. Because of an equality of lengths of straight and reverse signals velocity $|B - A_1| = |A_2 - B|$, so we have:

$$(a^1 + T)(a^2 + T)(a^3 + T) = (T - a^1)(T - a^2)(T - a^3),$$

where $a^i + T > 0$, $T - a^i > 0$, which after expanding takes form:

$$(a^1 + a^2 + a^3) \cdot T^2 + a^1 a^2 a^3 = 0. \tag{6}$$

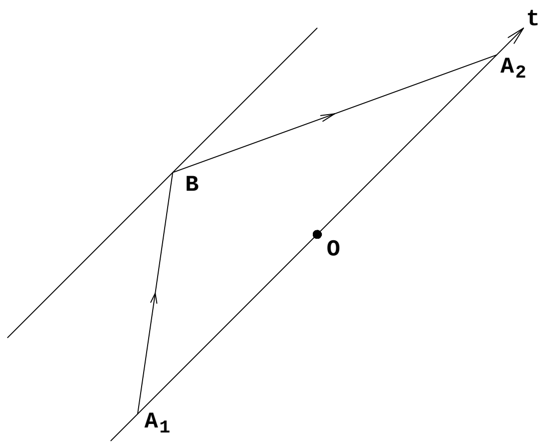

Figure 1: The measuring of a distance between the world lines by pre-light signals exchange.

The multitude of points-events satisfied to equation (6) form a surface of a simultaneity: it is for the observer on the real axis, being in the point with T coordinate, all these events are taking place in the same zero moment of time. Point $A = (0, 0, 0)$ belongs to the simultaneity surface, and the tangent plane to this surface in the origin has an equation:

$$a^1 + a^2 + a^3 = 0. \tag{7}$$

Substitution of (4) into (6) allows to obtain the equation of the simultaneity surface in form of the dependence of the time of the signal passing (on a clock of resting observer)

T from introduced coordinates $\{t, x, y\}$ of point of the simultaneity surface:

$$T^2 = \frac{1}{12}(x^2+y^2) - \frac{1}{3}\left\{t^2 + \frac{1}{t}\left[\frac{3}{4}xy(y\cdot\sin 3\varphi_0 - x\cdot\cos 3\varphi_0)\right.\right.$$
$$\left.\left. + x^3 \sin\varphi_0 \sin(\varphi_0+2\pi/3)\sin(\varphi_0+4\pi/3) + y^3\cos\varphi_0\cos(\varphi_0+2\pi/3)\cos(\varphi_0+4\pi/3)\right]\right\}.$$

According to this equation (and similar equations for other algebras, in particularly, H_4 algebra) the first items on the right side have an euclidian form, and then they dominate on other remaining items, square of travel time of signal depends linearly on square of the euclidian distance in the world lines space, which can be useful for the next physical interpretations.

1.4. *The system of curvilinear coordinates of the simultaneity surface and the transformations mapping it to itself*

Keeping in mind an important of an invariant transformations in modern physics, we shall briefly consider the topic of finding the transformations of the simultaneity surface, mapping it to itself. We introduce two-dimension coordinate system $\{\rho, \varphi\}$ on this surface, somewhat analogous to polar coordinate system on two-dimension plane to get:

$$\begin{cases} a^1 = (T-\rho)\cdot e^{R(\rho,\varphi)\sin(\varphi_0+\varphi)} - T, \\ a^2 = (T-\rho)\cdot e^{R(\rho,\varphi)\sin(\varphi_0+2\pi/3+\varphi)} - T, \\ a^3 = (T-\rho)\cdot e^{R(\rho,\varphi)\sin(\varphi_0+4\pi/3+\varphi)} - T, \end{cases} \tag{8}$$

where the function $R = R(\rho,\varphi)$ taken from transcendent equation is obtaining by using the coordinates (8) into (6):

$$\bar{Z}^3 - \bar{Z}^2\left[e^{-R\sin(\varphi_0+\varphi)} + e^{-R\sin(\varphi_0+2\pi/3+\varphi)} + e^{-R\sin(\varphi_0+4\pi/3+\varphi)}\right]$$
$$+2\bar{Z}\left[e^{R\sin(\varphi_0+\varphi)} + e^{R\sin(\varphi_0+2\pi/3+\varphi)} + e^{R\sin(\varphi_0+4\pi/3+\varphi)}\right] - 4 = 0,$$

where $\bar{Z} = (T-\rho)/T$.

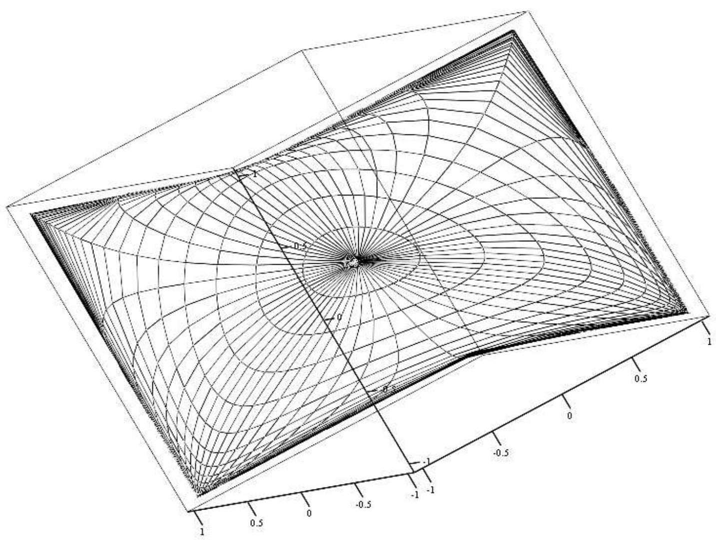

Figure 2: Curvilinear coordinates system ρ, ϕ on simultaneity surface.

In the vicinity of zero at $a^1, a^2, a^3 \ll 1,\ R \ll 1,\ \rho \ll 1$, the equations (8) are got simplified:

$$\begin{cases} a^1 \cong R \cdot T \cdot \sin(\varphi_0 + \varphi), \\ a^2 \cong R \cdot T \cdot \sin(\varphi_0 + 2\pi/3 + \varphi), \\ a^3 \cong R \cdot T \cdot \sin(\varphi_0 + 4\pi/3 + \varphi), \end{cases}$$

so that

$$a^1 + a^2 + a^3 \cong 0 \text{ and } (a^1)^2 + (a^2)^2 + (a^3)^2 \cong (R \cdot T)^2. \tag{9}$$

Thus, according to (9), the coordinate system (8) is distinguished: *in the vicinity of zero the parameter R is proportional to euclidian distance from a point, located on the simultaneity surface, to the center of this surface, in which $R = 0$.*

Then independent transformations of the simultaneity surface we seek are "rotations" by angle $\Delta\varphi(\varphi \to \varphi + \Delta\varphi)$ and "a similarity transformations" with a coefficient $K(\rho \to K \cdot \rho)$.

2. The representation a Lorentz transformations by rotations in the space, associated with H_4 algebra.

Following [10], we define the inner product of two arbitrary (with positive values of components) vectors A and B in the space under consideration by a symmetric four-form of H_4 space as:

$$(A, B) := \frac{(A, A, B, B)}{|A| \cdot |B|}.$$

The inner product of two vectors satisfying to properties of positiveness, homogeneity, and normality:
1. $(A, B) > 0$;
2. $(kA, B) = (A, kB) = k(A, B)$;
3. $(A, A) = |A|^2$.

The inner product of units vectors $a = A/|A|$ and $b = B/|B|$ may be regarded as an angle characteristic, setting a relation between two directions defined by these vectors – it is expressed via quotient components of these vectors ($d = b/a$):

$$(a, b) = (d_1 d_2 + d_1 d_3 + \ldots d_3 d_4)/6. \tag{10}$$

Consider a basis in the space associated with H_4 algebra, consisting of these vectors:

$$\begin{cases} 1 = e_1 + e_2 + e_3 + e_4, \\ j' = 3e_1 - e_2 - e_3 - e_4, \\ k' = \sqrt{2}(2e_2 - e_3 - e_4), \\ l' = \sqrt{6}(e_3 - e_4). \end{cases}$$

We denote coordinates of relation of two considered vectors in a new basis via t_d, x_d, y_d, z_d and expressing (10) via these components, we obtain:

$$(a, b) = t_d^2 - x_d^2 - y_d^2 - z_d^2.$$

We shall denote the nonlinear transformation of 4-space, associated with H_4 algebra, which remains all vectors in the direction setting by vector A in rest, and retains the introduced inner product, as a *rotation* of vector B round a vector A. Thus, in addition to the other representations of Lorentz group [13] the representation by rotations round arbitrary time-like axis in the space, associated with H_4 algebra, can be used.

Results and conclusions

The method of determination of the distances between the world lines introduced for the space associated with a commutative-associative H_3 algebra (or H_4) allows to distinguish "a euclidian part".

A new geometric interpretation of the Lorentz transformations as rotations in the space connected with algebra H_4 is obtained. Arbitrary setting of a rotation axis is possible; all said above gives a hope on the application of such new interpretation in relativity physics.

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GENERALIZED–ANALYTICAL FUNCTIONS OF POLY–NUMBER VARIABLE

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We introduce the notion of the generalized-analytical function of the poly–number variable, which is a non–trivial generalization of the notion of analytical function of the complex variable and, therefore, may turn out to be fundamental in theoretical physical constructions. As an example we consider in detail the associative-commutative hypercomplex numbers H_4 and an interesting class of corresponding functions.

1. Introduction

Let M_n be an n-dimensional elementary manifold and P_n denote the system of n-dimensional associative-commutative hypercomplex numbers (poly-numbers, n-numbers), and a one-to-one correspondence between the sets be assigned. Under these conditions, we choose in P_n the basis

$$e_1, e_2, ..., e_n; \quad e_i e_j = p_{ij}^k e_k, \tag{1}$$

$$X = x^1 \cdot e_1 + x^2 \cdot e_2 + \cdots + x^n e_n \in P_n, \tag{2}$$

where $e_1, e_2, ..., e_n$ – symbolic elements, p_{ij}^k stand for characteristic real numbers, and $x^1, x^2, ..., x^n$ – real coordinates with respect to the basis $(e_1 \equiv 1, e_2, \ldots, e_n)$. Obviously, the numbers $x^1, x^2, ..., x^n$ can be used not only as the coordinates in P_n, but also as coordinates in the manifold M_n , so that $(x^1, x^2, ..., x^n) \in M_n$. Though in M_n we can go over to any other curvilinear reference frame, the reference frame $\{x^i\}$, as being built by the help the basis of poly–numbers and a fixed one-to-one correspondence $M_n \leftrightarrow P_n$, ought to be considered preferable (as well as any other reference frame connected with this by non-degenerate linear transformation). The poly–number algebraic operations induce the same operations in the elementary manifold (formally) and in the tangent space at any point of the manifold (informally). Accordingly, the tangent spaces to M_n are isomorphic to P_n.

The function

$$F(X) := f^1(x^1, \cdots, x^n)e_1 + \cdots + f^n(x^1, \cdots, x^n)e_n \tag{3}$$

of the poly–number variable, where f^i are sufficiently smooth functions of n real variables, will be considered to be a vector (contravariant) field in M_n. Hence, apart from addition and multiplication by number, any operation of multiplication of vector fields

$$f_{(3)}^k = f_{(1)}^i \cdot f_{(2)}^j \cdot p_{ij}^k \tag{4}$$

can also defined in M_n. It is useful but not obligatory to consider the space M_n to be the main ("the examined") object and the space P_n to be a sort of an instrument with the help of which the space M_n is "examined". In the general case the parallel transportation of a vector in the space P_n does not correspond to the "parallel transportation" of the same vector in the space M_n, so that for a due definition of absolute differential (or the covariant derivative) we are to have the connection objects or the quantities which may replace them. If we avoid introducing the pair $\{M_n, P_n\}$, restricting the treatment only to associative-commutative hypercomplex numbers, then it is natural to introduce the definitions

$$dX := dx^i \cdot e_i \tag{5}$$

and

$$dF(X) := F(X + dX) - F(X) = \frac{\partial f^i}{\partial x^k} \cdot e_i \cdot dx^k. \tag{6}$$

The function $F(X)$ of poly-number variable X is called *analytical*, if such a function $F'(X)$ exists that

$$dF(X) = F'(X) \cdot dX, \tag{7}$$

where the multiplication in the right-hand part means the poly-number operation. From (7) it follows that

$$\frac{\partial f^i}{\partial x^k} = p^i_{kj} \cdot f'^j. \tag{8}$$

Since with respect to the basis e_i with the components $e_1 = 1$ the equalities

$$p^i_{1j} = \delta^i_j \tag{9}$$

hold, we have

$$f'^i = \frac{\partial f^i}{\partial x^1}. \tag{10}$$

Inserting (10) in (8) yields the Cauchy-Riemann relations

$$\frac{\partial f^i}{\partial x^1} - p^i_{kj} \cdot \frac{\partial f^j}{\partial x^1} = 0 \tag{11}$$

for the functions under study. The number $n(n-1)$ of these relations is growing quicker that the number n of components of analytical function. This leads to the *functional restriction* of the set of such functions at $n > 2$. The present work is just attempting to elaborate a non-trivial extension of the notion of analytical function of poly-number variable subject to the condition that number of the Cauchy–Riemann–type conditions does nor exceed the number of unknown function–components. The first step in this direction has been made above when introducing the pair $\{M_n, P_n\}$. Therefore it seems natural to replace the differential 6) by means of the absolute differential

$$DF(X) := \nabla_k f^i \cdot e_i \cdot dx^k, \tag{12}$$

where

$$\nabla_k f^i := \frac{\partial f^j}{\partial x^k} + \Gamma^i_{kj} \cdot f^j \tag{13}$$

is the covariant derivative, and Γ^i_{kj} means "the connection coefficients". Instead of the formulas (8) and (10) we get

$$\nabla_k f^i = p^i_{kj} \cdot f'^j \tag{14}$$

and

$$f'^i = \nabla_1 \cdot f^i, \tag{15}$$

and the Cauchy–Riemann conditions take on the form

$$\nabla_k f^i - p^i_{kj} \cdot \nabla_1 f^j = 0. \tag{16}$$

Of course, "the connection objects" Γ^i_{kj} in the formula (13) are not obligatory to be uniform for all the set of functions obeying the conditions (16).

2. Definitions and basic implications

Let us call the function $F(X)$ *generalized–analytical*, if such a function $F'(X)$ exists that

$$\widetilde{D}F(X) = F'(X) \cdot dX, \tag{17}$$

where

$$\widetilde{D}F(X) \equiv \widetilde{\nabla}_k f^i \cdot e_i \cdot dx^k \tag{18}$$

and the definition

$$\widetilde{\nabla}_k f^i := \frac{\partial f^i}{\partial x^k} + \gamma^i_k \tag{19}$$

has been used. It is assumed that under the transition from one (curvilinear) coordinate system to another coordinate system the involved objects γ^i_k are transformed according to the law

$$\gamma^{i'}_{k'} = \frac{\partial x^k}{\partial x^{k'}} \cdot \frac{\partial x^{i'}}{\partial x^i} \cdot \gamma^i_k - \frac{\partial x^k}{\partial x^{k'}} \cdot \frac{\partial^2 x^{i'}}{\partial x^k \partial x^i} \cdot f^i. \tag{20}$$

It will be noted that such a definition entails that $\widetilde{\nabla}_k f^i$ behaves like a tensor. The quantities γ^i_k will be called the *gamma-objects*. In general we do not assume the relations

$$\gamma^i_k = \Gamma^i_{kj} \cdot f^j \tag{21}$$

with a single "connection object" Γ^i_{kj} for generalized-analytical functions. It would be more precise to say of the pair $\{f^i, \gamma^i_k\}$, such that the analytical function of poly-number variable is the pair $\{f^i, 0\}$, but this pair transform to the pair $\{f^i, \gamma^{i'}_{k'} \neq 0\}$ under going over from the special coordinate system to another curvilinear one.

From the definition of generalized-analytical functions it follows that

$$\widetilde{\nabla}_k f^i = p^i_{kj} \cdot f'^j \tag{22}$$

and

$$f'^j = \widetilde{\nabla}_1 f^i; \tag{23}$$

the respective generalized Cauchy–Riemann relations take on the form

$$\widetilde{\nabla}_k f^j - p^i_{kj} \widetilde{\nabla}_1 f^j = 0. \tag{24}$$

The number of unknown functions in the pair $\{f^i, \gamma^i_k\}$ equals $n + n^2 = n(n+1)$, – which is more than the number $n(n-1)$ of the generalized Cauchy-Riemann relations (24). Thus, to use the notion of generalized-analytical function in theoretical-physical constructions it is necessary to additionally establish and formulate the set of requirements (possibly one requirement) which, when used in conjunction with the notion of generalized-analytical function, would lead unambiguously to equations of some field of physical meaning. Usually, they are n partial differential equations of second order for n independent function-component field.

If $\{f^i_{(1)}, \gamma^i_{(1)k}\}$ and $\{f^i_{(2)}, \gamma^i_{(2)k}\}$ – two generalized-analytical functions, then their arbitrary linear sum with real coefficients α, β is a generalized-analytical function. This ensues directly from the definition, on using also the formulae (22)–(24) and (20). Thus, we have

$$\alpha \cdot \{f^i_{(1)}, \gamma^i_{(1)k}\} + \beta \cdot \{f^i_{(2)}, \gamma^i_{(2)k}\} = \{\alpha \cdot f^i_{(1)} + \beta \cdot f^i_{(2)}, \alpha \cdot \gamma^i_{(1)k} + \beta \cdot \gamma^i_{(2)k}\}. \tag{25}$$

Now, let us consider the poly-number product of two generalized-analytical functions $f^i_{(1)}$ and $f^j_{(2)}$:

$$f^k_{(3)} = f^i_{(1)} \cdot f^j_{(2)} \cdot p^k_{ij} \tag{26}$$

and try to find the object $\gamma^i_{(3)k}$ such that the pair $\{f^i_{(3)}, \gamma^i_{(3)k}\}$ be generalized-analytical function. To this end we formally differentiate the left and right parts of (26) with respect to x^k and use the formula (22), obtaining

$$\frac{\partial f^i_{(3)}}{\partial x^k} + \gamma^i_{(3)k} = p^{i_1}_{kj} p^i_{i_1 i_2} f'^j_{(1)} f'^{j_2}_{(2)} + p^{i_2}_{kj} p^i_{i_1 i_2} f^{i_1}_{(1)} f'^j_{(2)}. \tag{27}$$

Owing to the formula

$$p^r_{im} \cdot p^m_{kj} = p^r_{km} \cdot p^m_{ij} \tag{28}$$

(which is an implication of the properties of associativity and commutativity of poly-numbers), we can write

$$\frac{\partial f^i_{(3)}}{\partial x^k} + \gamma^i_{(3)k} = p^i_{kj} p^j_{i_1 i_2} (f'^{i_1}_{(1)} f^{i_2}_{(2)} + f^{i_1}_{(1)} f'^{i_2}_{(2)}), \tag{29}$$

where

$$\gamma^i_{(3)k} = p^i_{i_1 i_2} \cdot (\gamma^{i_1}_{(1)k} f^{i_2}_{(2)} + f^{i_1}_{(1)} \gamma^{i_2}_{(2)k}). \tag{30}$$

The result (29) can conveniently be represented in terms of the absolute differential as follows:

$$D[F_{(1)}(X) \cdot F_{(2)}(X)] = [DF_{(1)}(X)] \cdot F_{(2)}(X) + F_{(1)}(X) \cdot [DF_{(2)}(X)] \tag{31}$$

or

$$D[F_{(1)}(X) \cdot F_{(2)}(X)] = [F'_{(1)}(X) \cdot F_{(2)}(X) + F_{(1)}(X) \cdot F'_{(2)}(X)] \cdot dX. \tag{32}$$

From the last formula we obtain the relation

$$[F_{(1)}(X) \cdot F_{(2)}(X)]' = F'_{(1)}(X) \cdot F_{(2)}(X) + F_{(1)}(X) \cdot F'_{(2)}(X). \tag{33}$$

It remains to clarify whether the transformation law of the objects $\gamma^i_{(3)k}$ under the transitions to arbitrary coordinate system is correct. With this aim the formula (30) should be written in a varied form:

$$\gamma^i_{(3)k} = p^i_{i_1 i_2} \cdot (\gamma^{i_1}_{(1)k} f^{i_2}_{(2)} + f^{i_1}_{(1)} \gamma^{i_2}_{(2)k}) + (\Gamma^i_{km} p^m_{i_1 i_2} - \Gamma^m_{ki_1} p^i_{mi_2} - \Gamma^m_{ki_2} p^i_{i_1 m}) \cdot f^{i_1}_{(1)} f^{i_2}_{(2)}, \tag{34}$$

where $\Gamma^j_{im} \equiv 0$ with the respect to our special coordinate system; however, under the transition to an arbitrary coordinate system the objects Γ^j_{ik} transform like ordinary connection objects and in general $\Gamma^{j'}_{i'k'} \neq 0$. The condition $\Gamma^j_{ik} \equiv 0$ can also be replaced to apply the more general condition

$$\Gamma^i_{km} p^m_{i_1 i_2} - \Gamma^m_{ki_1} p^i_{mi_2} - \Gamma^m_{ki_2} p^i_{i_1 m} \equiv 0 \tag{35}$$

and, moreover, the three coefficients Γ in (35) can be regarded as different. It is possible to restrict ourselves to but the class of generalized-analytical function obeying the property

$$({}^{(1)}\Gamma^i_{km} p^m_{i_1 i_2} - {}^{(2)}\Gamma^m_{ki_1} p^i_{mi_2} - {}^{(3)}\Gamma^m_{ki_2} p^i_{i_1 m}) \cdot f^{i_1}_{(1)} f^{i_2}_{(2)} \equiv 0. \tag{36}$$

Given the special coordinate system. If one has $\Gamma^i_{jk} \equiv {}^{(1)}\Gamma^i_{jk} \equiv {}^{(2)}\Gamma^i_{jk} \equiv {}^{(3)}\Gamma^i_{jk} \equiv 0$, then the tensor p^k_{ij} is transported "parallel" without any changes in components.

Thus, the poly-product of two generalized-analytical functions of poly-number variable is again a generalized-analytical function, and the formula (33) takes place for derivatives if one adopts that the "connection coefficients" associated to the tensor p^k_{ij} with respect to the special coordinate system vanishes identically over all three indices. In terms of the pairs $\{f^i, \gamma^i_k\}$ the poly-product of two generalized-analytical function can be written as follows:

$$\{f^{i_1}_{(1)}, \gamma^{i_1}_{(1)}\} \cdot \{f^{i_2}_{(2)}, \gamma^{i_2}_{(2)}\} = \{p^i_{i_1 i_2} f^{i_1}_{(1)} f^{i_2}_{(2)}, p^i_{i_1 i_2} \cdot (\gamma^{i_1}_{(1)k} f^{i_2}_{(2)} + f^{i_1}_{(1)}) \gamma^{i_2}_{(2)k})\}. \tag{37}$$

So, the polynomial or the converged series with real or poly-number coefficients of one or several generalized-analytical functions is a generalized-analytical function. The ordinary differentiation rules are operative for the respective derivative (which was denoted my means of the prime ($'$)) of such polynomials and series, whenever the tensor p^k_{ij} with respect to the special coordinate system vanishes identically over all three indices.

Since in such a theory of generalized-analytical functions of poly-number variable (in which the "connection objects" as well as the gamma-objects are different for each tensor and, generally speaking, for each index), the concept of "parallel transportation" is deprived of the geometrical simplicity that is characteristic of the spaces of affine connection, the Riemannian and pseudo-Riemannian spaces included. This notwithstanding, the concepts of absolute differential and covariant derivative can readily be extended on the basis of invariance of their form with respect to any curvilinear coordinate system. The covariant derivative $\widetilde{\bigtriangledown}_k$ for arbitrary tensor is defined quite similarly to the way which is followed to define the covariant derivative $\bigtriangledown_k$ in the spaces of affine connection; at the same time, for each tensor and probably for each index there exist, in general, their own "connection objects" or gamma-objects. The respective differential is constructed in accordance with the definition

$$\widetilde{D} := dx^k \cdot \widetilde{\bigtriangledown}_k. \tag{38}$$

Here, the converted indices can not be ignored, for "connection coefficients" correspond to them.

The Cauchy-Riemann relations (24) are necessary and sufficient conditions in order that f^i be a generalized-analytical function. Let us show that these relations can be written in an explicitly invariant form if the matrix composed of the numbers

$$q_{ij} = p^r_{im} p^m_{rj}, \tag{39}$$

is non-singular, that is if

$$q = \det(q_{ij}) \neq 0. \tag{40}$$

In this case the inverse matrix (q_{ij}) forms the tensor (q^{ij}) showing the properties

$$q_{jk} q^{ki} = q^{ik} q_{kj} = \delta^i_j. \tag{41}$$

Whence, when the formula (22) is applied instead of the formulae (23) and (24), we get the invariant expression for the derivative

$$f'^i = q^{is} p^r_{sm} \widetilde{\bigtriangledown}_r \cdot f^m \tag{42}$$

and for the Cauchy-Riemann relations

$$\widetilde{\bigtriangledown}_k f^i - p^i_{kj} \cdot q^{js} p^r_{sm} \widetilde{\bigtriangledown}_r f^m = 0. \tag{43}$$

Let us turn to the generalized-analytical functions $F_{(1)}(X)$ and $F_{(2)}(X)$, which are constrained by the relation

$$F_{(2)}(X) = F(X) \cdot F_{(1)}(X), \tag{44}$$

where $F(X)$ – some function of poly-number variable. The function is generalized-analytical in the field where the function $F_{(1)}(X)$ is not a divisor of zero. In this case

$$F(X) = \frac{F_{(2)}(X)}{F_{(1)}(X)}, \tag{45}$$

$$\widetilde{D}F(X) = \frac{F_{(1)}(X)\widetilde{D}[F_{(2)}(X)] - \widetilde{D}[F_{(1)}(X)]F_{(2)}(X)}{F^2_{(1)}(X)} \tag{46}$$

or

$$F'(X) = \frac{F_{(1)}(X)F'_{(2)}(X) - F'_{(1)}(X)F_{(2)}(X)}{F^2_{(1)}(X)}. \tag{47}$$

If

$$F(X) = F_{(2)}[F_{(1)}(X)], \tag{48}$$

then the function $F(X)$ is generalized-analytical with

$$F'(X) = F'_{(2)}(F_{(1)}) \cdot F'_{(1)}(X). \tag{49}$$

3. Similar geometries and conformal transformations

Actually, we are interested in not only the pair $\{M_n, P_n\}$ and generalized-analytical functions $\{f^i, \gamma^i_k\}$ but (eventually) possible ways of application of these notions to constructing physical models and solving new physical proiblems. Two spaces in which congruences of extremals (geodesics) coincide are similar in many respects. The extremals are meant to be solutions to set of equations for definition of curves over which the length of the curve acquires its extremum; alternatively, they are meant to be the curves which in a given geometry are defined to be geodesics (for example, geodesics in geometries of affine connection). However, for some physical as well as mathematical problems it is not of great importance which length element is used in applied space, – a real use is made to only the set of equations that define extremals (or to extremals proper). We shall say that two n–dimensional geometries are *similar*, if there exist such coordinate systems and parameters along curves that with respect to them the equations for extremals are equivalent and the initial and/or final date set forth in one space may also be given in another space.

All the set of generalized-analytic functions can be broken into the subsets $\{f^i, \Gamma^k_{ij}\}$ that involve the same connection coefficients Γ^k_{ij}, so that for all generalized-analytic functions from the subset the relation

$$\Gamma^i_{kj} f^j = \gamma^i_k \tag{50}$$

is fulfilled. It should be stressed (once more) that the coefficients Γ^k_{ij} are independent of any choice of functions in the subset $\{f^i, \Gamma^k_{ij}\}$. Generally speaking, the subset may be formed by only one generalized-analytic function. If f^i and γ^i_k are prescribed, then the relations (50) can be treated to be a set of equations for definition of the coefficients Γ^k_{ij}. Having find and fixed them, they can be applied for all tensors and indices, thereafter we get a due possibility to work with the space of affine connection $L_n(\Gamma^k_{ij})$ in which the set of equations for geodesics is of the form

$$\frac{d^2x^i}{d\sigma^2} = -\Gamma^i_{kj}\frac{dx^k}{d\sigma}\frac{dx^j}{d\sigma}. \tag{51}$$

Generally speaking, in this way we loose the possibility to use the poly–number product for construction of new generalized–analytical functions and should give up the simple differentiation rules (33). In the last case the covariant derivative $\widetilde{\nabla}_k$ in the special coordinate system must act on the tensor p^i_{kj}. In order to have simultaneously on the subset $\{f^i, \Gamma^k_{ij}\}$ the poly-number product of generalized–analytical functions and the rules (33), which application yields again a generalized–analytical function, we are to restrict ourselves to the functions subjected to the condition (36) with $\Gamma^i_{jk} \equiv {}^{(1)}\Gamma^i_{jk} \equiv {}^{(2)}\Gamma^i_{jk} \equiv {}^{(3)}\Gamma^i_{jk}$.

Let us require that the space $L_n(\Gamma^i_{jk})$ be similar to a Riemannian or pseudo–Riemannian one $V_n(g_{ij})$, where g_{ij} is a (fundamental) metric tensor. Then instead of (50) we get the system of equations

$$\left[\frac{1}{2}g^{im}\left(\frac{\partial g_{km}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^m}\right) + \frac{1}{2}(p_k\delta^i_j + p_j\delta^i_k) + S^i_{kj}\right]\cdot f^j = \gamma^i_k, \tag{52}$$

where S^i_{kj} stands for an arbitrary tensor (torsion tensor) obeying the property of skew–symmetry with respect to two indices, and p_i may be an arbitrary one–covariant tensor [1]. This system may be used to define the fundamental tensor g_{ij}.

There exist such Finslerian spaces which are not of Riemannian or pseudo–Riemannian type, but in which, however, one has the system of equations

$$\frac{d^2x^i}{d\sigma^2} = -\Gamma^i_{kj}[L(dx;x)]\cdot\frac{dx^k}{d\sigma}\frac{dx^j}{d\sigma}, \tag{53}$$

where the coefficients $\Gamma^i_{kj}[L(dx;x)]$ are defined by means of a relevant metric function $L(dx^1,\dots,dx^n;x^1\dots,x^n)$ of Finsler type. The corresponding Finsler spaces are similar to spaces of affine connection endowed with the connection coefficients Γ^i_{kj} deviated possibly from the coefficients $\Gamma^i_{kj}[L(dx;x)]$ by occurrence of an additive torsion tensor and/or an additive tensor $\frac{1}{2}(p_k\delta^i_j + p_j\delta^i_k)$ [1].

Let a generalized–analytical functions define spaces of the affine connection $L_n({}^{(1)}\Gamma^k_{ij})$ and $L_n({}^{(2)}\Gamma^k_{ij})$ similar to corresponding Riemannian or pseudo–Riemannian spaces $V_n(g_{ij})$ and $V_n(K_V^2 g_{ij})$ and/or the Finslerian spaces $F_n[L(dx;x)]$ and $F_n[K_F L(dx,x)]$, where $K_V(x^1,..,.x^n) > 0$, $K_F(x^1,...,x^n) > 0$ – scalar functions (invariants). Then the transformation (coordinate and/or in the space of generalized–analytical functions) going over the set $f^i_{(1)}$ in the set $f^i_{(2)}$, can be called conformal, for under this procedure one has

$$g_{ij}(x) \to K_V^2(x)\cdot g_{ij} \tag{54}$$

and

$$(dx;x) \to K_F(x)\cdot L(dx;x). \tag{55}$$

4. Possible additional requirements

From the definition of a generalized–analytical function it follows that it is possible to present the function by choosing two arbitrary one–covariant fields $f^i(x^1,\dots,x^n)$ and $f'^i(x^1,\dots,x^n)$. Then the formula (23) entails the following representation for the gamma–objects:

$$\gamma^i_k = -\frac{\partial f^i}{\partial x^k} + p^i_{kj}f'^j \tag{56}$$

The Cauchy–Riemann conditions are fulfilled automatically. So, to get the field equations for the unknown function–components $f^i(x^1,\dots,x^n)$ and $f'^i(x^1,\dots,x^n)$, it is nesessary to set forth at least $2n$ additional relations, for example, some partial differential equations of the first–order with respect to $f^i(x^1,\ldots,x^n)$ and $f'^i(x^1,\ldots,x^n)$.

(1): Let us consider the subset of generalized–analytical functions f^i such that

$$\tilde{D}F(x) \equiv 0, \quad \leftrightarrow \quad \tilde{\nabla}_k f^i \equiv 0, \quad \leftrightarrow \quad f'^i \equiv 0 \tag{57}$$

In this case the Cauchy–Riemann conditions are fulfilled automatically and arbitrary vector–function coupled with $\gamma^i_k = -\frac{\partial f^i}{\partial x^k}$, that is the pair $\{f^i, -\frac{\partial f^i}{\partial x^k}\}$, is a generalized–analytical function. It is important to note that the properties of poly–numbers do not influence this procedure. In other words, this subset (treated on the level of the Cauchy–Riemann conditions) are independent of any choice of the system of poly–numbers.

(2): If instead of the conditions (57) we assume the relations

$$\tilde{D}F(X) = \lambda \cdot F(X)\cdot dX, \quad \leftrightarrow \quad \tilde{\nabla}_k f^i = \lambda \cdot p^i_{kj}\cdot f^j, \quad \leftrightarrow \quad f'^i = \lambda \cdot f^i, \tag{58}$$

where λ is a real number, then the pairs $\{f^i, -\dfrac{\partial f^i}{\partial x^k} + \lambda p^i_{kj} f^j\}$ with arbitrary vector–functions f^i will form the subset of the generalized-alanytical functions which to some extent account for properties of poly–numbers.

(3): Farther generalizing the requirements (57) and (58) can be formulated in the form

$$F'(X) = \Lambda \cdot F(X), \tag{59}$$

where

$$\Lambda = \lambda^1 e_1 + \lambda^2 e_2 + ... + \lambda^n e_n \tag{60}$$

an arbitrary poly–number. In this case the pair

$$\left\{ f^i, -\frac{\partial f^i}{\partial x^k} + p^i_{kj} p^j_{mr}\lambda^m f^j \right\} \tag{61}$$

will be the generalized-alanytical functions.

(4): Using the formulas (23) and (24), we can prove the following statement. If the relations

$$1)\ \Gamma^i_{kj} f^j = \gamma^i_k, \tag{62}$$

$$2)\ \Gamma^i_{1j} p^j_{kr} - p^i_{kj}\Gamma^j_{1r} = 0, \tag{63}$$

$$3)\ \frac{\partial \Gamma^i_{1r}}{\partial x^k} - \frac{\partial \Gamma^i_{kr}}{\partial x^1} + \left[(\Gamma^i_{kj} - p^i_{km}\Gamma^m_{1j})\Gamma^j_{1r} - \Gamma^i_{1j}(\Gamma^j_{kr} - p^j_{km}\Gamma^m_{1r})\right] = 0 \tag{64}$$

hold, then together with the generalized-alanytical pair $\{f^i, \gamma^i_k\}$, the pair

$$\{f'^i, \Gamma^i_{kj} f'^j\}, \{f''^i, \Gamma^i_{kj} f''^j\}, \ldots, \{f^{(m)i}, \Gamma^i_{kj} f^{(m)j}\}, \ldots \tag{65}$$

are also generalized-alanytical. In the last formulas the notation

$$f^{(m)i} \equiv \frac{\partial f^{(m-1)j}}{\partial x^1} + \Gamma^i_{1j} f^{(m-1)j} \tag{66}$$

has been used.

(5): One additional requirements can sound: for the subset $\{f^i, \Gamma^i_{kj}\}$ of generalized-alanytical functions a Riemannian or pseudo–Riemannian geometry $V_n(g_{ij})$ similar to the affine connection geometry $L_n(\Gamma^i_{jk})$ can be found.

(6): If a Finsler space $F_n[L(dx;x)]$ is similar to a space of affine connection, then one among possible requirements can claim that the subset $\{f^i, \Gamma^i_{jk}\}$ give rise to an affine connection geometry similar to the Finsler geometry $F_n[L(dx;x)]$.

(7): Let

$$x^i = x^i(\tau) \tag{67}$$

be a parametric presentation of some curve joining two points $x^i_{(0)} = x^i(0)$, $x^i_{(0)} = x^i(0)$, that is, the parameter along curves varies in the limits $\tau \in [0;1]$. Let us consider the functional with integration along indicated curve

$$\begin{aligned} I[x^i(\tau)] &= \int_0^1 F(X)\,dX = \left[\int_0^1 p^i_{kj} f^k(x^1(\tau),\dots,x^n(\tau))dx^j\right]\cdot e_i \\ &= \left[\int_0^1 p^i_{kj} f^k \frac{dx^j}{d\tau}\right]\cdot e_i, \end{aligned} \tag{68}$$

where $F(X)$ – some generalized-alanytical function, and require that value of the integral (68) be independent of integration way, in which case the variation of this functional at fixed ends of curves should vanish, that is the Euler conditions

$$\frac{d}{d\tau}\left(p^i_{kj} f^j\right) - p^i_{mj}\frac{\partial f^j}{\partial x^k}\frac{dx^m}{d\tau} = 0 \tag{69}$$

or

$$\left(p^i_{kj}\frac{\partial f^j}{\partial x^m} - p^i_{mj}\frac{\partial f^j}{\partial x^k}\right)\cdot\frac{dx^m}{d\tau} = 0 \tag{70}$$

must be valid. Assuming that $x^i(\tau)$ are arbitrary smooth functions, from these equations we get

$$p^i_{kj}\frac{\partial f^j}{\partial x^m} - p^i_{mj}\frac{\partial f^j}{\partial x^k} = 0, \tag{71}$$

or, recollecting that $\{f^i, \gamma^i_k\}$ is a generalized–analytic pair,

$$p^i_{kj}\gamma^i_m - p^i_{mj}\gamma^j_k = 0. \tag{72}$$

From these relations it ensues that for the functions f^i the Cauchy–Riemann conditions (11) hold fine.

Thus, the assumption of independence of the integral (68) of the path leads to the conclusion that the function $F(X)$ is analytical, that is such an assumption is superfluous for non–trivial generalization of the concept of analyticity.

5. Case H_4

It is convenient to work with the associative-commutative hypercomplex numbers in term of the ψ-basis which relates to the basis

$$e_1 = 1, e_2 = j, e_3 = k, e_4 = jk, \qquad j^2 = k^2 = (jk)^2 = 1 \tag{73}$$

by means of the linear dependence

$$e_i = s^j_i \cdot \psi_j, \tag{74}$$

where

$$s_i^j = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad s_i^k \cdot s_k^j = 4 \cdot \delta_i^j. \tag{75}$$

For the basis elements $\psi_1, \psi_2, \psi_3, \psi_4$ the multiplication law

$$\psi_i \cdot \psi_j = p_{ij}^{(\psi)k} \cdot \psi_k \tag{76}$$

involves the characteristic numbers

$$p_{ij}^{(\psi)k} = \begin{cases} 1, & \text{if } i = j = k, \\ 0, & \text{in other cases} \end{cases} \tag{77}$$

We shall use the following notation:

$$X = x^1 e_1 + ... + x^4 e_4 = \xi^1 \psi_1 + ... + \xi^4 \psi_4 \tag{78}$$

and

$$F(X) = \varphi^1(\xi^1, ..., \xi^4) \cdot \psi_1 + \varphi^4(\xi^1, ..., \xi^4) \cdot \psi_4. \tag{79}$$

Thus, if $\varphi^i(\xi^1, ..., \xi^4)$ – a generalized-analytical function of the H_4-variable used, then such a vector-function $\varphi'^i(\xi^1, ..., \xi^4)$ can be found that

$$\frac{\partial \varphi^i}{\partial \xi^k} + \gamma_k^{(\psi)i} = p_{kj}^{(\psi)i} \cdot \varphi'^j. \tag{80}$$

Taking into account (77), we get

$$\left.\begin{array}{llll}
\dfrac{\partial \varphi^1}{\partial \xi^1} + \gamma_1^{(\psi)1} = \varphi'^1, & \dfrac{\partial \varphi^1}{\partial \xi^2} + \gamma_2^{(\psi)1} = 0, & \dfrac{\partial \varphi^1}{\partial \xi^3} + \gamma_3^{(\psi)1} = 0, & \dfrac{\partial \varphi^1}{\partial \xi^4} + \gamma_4^{(\psi)1} = 0, \\
\dfrac{\partial \varphi^2}{\partial \xi^1} + \gamma_1^{(\psi)2} = 0, & \dfrac{\partial \varphi^2}{\partial \xi^2} + \gamma_2^{(\psi)2} = \varphi'^2, & \dfrac{\partial \varphi^2}{\partial \xi^3} + \gamma_3^{(\psi)2} = 0, & \dfrac{\partial \varphi^2}{\partial \xi^4} + \gamma_4^{(\psi)2} = 0, \\
\dfrac{\partial \varphi^3}{\partial \xi^1} + \gamma_1^{(\psi)3} = 0, & \dfrac{\partial \varphi^3}{\partial \xi^2} + \gamma_2^{(\psi)3} = 0, & \dfrac{\partial \varphi^3}{\partial \xi^3} + \gamma_3^{(\psi)3} = \varphi'^3, & \dfrac{\partial \varphi^3}{\partial \xi^4} + \gamma_4^{(\psi)3} = 0, \\
\dfrac{\partial \varphi^4}{\partial \xi^1} + \gamma_1^{(\psi)4} = 0, & \dfrac{\partial \varphi^4}{\partial \xi^2} + \gamma_2^{(\psi)4} = 0, & \dfrac{\partial \varphi^4}{\partial \xi^3} + \gamma_3^{(\psi)4} = 0, & \dfrac{\partial \varphi^4}{\partial \xi^4} + \gamma_4^{(\psi)4} = \varphi'^4.
\end{array}\right\} \tag{81}$$

These relations involve the expression for the derivative

$$\varphi'^i = \frac{\partial \varphi^i}{\partial \xi^{i_-}} + \gamma_{i_-}^{(\psi)i} \tag{82}$$

($i = i_-$, for which no summation is assumed), and also the Cauchy-Riemann relations

$$\frac{\partial \varphi^i}{\partial \xi^k} + \gamma_k^{(\psi)i} = 0, \; i \neq k. \tag{83}$$

The space H_4 is the metric (Finslerian) space in which the length element ds is expressible through the form $d\xi^1 d\xi^2 d\xi^3 d\xi^4$ in a conic region defined possibly in various ways. Let us stipulate that

$$ds = \sqrt[4]{d\xi^1 d\xi^2 d\xi^3 d\xi^4}, \tag{84}$$

assuming that the region is prescribed by the inequalities

$$d\xi^1 \geq 0,\ d\xi^2 \geq 0,\ d\xi^3 \geq 0,\ d\xi^4 \geq 0\ . \tag{85}$$

Let us consider the four-dimensional Finslerian geometry with the length element of the form

$$ds = \sqrt[4]{\kappa^4 \cdot d\xi^1 d\xi^2 d\xi^3 d\xi^4}, \tag{86}$$

where $\kappa \equiv \kappa(d\xi^1 d\xi^2 d\xi^3 d\xi^4) > 0$. Such a geometry is not Riemannian or pseudo-Riemannian. Let us show that such a geometry is similar (according to terminology adopted above) to some affine geometry with a connection $L_4(\Gamma^i_{kj})$. Let us write equations for extremals of this Finslerian space by using the tangential equation of indicatrix [2]:

$$\Phi(p_1, ..., p_4; \xi^1, ..., \xi^4) = 0\ , \tag{87}$$

where

$$\Phi(p;\xi) = p_1 p_2 p_3 p_4 - \left(\frac{\kappa}{4}\right)^4\ , \tag{88}$$

and

$$p_i = \frac{\partial(ds)}{\partial(d\xi^i)} = \frac{1}{4} \cdot \frac{\sqrt[4]{\kappa^4 \cdot d\xi_1 d\xi_2 d\xi_3 d\xi_4}}{d\xi^i}. \tag{89}$$

Then the set of equations for definition of extremals reads

$$\left.\begin{array}{c} \dfrac{d\xi^1}{\dfrac{\partial\Phi}{\partial p_1}} = ... = \dfrac{d\xi^4}{\dfrac{\partial\Phi}{\partial p_4}} = \dfrac{dp_1}{-\dfrac{\partial\Phi}{\partial\xi^1}} = ... = \dfrac{dp_4}{-\dfrac{\partial\Phi}{\partial\xi^4}}, \\ \Phi(p,\xi) = 0; \end{array}\right\} \tag{90}$$

or

$$d\xi^i = \frac{\partial\Phi}{\partial p_i} \cdot \lambda \cdot d\tau,\ dp_i = -\frac{\partial\Phi}{\partial\xi^i} \cdot \lambda \cdot d\tau,\ \ \Phi(p;\xi) = 0, \tag{91}$$

where τ – a parameter along extremals, and $\lambda \equiv \lambda(p;\xi) \neq 0$ – a function. For the tangential equation of the indicatrix (87), (88) the set of equations (91) takes on the form

$$\dot{\xi^i} = \frac{p_1 p_2 p_3 p_4}{p_i} \cdot \lambda, \quad \dot{p^i} = (\frac{1}{4})^4 \frac{\partial k^4}{\xi^i} \cdot \lambda, \quad p_1 p_2 p_3 p_4 = \left(\frac{k}{4}\right)^4, \tag{92}$$

with

$$\dot{\xi^i} = \frac{d\xi^i}{d\tau},\ \ \dot{p_i} = \frac{dp_i}{d\tau}. \tag{93}$$

Let us consider $\lambda = \lambda(\xi) > 0$ to be a function of only coordinates. Then, by explicating p_i, we get the set of equations for definition of extremals in the Finslerian space (86) in the form

$$\ddot{\xi^i} = -\Gamma^i_{kj}\ \dot{\xi^k}\dot{\xi^j}, \tag{94}$$

where

$$\Gamma^i_{kj} = -\left\{\begin{array}{ll} \dfrac{\partial ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial\xi^j}, & \text{if } i = j = k, \\ \delta^i_k \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^j}, & \text{in other cases;} \end{array}\right. \tag{95}$$

$$\sigma = \left(\frac{\kappa}{4}\right)^4 \cdot \lambda, \tag{96}$$

λ_0 and σ_0 are constants of relevant dimensions. Let us write down explicitly the coefficients Γ^i_{kj}:

$$(\Gamma^1_{kj}) = -\begin{pmatrix} \frac{\partial ln\left(\frac{\lambda}{\lambda_0}\right)}{\partial\xi^1} & \frac{\partial ln\left(\frac{\sigma}{\sigma_0}\right)}{\partial\xi^2} & \frac{\partial ln\left(\frac{\sigma}{\sigma_0}\right)}{\partial\xi^3} & \frac{\partial ln\left(\frac{\sigma}{\sigma_0}\right)}{\partial\xi^4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{97}$$

$$(\Gamma^2_{kj}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\partial ln\left(\frac{\sigma}{\sigma_0}\right)}{\partial\xi^1} & \frac{\partial ln\left(\frac{\lambda}{\lambda_0}\right)}{\partial\xi^2} & \frac{\partial ln\left(\frac{\sigma}{\sigma_0}\right)}{\partial\xi^3} & \frac{\partial ln\left(\frac{\sigma}{\sigma_0}\right)}{\partial\xi^4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{98}$$

$$(\Gamma^3_{kj}) = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\partial ln\left(\frac{\sigma}{\sigma_0}\right)}{\partial\xi^1} & \frac{\partial ln\left(\frac{\sigma}{\sigma_0}\right)}{\partial\xi^2} & \frac{\partial ln\left(\frac{\lambda}{\lambda_0}\right)}{\partial\xi^3} & \frac{\partial ln\left(\frac{\sigma}{\sigma_0}\right)}{\partial\xi^4} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{99}$$

$$(\Gamma^4_{kj}) = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\partial ln\left(\frac{\sigma}{\sigma_0}\right)}{\partial\xi^1} & \frac{\partial ln\left(\frac{\sigma}{\sigma_0}\right)}{\partial\xi^2} & \frac{\partial ln\left(\frac{\sigma}{\sigma_0}\right)}{\partial\xi^3} & \frac{\partial ln\left(\frac{\lambda}{\lambda_0}\right)}{\partial\xi^4} \end{pmatrix}. \tag{100}$$

It will be noted that instead of the matrices (97) – (100) one can take their transforms. Thus, the Finslerian geometry with the length element (86) is similar to the geometry of the affine connection $L_4[\Gamma^i_{kj} + S^i_{kj} + \frac{1}{2}(p_k\delta^i_j + p_j\delta^i_k)]$, where S^i_{kj} – a tensor which is assumed to be skew-symmetric with respect to the subscripts, and p_k stands for an arbitrary one-covariant tensor.

Let us consider the generalized-analytical functions φ^i of H_4-variable that obey the additional condition 3), that is the pair

$$\left\{\varphi^i, -\frac{\partial\varphi^i}{\partial\xi^k} + p^{(\psi)i}_{kj}\mu^j\varphi^j\right\}, \tag{101}$$

where

$$\Lambda = \lambda^i \cdot e_i = \mu^j \cdot \psi_j. \tag{102}$$

Let us select from such pairs a subset $\{\varphi^i, \Gamma^i_{kj}\}$, where Γ^i_{kj} are defined by the matrices transposed to the matrices (97) – (100). In this way, the requirement 6) is retained. Then the pair (101) should fulfill the 16 relations (50) the first four of which are

$$\frac{\partial \varphi^1}{\partial \xi^1} = \mu^1 \varphi^1 + \frac{\partial\, ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial \xi^1}\varphi^1, \qquad \frac{\partial \varphi^1}{\partial \xi^2} = \frac{\partial\, ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial \xi^2}\varphi^1, \tag{103}$$

$$\frac{\partial \varphi^1}{\partial \xi^3} = \frac{\partial\, ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial \xi^3}\varphi^1, \qquad \frac{\partial \varphi^1}{\partial \xi^4} = \frac{\partial\, ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial \xi^4}\varphi^1.$$

For the compatibility it is necessary and sufficient that the mixed derivatives obtained with the help of the formulae (103) be equal. A part of these equations, except for three ones, is automatically satisfied. If we consider all the 16 equations, not confining ourselves to the first four equations, we get the following 12 conditions:

$$\frac{\partial^2 ln\left(\dfrac{\kappa}{\kappa_0}\right)^4}{\partial \xi^i \xi^j} = 0,\ i \neq j; \tag{104}$$

from which it ensues that

$$ln\left(\frac{\kappa}{\kappa_0}\right)^4 = a_1(\xi^1) + a_2(\xi^2) + a_3(\xi^3) + a_4(\xi^4) \tag{105}$$

or

$$\kappa = \kappa_0 \cdot \exp\{[a_1(\xi^1) + a_2(\xi^2) + a_3(\xi^3) + a_4(\xi^4)]/4\}, \tag{106}$$

where a_i are four arbitrary functions of one real argument. Then from equations (103) and relevant equations for other components of the generalized-analytical function, we get

$$\varphi^i = \varphi^i_{(0)}\left(\frac{\kappa}{\kappa_0}\right)^4\left(\frac{\lambda}{\lambda_0}\right) b_i(\xi^{i-}) \cdot exp(\mu^{i-}\xi^i), \tag{107}$$

where

$$a_i(\xi^{i-}) = ln\left|b_i(\xi^{i-})\right|. \tag{108}$$

Thus, despite of two additional requirement, the generalized-analytical function (107) in general case is not reducible to an analytical function of H_4-variable, and besides we obtain the expression (106) for the coefficients κ in the metric function of the Finslerian space with the length element (86). If $\frac{\lambda}{\lambda_0} = \left(\frac{\kappa_0}{\kappa}\right)^4$, then φ^i is an analytical function.

If

$$\kappa = \kappa_0 \cdot \exp\{[(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + (\xi^4)^2]/4\}, \tag{109}$$

then with respect to the coordinates x^i

$$\kappa = \kappa_0 \cdot exp\{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\}. \tag{110}$$

Conclusion

Having introduced the concept of the generalized-analytical function of poly-number variable in the present work, we have made the first step in the direction of constructing a relevant theory aiming to develop theoretical-physical models. An important ingredient of such investigations must be search for additional requirements to be obeyed by the generalized-analytical functions and for the consequences implied by the requirements. The conditions that lead to trivial results — to analytical functions — should especially be analyzed. This may admit formulating the properties that are forbidden to attribute proper generalized-analytical functions of poly-number variable (in contrast to analytical functions proper). As it has been shown above, the independence of integral of integration path relates to such properties. Of course, it is necessary to carry out a particular attentive study to compare the properties of analytical functions of complex variable and generalized-analytical functions of poly-number variable in case of the dimension exceeding 2. It can be hoped, therefore, that the concepts and results of the present work may face future novel theoretical-physical applications.

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THE ALGEBRODYNAMICS: PRIMODIAL LIGHT, PARTICLES-CAUSTICS AND FLOW OF TIME

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In the field theories with twistor structure particles can be identified with (spacially bounded) caustics of null geodesic congruences defined by the twistor field. As a realization, we consider the "algebrodynamical" approach based on the field equations which originate from noncommutative analysis (over the algebra of biquaternions) and lead to the complex eikonal field and to the set of gauge fields associated with solutions of the eikonal equation. Particle-like formations represented by singularities of these fields possess "elementary" electric charge and other realistic "quantum numbers" and manifest self-consistent time evolution including transmutations. Related concepts of generating "World Function" and of multivalued physical fields are discussed. The picture of Lorentz invariant light-formed aether and of matter born from light arises then quite naturally. The notion of the Time Flow identified with the flow of primodial light ("pre-Light") is introduced in the context.

Introduction. The algebrodynamical field theory

Theoretical physics has arrived to the crucial point at which it should fully reexamine the sense and the interrelations of the three fundamental entities: fields, particles and space-time geometry. *String theory* offers a way to derive the low-energy phenomenology from the unique physics at Plankian scale. However, it doesn't claim to find the *origin* of physical laws, the *Code of Universe* and is in fact nothing but one more attempt to *describe* Nature (in a possibly the most effective way) but not at all to *understand* it.

Twistor program of R. Penrose [1, 2] suggests an alternative to string theory in the framework of which one can hope, in principle, to explain the origin of basic physical entities. For this, one only assumes the existence of the primary *twistor space* $\mathbb{C}P^3$ which underlies the physical space-time and predetermines its Minkowsky geometry and, to some extent, the set of physical fields.

The most interesting manifestation of twistor structure is its ability to reduce the resolution of free massless (conformally invariant) equations (both linear and nonlinear ones, specifically of the Yang-Mills type) either to explicit integration in twistor space (the so called Penrose transform) or to resolution of purely algebraic problems (the Kerr theorem, the Ward construction etc. [2]). Making use of the Kerr theorem and of the Penrose's "nonlinear graviton construction", one can also obtain, in a purely algebraic way, the whole set of the self-dual solutions to (complex) Einstein equations.

However, general concept of twistor program as a unified field theory is not at all clear or formulated up to now. Which equations are really fundamental, in which way can the massive fields be described and in which way the particles' spectrum can be obtained? And, finally, why precisely twistor, a rather refined mathematical object, should be taken as a basis of fundamental physics?

In the interim, twistor structure arises quite naturally in the so called *algebrodynamics* of physical fields which has been developed in our works. From general viewpoint, the paradigm of algebrodynamics can be thought of as a revive of Pithagorean or Platonean ideas about *"Numbers governing physical laws"*. As the only (!) postulate of algebrodynamics one admits the existence of a certain unique and exeptional structure, of purely abstract (algebraic) nature, the internal properties of which completely determine both the geometry of physical space-time and the dynamics of physical fields (the latters being also algebraic in nature).

In the most successful realization of algebrodynamics principal structure of the "World algebra" has been introduced via generalization of complex analysis to exeptional noncommutative algebras of quaternion ($\mathbb{Q}$) type [15, 16, 17, 25, 27]. In particular, it was demonstrated that explicit account of noncommutativity in the very definition of functions "differentiable" in $\mathbb{Q}$ inevitably results in the *non-linearity* of the generalized Cauchy-Riemann equations (GCRE) which follow. This makes it possible to regard the GCRE as fundamental dynamical equations of *interacting* physical fields represented by (differentiable) functions of the algebraic $\mathbb{Q}$-type variable.

A wide class of such fields-functions exists only for the complex extension of $\mathbb{Q}$-algebra, i.e. for the algebra of complex quaternions $\mathbb{B}$ (*biquaternions*). Over the $\mathbb{B}$-algebra, the GCRE turn to be Lorentz invariant and acquire, moreover, the gauge and the spinor structures. On this base a self-consistent and unified *algebrodynamical field theory* has been constructed in our works [15, 16, 24, 25, 27, 26, 28, 30].

From the physical viewpoint, the most important property of GCRE is their direct correspondence to a fundamental *light-like* structure. The latter manifests itself in the fact that every (spinor) component $S(x, y, z, t) \in \mathbb{C}$ of the primary $\mathbb{B}$-field must satisfy the *complex eikonal equation* (CEE) [14, 15]

$$\eta^{\mu\nu}\partial_\mu S \partial_\nu S = (\partial_t S)^2 - (\partial_x S)^2 - (\partial_y S)^2 - (\partial_z S)^2 = 0, \tag{1}$$

where $\eta_{\mu\nu} = diag\{1, -1, -1, -1\}$ is the Minkowsky metric and ∂ stands for the partial derivative by respective coordinate. The CEE (1) is Lorentz invariant, nonlinear and plays the role similar to that of the *Laplace equation* in complex analysis. Each solution to GCRE can be reconstructed from a set of (four or less) solutions to CEE.

In the meantime, in [30] the intrinsic *twistor* structure of CEE has been discovered, and on its base the general solution of the nonlinear eikonal equation has been obtained. It was proved that, in this respect, every CEE solution belongs to one of two classes which both can be obtained from a twistor generating function via a simple and purely algebraic procedure. This construction allows also for definition of singular loci of the null geodesic congruences correspondent to the eikonal field – the *caustics*. Just at the caustics – the envelopes of the congruences – the neighbouring rays intersect each other, and the associated physical fields turn to infinity forming, thus, a unique *particle-like* object – a common source of the fields and of the congruence itself. Thus, in the algebrodynamical theory *the particles can be considered as (spacially bounded) caustics of the primodial null congruences.*

On the other hand, null congruences naturally define the universal local "transfer" of the basic twistor field with fundamental constant velocity "c" (in full analogy with the transfer of field by an electromagnetic wave) and point thus to exceptional role of the time coordinate in the algebrodynamical scheme and in twistor theory in general. Existence of the "Flow of Time" becomes therein a direct consequence of the existence of Lorentz invariant "aether" formed by the primodial light-like congruence ("preLight"). In the paper, we underline the principal property of *multivaluedness* of the fundamental complex solution to CEE ("World solution") and of the physical fields associated with it. As a result, at each space-time point one has a *superposition* of a great number of rays which belong to locally distinct null congruences, and the Time Flow turns to be *multi-directional*, i.e. consists of a number of superposed "subflows" (linked globally by complex structure into a unique physical "corpuscular-field" dualistic complex).

In section 2 we consider the twistor structure of CEE and the procedure of algebraic construction of its two classes of solutions. A few simple illustrative examples are presented. In section 3 we discuss the caustic structure of the CEE solutions, in particular of spatially bounded type (particle-like singular objects), and the properties of associated physical fields. In section 4, we introduce the "World function" responsible for generation of the "World solution" to CEE and discuss the related concept of multivaluedness of physical fields. Final section 5 is devoted to some general issues which bear on the nature of physical time. The notions of the *primodial light* ("pre-Light") and of the *light-formed aether* are introduced, and the Time Flow is actually identified with the Flow of preLight. Intrinsic structure of these fundamental flows is studied which relates to the property of multivaluedness of the basic twistor field.

The article is an extended version of the preceding paper [41] and, as to description of physical picture of the World, continues our paper [42]. In order to simplify the presentation, we avoid to apply the 2-spinor and the other refined mathematical formalisms, for this refering a prepared reader to our recent papers [25, 27, 28, 30].

The two classes of solutions to the complex eikonal equation

The eikonal equation describes the process of propagation of wave fronts (field discontinuities) in any relativistic theory, in Maxwell electrodynamics in particular [4, 5]. Physical and mathematical problems related to the eikonal equation were dealt with in a lot of works, see e.g. [6, 7, 8, 10, 11, 12].

The complex eikonal equation (CEE) arises naturally in problems of propagation of restricted light beams [13] and in theory of congruences related to solutions of Einstein or Einstein-Maxwell system of equations [14]. We, however, interpret the complex eikonal, to the first turn, as a fundamental physical field which describes, in particular, the interacting and "self-quantized" particle-like objects formed by singularities of the CEE solutions. By this, the electromagnetic and the other conventional physical fields can be associated with any solution of the CEE; they are responsible for description of the process of interaction of particles-singularities. Note that particle-like properties of field singularities related to the *5-dimensional* real eikonal field have been studied in [9]; the concept of particles as singularities of electromagnetic and eikonal fields has been incidentically discussed by many authors, in particular by H. Bateman [6] af far as in 1915.

We start with a definition, together with Cartesian space-time coordinates $\{t, x, y, z\}$, of the so called *spinor* or *null* coordinates $\{u, v, w, \bar{w}\}$ (the light velocity is taken to be unity, $c = 1$)

$$u = t + z, \quad v = t - z, \quad w = x - iy, \quad \bar{w} = x + iy \tag{2}$$

which form the *Hermitian* 2×2 matrix $X = X^+$ of coordinates

$$X = \begin{pmatrix} u & w \\ \bar{w} & v \end{pmatrix} \tag{3}$$

In the representation using spinor coordinates the CEE (1) looks as follows:

$$\partial_u S \partial_v S - \partial_w S \partial_{\bar{w}} S = 0. \tag{4}$$

The CEE possesses a remarkable *functional invariance* [15, 16]: for every $S(X)$ being its solution any (differentiable) function $f(S(X))$ is also a solution. The eikonal equation is known also [6] to be invariant under transformations of the full 15-parameter *conformal group* of the Minkowsky space-time.

Let us take now an *arbitrary homogeneous* function Π of two pairs of complex variables $\{\xi, \tau\}$

$$\Pi = \Pi(\xi_0, \xi_1, \tau^0, \tau^1) \tag{5}$$

which are **linearly dependent** at any space-time point via the so called *incidence relation*

$$\tau = X\xi \quad \Leftrightarrow \quad \tau^0 = u\xi_0 + w\xi_1, \quad \tau^1 = \bar{w}\xi_0 + v\xi_1, \tag{6}$$

and which transform as *2-spinors* under Lorentz rotations [1]. The pair of 2-spinors $\{\xi(X), \tau(X)\}$ linked through Eq.(6) is known as a (null projective) *twistor* of the Minkowsky space-time [2].

Let us assume now that one of the components of the spinor $\xi(X)$, say ξ_0, is not zero. Then, by virtue of homogeneity of the function Π, we can reduce the number of its arguments to *three projective twistor variables*, namely to

$$\Pi = \Pi(G, \tau^0, \tau^1), \qquad G = \xi_1/\xi_0, \quad \tau^0 = u + wG, \quad \tau^1 = \bar{w} + vG \tag{7}$$

Now we are in order to formulate the main result proved in our paper [30].

Theorem. *Any (analytical) solution of CEE belongs, with respect to its twistor structure, to one of two and only two classes and can be obtained from some generating twistor function of the form (7) via one of the two simple algebraical procedures (described below).*

To obtain the first class of solutions, let us simply resolve the algebraic equation defined by the function (7)

$$\Pi(G, u + wG, \bar{w} + vG) = 0 \tag{8}$$

with respect to the only unknown G. In this way we come to a complex field $G(X)$ which necessarily satisfies the CEE. Indeed, after substitution $G = G(X)$ Eq.(8) becomes an *identity* and, in particular, can be differentiated with respect to the spinor coordinates $u, v, w, \bar{w}$. Then we get

$$P\partial_u G = -\Pi_0, \quad P\partial_w G = -G\Pi_0, \quad P\partial_{\bar{w}} G = -\Pi_1, \quad P\partial_v G = -G\Pi_1, \tag{9}$$

where Π_0, Π_1 are the partial derivatives of Π with respect to its twistor arguments τ^0, τ^1 while P is its *total* derivative with respect to G,

$$P = \frac{d\Pi}{dG} = \partial_G \Pi + w\Pi_0 + v\Pi_1 \ , \tag{10}$$

which we thus far assume to be nonzero in the space-time domain considered. Multiplying then Eqs.(9) we prove that $G(X)$ satisfies the CEE in the form (4). It is easy to check that *arbitrary* twistor function $S = S(G, u + wG, \bar{w} + vG)$, under substitution of the obtained $G = G(X)$, also satisfies the CEE (owing to the functional constraint (8) it depends in fact on only *two* of three twistor variables).

To obtain the second class of CEE solutions, we have from the very beginning to differentiate the function Π with respect to G and only after this to resolve the resulting algebraic equation

$$P = \frac{d\Pi}{dG} = 0 \tag{11}$$

[1] To simplify the notation, we do not distinguish between the primed and unprimed spinor indices. In the incidence relation (6) the standard factor "i" (imaginary unit) is omitted what is admissible under the proper redefinition of the twistor norm

with respect to G again. Now the function $G(X)$ does not satisfy the CEE; however, if we substitute it into (7) the quantity Π becomes an explicit function of space-time coordinates and necessarily satisfies the CEE (as well as any function $f(\Pi(X))$ by virtue of functional invariance of the CEE). Indeed, differentiating the function Π with respect to the spinor coordinates we get

$$\partial_u \Pi = \Pi_0 + P\partial_u G, \;\; \partial_w \Pi = G\Pi_0 + P\partial_w G, \;\; \partial_{\bar{w}} \Pi = \Pi_1 + P\partial_{\bar{w}} G, \;\; \partial_v \Pi = G\Pi_1 + P\partial_v G, \quad (12)$$

and, taking into account the generating condition (11), we immegiately find that the function Π itself obeys the CEE (4).

The functional condition (8) and, therefore, the CEE solutions of the first class are in fact well known. Indeed, apart from the CEE, the field $G(X)$, if it is obtained by the resolution of Eq.(8), satisfies (as it is easily seen from Eqs.(9) for derivatives), the *over-determined* system of differential constraints

$$\partial_u G = G\partial_w G, \quad \partial_{\bar{w}} G = G\partial_v G \quad (13)$$

which define the so called *shear-free (null geodesic) congruences* (SFC). By this, algebraic Eq.(8) represents (in implicit form) *general solution* of Eqs.(13), i.e. describes the whole set of SFC in the Minkowski space-time. This remarkable statement proved in [18] is known as the *Kerr theorem.*

The second class of CEE solutions generated by algebraic constraint (11), to our knowledge, hasn't been considered in literature previously [2]. It is known, however, that condition (11) defines the *singular locus* for SFC, i.e. for the CEE solutions obtained from the Kerr constraint (8). Precisely, condition (11) fixes the *branching points* of the principal complex field $G(X)$ or, equivalently, – the space-time points where Eq.(8) has *multiple roots.* As to the CEE solutions of the second class themselves, their branching points occure at the locus defined by another condition which evidently follows from generating Eq.(11) and has the form

$$\Lambda = \frac{d^2\Pi}{dG^2} = 0. \quad (14)$$

The null congruences (especially the congruences with zero shear), as well as their singularities and branching points, play crucial role in the algebrodynamical approach. They will be discussed below in more details. Here we only repeat that, as it has been proved in [30],

the two simple generating procedures described above exhaust all the (analytical) solutions to the CEE representing, thus, its general solution

(note only that for solutions with zero spinor component, $\xi_0 = 0$, another gauge, in compare with the one used above, should be choosed). The obtained result can be thought of as *a direct generalization of the Kerr theorem.*

To make the exposition more clear, we present below several examples of the described construction.

1. Static solutions. Let the generating function Π depends on its twistor variables in the following way:

$$\Pi = \Pi(G, H), \quad H = G\tau^0 - \tau^1 = wG^2 + 2zG - \bar{w}, \quad (15)$$

where $z = (u - v)/2$, and the time coordinate $t = (u + v)/2$ is, in this way, eliminated. It is evident that the generating ansatz (15) covers the whole class of *static* CEE solutions.

[2]Study of solutions of the *real* eikonal equation by differentiation of generating functions depending on coordinates as parameters is used in general theory of sungularities of caustics and wavefronts [11]

In [21, 14] it was proved that static solutions to the SFC equations (and, therefore, static solutions to the CEE too) with *spacially bounded* singular locus are exhausted, up to 3D translations and rotations, by the *Kerr solution* [18] which follows from generating function of the form

$$\Pi = H + 2iaG = wG^2 + 2z^*G - \bar{w}, \quad (z^* = z + ia) \tag{16}$$

with a real constant parameter $a \in \mathbb{R}$. Explicitly resolving equation $\Pi = 0$ which is quadratic in G we obtain the two "modes" of the field $G(X)$

$$G = \frac{\bar{w}}{z^* \pm r^*} = \frac{x + iy}{z + ia \pm \sqrt{x^2 + y^2 + (z + ia)^2}} \tag{17}$$

which in the case $a = 0$ correspond to the ordinary *stereographic projection* $S^2 \mapsto \mathbb{C}$ from the North or the South pole respectively. It is easy to check that this solution and also its twistor counterpartners

$$\tau^0 = t + r^*, \quad \tau^1 = G\tau^0, \tag{18}$$

satisfy the CEE (as well as any function of them). Correspondent SFC is in the case $a = 0$ *radial* with a point singularity; in general case $a \neq 0$ the SFC is formed by the rectilinear constituents of a system of hyperboloids and has a *ring-like* singularity of a radius $R = |a|$. Using this SFC, a Riemannian metric (of the "Kerr-Schild type") and an electric field can be defined which satisfy together the electrovacuum Einstein-Maxwell system. In the case $a = 0$ this is the Reissner-Nordström solution with Coulomb electric field, in general case – the Kerr-Newman solution with three characteristical parameters: the mass M, the electric charge Q and the angular momentum (spin) Mca, – for which the field distribution possesses also the proper magnetic moment Qa which corresponds to the gyromagnetic ratio specific for the Dirac particle [19, 20]. In the algebrodynamical scheme, moreover, *electric charge of the point or the ring singularity is necessarily fixed in modulus*, i.e. "elementary" [15, 16, 28, 29] (see also [25] where a detailed discussion of this solution in the framework of algebrodynamics can be found).

Now let us obtain, from the same generating function, a solution to CEE of the second class. Differentiating Eq.(16) with respect to G and equating derivarive to zero, we get $G = -z^*/w$ and, substituting this expression into Eq.(16), obtain finally the following solution to CEE (which is univalued everywhere on 3D-space):

$$\Pi = -\frac{(r^*)^2}{w} = -\frac{x^2 + y^2 + (z + ia)^2}{x - iy}. \tag{19}$$

It is instructive to note that equation $\Pi = 0$, being equivalent to two real-valued constraints $z = 0,\ x^2 + y^2 = a^2$, defines here the ring-like singularity for the Kerr solution (17), as it should be in account of the theorem above presented (for this, see also section 4).

Static solutions of the II class with spatially bounded singularities are not at all exhausted by the solution (19). Consider, for example, solutions generated by the functions

$$\Pi = \frac{G^n}{H}, \quad n \in \mathbb{Z}, \quad n > 2. \tag{20}$$

We'll not write out correspondent solutions in explicit form and shall restrict ourselves by examination of the spacial structure of their singularities which can be obtained from the joint system of equations $P = 0,\ \Lambda = 0$, see Eqs. (11), (14). Eliminating from the latter the unknown field G we find that singularities (branching points of the eikonal field) have again the ring-like form $z = 0,\ x^2 + y^2 = R^2$ with radii equal to

$$R_n = \frac{a(n-1)}{\sqrt{n(n-2)}} \tag{21}$$

The cases $n = 1, 2$ evidently need special consideration. For $n = 1$ equating to zero derivative of the function G/H we find $G = \pm i\bar{w}/\rho$ with $\rho = \sqrt{x^2 + y^2}$. This brings us after substitution to the following solution of the CEE:

$$\Pi = (z + ia \pm i\sqrt{x^2 + y^2})^{-1} \tag{22}$$

which has the pole at the ring $z = O, x^2 + y^2 = a^2$ but *has a branching point only on the origin* $r = 0$, i.e. *which under any a corresponds to the point singularity.*

In the case $n = 2$ via analogous procedure we get $G = \bar{w}/z^*$ and after substitution come to the following solution of the CEE [30]:

$$\Pi = \frac{\bar{w}}{r^*} = \frac{x + iy}{x^2 + y^2 + (z + ia)^2} \tag{23}$$

which is of the same structure as (the inverse of) the solution (19). As the latter, it has no branching points on the real space-time slice while its pole corresponds to the Kerr ring. Let us take for simplicity $a = -1$; then solution (23) can be rewritten in the following familiar form:

$$\Pi = i\frac{x + iy}{2z + i(r^2 - 1)} \tag{24}$$

which can be easily identified as the standard *Hopf map*. As the solution of the CEE it has been studied in [22] and especially in the recent paper [23] where its geometrical and topological nature has been examined in detail. We suspect also that generalized Hopf maps considered therein relate (in the case $m = 1$) to the CEE solutions generated by the functions (20) and, as the latters, has the ring singularities correspondent to those represented by Eq.(21). However, this should be verified by direct calculations.

2. Wave solutions. Consider also the class of generating functions dependent on one of the two twistor variables τ^0, τ^1 only, say on τ^0:

$$\Pi = \Pi(G, \tau^0) = \Pi(G, u + wG). \tag{25}$$

Both classes of the CEE solutions obtained via functions (25) will then depend on only two spinor coordinates $u = t + z,\ w = x - iy$. This means, in particular, that the fields propagate along the Z-axis with fundamental (light) velocity $c = 1$. A "photon-like" solution of this type, with singular locus spacially bounded in all directions, was presented in [29].

Notice also that an example of the CEE solution with a considerably more rich and realistic structure of singular locus is presented below in section 4 (see also [29]).

Particles as caustics of the primodial light-like congruences

It's well known that a null congruence of rays corresponds to any solution of the eikonal equation; it is orthogonal to hypersurfaces of constant eikonal $S = const$ and directed along the 4-gradient vector $\partial_\mu S$. Usually, these two structures define the *characteristics* and *bicharacteristics* of a (linear) hyperbolic-type equation, e.g. of the wave equation $\Box\Psi = 0$.

In the considered complex case, i.e. in the case of CEE, the hypersurfaces of constant eikonal and the 4-gradient null congruences belong geometrically to the *complex extension* $\mathbb{C}M^4$ of the Minkowski space-time which looks here quite natural in account of the complex structure of the primary biquaternion algebra $\mathbb{B}$. The problem of physical sense of the additional (imaginary) dimensions is much important and nontrivial, and we hope to discuss it in the forthcoming paper.

Here we use another interesting property: existence of a null geodesic congruence defined on a *real* space-time for every of the *complex-valued* solutions to CEE. This remarkable property follows directly from the twistor structure inherent to CEE. Indeed, according to the theorem above-presented, any of the CEE solutions (both of the I and the II classes) is fully determined by a (null projective) twistor field $\{\xi(X), \tau(X)\}$ (in the choosed gauge one has $\xi_0 = 1$; $\xi_1 = G(x)$) subject to the incidence relation (6). This latter "Penrose equation" can be explicitly resolved with respect to the space coordinates $\{x_a, \ a = 1, 2, 3\}$ as follows:

$$x_a = \frac{\Im(\tau^+\sigma\xi)}{\xi^+\xi} - \frac{\xi^+\sigma\xi}{\xi^+\xi}\ t \ , \tag{26}$$

with $\{\sigma_a\}$ being the Pauli matrices and the time t remaining a *free parameter*. Eq.(26) manifests that the primodial spinor field $\xi(X)$ *reproduces its value along the 3D rays formed by the unit "director vector"*

$$\vec{n} = \frac{\xi^+\vec{\sigma}\xi}{\xi^+\xi}, \quad \vec{n}^2 = 1 \ , \tag{27}$$

and propagates along these locally defined directions with fundamental constant velocity $c = 1$. In the choosed gauge we have for Cartesian components of the director vector (27)

$$\vec{n} = \frac{1}{(1 + GG^*)}\{(G + G^*), -i(G - G^*), (1 - GG^*)\}, \tag{28}$$

the two its real degrees of freedom being in one-to-one correspondence with the two components of the complex function $G(X)$.

Thus, for every solution of the CEE the space is foliated by a congruence of *rectilinear* light rays, i.e by a *null geodesic* [3] *congruence* (NGC). Notice that the director vector obeys the *geodesic equation* [42]

$$\partial_t\vec{n} + (\vec{n}\vec{\nabla})\vec{n} = 0 \ . \tag{29}$$

The basic field $G(X)$ of the NGC can be always extracted from one of the two algebraic constraints (8) or (11) which at any space-time point possess, as a rule, not one but rather a *finite (or even infinite) set* of different solutions. Suppose that generating function Π is *irreducible*, i.e. can't be factorized into a number of twistor functions of the same structure (otherwise, we should make a choice in favour of one of the multiplies). Then a generic solution of the constraints will be nothing but a *multivalued complex function* $G(X)$. Choose locally (in the vicinity of a particular point X) one of the continious *branches* of this function. Then a particular NGC and a set of physical fields can be associated with this branch, i.e. with one of the "modes" of the multivalued field distribution.

Specifically, for any of the I class CEE solutions the spinor $F_{(AB)}$ of *electromagnetic field* can be defined explicitly in terms of twistor variables of the solution [27, 28, 29]:

$$F_{(AB)} = \frac{1}{P}\left\{\Pi_{AB} - \frac{d}{dG}\left(\frac{\Pi_A\Pi_B}{P}\right)\right\}. \tag{30}$$

[3] On the flat Minkowsky background the geodesics are evidently rectilinear

where Π_A, Π_{AB} are the first and the second order derivatives of the generating function Π with respect to its two twistor arguments τ^0, τ^1. For every branch of the solution $G(X)$ *this field locally satisfies Maxwell homogeneous ("vacuum") equations.* Moreover, as it has been demonstrated in [16, 24, 27], a complex-valued $SL(2,\mathbb{C})$ *Yang-Mills field* and a *curvature field* (of some effective Riemannian metric) can be also defined through only the same principal function $G(X)$ for any of the CEE solution of the first class.

Consider now analytical continuation of the function $G(X)$ up to one of its branching points which corresponds to a multiple root of Eq.(8) (or, alternatively, of Eq.(11) for solutions of the II class). At this point $P = 0$, and the strength of electromagnetic field (30) turns to infinity. The same holds for the other associated fields, for curvature field [4] in particular [21]. Thus, the locus of branching points (which can be 0-, 1- or even 2-dimensional, see section 4) manifests itself as a *common source* of a number of physical fields and can be identified (at least, in the case when it is bounded in 3-space) as a unique *particle-like* object.

Such formations are capable of much nontrivial evolution simulating physical interactions or even mutual *transmutations* represented by *bifurcations* of the field singularities (see, e. g., the example in section 4). They possess also a realistic set of "quantum numbers" including a *self-quantized electric charge* and a *Dirac-type gyromagnetic ratio* (equal to that for a spin 1/2 fermion) [19, 20, 25]. Numerous examples of such solutions and their singularities can be found in our works [24, 25, 26, 27].

On the other hand, for the light-like congruences – NGC – associated with CEE solutions via the guiding vector (28) the locus of branching points coincides with that of the principal G-field and represents the familiar *caustic* structure, i.e. the envelope of the system of rays at which the neighbouring rays intersect each other ("focusize"). From this viewpoint, within the algebrodynamical theory *the "particles" are nothing but the caustics of null rectilinear congruences associated with the CEE solutions.*

The World function and the multivalued physical fields

At this point we have to decide which of the two types of the CEE solutions can be in principle taken in our scheme as a representative for description of the Universe structure as a whole. *As a "World solution" we choose a CEE solution of the first class* because a lot of peculiar geometrical structures and physical fields can be associated with any of them [16, 25, 27]. Such a solution can be obtained algebraically from the Kerr functional constraint (8) and a generating twistor "World function" Π which is *exceptional* with respect to its internal properties; geometrically it gives rise to an NGC with a special property – zero shear [2, 3].

Moreover, a *conjugated* CEE solution of the II class turns then also to be involved into play since it defines a characteristic hypersurface of the (I class) "World solution". In fact, this is determined as a solution of the joint algebraic system of Eqs.(8),(11). Precisely, if we resolve Eq.(11) with respect to G and substitute the result into (8), equation $\Pi(G(X)) = 0$ would define then the singular locus (the characteristic hypersurface) of the World solution. On the other hand, the function $\Pi(G(X))$ would necessarily satisfy the CEE representing its II class solution in account of the theorem presented in section 2. Thus, *the eikonal field here carries out two different functions being a fundamental physical field (as a CEE solution of the I class) and, at the same time, a characteristic field (as a solution of the II class) which describes the locus of branching points of the basic field (i.e., the discontinuties of its derivatives).*

[4] Associated Yang-Mills fields possess, generically, additional *string-like* singularities

Let us conjecture now that the World function Π is *an irreducible polinomial of a very high but finite order* [5] so that Eq.(8) is an algebraic (not a transcendental) one. Note that in this case Eq.(8) defines an algebraic surface in the projective twistor space $\mathbb{C}P^3$.

The World solution consists then of a finite number of modes – branches of multivalued complex G-field. A finite number of null directions (represented in 3-space by the director vector (28)) and an equal number of locally distinct NGC would exist then *at every point.*

Any pair of these congruences at some fixed moment of time will, generically, has an envelope consisting of a number of connected one-dimensional components-caustics [6]. Just these spacial structures (in the case they are bounded in 3-space) represent here the "particles" of generic type. Other types of particle-like structures are formed at the focal points of *three or more* NGC where Eq.(8) has a root of higher multiplicity. Formations of the latter type would, of course, meet rather rarely, and their stability is problematic. One can speculate on their possible relation to particle's excitations – *resonances.*

Nonetheless, we can model both types of particles-caustics in a simple example based on generating twistor function of the form [29]

$$\Pi = G^2(\tau^0)^2 + (\tau^1)^2 - b^2G^2 = 0, \quad b = const \in \mathbb{R}\ , \tag{31}$$

which leads to the 4-th order polinomial equation for the G-field. At initial moment of time $t = 0$, as it can be obtained analytically, the singular locus consists of a pair of point singularities (with opposite and equal in modulus "elementary" electric charges) and of a neutral 2-surface (ellipsoidal *cocoon*) covering the charges (see [29] for more details). The latter corresponds to the intersection of all of the 4 modes of the multivalued solution while each of the point charges is formed by intersection of a particular pair of (locally radial, Coulomb-like) congruences [29]. Time evolution of the solution and of its singularities is very peculiar: for instance, at $t = b/\sqrt{2}$ the point singularities cancel themselves at the origin $r = 0$ simulating thus the process of *annihilation* of elementary particles. Moreover, this process is accompanied by emission of the *singular light-like wavefront* represented by another 2-dimensional component of connection of the caustic structure.

Thus, we see that the multivalued fields are quite necessary for to ensure the self-consistent structure and evolution of a complicated (realistic) system of particles-singularities. One only should not be confused by such, much unusual, property of the principal G-field and, especially, by multivalued nature of the other associated fields including the electromagnetic one.

Indeed, in convinient classical theories, the fields are in fact only a tool which serves for adequate description of particle dynamics (including the account of retardation etc.) and for nothing else. In nonlinear theories, as well as in our algebrodynamical scheme, the fields are moreover responsible for *creation* and structure of particles themselves, as regular *solitons* or *singularities* of fields respectively. In the first, more familiar case we, apparently, should consider the fields to be univalued. The same situation occures in the framework of quantum mechanics where the quantization rules often follow from the requirement for the wave function to be univalued.

However, as we have seen above, in the algebrodynamical construction *the field distributions must not necessarily be univalued!* On the other hand, acception of fields'

[5] This conjecture is, in fact, not at all necessary. Indeed, one can easily imagine that the World function leads to the Kerr Eq.(8) which possesses an **infinite number of roots** for complex-valued field function $G(X)$ at any space-time point X

[6] In fact, the caustics of **generic type** are determined by one complex condition $\Pi(G(X)) = 0$ (i.e., by two real equations) on three coordinates and, at a fixed moment of time $t = t_0$, correspond to a number of one-dimensional curves ("strings")

multivaluedness does not at all prevent to obtain the discrete spectrum of characteristics in a full analogy with quantum mechanics. For example, the requirement of univaluedness of a **particular, locally choosed mode** of the principal G-field and of the associated electromagnetic field (far from the branching points of the first and, consequently, from the infinities of the second!) leads to the general property of *quantization of electric charge of singularities* in the framework of algebrodynamical theory [28, 29].

As to the process of "measurement" of the field strength, say, of electromagnetic field, it directly relates to only the measurements of particles' accelerations, currents etc., and only after the measurements the results are translated into conventional field language. However, this is not at all necessary (in recall, e.g, of the Wheeler-Feynman electrodynamics and of numerous "action-at-a-distance" approaches [31, 32]). In fact, "we never deal with fields but only with particles" (F. Dyson).

In particular, on the classical (nonstochastic) level we can deal, effectively, with the *mean value* of the set of field modes at a point; similar concept based on purely quantum considerations has been recently developed in the works [33]. In our scheme, the true role of the multivalued field will become clear only after the spectrum and the effective mechanics of particles-singularities will be obtained in a general and explicit form.

We hope that a sort of psycological barrier for acception of general idea of the field multivaluedness will be get over as it was with possible *multidimensionality* of physical space-time. The advocated concept seems indeed very natural and attractive. In the purely mathematical framework, multivalued solutions of PDEs are the most common in comparison with the familiar δ-type distributions [34, 8]. From physical viewpoint, this makes it possible to naturally define a dualistic "corpuscular-field" complex of a very rich structure which, actually, gathers all the particles in the Universe into a unique object. The caustics-singularities are well-defined themselves and undergo a collective self-consistent motion free of any ambiguity or divergence (the latters can arise here only in result of incorrect discription of the evolution process and can be removed, if arise, on quite legal grounds, contrary, say, to the renormalization procedure in the quantum field theory). Note also that recently accomplished universal local classification of singularities of differentiable maps, in particular of caustics and wavefronts [11], can explicitly bear on the characteristics of elementary particles if the latters are treated in the framework of the algebrodynamical theory.

As to the principal problem of the choice of a particular representative of the generating *World function* Π *of the Universe* we are ready to offer an interesting candidature being in hope to discuss it elsewhere.

The light-formed relativistic aether and the nature of time

Light-like congruences (NGC) are the basic elements of the picture of physical world which arises in the algebrodynamical scheme and, to some extent, in twistor theory in general. The rays of the NGC densely fill the space and consist of a great number of branches – components superposed at each space point and propagating in different directions with constant in modulus and universal (for any branch of multivalued solution, any point and any system of reference) fundamental velocity. *There is nothing in the Universe exept this primodial light flow ("pre-Light Flow")* because *the whole Matter is born by pre-Light and from pre-Light* at the caustic regions of "condensation" of the pre-light rays.

In a sense, one can speak here about an exeptional form of *relativistic aether* which is formed by a flow of pre-Light. Such an exeptional form of the World aether has nothing in common with old models of the *light-carrying* aether which had been considered as a sort of elastic medium. Here, the *light-formed* aether consists of structureless "light elements" and is, obviously, in full correspondence with special theory of relativity [7].

At the same time, notions of the aether formed by pre-Light and of the matter formed by its "thickenings" evoke numerous associations with the Bible and with ancient Eastern philosophy. Certainly, there were teologists, philosophers or mystics who were brought to imagine a similar picture of the World. However, in the framework of successive physical theory this picture becomes more trushworthy and, to our knowledge, has not been yet discussed in literature [8].

On the other hand, existence of the primodial light-formed aether and manifestation of universal property of local "transfer" of the *aether – generating field* $G(X)$ with constant fundamental velocity $c = 1$ points to different status of space and time coordinates and *offers a new approach to the problem of physical time* as a whole. By this, it is noteworthy that since in 1908 H. Minkowsky has joined space and time into a unique 4-dimensional continuum, no further understanding of the nature of time has been achieved in fact. Moreover, this synthesis has "shaded" the principal distinction of space and time entities and clarified none of such problems as (micro/macro)irreversibility, (in)homogeneity and (non)locality of time, its dependence on material processes etc.

In the interim, the key problem of Time can be formulated in a rather simple way. *Subjectively*, we perceive time as a continious intrinsic motion, a latent *flow*. Everybody comprehends in a moment, as the ancient Greeks did, what is meant by the "River of Time", the "Flow of Time". As a rule, we consider this intrinsic motion to be independent on our will and on material processes and uniform: not for nothing, in physics the flow of time is modelled by the uniform motion of, say, the record tape etc. Moreover, under variations in time one does not only observe the *conservation* of a particular set of *integral* quantities (which is widely used in the orthodox physics) but perceives subjectively the complete *repetition, reproduction of the* **local states** *of any system;* that's why for measurements of time itself we use *clocks* whose principle of operation is based on reproducible, periodical processes. In other words, whereas one has much ambiguious and diverse distributions of spacial positions of physical bodies, all they and we all have always *one and the same monotonically increasing time coordinate*, i.e. are in a common and permanent motion together with the "Time River".

Surprisingly, almost all these considerations are absent in the structure of theoretical physics and, in particular, in relativity theory. To bring into correspondence the results of calculations with practice (e. g. for the Cauchy problem etc.) one chooses a "time orthogonal hypersurface", i. e. quite ambiguiously fixes the unity of the present moment of time, of the moment "now", perceived subjectively by everybody; however, *there are no intrinsic reasons for this choice in the very structure of theoretical physics, including the STR.*

At least partially, such a situation is caused by the following. The notion of everywhere existing, eternal Flow of Time immegiately leads to the problem of its (material? pre-material?) carrier. In this connection, the works of N. A. Kozy'rev [39] should be marked, of course, in which the concept of the "active" Flow of Time influencing directly the material processes has been proposed. To our opinion, however, there are no reliable physical grounds at present which confirm the Kozy'rev's ideas, and no mechanism of "interaction" of this exotic form of matter with the ordinary ones. As to the algebrody-

[7] At present, it seems rather strange that A. Einstein didn't come himself to the concept of relativistic aether so consonant with the ideas of STR and with his favourite *Mach principle*. Suprisingly, R. Penrose also overlooked this opportunity which follows naturally from his twistor theory

[8] Similar in some aspects ideas have been advocated in the works [36, 37, 38]. Note, in particular, the concept of the "radiant particle" offered by L. S. Shikhobalov [35]

namical paradigm, the Time Flow is non-material therein: it does not *interact* or *influence* the Matter at all but just *forms* it. In distinction from the Kozy'rev's concept, we do not deal here with various material entities only one of them being the Time itself: on the contrary, here we have one triply-unique entity – preLight-Time-Matter. Note that more close the approach turns to be to the concept of "Time-generating Flows" developed by A. P. Levich [40].

On the other hand, under consideration of the problem of the carrier of the Time Flow, we inevitably return back to the notion of some form of the *World aether* which has been exiled from physics after the triumph of Einstein's theory. To do without aether, none Flow of Time can be successively included into the structure of theoretical physics and none subjectively perceived properties of time can be precisely formulated and described.

However, in a paradoxical way, just the STR with its postulate of the invariance of light velocity justifies the introduction of the *dynamical Lorentz invariant aether* formed by the light-like congruences as the primary element of physical World. Specifically, the Time Flow can be naturally identified now with the Flow of Primodial Light (pre-Light), and the "River of Time" turns to be nothing but the "River of Light". Moreover, it is the universality of light velocity which explains our subjective perception of uniformity and homogeneity of the Time Flow.

There is, however, another, the most striking and unexpected feature of the introduced concept of physical time. *The Time Flow manifests here itself as a superposition of a great number of distinctly directed and locally independent components – "subflows"*. At any point of 3-dimensional space there exists a (finite) set of directions: each mode of the primodial multivalued field $G(X)$ defines one of these directions and *propagates (reproduces its value) along it* forming thus one of the constituents of the (globally unique) Flow of pre-Light identical to the Flow of Time.

One can conjecture that just by virtue of the local multivaluedness we are not capable of to perceive the particular local *direction* of the Time Flow. Apart from this, it is natural to assume that in the tremendously complicated structure of the World solution a *stochastic* component is necessarily present, particularly in the structure of the primodial Light-Time Flow. This results in chaotic variations of local directions of the light-like congruences which are certainly inaccessible for perception. On the other hand, it is the existence of (constant in modulus and the same for all of the branches of the multivalued World solution) *fundamental propagation velocity of the pre-Light rays* which makes it possible to feel the Flow of Time in general and to subjectively regard it as uniform and homogeneous in particular.

Conclusion

Thus, we have examined the realization of the algebrodynamical approach in which, as a base of unified physical theory, the only structure of a purely abstract nature is choosed, namely the algebra of complex quaternions and the generalized CR-equations – the conditions of differentiability in this algebra. Very the same structure can be successively expressed, in fact, on a number of equivalent geometrical languages (of covariantly constant fields, twistor geometry, shear-free congruences etc.).

Primary GCR-equations result directly in the field of complex eikonal which is regarded as a fundamental physical field (alternative in a sense to the linear fields of quantum mechanics). In its turn, the eikonal field is here closely related to the fundamental 2-spinor and twistor fields, on whose language, in particular, the general solution of the complex eikonal equation is formulated. Through the eikonal field also the other ones are defined, namely the electromagnetic and Yang-Mills fields. Singularities of the eikonal and of correspondent null congruences are considered as particle-like formations ("self-quantized" and effectively interacting).

In result, physical picture of the World which arises as a consequence of the only algebraic structure appears to be very beatiful and unexpected. As its basic elements it contains the primodial light flow – "pre-Light" – and the relativistic aether formed by the latter, multivalued physical fields and prelight-born matter (consisting of particles-caustics formed by the superposition of individual branches of the unique pre-light congruence in the points of their "focusization").

As very natural and deep seems to be the here arising connection between the existence of universal velocity (velocity of "light") and of the time flow; connection which permits to understand, in a sense, the origin of the Time itself. *Time is nothing but the primodial Light*; these two entities are undividible. On the other hand, *there is nothing in the World except the preLight Flow* which gives rise to all the "dense" Matter in the Universe.

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ON SOME QUESTIONS OF FOUR DIMENSIONAL TOPOLOGY: A SURVEY OF MODERN RESEARCH

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Introduction

Our physical intuition distinguishes four dimensions in a natural correspondence with material reality. Four dimensionality plays special role in almost all modern physical theories. High dimensional quantum fields theory and string theory are considered together with their compactifications, i. e. the main space, describing the reality is a product of a four-dimensional manifold with some compact high-dimensional space. In this way we come to the well-known Kaluza-Klein model and ten-dimension superstring theory.

It is an interesting fact that the dimension four is a more complicated dimension from pure mathematical point of view. It seems that there is a contradiction with our intuition in understanding of the dimension concept, really, new dimensions give us new complexity. But it is not true in general. Additional dimensions often give a new freedom. It is natural that we must have some golden mean in this approach, in which we don't have a necessary freedom, but low-dimensional methods weakly work. In topology this mean is dimension four.

The goal of this note is to give a small survey of some problems in four-dimensional topology.

S-cobordism problem

One of the main questions of geometric topology is the problem to classify manifolds lying in a given category with respectively chosen equivalence relation. Working in the topological category, the question about classification of topological manifolds up to homeomorphism rises, for example, assuming compactness, connecteness and closedness. In dimension one we have only circles, in dimension two we come to the complete classification: every connected closed compact manifold is homeomorphic to the two-dimensional sphere with handles and Mobius bands. In this case, the fundamental group is a complete topological invariant. In the dimension three the question about classification becomes a hard problem, the existence of the connected 1-connected three-dimensional manifold, which is not homeomorphic to the three-dimensional sphere, is a well-known Poincare Problem. It is interesting that in dimension ≥ 5 many difficulties, occurring place in low dimensions, are disapear. First of all, this fact is connected with the concept of general position in high-dimensional spaces. Roughly speaking, in many important cases small deformations give possibility to cancel self-intersections of complexes. But in low-dimensional case we can not do the same.

Let's introduce one of the central equivalence relation in the topology of manifolds, so called s-cobordism relation. Let M_1 and M_2 be n-dimensional manifolds. We say that they are *cobordant* if there exists a $(n+1)$-dimensional manifold W, such that $\partial W = M_1 \cup M_2$. Further, if the embeddings $M_i \to W, i = 1, 2$ are homotopical equivalences, then this cobordism is called $h-cobordism$ (and manifolds are $h-cobordant$). Every homotopical equivalence defines an element from the Whitehead group, which depends only on the fundamental group of a given manifold (or in general, fundamental group of cell complex). The Whitehead group can be defined as a quotient of the K_1-functor of the integral group ring of the fundamental group by the natural action of group. In this way, the homotopical equivalence represents a trivial element of the Whitehead group if and only if it is homotopic to the composition of elementary cell extensions and collapsings, i.e. so-called simple homotopy equivalence. H-cobordism with simple homotopy equivalence is called *s-cobordism.* In particular, every homotopy equivalence between 1-connected manifolds is homotopic to the simple one.

The main result of the high-dimensional topology is following Theorem (see [1], [2]).

s-cobordism Theorem. Let $n \geq 5$. The connected h-cobordism W between n-dimensional manifolds M_1 and M_2 is homeomorphic to the direct product $W \equiv M_1 \times I$, if and only if this cobordism is an s-cobordism.

In particular, if we consider only 1-connected manifolds then arbitrary h-cobordism between them is a direct product. The higher-dimensional Poincare Conjecture then follows from this, i.e. every homotopical sphere is homeomorphic to the standard one in dimension ≥ 5. The proof of the s-cobordism Theorem fails in the case of dimension four and analogical statement presents an open problem:

Problem. Does the s-cobordism Theorem hold in the dimension 4?

The proof of the high-dimensional s-cobordism Theorem is based on the handlebody decomposition of the manifold W and reduction of a given manifold to the structure of the direct product of M_i with interval. The crucial point in this method is so-called Whitney trick. It gives a possibility to cancel the intersection points of the immersed submanifolds due to the embedding of a 2-dimensional disk (Whitney's disk), (see [1]). The main obstruction to extend the proof on the case of dimension four is the fact that Whitney trick does not work in dimension four. Actually, it is well-known that every 2-dimensional complex can be isotopically reduced to the embedded one in the 5-dimensional manifold. But in dimension four it is not true in general and we can consider the Whitney's disk only as immersed one. This easy fact destroys all prove of the s-cobordism theorem in the case of dimension 4.

To get over the difficulties related to the immersed Whitney disc, some new methods have been developed. The method given by A. Casson is most effective. The meaning of this method is to paste a self-intersections step by step by new immersed discs. This process can be extended infinitely long but the neighborhood of the final 2-complex is a handle, which is homotopically equivalent to the standard one. This idea was used by M. Freedman in the proof of the topological Poincare Conjecture in dimension four.

In general, as it was mentioned above, the s-cobordism problem in dimension 4 is still open. The analog of the s-cobordism Theorem was proved by M. Freedman and P. Teichner in 1996 in the class of 4-dimensional manifolds with fundamental groups of the subexponential growth (more precisely, of the growth $\leq 2^n$) [4].

False and exotic 4-dimensional manifolds

There is a natural question of comparison of given equivalence relations, i.e. homotopical equivalence, homeomorphisms, diffeomorphisms, in the class of manifolds of a fixed dimension. So, any two continously homeomorphic smooth manifolds are diffeomorphic in the dimension less than four. The situation in dimension four is much more complicated.

A manifold N is called a *false copy* of the manifold M if N is homotopically equivalent to M but not homeomorphic to M. N is called an *exotic copy* of M if N and M are homeomorphic, but not diffeomophic as manifolds.

The existence of the false and exotic spheres is connected with the topological and smooth versions of the Poincare Conjecture respectively. The smooth Poincare Conjecture is true in the dimensions less than four: there are no exotic three (and less) dimensional spheres. The analysis of the high-dimensional question leads to the beautiful theory of exotic spheres: there exist 28 7-dimensional manifolds, which are homeomorphic to the standard 7-dimensional sphere, but not diffeomorphic, due to the wonderful result of Milnor. The most intriging case is again dimension four. This is the only dimension, in which the existence of the exotic spheres is still open.

The situation with exotic copies of $\mathbb{R}^4$ is also very surprising. It is known that there does not exist any exotic $\mathbb{R}^n$ in dimension $n \neq 4$ and the analogical question was open for a long time in dimension four. In eighties due to the results of Freedman and Donaldson it was proved that there exist infinitely many smooth pair-wise nondiffeomorphic four-dimensional manifolds, such that each of them is homeomorphic to $\mathbb{R}^4$. The proof of this fact essentially used the methods of mathematical physics: instantons, Yang-Mills connections etc (see [5]). One of the main invariants of 1-connected four-dimensional manifolds is so-called intersection form, i.e. symmetric bilinear form, define on the second cohomologies of a given manifold. Classical Whitehead's theorem says that two given 1-connected oriented closed smooth four-dimensional manifolds are homotopically equivalent if and only if they have isomorphic intersection forms. In this connection, there is an actual question to classify all symmetric bilinear forms which can be realized as intersection form for some four-dimensional manifold. M. Freedman has shown that every symmetric bilinear form can be realized as an intersection form of some compact 1-connected four-dimensional manifolds and that there exist no more than two manifolds with given form. Donaldson classified all intersection forms of smooth manifolds and concluded from this the existence of the exotic structures on $\mathbb{R}^4$. The structure of exotic $\mathbb{R}^4$ is very complicated and takes important place in modern research. There are still many open questions related to such manifolds. In particular, does there exist any exotic $\mathbb{R}^4$ such that it can not be divided by properly embedded $\mathbb{R}^3$ onto two exotic pieces (Problem 4.43 (D), [6]).

The false four-dimensional manifolds construction requires an application of other techniques. As it was mentioned above, there are no false four-dimensional spheres (four-dimensional topological Poincare Conjecture). Very often the question about homeomorphicity of a given homotopic four-dimensional manifolds is very difficult. One of the first such type examples of four-dimensional manifolds is Cappell-Shaneson construction (see [2]): there exists a false projective $\mathbb{R}P^4$, which is homotopically equivalent but not diffeomorphic to $\mathbb{R}P^4$. This space is not PL-homeomorphic to $\mathbb{R}P^4$.

Finishing this section let's present more open problems in dimension four, related to exotic structures. The reader can find many classical and modern problems of this type the Kirby Problem List [6] (see also [7]).

Problem (4.77 [6]): An exotic smooth structure on $\mathbb{R}^4$ with $\mathbb{R}^1$ is diffeomorphic to $\mathbb{R}^5$. How can we usefully see the exotic $\mathbb{R}^4$ in $\mathbb{R}^5$?

Problem (4.86 [6]): Do all closed, smooth 4-manifolds have more than one smooth structure? (The generalization of the smooth 4-dimensional Poincare Conjecture).

Problem (4.87 [6]): Does every non-compact, smooth 4-manifold have an uncountable number of smoothings?

Schoenflies Conjecture

Consider one more problem, which has the solution in all dimensions besides four. This problem is about knotting in codimension equal to one. Recall that the embedding $f : M^m \to N^{n+m}$ is called *locally-flat* if the image of each point in N^{m+n} has neighborhood U such that the pair $(Im(f) \cap U, U)$ is homeomorphic (piecewise-linearly, in the case we work in this category) to the pare $(D^m \times D^n, D^m \times \{0\})$.

Conjecture Let $f : S^n \to S^{n+1}$ be a piece-linear locally-flat embedding. Then $S^{n+1} \setminus im(f)$ is 2-component and the closure of each of the components is a piece-linear n-dimensional ball.

Roughly speaking, this conjecture states that n-dimensional sphere can not knot in $(n+1)$-dimensional one. This conjecture turn out to be true in dimensions $n+1 \neq 4$. But in the case of dimension four, again we can not apply the methods which we use in other dimensions.

Finishing this note, we want to mention one more time that there exist not so much fields in mathematics which use so different methods as four-dimensional topology. The problems of four-dimensional topology lead to the difficult questions of group theory. This is a theory of growth in groups, Andrews-Curtis-type problems, lower central series in groups etc. Also we can see many applications of high-dimensional methods in dimension four, for example, surgery exact sequences, methods of the link and knot theory. The dimension four is the unique dimension from the topological point of view, where we can find so many application of different techniques and which has so many open problems, the development of new tecniques of algebra and topology will be needed for their solution.

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QUATERNIONS: ALGEBRA, GEOMETRY AND PHYSICAL THEORIES

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A review of modern study of algebraic, geometric and differential properties of quaternionic (Q) numbers with their applications. Traditional and "tensor" formulation of Q-units with their possible representations are discussed and groups of Q-units transformations leaving Q-multiplication rule form-invariant are determined. A series of mathematical and physical applications is offered, among them use of Q-triads as a moveable frame, analysis of Q-spaces families, Q-formulation of Newtonian mechanics in arbitrary rotating frames, and realization of a Q-Relativity model comprising all effects of Special Relativity and admitting description of kinematics of non-inertial motion. A list of "Quaternionic Coincidences" is presented revealing surprising interconnection between basic relations of some physical theories and Q-numbers mathematics.

Introduction

The discovery of quaternionic (Q) numbers dated by 1843 is usually attributed to Hamilton [1, 2], but in the previous century Euler and Gauss made a contribution to mathematics of Q-type objects; moreover Rodriguez offered multiplication rule for elements of similar algebra [3-5]. Active opposition of Gibbs and Heaviside to Hamilton's disciples gave a start to the modern vector algebra, and later to vector analysis, and quaternions practically ceased to be a tool of mathematical physics, despite of exclusive nature of their algebra confirmed by Frobenius theorem. At the beginning of 20 century last bastion of Q-numbers amateurs, "Association for the Promotion of the Study of Quaternions", was ruined. The only reminiscence of once famous hypercomplex numbers was the set of Pauli matrices. Later on quaternions appeared incidentally as a mathematical mean for alternative description of already known physical models [6, 7] or due to surprising simplicity and beauty they were used to solve rigid body cinematic problems [8]. An interest to quaternionic numbers essentially increased in last two decades when a new generation of theoreticians started feeling in quaternions deep potential yet undiscovered (e. g. [9 − 11]).

This work is an attempt to give more systematic overview of contemporary state of Q-number mathematics, its applications to physical theories and possible perspectives in this area. In the context some quite specific even surprising physical models, but worth to pay attention to, are shortly discussed.

The review arranged as follows. In section 1 general relations of the quaternionic algebra are briefly described in the traditional hamiltonian formulation as well as in tensor-like format. Section 2 is devoted to description of structure of three "imaginary" quaternionic units. In section 3 the elements of differential Q-geometry are given with examples of their mathematical application. Section 4 comprises Q-formulation of Newtonian mechanics in the rotating frames of reference. Quaternionic Relativity Theory with a number of cinematic relativistic effects is found in section 5. Section 6 contains the list of "Great Quaternion Coincidences" and final discussion.

1. Algebra of quaternions

Traditional approach

According to Hamilton, a quaternion is a mathematical object of the form

$$Q \equiv a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},$$

where a, b, c, d are real numbers, a is a coefficient at real unit "1", and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ – three imaginary quaternion units. The multiplication rule for these units given by Hamilton and often used in literature is

$$1\mathbf{i} = \mathbf{i}1 \equiv \mathbf{i}, \qquad 1\mathbf{j} = \mathbf{j}1 \equiv \mathbf{j}, \qquad 1\mathbf{k} = \mathbf{k}1 \equiv \mathbf{k},$$
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$
$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \qquad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

These very cumbersome equations mean, that Q-multiplication loses a commutativity.

$$Q_1 Q_2 \neq Q_2 Q_1,$$

so that a notion of the right and the left multiplication appears, but it remains associative.

$$(Q_1 Q_2) Q_3 = Q_1 (Q_2 Q_3).$$

Two rather different algebraic parts are separated naturally in a quaternion, these once could be denoted as scalar:

$$scal\ Q = a,$$

and vector

$$vect\ Q = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}.$$

Addition (subtraction) of quaternions is performed by components, scalar and vector parts are added (subtracted) separately. With respect to addition the Q-algebra is commutative and associative.

Further step is quaternion conjugation introduced similarly to that of the complex numbers

$$\bar{Q} \equiv scal\ Q - vect\ Q = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k},$$

modulus of a Q-number is defined as

$$|Q| \equiv \sqrt{Q\bar{Q}} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

This permit to formulate a quaternionic division being as multiplication "right" and "left"

$$Q_L = \frac{Q_1 \bar{Q}_2}{|Q_2|^2}, \qquad Q_R = \frac{\bar{Q}_2 Q_1}{|Q_2|^2}.$$

Definition of Q-modulus enhances the famous four squares identity

$$|Q_1 Q_2|^2 = |Q_1|^2 |Q_2|^2.$$

Due to the properties mentioned above the Q-numbers form the algebra, which belongs to the elite group of four the so-called exclusive – "very good" – algebras: of real, complex, quaternionic numbers and the octonions (Frobenious and Horwits theorems of 1878-1898 [12]).

Special attention should be paid to Q-units representations. In terms of Hamilton real unit is simply 1 while three imaginary units similarly to complex numbers algebra are denoted as **i, j, k**. Later a simple 2×2 matrices representation of these units was revealed

$$\mathbf{i} = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{j} = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \mathbf{k} = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This representation of course is not unique. Here is a simple example. If in the above expressions imaginary unit i of complex numbers is represented as 2×2 with real elements

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then three vector Q-units turn out to be represented by real 4×4 matrices. The procedure of the matrix rank duplication can obviously be continued further.

"Tensor" form and representations

If each Q-unit is endowed with its proper number (as components of a tensor)

$$(\mathbf{i}, \mathbf{j}, \mathbf{k}) \to (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \mathbf{q}, \qquad k, j, k, l, m, n, \ldots = 1, 2, 3,$$

then quaternionic multiplication rule acquires compact form

$$1\mathbf{q}_k = \mathbf{q}_k 1 = \mathbf{q}_k, \qquad \mathbf{q}_j \mathbf{q}_k = -\delta_{jk} + \varepsilon_{jkn} \mathbf{q}_n,$$

where δ_{kn} and ε_{knj} – respectively, 3-dimension (3D) symbols Kronecker and Levi-Chivita.

It is easy to show that a number of the Q-units representations even only by 2×2 matrices is infinite. Indeed for any 2×2 matrices with properties

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \qquad B = \begin{pmatrix} d & e \\ f & -d \end{pmatrix}, \qquad TrA = TrB = 0,$$

the first two Q-units can be constructed as follows

$$\mathbf{q}_1 = \frac{A}{\sqrt{\det A}}, \qquad \mathbf{q}_2 = \frac{B}{\sqrt{\det B}},$$

while the third one is

$$\mathbf{q}_3 \equiv \mathbf{q}_1 \mathbf{q}_2 = \frac{AB}{\sqrt{\det A \det B}} \qquad \text{provided that } \; Tr(AB) = 0.$$

The scalar unit is always invariant:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Transformations of Q-units and invariancy of the multiplication rule

a. Spinor-type transformations

If U is an operator changing at once all the units, and there is an inverse operator $U^{-1}: \quad UU^{-1} = E$, then transformations

$$\mathbf{q}_{k'} \equiv U\mathbf{q}_k U^{-1} \quad \text{and} \quad 1' \equiv U1U^{-1} = E1 = 1$$

retain the multiplication rule

$$1\mathbf{q}_k = \mathbf{q}_k 1 = \mathbf{q}_k, \quad \mathbf{q}_j\mathbf{q}_k = -\delta_{jk} + \varepsilon_{jkn}\mathbf{q}_n$$

form-invariant

$$\mathbf{q}_{k'}\mathbf{q}_{n'} = U\mathbf{q}_k U^{-1}U\mathbf{q}_n U = U\delta_{kn}U^{-1} + \varepsilon_{knj}U\mathbf{q}_j U^{-1} = \delta_{kn} + \varepsilon_{knj}\mathbf{q}_{j'}.$$

Such operator can be represented for example by 2×2 matrix

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det U = 1,$$

or unimodular quaternion,

$$U = \frac{a+d}{2} + \sqrt{1 - \left(\frac{a+d}{2}\right)^2}\,\mathbf{q},$$

where

$$\mathbf{q} \equiv \left(\sqrt{1 - \left(\frac{a+d}{2}\right)^2}\right)^{-1} \begin{pmatrix} \frac{a-d}{2} & b \\ c & -\frac{a-d}{2} \end{pmatrix}.$$

In general this transformation contains 3 independent complex parameter (or 6 real ones), then $U \in SL(2, C)$. In special case of only three real parameters, then $U \in SU(2)$.

b. Vector type transformations

Vector Q-units can be transformed by 3×3 matrix $O_{k'n}$

$$\mathbf{q}_{k'} = O_{k'n}\mathbf{q}_n.$$

The requirement of Q-multiplication form-invariance forces the transformation matrix to be orthogonal and unimodular

$$O_{k'n}O_{j'n} = \delta_{kn} \Rightarrow O^{-1}_{nk'} = O_{k'n}, \quad \det O = 1.$$

This transformation in general has 6 independent real parameters, then $O \in SO(3, C)$. In the special case of three parameters $O \in SO(3, R)$. Below a variant of representation of the transformation matrix O is given with x, y, z being arbitrary real or complex functions

$$O = \begin{pmatrix} \sqrt{1-x^2-z^2} & -\frac{x\sqrt{1-y^2-z^2}+yz\sqrt{1-x^2-z^2}}{1-z^2} & \frac{xy-z\sqrt{1-x^2-z^2}\sqrt{1-y^2-z^2}}{1-z^2} \\ x & \frac{\sqrt{1-x^2-z^2}\sqrt{1-y^2-z^2}-xyz}{1-z^2} & \frac{-y\sqrt{1-x^2-z^2}-xz\sqrt{1-y^2-z^2}}{1-z^2} \\ z & y & \sqrt{1-y^2-z^2} \end{pmatrix}.$$

This matrix can be represented as a product of three irreducible multipliers

$$O=\begin{pmatrix}\sqrt{\frac{1-x^2-z^2}{1-z^2}} & -\frac{x}{\sqrt{1-z^2}} & 0\\ \frac{x}{\sqrt{1-z^2}} & \sqrt{\frac{1-x^2-z^2}{1-z^2}} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{1-z^2} & 0 & -z\\ 0 & 1 & 0\\ z & 0 & \sqrt{1-z^2}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \sqrt{\frac{1-y^2-z^2}{1-z^2}} & -\frac{y}{\sqrt{1-z^2}}\\ 0 & \frac{y}{\sqrt{1-z^2}} & \sqrt{\frac{1-y^2-z^2}{1-z^2}}\end{pmatrix}.$$

after substitutions $z\equiv\sin\mathrm{B}$, $x\equiv-\sin\mathrm{A}\cos\mathrm{B}$, $y\equiv-\sin\Gamma\cos\mathrm{B}$, where A,B,Γ – are complex "angles", it takes the form

$$O=\begin{pmatrix}\cos\mathrm{A} & \sin\mathrm{A} & 0\\ -\sin\mathrm{A} & \cos\mathrm{A} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\mathrm{B} & 0 & -\sin\mathrm{B}\\ 0 & 1 & 0\\ \sin\mathrm{B} & 0 & \cos\mathrm{B}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\Gamma & \sin\Gamma\\ 0 & -\sin\Gamma & \cos\Gamma\end{pmatrix}=O_3^{\mathrm{A}}O_2^{\mathrm{B}}O_1^{\Gamma}.$$

If the angles are real: $\mathrm{A}=\alpha$, $\mathrm{B}=\beta$, $\Gamma=\gamma$, then this transformation is an ordinary vector rotation consisting of three simple rotations around numbered orthogonal axes: $O\Rightarrow R, R=R_3^{\alpha}R_2^{\beta}R_1^{\gamma}$. Correlation between related "spinor" and "vector" transformations is easily determined:

$$O_{k'n}=-\frac{1}{2}Tr(U\mathbf{q}_kU^{-1}\mathbf{q}_n),\qquad U=\frac{1-O_{k'n}\mathbf{q}_k\mathbf{q}_n}{2\sqrt{1+O_{mm'}}}.$$

Q-geometry in three dimensional space

Hamilton was the first to note that triad of Q-units behaves as three strictly tied unit vectors (with length i) initiating Cartesian coordinate system, somewhat exotic because of its "imaginarity". Due to the fact the Q-triad in 3D-space $(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)$ will be called 'quaternionic basis' (Q-basis). Now Q-units transformations have apparent geometrical sense of various rotations of the Q-basis. An example: a simple rotation by real angle α around axis # 3

$$\mathbf{q}'=R_3^{\alpha}\mathbf{q}.$$

Notion of Q-basis helps to introduce 3D quaternionic vectors (Q-vectors), defined as

$$\mathbf{a}=a_k\mathbf{q}_k,$$

here all its components a_k are real. The most important property of Q-vector – is its invariancy with respect to vector transformations from the group SO(3,R)

$$\mathbf{a}'=a_{k'}\mathbf{q}_{k'}=a_{k'}R_{k'j}\mathbf{q}_j=a_j\mathbf{q}_j=\mathbf{a}.$$

The projection of Q-vector onto arbitrary coordinate axis (represented by any different Q-unit) can be found in two ways. First, if at least one set of projections of Q-vector and rotation matrices $R_{nk'}$ are known then projections of this vector on rotated axis are immediately found

$$a_{k'}=a_nR_{nk'}.$$

The second approach is related to existence of internal structure of the Q-units; a brief analysis of it is given in the next section.

2. Structure of quaternionic "imaginary" units

Eigenfunctions of Q-units [13]

Each vector Q-unit can be thought of as operator, so eigenfunctions and eigenvalues problem can be formulated for it

$$\mathbf{q}\psi = \lambda\psi, \qquad \varphi\mathbf{q} = \mu\varphi.$$

The solution of this problem are the eigenvalues ("imaginary length" of Q-unit with division by parity)

$$\lambda = \mu = \pm i,$$

and two sets of eigenfunctions(one for each parity), possible given by columns $\psi^{\pm}$ and rows $\varphi^{\pm}$, being the functions of components $\mathbf{q}$.

Here is an example explicit form of eigenfunction: for the Q-unit represented by matrix

$$\mathbf{q} = -\frac{i}{T}\begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

where $T \equiv a^2 + bc \neq 0, b \neq 0, c \neq 0$, its eigenfunctions are defined as

$$\varphi^{\pm} = x\begin{pmatrix} 1 & \pm\frac{b}{T\pm a} \end{pmatrix}, \qquad \psi^{\pm} = y\begin{pmatrix} 1 \\ \mp\frac{c}{T\pm a} \end{pmatrix},$$

where x, y are arbitrary complex factors.

The freedom of components, arising in the calculations is reduced by convenient normalization condition

$$\varphi^{\pm}\psi^{\pm} = 1,$$

while the eigenfunctions orthogonality (by parity) is an inherited property

$$\varphi^{\mp}\psi^{\pm} = 0.$$

One can construct tensor products of eigenfunctions and obtain 2×2 matrices

$$C^{\pm} \equiv \psi^{\pm}\varphi^{\pm},$$

possessing a properties reciprocal with respect to the ones of vector $\mathbf{q}$:

$$\det C = 0, \quad Tr\, C = 1,$$

whereas

$$\det \mathbf{q} = 1, \quad Tr\, \mathbf{q} = 0.$$

Matrix C is idempotent

$$C^n = C,$$

and can be expressed throw their own unit Q-vector

$$C^{\pm} = \frac{1 \pm i\mathbf{q}}{2}.$$

When inversed the latter expression gives information about internal structure of Q-unit

$$\mathbf{q} = \pm i(2C^{\pm} - 1) = \pm i(2\psi^{\pm}\varphi^{\pm} - 1),$$

which turns out to consist of a combination of its eigenfunctions and scalar units.

Since each Q-unit has its own eigenfunctions the Q-triad as a whole possesses unique set of eigenfunctions $\{\varphi^{\pm}_{(k)}, \psi^{\pm}_{(k)}\}$. There is an interesting algebraic observation concerning this set. Three Q-units are interrelated by obviously nonlinear combination – multiplication e. g.

$$\mathbf{q}_3 = \mathbf{q}_1\mathbf{q}_2,$$

but it is easy to show that corresponding eigenfunctions depend on each other linearly:

$$\varphi^{\pm}_{(3)} = \sqrt{\mp i}\varphi^{\pm}_{(1)} \pm \sqrt{i}\varphi^{\pm}_{(2)}, \quad \psi^{\pm}_{(3)} = \sqrt{\pm i}\psi^{\pm}_{(1)} \pm \sqrt{-i}\psi^{\pm}_{(2)}.$$

Q-eigenfunctions help to represent a spinor-type transformation of Q-units retaining Q-multiplication invariant in the familiar form

$$\psi^{\pm}_{(k')} = U\psi^{\pm}_{(k)}, \quad \varphi^{\pm}_{(k')} = \varphi^{\pm}_{(k)}U^{-1},$$

so that the eigenfunctions can be regarded as a set of specific spinor functions, allowing in subject in general to $SL(2C)$ transformations. Yet another mathematical observation should be noted: from pairs of eigenfunctions, belonging to different Q-units of one triad and having one parity, one can construct 24 scalar invariants $SL(2C)$ group; these invariants are real or complex numbers, e. g.:

$$\sigma^{+}_{12} \equiv \varphi^{+}_{(1)}\psi^{+}_{(2)} = \sqrt{-\frac{i}{2}} = \frac{1-i}{2}.$$

Quaternionic eigenfunctions as projectors

Eigenfunctions act on their own Q-basis as following

$$\varphi^{\pm}_{(1)}\mathbf{q}_1\psi^{\pm}_{(1)} = \pm i, \quad \varphi^{\pm}_{(1)}\mathbf{q}_2\psi^{\pm}_{(1)} = 0, \quad \varphi^{\pm}_{(1)}\mathbf{q}_3\psi^{\pm}_{(1)} = 0,$$

or in general

$$\varphi^{\pm}_{(k)}\mathbf{q}_n\psi^{\pm}_{(k)} = \pm i\delta_{kn} \qquad \text{(no summation by k)}.$$

It looks like that eigenfunctions select a projection of the unit Q-vector, generating them. This idea is confirmed by an example of an action of eigenfunctions of one Q-basis onto the vectors of the rotated Q-basis

$$\varphi^{\pm}_{(k)}\mathbf{q}_{n'}\psi^{\pm}_{(k)} = \varphi^{\pm}_{(k)}R_{n'm}\mathbf{q}_m\psi^{\pm}_{(k)} = \pm iR_{n'k} = \pm i\cos\angle(\mathbf{q}_{n'}, \mathbf{q}_k) \quad \text{(no summation by k)},$$

the result of the action is 'nearly' projection of Q-basis $\mathbf{q}'$ on $\mathbf{q}$. It is convenient to denote precise projection as

$$\langle \mathbf{q_{n'}} \rangle_k \equiv \mp i\varphi^{\pm}_{(k)}\mathbf{q}_{n'}\psi^{\pm}_{(k)} = \cos\angle(\mathbf{q}_{n'}, \mathbf{q}_k) \qquad \text{(no summation by k)}.$$

It is now easy to formulate rule of calculation of projection of a Q-vector a onto arbitrary direction, defined by vector $\mathbf{q}_j$ (e. g. with help of eigenfunctions of positive parity)

$$\langle \mathbf{a} \rangle^{+}_{j} \equiv -ia_{k'}\varphi^{+}_{(j)}\mathbf{q}_{k'}\psi^{+}_{(j)} = a_{k'}R_{k'j} = a_j \qquad \text{(no summation by j)}.$$

Thus quaternionic eigenfunctions with their own interesting properties are more fundamental mathematical objects then Q-units and too can serve as useful tool for practical purposes such as computing projections of Q-vectors.

4. Differential Q-geometry

Quaternionic connection

If vectors of Q-basis are smooth functions of parameters $\mathbf{q}_k(\Phi_\xi)$ (index ξ enumerates parameters), then

$$d\mathbf{q}_k(\Phi) = \omega_{\xi\ kj}\mathbf{q}_j d\Phi_\xi,$$

where an object $\omega_{\xi\ kj}$ is called quaternionic connection. Q-connection is antisymmetric in vector indices

$$\omega_{\xi\ kj} + \omega_{\xi\ jk} = 0,$$

and has the following number of independent components

$$N = Gp(p-1)/2,$$

where G is an number of parameters and p = 3 – is a number of space dimensions. If G = 6 [a case of group SO(3,C)], then N = 18; if G = 3 [a case of group SO(3,R)], then N = 9. Q-connection can be calculated at least in three ways:

$$\text{using vectors of Q-basis} \qquad \omega_{\xi\ kn} = \left\langle \frac{\partial \mathbf{q}_k}{\partial \Phi_\xi} \right\rangle_n^+,$$

using matrices U from the group SL(2C) (general case) and special representation of constant Q-units $\mathbf{q}_{\tilde{k}} = -i\sigma_k$, where σ_k – Pauli matrices

$$\omega_{\xi\ kn} = \left\langle U^{-1}\frac{\partial U}{\partial \Phi_\xi}\mathbf{q}_{\tilde{k}} - \mathbf{q}_{\tilde{k}} U \frac{\partial U^{-1}}{\partial \Phi_\xi} \right\rangle_n^+,$$

and, finally, using matrices O from $SO(3, C)$ (in a general case)

$$\omega_{\xi\ kn} = \frac{\partial O_{k\tilde{j}}}{\partial \Phi_\xi} O_{n\tilde{j}}.$$

All the formulae of course provide same result.

From the point of view of vector transformations a Q-connection is not a tensor. If $\mathbf{q}_k = O_{kp'}\mathbf{q}_{p'}$, then transformed components of connection are expressed throw original ones with addition of inhomogeneous term

$$\omega_{\xi\ kj} = O_{kp'}O_{jn'}\omega_{\xi\ p'n'} + O_{jp'}\frac{\partial O_{kp'}}{\partial \Phi_\xi}.$$

In 3D space Q-connectivity has clear geometrical and physical treatment as moveable Q-basis with behavior of Cartan 3-frame. Parameters of its ordinary rotations can depend on spatial coordinates $\Phi_\xi = \Phi_\xi(x_k)$, then $\partial_n \mathbf{q}_k = \Omega_{nkj}\mathbf{q}_j$, then components of slightly modified Q-connection

$$\Omega_{nkj} \equiv \omega_{\xi\ kj}\partial_n \Phi_\xi$$

have a sense of Ricci rotation coefficients. Parameters can also depend on the length of line of motion of the Q-basis or on the observer's time. Then $\Phi_\xi = \Phi_\xi(t), \partial_t \mathbf{q}_k = \Omega_{kj}\mathbf{q}_j$, and components of Q-connection

$$\Omega_{kj} \equiv \omega_{\xi\ kj}\partial_t \Phi_\xi$$

became generalized angular velocities of rotations of the frame.

The typical examples of Q-frames and Q-connection are

a) Frene frame. For the smooth curve $x_{\tilde{k}}(s)$ defined in constant basis the Frene frame is represented by the triad $\mathbf{q}_k$, obeying the equations

$$\frac{d}{ds}\mathbf{q}_1 = R_I(s)\mathbf{q}_2, \ \frac{d}{ds}\mathbf{q}_2 = -R_I(s)\mathbf{q}_1 + R_{II}(s)\mathbf{q}_3, \ \frac{d}{ds}\mathbf{q}_3 = -R_{II}(s)\mathbf{q}_2,$$

where the first and the second curvatures are

$$R_I = \Omega_{12}, R_{II} = \Omega_{23}.$$

b) Twisted straight line. For a given straight line $x_{\tilde{1}} = u, \ x_{\tilde{2}} = x_{\tilde{3}} = 0$, one can construct a Q-basis associated with it so that one vector is tangent to the line. In this case Q-connection is not zero and represented the only component describing torsion (or rather twist) of the line about itself.

$$\mathbf{q}_1 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{q}_2 = -i\begin{pmatrix} 0 & -ie^{-i\gamma(u)} \\ ie^{i\gamma(u)} & 0 \end{pmatrix}, \qquad \Omega_{23} = \frac{d\gamma}{du},$$

here $\gamma(u)$ is the angle, which is an arbitrary but smooth function of the line length.

Quaternionic spaces

Tangent Q-space [15]. It is known that on every N-dimensional differentiable manifold U_N with coordinates $\{y^A\}$ one can construct a tangent space T_N with coordinates $\{X^{(A)}\}$ so that $dX^{(A)} = g_B^{(A)}dy^B$, where $g_B^{(A)}$ – Lame coefficients. By an extra rotation one can construct a tangent Q-space $T(U, \mathbf{q})$, with coordinates $\{x_k\}$, k = 1,2,3, which associated with Q-frame vectors.

$$dx_k = h_{k(A)}dX^{(A)} = h_{k(A)}g_B^{(A)}dy^B,$$

where $h_{k(A)}$ are in general non-square matrices normalized by projectors of the basic space onto 3D one or vice versa.

Proper quaternionic space itself $\mathbf{U}_3$ is defined as 3D-space, locally identical to own tangent space $T(\mathbf{U}_3, \mathbf{q})$. The Q-space has the following basic features. Its Q-metric represented by vector part of the Q-multiplication rule $\mathbf{q}_j\mathbf{q}_k = -\delta_{jk} + \varepsilon_{jkn}\mathbf{q}_n$ is non-symmetric, its antisymmetric part is Q-operator (matrix), so that every point $\mathbf{U}_3$ has internal quaternionic structure. Q-connection $\mathbf{U}_3$ can be: (i) proper (metric) $\Omega_{nkj} \equiv \omega_{\xi kj}\partial_n\Phi_\xi$, for variable Q-basis it is always non zero, and (ii) affine (non-metric), independent from Q-basis. Q-torsion does not vanish in both cases, whereas Q-curvature $r_{knab} = \partial_a\Omega_{bkn} - \partial_b\Omega_{akn} + \Omega_{ajn}\Omega_{bjk} - \Omega_{bjk}\Omega_{ajn}$ for the metric Q-connection identically equals zero, but can be present in the space of affine Q-connection.

Once Q-space is introduced, there appears a new field of investigation of differential manifolds and spaces. Thus in the preliminary classification of Q-spaces based on presence and nature of curvature, torsion and non-metricity at least 10 different families can be distinguish [15]. In addition Q-spaces can be a nontrivial background for classical and quantum theories and problems.

4. Newton mechanics in Q-basis

Dynamics equations in rotating frame [16]

The Q-basis endowed with clock becomes a classical (non-relativistic) reference system. For an inertial observer the dynamic equations of classical mechanics can be written in constant Q-basis

$$m\frac{d^2}{dt^2}x_{\tilde{k}}\mathbf{q}_{\tilde{k}} = F_{\tilde{k}}\mathbf{q}_{\tilde{k}}.$$

$SO(3,R)$-invariance of two Q-vectors, the radius-vector $\mathbf{r} \equiv x_k\mathbf{q}_k$ and force $\mathbf{F} \equiv F_k\mathbf{q}_k$ allow to represent these equations in Q-vector form

$$m\frac{d^2}{dt^2}(x_k\mathbf{q}_k) = F_k\mathbf{q}_k, \qquad \text{or} \quad m\ddot{\mathbf{r}} = \mathbf{F}$$

In explicit form these equations possess enough complicated structure

$$m(\frac{d^2}{dt^2}x_n + 2\frac{d}{dt}x_k\Omega_{kn} + x_k\frac{d}{dt}\Omega_{kn} + x_k\Omega_{kj}\Omega_{jn}) = F_n$$

which nevertheless can be simplified and interpreted from physical points of view. Due to antisymmetry of the connection (generalized angular velocity)

$$\Omega_j \equiv \Omega_{kn}\frac{1}{2}\varepsilon_{knj}, \qquad \Omega_{kn} = \Omega_j\varepsilon_{knj},$$

the dynamic equations can be rewritten in vector components

$$m(a_n + 2v_k\Omega_j\varepsilon_{knj} + x_k\frac{d}{dt}\Omega_j\varepsilon_{knj} + x_k\Omega_j\Omega_m\varepsilon_{jkp}\varepsilon_{mpn}) = F_n$$

or by conventional vector notation

$$m(\vec{a} + 2\vec{\Omega} \times \vec{v} + \dot{\vec{\Omega}} \times \vec{r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})) = \vec{F}.$$

Among left hand side terms one easily recognizes 4 classical accelerations: linear, Coriolis, angular and centripetal. However this traditional interpretation is good only for simple rotation; in the case of combination of many Q-frame rotations number of components of generalized accelerations highly increases, and the equations become much more complicated. However it is worth noting that derivation of these equations for the most complicated rotations with the help of Q-basis and Q-connection is extremely simple.

Samples of Q-formulation of problems of classical mechanics

'Chasing' Q-basis – is a frame with one of its vectors, say $\mathbf{q}_1$ is always directed to observed particle. Dynamic equations for this case are written in explicit form in following manner

$$\ddot{r} - r(\Omega_2^2 + \Omega_3^2) = F_1/m,$$
$$2\dot{r}\Omega_3 + r\dot{\Omega}_3 + r\Omega_2\Omega_1 = F_2/m,$$
$$2\dot{r}\Omega_2 + r\dot{\Omega}_2 + r\Omega_1\Omega_3 = -F_3/m.$$

Components of Q-connection are defined as functions of angles of two rotations, the first (an angle α) – around vector $\mathbf{q}_3$, the second (an angle β) – around $\mathbf{q}_2$

$$\Omega_1 = \dot{\alpha}\sin\beta, \qquad \Omega_2 = -\dot{\beta}, \qquad \Omega_3 = \dot{\alpha}\cos\beta.$$

The chasing Q-basis approach is convenient to solve a number of mechanical problems related to rotations, some times very complicated, of observed objects and systems of reference. Here is an illustration.

Rotating oscillator. One seeks for motion law $r(t)$ of a harmonic oscillator (mass m, spring elasticity k) which has a freedom of motion along rigid smooth rod rotating in the plane around one of its ends (here one end of the spring is fixed) with angular velocity ω; the equilibrium point is located at the distance l from the rotation center, there is no gravity. Radial and tangent dynamic equations in the chasing Q-basis (F is unknown rod reaction force)

$$\ddot{r} - r\omega^2 = -\frac{k}{m}(r-l), \qquad 2\dot{r}\omega = \frac{1}{m}F,$$

admit the following family of solutions:

$$\text{(i)} \qquad r(t) = r_0 + v_0 t + at^2$$

mass moves away from the center of rotation with quadratic (or linear) law,

$$\text{(ii)} \qquad r(t) = const + Ae^{iwt} + Be^{-iwt}, \qquad w \equiv \sqrt{k/m - \omega^2}$$

here are three different situations depending on a relation of the quantities under the square root:

– r = const,

– harmonic oscillators,

– exponential motion away from the center of rotation.

It is interesting that the variants of rotating classical oscillator behavior with $l = 0$ are precisely similar to behavior of four known cosmological models of Einstein-DeSitter-Friedman considered in the General Relativity.

5. Construction of Quaternionic Relativity

Hyperbolic rotations and biquaternions [17]

It was noted above, that SO(3,C)-transformations of Q-units admit pure imaginary parameters. In this case rotations become hyperbolical (H – from hyperbolic); e. g. simple H-rotation $\mathbf{q}' = H_3^{\psi}\mathbf{q}$ is performed by matrix of the form

$$H_3^{\psi} = \begin{pmatrix} \cosh\psi & -i\sin\psi & 0 \\ i\sin\psi & \cosh\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and 2×2 -matrices of Q-units representation are no longer hermitian:

$$\mathbf{q}_{1'} = -i \begin{pmatrix} 0 & e^{\psi} \\ e^{-\psi} & 0 \end{pmatrix}.$$

This is the time to recall the notion of so called biquaternionic vectors (BQ). BQ-vector is defined as Q-vector with complex components $\mathbf{u} = (a_k + ib_k)\mathbf{q}_k$. Obviously for vectors of this type the norm (or modulus) in general can not be defined; but among all BQ-vectors there is a subset of "good" elements with well definable norm by $\mathbf{u}^2 = b^2 - a^2$. These vectors appear to be form-invariant with respect to transformations of subgroup

$SO(2,1) \subset SO(3,C)$, and in particular, with respect to simple H-rotations $\mathbf{q}' = H\mathbf{q}\mathbf{u} = u_k\mathbf{q}_k = u_{k'}\mathbf{q}_{k'}$, but only when reciprocally imaginary components $a_k b_k = 0$ are orthogonal to each other.

Quaternionic Relativity

The made above observation allows to suggest a space-time BQ-vector "interval"

$$d\mathbf{z} = (dx_k + idt_k)\mathbf{q}_k,$$

with specific properties:

(i) Temporal interval is defined by imaginary vector,

(ii) space-time of the model appears to have six-dimensional (6D),

(iii) vector of the displacement of the particle and vector of corresponding time change must always be normal to each other $dx_k dt_k = 0$.

In this case BQ-vector-interval is invariant under group $SO(2,1) \subset SO(3,C)$, as well as of course its square (which differs from the square of norm only by sign) $d\mathbf{z}^2 = dt^2 - dr^2$, the latter has precisely the same form as a space-time interval of Special Relativity of Einstein. This 6D-model was initially named the Quaternionic Relativity. Temporal and spatial variables symmetrically enter the expression of BQ-vector-interval, and the Q-triad related to it describes relativistic system of reference $\Sigma \equiv (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$. Transition from one reference system to another is performed with the help of 'rotational equations' of the type $\Sigma' = O\Sigma$ with matrix O from the group $SO(2,1)$ is a product of matrices of real and hyperbolic rotations. So the theory could also be named (may be more correctly) 'Rotational Relativity'. The meaning of a simple H-rotation is immediately revealed from the first line of equation $\Sigma' = H_3^{\psi}\Sigma$ in the explicit form

$$i\mathbf{q}_{1'} = i\cosh\psi(\mathbf{q}_1 + \tanh\psi\mathbf{q}_2).$$

If like in Special Relativity $\cosh\psi = dt/dt'$, then

$$idt'\mathbf{q}_{1'} = idt(\mathbf{q}_1 + V\mathbf{q}_2),$$

which describes motion of reference system Σ' relative to Σ with velocity V along direction $\mathbf{q}_2$. It is easy to show that $SO(2,1)$-rotations of Q-reference system enhance Lorenz coordinate transformations and therefore all cinematic effects of Special Relativity.

It should be noted here that parameters of real and hyperbolic rotations can be variable for instance dependent on observer's time. This hints to expect of the discussed theory a possibility to describe non-inertial motions. Analysis of the rotational equations confirms the expectation. Well-known relativistic model of reference system constantly accelerated with respect to the inertial one (hyperbolic motion), frequently found in literature and normally regarded with use of assumption beyond frames of Special Relativity, in quaternionic theory is solved naturally and fast not only from the inertial observer viewpoint, but from position of accelerated frame too [18].

The kinematic problem of other non-inertial motion – relativistic circular motion – can be completely and precisely resolved by means of the rotation equation $\Sigma' = H_2^{\psi(t)} R_1^{\alpha(t)}\Sigma$, where Σ' is reference system rotating along the circle around the immobile frame Σ. This problem also can be solved both from the point of view of inertial observer, in this case the result has the form

$$t = \int dt' \cosh\psi(t'), \qquad \alpha(t) = \frac{1}{R}\int dt' \tanh\psi(t'),$$

$$a_{\tan}(t) = \frac{1}{\cosh^2\psi}\frac{d\psi}{dt}, \qquad a_{norm}(t) = R\left(\frac{d\alpha(t)}{dt}\right)^2,$$

and from the point of view of the observer in the reference system arbitrary moving along circular orbit.

The solution of the problem of "classical" Thomas precession in the framework of Special Relativity also needs additional assumptions, while in the quaternionic theory has a single line form – the first row of the matrix of rotation equation $\Sigma'' = R_1^{-\alpha(t)} H_2^{\psi} R_1^{\alpha(t)}\Sigma$, in this case of course correct value of precession frequency is obtained

$$\omega_T = (1 - \cosh\psi) \approx -\frac{1}{2}\omega V^2.$$

Moreover, the quaternionic theory of relativity appears to be able to describe Thomas precession for the vectors moving along trajectories of general type. The basic rotational equation in this case naturally generalized: $\Sigma'' = R^{-\theta(t)} H^{\psi(t)} R^{\theta(t)}\Sigma$, here $\theta(t)$ – an angle of instant rotation. Requirement that an axis of hyperbolic rotation be normal to the plane formed by the radius-vector of observed frame and its velocity vector, is also significant. In this case formula of variable frequency of general Thomas precession has the form

$$\Omega_T = \frac{d}{dt}(\theta - \theta').$$

An example of such Thomas precession is an apparent displacement of mercurial perihelion, for which calculations give a value $\Delta\varepsilon = 2,7''/100$ years.

Universal character of motion of the bodies (including non-inertial motions) in the Quaternionic Relativity suggests seeking for new cinematic relativistic effects. One is found in Solar System planets' satellites motion. Relative velocity of the Earth and other planets changes with time and sometimes achieves significant value comparable somehow to value of the fundamental velocity. This can lead to discrepancy between calculated and observed from the Earth cinematic magnitudes characterizing cyclic processes on this planet or near it. In particular there must be a deviation of the planetary satellite position. Such an angular difference is surprisingly found to be linearly dependent upon the time of observation

$$\Delta\varphi \approx \frac{\omega V_E V_P}{c^2} t,$$

here ω is an angular velocity of satellite motion around the planet, V – are linear velocities of the Earth and the planet around the sun. The magnitude of the effect is the following for the closest to the Jupiter and "the fasters" Jupiter satellite $\Delta\varphi \cong 12'$ for 100 terrestrial years; for the Mars satellite (Phobos) $\Delta\varphi \cong 20'$ for 100 terrestrial years [19]. Both values are big enough for the effect to be noticed in prolonged and precise observations.

One can say that space-time model and kinematics of the Quaternionic Relativity are nowadays studied in enough details and can be used as an effective mathematical tool for calculation of many relativistic effects. But respective relativistic dynamic has not been yet formulated, there are no quaternionic field theory; Q-gravitation, electromagnetism, weak and strong interactions are still remote projects. However, there is a hope that it is only beginning of a long way, and the theory will "mature". This hope is supported by observation of number of remarkable "Quaternionic Coincidences" forming a discrete mosaic of physical and mathematical facts; probably one day it will turn into a logically consistent picture providing new instruments and extending our insight of physical laws.

6. Remarkable "quaternionic coincidences"

There are, at least, five such coincidences (all of them given below), noted by different authors in various time.

1. *The Maxwell equations as an conditions of the analyticity of functions of quaternionic variable.*

In 1937 year Fueter [20] noted, that Cauchy-Riemann $\partial f/\partial z^* = 0$ equations defining the differentiability of complex variable function and modeling physically a flat motion of liquid without sources and whirls, have the following quaternionic analogue

$$\left(i\frac{\partial}{\partial t} - \mathbf{q}_{\tilde{k}}\frac{\partial}{\partial x_{\tilde{k}}}\right)\mathbf{H} = 0, \qquad \mathbf{H} = (B_{\tilde{n}} + iE_{\tilde{n}})\mathbf{q}_{\tilde{n}}.$$

Surprising fact is that the equations of classic Maxwell electrodynamics in vacuum prove to be corresponding physical model

$$div\vec{E} = 0, \quad div\vec{B} = 0, \quad rot\vec{E} - \frac{\partial\vec{B}}{dt} = 0, \quad rot\vec{B} + \frac{\partial\vec{E}}{dt} = 0.$$

2. *Classical mechanics in the rotating reference systems.*

The compact form of Newton equations in quaternion frame is described above in section 4. Finally it should be stressed that the form of dynamics equations naturally arising and externally primitive

$$m\ddot{\mathbf{r}} = \mathbf{F}$$

hides all possible combinations of rotations of reference systems or observed bodies. Using differential quaternionic objects helps to easily obtain explicit form of the equations whose elements have obvious physical meaning.

3. *The quaternionic theory of relativity.*

1:1 isomorphism of the Lorenz group of Special Relativity and the group of invariance of quaternionic multiplication $SO(3, C)$ leads to non-standard theory of relativity with symmetric six-dimensional space-time. This theory significantly differs from Einstein Special Relativity in origin, model, possibilities and mathematical tools, but predicts absolutely similar cinematic effects. Invariance of specific biquaternionic vector "interval" $d\mathbf{z} = (dx_k n + i\, dt_k)\mathbf{q}_k$ under subgroup $SO(2, 1)$ with in general variable parameters admits calculation of relativistic effects for non-inertial motion of reference systems.

4. *Pauli equations* [21].

Consider the quantum particle with electric charge e, mass m, and generalized momentum

$$P_k \equiv -i\hbar\frac{\partial}{\partial x_k} - \frac{e}{c}A_k$$

in the simplest quaternionic space (all the parameters are constant, connection, non-metricity, torsion and curvature equal to zero). Hamiltonian of such particle in Q-metrics

$$H \equiv -\frac{1}{2m}P_k P_m \mathbf{q}_k \mathbf{q}_m$$

is the exact copy of Hamilton function of Pauli equation

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{2mc}\vec{B}\cdot\vec{\sigma},$$

and the spin term "automatically" acquires a coefficient equal to Bohr magneton.

5. *Young-Mills field strength.*

If one constructs a "potential" vector in an arbitrary quaternionic space from Q-connection components Ω_{amn} (indices a, b, c enumerate coordinates of basic Q-space, indices j, k, m, n enumerate vectors of tangent triad)

$$A_{ka} \equiv \frac{1}{2}\varepsilon_{kmn}\Omega_{amn},$$

and similarly construct a "field strength" vector

$$F_{kab} \equiv \frac{1}{2}\varepsilon_{kmn}r_{mnab},$$

from quaternionic curvature components

$$r_{knab} = \partial_a\Omega_{bkn} - \partial_b\Omega_{akn} + \Omega_{ajn}\Omega_{bjk} - \Omega_{bjk}\Omega_{ajn}$$

then these two geometrical objects are interconnected in the similar manner as the field strength and potential of the Young-Mills field

$$F_{kab} \equiv \partial_b A_{ka} - \partial_a A_{kb} + \varepsilon_{kmn}A_{ma}A_{nb}.$$

(formula) It should be stressed that for the Q-spaces with metric (not affine) connection curvature (field strength) identically vanish.

Discussion

Quaternionic numbers of course are first of all mathematical objects, so the problem of development of their algebra, analysis and geometry is self-consistent. But history of modern science states that once the geometry, in particular differential geometry, is discussed the presence of physics is unavoidable. There is a known point of view that Einstein who suggested General Relativity was a pioneer in geometrization of physics. But it is also known that quite earlier Maxwell formulated his electrodynamics in terms of quaternions convenient for description of 'etheric tensions' which were thought to represent field strength vectors. But since that the geometrical language has not been utilized for many decades.

The aspects of quaternionic mathematics given in this review once again draw attention to 'genetic relations' between physics and geometry: from description of frames rotations to quaternionic field structure phenomena in Pauli equations and Young-Mills theory.

Wide variety of possibilities provided by Q-approach and derived within it non-traditional physical models, like six-dimensional space-time or mentioned above coincidences may lead to opinion that quaternions are still a mathematical play, something like 'lego' elements, from which one can build many exotic constructions.

As a comment there are the following two observations.

1. Producing non-standard physical models Q-method nonetheless allows to successfully solve physical problems thus being a useful tool for practical purposes. A typical example: inherited exponential character of representation of simple rotations helps to simply formulate summation of different rotations, including, of course, imaginary rotations, describing relativistic boosts. Recall that in classical mechanics summation of ordinary rotations is quite a task.

2. All physical quaternionic theories are not heuristically invented, but appear naturally from fundamental mathematical lows, as though confirming Pythagorean idea on "world – number" dependence. Indeed, Q-algebra, the last associative algebra, describes well physical quantities, all of them up to our knowledge being associative with respect to multiplication: from observable kinematic and dynamic one, to tensors and spinors incorporated in the theories. All this gives a hope that further efforts in the research "quaternions – physical laws" relations will once grow into wide scientific programme. Yet another small, but persevering step in this direction has been recently made, when the author of this review succeeded to found an exact solution for relativistic oscillator problem in the framework Quaternionic Relativity. Details of the solution will be published elsewhere.

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THREE-NUMBERS WHICH CUBE OF NORM IS NONDEGENERATE THREE-FORM

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Arbitrary three-form can be put in a canonical form. The requirement of existence of two-parametric Abelian Lie group to play the role of group of symmetry for three-form admits selecting the three-forms that correspond to three-numbers and finding all the three-numbers which cube of norm is a non-degenerate three-form with respect to a special coordinate system. There are exactly two (up to isomorphism) such sets of hypercomplex numbers, namely the sets: C_3, H_3. They can be regarded as generalizations of complex and binary (hyperbolic) bi-numbers to the case of three-numbers.

1. Introduction

The real number is a stoneconcept for both the mathematics and physics. The associative-commutative n-dimensional hypercomplex numbers over the field of real numbers, – which we shall call the n-numbers for short, – comprises an attractive extension of this concept. The complex numbers are well adopted in solving problems of mathematical and theoretical physics and present actually a particular case of such hypercomplex numbers, bi-numbers. Regrettably, the n-numbers at $n > 2$ have not been studied in great detail. It can be hoped that, possessing such simplified properties as associativity and commutativity and showing sufficient complexity in some cases, the associative-commutative hypercomplex numbers shall find their non-trivial application. At $n > 2$ the very classification and choice of the n-numbers for mathematical studies with the aim of farther application in physics is a non-trivial problem. The formulation of additional conditions to specify a narrow (but significant) class in all the set of n-numbers seems to be a convenient way to attack the problem. The stipulating of a special basis in term of which the coordinates of the n-numbers be similar (for example, the norm would be independent of permutation of coordinate labels, the more strength condition insists of fulfilling the requirement that n-th degree of norm of the n-number be non-degenerate with respect to such coordinates) can play the role of such a condition. For the sake of brevity, in the present work the n-form of the coordinates of the n-dimensional linear space is meant to be the highsymmetric poly-linear form of n-th degree, all the arguments of which being equal to a fixed vector. *Highsymmetry of form* means existence of such a basis that the relevant representation of the symmetric form of n-vector arguments does not change under permutation of of coordinates. *The non-degeneracy of form* means the impossibility to express the form as an integer degree of a form of lower degree. Below we shall often omit the term "non-degenerate" , implying merely an n-form.

The present work is devoted to studying the three-numbers, that is, the associative-commutative hypercomplex numbers of the form

$$X = x_1 + x_2 \cdot e_2 + x_3 \cdot e_3, \tag{1}$$

where e_2, e_3 are symbolic elements, and x_1, x_2, x_3 are real numbers applying as the coordinates with respect to the basis $e_1 \equiv 1, e_2, e_3$. If a number X admits the exponential representation

$$X = \rho \cdot \exp(\alpha \cdot e_2 + \beta \cdot e_3), \tag{2}$$

where $\rho > 0, \alpha, \beta$ – real numbers, then the quantity ρ can naturally be called the modulus of the three-number X. Let us search for only the three-numbers that with respect to a special basis (the latter is not necessary the basis $e_1 \equiv 1, e_2, e_3$) the cube of the norm $\rho(x_1, x_2, x_3)$ is a non-degenerate three-form of coordinates, that is,

$$\rho^3 = \Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3), \tag{3}$$

where the three-form of the general type

$$\Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3) = \omega_1 \Omega_1(x_1, x_2, x_3) + \omega_2 \Omega_2(x_1, x_2, x_3) + \omega_3 \Omega_3(x_1, x_2, x_3) \tag{4}$$

is an arbitrary linear combination with real numbers $\omega_i \quad (i = 1, 2, 3)$ at the basis three-forms:

$$\Omega_1(x_1, x_2, x_3) \equiv x_1^3 + x_2^3 + x_3^3, \tag{5}$$

$$\Omega_2(x_1, x_2, x_3) \equiv x_1 x_2^2 + x_1 x_3^2 + x_1^2 x_2 + x_2 x_3^2 + x_1^2 x_3 + x_2^2 x_3, \tag{6}$$

$$\Omega_3(x_1, x_2, x_3) \equiv x_1 x_2 x_3. \tag{7}$$

It will be noted that in the three-dimensional space the symmetric cubic form of three vector arguments, assuming the linearity in each argument, contains not three but ten arbitrary real parameters; that is, the form is a more general notion then the high-symmetric three-form and hence leads to the form which is more general then (4). The requirement of non-degeneracy of three-form reads

$$\Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3) \neq \Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3) \equiv \omega \cdot (x_1 + x_2 + x_3)^3. \tag{8}$$

In the sequel we shall assume the non-degenerate type, unless otherwise stated explicitly.

The multiplication of the number X by a unimodular number X_1 yields the number

$$Y = X_1 \cdot X, \tag{9}$$

which modulus is equal to the modulus of the number X, so that for such three-numbers we have

$$\Omega(y_1, y_2, y_3; \omega_1', \omega_2', \omega_3') = \Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3). \tag{10}$$

Thus in order that the cube of norm be three-form, the set of unimodular numbers of this hypercomplex system must form a two-parametrical continuous Abelian Lie group (the symmetry group which retains the form of the three-form) consisted of linear transformations (9) of the coordinate space of considered three-forms.

Let us assume that for definite values of parameters of three-form (4) we find the symmetry group which is two-parametric Abelian group of continuous linear transformations with generators E_2, E_3 given by real quadratic matrices 3×3. Then, as is known, the linear transformations themselves can be defined through generators of matrix $\hat{A}$ according to the formula

$$\hat{A} = \exp(\alpha \cdot \hat{E}_2 + \beta \cdot \hat{E}_3), \tag{11}$$

where α, β are real parameters. Let in this way the multiplication rules

$$\hat{E}_i \cdot \hat{E}_j = p_{ij}^k \cdot \hat{E}_k \tag{12}$$

obey for generators, where $i, j, k = 1, 2, 3$; $\hat{E}_1$ stands for the unit matrix (the generator of general scale transformation), p_{ij}^k is some real number; summation over repeated indices is assumed. Then $\hat{E}_1\hat{E}_2, \hat{E}_3$ can be regarded as a representation of the basis elements $e_1 \equiv 1, e_2, e_3$ of some set of three-numbers, whence the representation of the set of such numbers in the coordinate linear three-dimensional space x_1, x_2, x_3 in the form of linear quadratic matrices 3×3. It is obvious that the multiplication law for the basis elements $e_1 \equiv 1, e_2, e_3$ will be of the same form (12) with the same characteristic numbers p_{ij}^k

$$e_i \cdot e_j = p_{ij}^k \cdot e_k. \tag{13}$$

Now we can write the numbers representable in the exponential form (2). The coordinate linear space x_1, x_2, x_3 is not obliged to be introduced in the same basis, that is in accordance with the formula (1). Therefore, in general case there appears the following relation for numbers representable in exponential form:

$$x_1 \cdot e'_1 + x_2 \cdot e'_2 + x_3 \cdot e'_3 = \rho \cdot \exp(\alpha \cdot e_2 + \beta \cdot e_3), \tag{14}$$

where e'_1, e'_2, e'_3 is a basis differed in general case from $e_1 \equiv 1, e_2, e_3$, and such that e'_1 may differ from real unity. Using three coordinate relations (14) and finding two real parameter α, β, we get the expression for the cube of norm through the coordinates x_1, x_2, x_3:

$$\rho^3 = f(x_1, x_2, x_3). \tag{15}$$

If an initial three-form enters the right-hand part of this formula, then the relevant three-numbers are found.

2. Transformation of three-form to a canonical type

Apart of general scale transformation, there exists but one linear coordinate transformation connected continuously with the identity by means of which an arbitrary three-form goes over again in a three-form. Let us write the transformation in the matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{3q} \begin{pmatrix} p+2 & p-1 & p-1 \\ p-1 & p+2 & p-1 \\ p-1 & p-1 & p+2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \tag{16}$$

where q is an arbitrary positive real number, and

$$p \equiv q^3. \tag{17}$$

With respect to new variables, the three-form $\Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3)$ transformed by (16) takes on the form

$$\Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3) = \Omega(y_1, y_2, y_3; \omega'_1, \omega'_2, \omega'_3), \tag{18}$$

where

$$\left.\begin{aligned} \omega'_1 &\equiv u \cdot (w_1 p^3 + 3w_2 p + 2w_3), \\ \omega'_2 &\equiv 3u(w_1 p^3 - w_3), \\ \omega'_3 &\equiv 3u(2w_1 p^3 - 3w_2 p + 4w_3), \end{aligned}\right\} \tag{19}$$

$$u \equiv \frac{1}{27p}, \tag{20}$$

$$\left.\begin{aligned} w_1 &\equiv 3\omega_1 + 6\omega_2 + \omega_3, \\ w_2 &\equiv 6\omega_1 - \omega_3, \\ w_3 &\equiv 3\omega_1 - 3\omega_2 + \omega_3. \end{aligned}\right\} \tag{21}$$

Certainly, the classification of three-forms (by transforming to canonical type) can be performed in various ways. Let us start with stipulating that the three-forms connected by the linear non-degenerate coordinate transformation that does not affect the values of the three-form itself, are equivalent, — in the sense that they differ by only the choice of basis in three-dimensional linear space of x_1, x_2, x_3, that is by the choice of basis (symbolic) element in the space of three-numbers. When transforming three-form to a canonical form, we shall consider not all linear non-degenerate transformations but only the possible triple, namely, the transformation (16); the discrete transformation (changing simultaneously the sign for all three coordinates); general scale transformation (multiplying simultaneously all three coordinates by a fixed real positive number). The basis forms (5) – (7), because of their preferable type, are certainly regarded as canonical.

So, let us consider three-form of the general type (4) and go over by the help of the linear transformation (16) to new coordinates. Since the relationship between the quantities w_i and the parameters of the three-form ω_i is one-to-one, we shall try to diminish the number of parameters of three-form with respect to new coordinates, considering various variants and using the quantities w_i and the formulas (19).

1). If

$$\operatorname{sign}(w_1) = \operatorname{sign}(w_2) \neq 0, \tag{22}$$

then by the help of the coordinate transformation (16) with the parameter value

$$p = \sqrt[3]{\frac{w_3}{w_1}} \tag{23}$$

the initial three-form can be reduced to the form $\Omega(y_1, y_2, y_3; \omega'_1, 0, \omega'_3)$.

2). If

$$\operatorname{sign}(w_1) = -\operatorname{sign}(w_3) \neq 0, \tag{24}$$

then the two transformations (16) can always be found such that ω'_1 can be nullified by using one of them, whereas ω'_3 can be nullified by using another member, and in both cases the parameter ω'_2 gets strongly not equal to zero at any value w_2. Thus as a result, one is to choose either the form $\Omega(y_1, y_2, y_3; 0, \omega'_2, \omega'_3)$ or thereto equivalent three-form $\Omega(y_1, y_2, y_3; \omega'_1, \omega'_2, 0)$. In order to exclude ambiguity, we shall always choose the first version, that is the three-form $\Omega(y_1, y_2, y_3; 0, \omega'_2, \omega'_3)$. On so doing, the parameter p in the transformation (16) is a real positive root of the cubic equation

$$w_1 p^3 + 3w_2 p + 2w_3 = 0. \tag{25}$$

There remains to consider the case when the quantities vanish either separately or totally.

3). If

$$w_1 = 0, \quad \operatorname{sign}(w_2) = -\operatorname{sign}(w_3) \neq 0, \tag{26}$$

then by the help of the transformation (16) the three-form can be reduced to the canonical form $\Omega(y_1, y_2, y_3; 0, \omega'_2, \omega'_3)$, with $\omega'_2 \neq 0$ and $\omega'_3 \neq 0$, as well as

$$p = -\frac{2w_3}{3w_2}. \tag{27}$$

4). If

$$w_1 = 0, \quad \operatorname{sign}(w_2) = \operatorname{sign}(w_3) \neq 0, \tag{28}$$

then by the help of the transformation (16) the three-form can be reduced to the canonical form $\Omega(y_1, y_2, y_3; \omega'_1, \omega'_2, 0)$, with ω'_1 and $\omega'_2 \neq 0$, and

$$p = \frac{4w_3}{3w_2}. \tag{29}$$

5). If

$$w_1 = 0, \quad w_2 = 0, \quad w_3 \neq 0, \tag{30}$$

then

$$\omega'_1 = 2uw_3, \quad \omega'_2 = -3uw_3, \quad \omega'_3 = 12uw_3. \tag{31}$$

In this case the three-form can be presented by $\Omega(x_1, x_2, x_3; \omega_1, -\frac{3}{2}\omega_1, 6\omega_1)$, so that the coordinate transformation (16) leads to the representation $\Omega(y_1, y_2, y_3; \omega'_1, -\frac{3}{2}\omega'_1, 6\omega'_1)$ with $\omega'_1 \neq 0$, that is, the transformation (16) degenerates to a general scale transformation.

6). If

$$\operatorname{sign}(w_1) = -\operatorname{sign}(w_2) \neq 0, \quad w_3 = 0, \tag{32}$$

then the transformation (16) with the parameter

$$p = \sqrt{-\frac{3w_2}{w_1}} \tag{33}$$

transfers the initial three-form into the three-form $\Omega(y_1, y_2, y_3; 0, \omega'_2, \omega'_3)$, with $\omega'_2 \neq 0$ and $\omega'_3 \neq 0$.

7). If

$$\operatorname{sign}(w_1) = \operatorname{sign}(w_2) \neq 0, \quad w_3 = 0, \tag{34}$$

then under the linear transformation (16) with

$$p = \sqrt{\frac{3w_2}{2w_1}} \tag{35}$$

the three-form is reduced to $\Omega(y_1, y_2, y_3; \omega'_1, \omega'_2, 0)$, with $\omega'_1 \neq 0 and \omega'_2 \neq 0$.

8). If

$$w_1 \neq 0, \quad w_2 = 0, \quad w_3 = 0, \tag{36}$$

then

$$\omega'_1 = uw_1p^3, \quad \omega'_2 = 3uw_1p^3, \quad \omega'_3 = 6uw_1p^3, \tag{37}$$

and hence under the transformation (16) the three-form $\Omega(x_1, x_2, x_3; \omega_1, 3\omega_1, 6\omega_1)$ becomes $\Omega(y_1, y_2, y_3; \omega'_1, 3\omega'_1, 6\omega'_1)$, where $\operatorname{sign}(\omega'_1) = \operatorname{sign}(\omega_1)$. Thus, the transformation (16) in such a case is reduced to multiplication of the initial three-form by a real positive number, that is, to a general scale transformation. We exclude such case in constructing three-numbers, for the case is degenerate (see (8)).

9). Lastly, we are to consider the variant

$$w_1 = 0, \quad w_2 \neq 0, \quad w_3 = 0, \tag{38}$$

in which

$$\omega'_1 = 3 \cdot u \cdot w_2 \cdot p = \frac{w_2}{9} = \omega_1, \quad \omega'_2 = \omega_2 = 0, \quad \omega'_3 = -9 \cdot u \cdot w_2 \cdot p = -\frac{w_2}{3} = -3\omega_1, \tag{39}$$

that is, applying (16) with arbitrary p transforms the three-form $\Omega(x_1, x_2, x_3; \omega_1, 0, -3\omega_1)$ to $\Omega(y_1, y_2, y_3; \omega_1', 0, -3\omega_1')$. Thus, in the given case the transformation (16) does not influence parameters of three-form, so that we can conclude that this transformation is a symmetry transformation for three-form $\Omega(x_1, x_2, x_3; \omega_1, 0, -3\omega_1)$.

Subsequent simplifying the three-form can be performed by multiplying it by an arbitrary real number deviating from zero. Such an operation is reduced to the following two ones: the changing of sign for all the coordinates and the general scale transformation. As a result, one of the coefficients $\omega_i' \neq 0$ of the three-form can be put to be unity, that is, the normalization can be performed. The proposed scheme 1) – 9) together with the normalization does not contradict to selecting three basis forms to play the role of canonical coordinates and introducing the notion of non-degeneracy, for the given algorithm goes over the basis forms (5) – (7) to the same basis forms, and any degenerate form to a degenerate one.

Thus, we have arrived at the following conclusion. Studying three-form of the general type $\Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3)$ reduces to studying the 8 canonical three-forms:

$$\Omega(x_1, x_2, x_3; 1, 0, 0) \equiv \Omega_1(x_1, x_2, x_3); \tag{40}$$

$$\Omega(x_1, x_2, x_3; 0, 1, 0) \equiv \Omega_2(x_1, x_2, x_3); \tag{41}$$

$$\Omega(x_1, x_2, x_3; 0, 0, 1) \equiv \Omega_3(x_1, x_2, x_3); \tag{42}$$

$$\Omega(x_1, x_2, x_3; 1, -\frac{3}{2}, 6); \tag{43}$$

$$\Omega(x_1, x_2, x_3; 1, 3, 6) \equiv (x_1 + x_2 + x_3)^3, \qquad \text{(degenerate)}; \tag{44}$$

$$\Omega(x_1, x_2, x_3; 1, \omega, 0), \qquad \omega \in [-\frac{1}{2}; 0) \cup (0; 1;] \tag{45}$$

$$\Omega(x_1, x_2, x_3; 1, 0, \omega), \qquad \omega \neq 0; \tag{46}$$

$$\Omega(x_1, x_2, x_3; 0, 1, \omega), \qquad \omega \neq 0. \tag{47}$$

The condition on the parameter ω (45) for the sixth canonical three-form is necessary in order that the uncertainty be avoided that does exists under consideration of the variant 2) of values of parameters of the general-type three-form. The condition $\omega \neq 0$ for the sixth, seventh, and eighth canonical three-forms is necessary to exclude the basis three-forms that have been ascribed to a canonical type.

3. Three-forms which may relate to three-numbers

Instead of searching directly the linear transformations leaving the three-forms 1 (40) − 8 (47) unchanged, we shall try to find the linear transformations which are infinitely near to identical ones. This problem is reduced to finding relevant generators.

1. There does not exist any continuous two-parametric Abelian Lie group which leave the form of the first canonical three-from (40) unchanged.

2. There does not exist any continuous two-parametric Abelian Lie group which leave the form of the second canonical three-from (41) unchanged.

3. The third canonical three-form (42) has a two-parametric non-Abelian group Lie to act as a symmetry group with the generators

$$\hat{a}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{a}_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{48}$$

4. The fourth canonical three-form (43) has a three-parametric non-Abelian group Lie to act as a symmetry group with the generators

$$\hat{a}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{a}_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{a}_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{49}$$

It is necessary to verify that whether a two-parametric Abelian sub-group exists in this group.

5. The fifth canonical three-form (44) is non-degenerate and, therefore, is excluded from searching the three-numbers corresponding thereto.

6. The sixth canonical three-form does involve any two-parametric Lie group (for any admissible-type parameter (45)), although at $\omega = 1$ this three-from has one-parametric group of symmetry. Therefore the sixth canonical three-from cannot relates to three-numbers.

7. $\omega = -3$ 7th is the only parameter value at which the three-form has a two-parametric Abelian Lie group to serve as a symmetry group with the generators

$$\hat{a}_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \hat{a}_7 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{50}$$

It will be noted that the transformation (16) with the generator presented by the sum of the generators (50) enters this symmetry group, so that the three-form $\Omega(x_1, x_2, x_3; 1, 0, -3)$ should related to special cases.

8. The eight canonical three-form (47) at $\omega = 3$ has a one-parametric symmetry group which cannot relate to three-numbers, and at $\omega = 2$ has a two-parametric Abelian symmetry group with the generators

$$\hat{a}_8 = \begin{pmatrix} -\frac{2}{3} & -1 & -1 \\ 0 & \frac{1}{3} & 1 \\ 0 & 1 & \frac{1}{3} \end{pmatrix}, \qquad \hat{a}_9 = \begin{pmatrix} \frac{1}{3} & 1 & 0 \\ 1 & \frac{1}{3} & 0 \\ -1 & -1 & -\frac{2}{3} \end{pmatrix}. \tag{51}$$

Thus among the canonical three-forms we are able to find the four non-degenerate types that may relate to three-numbers. Retaining the numeration of canonical three-forms, let us write down these four forms indicating the related generators of symmetry group:

1. $- - - - - - - - -$;

2. $- - - - - - - - -$;

$$3. \ \Omega(x_1, x_2, x_3; 0, 0, 1) \equiv \Omega_3(x_1, x_2, x_3), \quad \{\hat{a}_1, \hat{a}_2\}; \tag{52}$$

$$4. \ \Omega(x_1, x_2, x_3; 1, -\frac{3}{2}, 6) \quad \{\hat{a}_3, \hat{a}_4, \hat{a}_5\}; \tag{53}$$

5. $- - - - - - - - -$;

6. $- - - - - - - - -$;

$$7. \ \Omega(x_1, x_2, x_3; 1, 0, -3), \quad \{\hat{a}_{12}, \hat{a}_{13}\}; \tag{54}$$

$$8. \ \Omega(x_1, x_2, x_3; 0, 1, 2), \quad \{\hat{a}_{14}, \hat{a}_{15}\}; \tag{55}$$

4. Three-forms $\Omega_3(x_1, x_2, x_3), \Omega(x_1, x_2, x_3; 0, 1, 2)$ and three-numbers

Let us consider the three-form $\Omega_3(x_1, x_2, x_3)$ which, as have been clarified above, possesses a two-parametric continuous Lie group — namely the symmetry group with the generators $\hat{a}_1, \hat{a}_2$ (48). Juxtaposing to the unit matrix and generators $\hat{a}_1, \hat{a}_2$ the basis elements $e_1 \equiv 1, e_2, e_3$ of sought system of three-numbers, we get for them the following multiplication table:

$\times$	1	e_2	e_3
1	1	e_2	e_3
e_2	e_2	$\frac{1}{3}(2 - 2e_2 + e_3)$	$\frac{1}{3}(1 - e_2 - e_3)$
e_3	e_3	$\frac{1}{3}(1 - e_2 - e_3)$	$\frac{1}{3}(2 + e_2 - 2e_3)$

Table 1.

So the three-numbers which can relate to the three-form $\Omega_3(x_1, x_2, x_3)$ have been found. It remains to verify whether a system of linear coordinates exists with respect to which the cube of the found three-numbers is a three-form $\Omega_3(x_1, x_2, x_3)$.

From the form of generators (48) it is obvious that the obtained system of three-numbers is isomorphic to the algebra of the diagonal matrices 3×3; therefore, we denote such numbers as H_3 and introduce a linear coordinates x_1, x_2, x_3 in terms of the basis

$$\psi_1 = \frac{1}{3}(1 - e_2 - e_3), \quad \psi_2 = \frac{1}{3}(1 + 2e_2 - e_3), \quad \psi_3 = \frac{1}{3}(1 - e_2 + 2e_3) \tag{56}$$

with the multiplication table

$\times$	ψ_1	ψ_2	ψ_3
ψ_1	ψ_1	0	0
ψ_2	0	ψ_2	0
ψ_3	0	0	ψ_3

Table 2.

Whence,

$$x_1\psi_1 + x_2\psi_2 + x_3\psi_3 = \rho \cdot \exp(\alpha \cdot e_2 + \beta \cdot e_3) \tag{57}$$

or

$$x_1\psi_1 + x_2\psi_2 + x_3\psi_3 = \rho \cdot \exp[(-\alpha - \beta) \cdot \psi_1 + \exp(\alpha) \cdot \psi_2 + \exp(\beta) \cdot \psi_3] \tag{58}$$

Thus the exponential representation of the numbers H_3 is possible, if $x_i > 0$ for the coordinates. If the angles α, β are excluded from three relations, then we obtain the expression for the cube of norm

$$\rho^3 = x_1 \cdot x_2 \cdot x_3 \tag{59}$$

This is not a unique possibility of symmetric introducing linear coordinates. For the numbers H_3 there exists the basis involving two hyperbolic unities

$$1 = \psi_1 + \psi_2 + \psi_3, \quad j = -\psi_1 - \psi_2 + \psi_3, \quad k = -\psi_1 + \psi_2 - \psi_3 \tag{60}$$

$\times$	1	j	k
1	1	j	k
j	j	1	$-1 + j + k$
k	k	$-1 + j + k$	1

Table 3.

If linear coordinates are introduced with respect to this basis then the cube of norm of the numbers H_3 reads

$$\rho^3 = \Omega(x, x, x; 1, -1, 2) \tag{61}$$

A noncanonical form enters the right-hand part of the formula (61). By the help of the transformation (16) at $p = 4$, wich changing signs simultaneously for all the coordinates and applying the general scale transformation, the three-form $\Omega(x_1, x_2, x_3; 1, -1, 2)$ can be sent into the eighth canonical three-form $\Omega(x, x, x; 0, 1, 2)$. The linear coordinates x_i for the numbers H_3 can be introduce alternatively as

$$(x_2 + x_3)\psi_1 + (x_1 + x_3)\psi_2 + (x_1 + x_2)\psi_3 = \rho \cdot \exp(\alpha \cdot e_2 + \beta \cdot e_3), \tag{62}$$

in which case

$$\rho^3 = \Omega(x_1, x_2, x_3; 0, 1, 2) \tag{63}$$

is again the eighth canonical form (55).

On so doing, the three-forms $\Omega(x_1, x_2, x_3; 0, 0, 1) \equiv \Omega_3(x_1, x_2, x_3)$, $\Omega(x_1, x_2, x_3; 1, -1, 2)$, $\Omega(x_1, x_2, x_3; 0, 1, 2)$ relate to one and same three-numbers H_3 which isomorphic to the algebra of quadratic diagonal matrices 3×3. Although the three-forms $\Omega(x_1, x_2, x_3; 0, 0, 1) \equiv \Omega_3(x_1, x_2, x_3)$, $\Omega(x_1, x_2, x_3; 0, 1, 2)$ cannot be obtained one from another by applying continuous linear transformation (16) in conjunction with scale-general transformation and probably also changing the sign of all the three coordinates, the forms are nevertheless connected by discrete linear transformation of corrdinates:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{64}$$

5. Three-form $\Omega(x_1, x_2, x_3; 1, -\frac{2}{3}, 6)$

Let us consider the generators $\hat{a}_3, \hat{a}_4, \hat{a}_5$ of linear transformations which leave the three-form $\Omega(x_1, x_2, x_3; 1, -\frac{3}{2}, 6)$ unchanged. These generators does not commute with one another. To single out two commuting generators, let us comprise the following linear combinations for these operators:

$$\hat{E}_0 = \hat{a}_3 + \hat{a}_4 + \hat{a}_5, \qquad \hat{E}_2 = -\hat{a}_3 + \hat{a}_4, \qquad \hat{E}_3 = -\hat{a}_3 + \hat{a}_5. \tag{65}$$

For them the following multiplication table is operative:

$\times$	$\hat{E}_0$	$\hat{E}_2$	$\hat{E}_3$
$\hat{E}_0$	$3\hat{E}_0$	$3\hat{E}_2$	$3\hat{E}_3$
$\hat{E}_2$	0	0	0
$\hat{E}_3$	0	0	0

Table 4.

Thus,$\hat{E}_2, \hat{E}_3$ or arbitrary two linear-independent combination thereof can be taken to serve as a pair of commuting generators. Using Table 4, , it can readily be shown that apart of $\hat{E}_2, \hat{E}_3$ and their linear combinations, no linear combinations of three operators $\hat{E}_0, \hat{E}_2, \hat{E}_3$, that is, the operators $\hat{a}_3, \hat{a}_4, \hat{a}_5$, may exist which commute with one another.

Let us relate to $\hat{E}_2, \hat{E}_3$ the symbolic elements e_2, e_3 of the hypercomplex number. Then for the basis elements $e_1 \equiv 1, e_2, e_3$ we obtain the Kely table

$\times$	1	e_2	e_3
1	1	e_2	e_3
e_2	e_2	0	0
e_3	e_3	0	0

Table 6.

Three-numbers with such a multiplication table of symbolic elements may naturally be called dual and denoted by D_3. For such three-numbers,

$$\rho \cdot \exp(\alpha \cdot e_2 + \beta \cdot e_3) = \rho \cdot (1 + \alpha \cdot e_2 + \beta \cdot e_3). \tag{66}$$

Up to the nummeration order, the unique possibility to introduce linear coordinates x_i in a symmetric fashion is

$$X = x_1 + x_2 \cdot (1 + e_2) + x_3 \cdot (1 + e_3), \tag{67}$$

so that the three-form

$$\rho^3 = (x_1 + x_2 + x_3)^3 \equiv \Omega(x_1, x_2, x_3; 1, 3, 6) \tag{68}$$

is non-degenerate.

Thus no three-number which cube of norm is equal to this three-form $\Omega(x_1, x_2, x_3; 1, -\frac{3}{2}, 6)$ can be found.

6. Three-form $\Omega(x_1, x_2, x_3; 1, 0, -3)$

The generators $\hat{a}_6, \hat{a}_7$ of the group symmetry under which actions the form $\Omega(x_1, x_2, x_3; 1, 0, -3)$ leaves unchanged possess the following multiplication rules:

$$\hat{a}_6 \cdot \hat{a}_6 = \hat{a}_7, \qquad \hat{a}_7 \cdot \hat{a}_7 = \hat{a}_6, \qquad \hat{a}_6 \cdot \hat{a}_7 = \hat{a}_7 \cdot \hat{a}_6 = 1. \tag{69}$$

Juxtaposing with them the symbolic elements e_2, e_3 of the system of three-numbers, we obtain the following Kely table:

$\times$	1	e_2	e_3
1	1	e_2	e_3
e_2	e_2	e_3	1
e_3	e_3	1	e_2

Table 7.

The hypercomplex associative-commutative three-dimensional numbers with the multiplication law for basis elements that is indicated by Table 7 will be denoted as C_3. Using this Kely table, we get the formula

$$\begin{aligned}\exp(\alpha \cdot e_2 + \beta \cdot e_3) = \frac{1}{3}\, e^{\alpha+\beta}\{1 + 2e^{-\frac{3}{2}(\alpha+\beta)} \cdot \cos[\frac{\sqrt{3}}{2}(\alpha - \beta)]\} + \\ + \frac{1}{3} e^{\alpha+\beta}\{1 - 2e^{-\frac{3}{2}(\alpha+\beta)} \cdot \cos[\frac{\sqrt{3}}{2}(\alpha - \beta) + \frac{\pi}{3}]\} \cdot e_2 + \\ \frac{1}{3} e^{\alpha+\beta}\{1 - 2e^{-\frac{3}{2}(\alpha+\beta)} \cdot \cos[\frac{\sqrt{3}}{2}(\alpha - \beta) - \frac{\pi}{3}]\} \cdot e_3.\end{aligned} \tag{70}$$

Let us introduce a coordinate system x_1, x_2, x_3 with respect to the same basis as follows:

$$x_1 + x_2 \cdot e_2 + x_3 \cdot e_3 = \exp(\alpha \cdot e_2 + \beta \cdot e_3). \tag{71}$$

Using the formula (70) and three coordinate relations (71), we get two relations

$$x_1 + x_2 + x_3 = \rho \cdot e^{(\alpha+\beta)}, \qquad x_1^2 + x_2^2 + x_3^2 = \frac{1}{3}\rho^2 \cdot e^{2(\alpha+\beta)}\{1 + 2 \cdot e^{-3(\alpha+\beta)}\}, \tag{72}$$

which are no more involving any difference of parameters $(\alpha - \beta)$. Expressing the sum of parameters $(\alpha + \beta)$ from (70), we get two relations

$$\rho^3 = \frac{3}{2} \cdot (x_1 + x_2 + x_3\} \cdot (x_1^2 + x_2^2 + x_3^2\} - \frac{1}{2} \cdot (x_1 + x_2 + x_3\}^3 \equiv \Omega(x_1, x_2, x_3; 1, 0, -3\}. \tag{73}$$

Thus we observe that for the three-numbers C_3 the cube of modulus is a three-form $\Omega(x_1, x_2, x_3; 1, 0, -3)$.

Although for the numbers C_3, by using symbolic element and unity, one can comprise the linear combination

$$j = \frac{1}{3}[1 - 2(e_2 + e_3)], \qquad j^2 = 1, \tag{74}$$

which is a hyperbolic unity ($j^2 = 1$), that is the numbers C_3 really present a generalization of hyperbolic (binary) numbers, it proves impossible to form a linear combination which would be the elliptic unity (with $i^2 = -1$); in a sense, the three-numbers C_3 present a generalization also for complex numbers for which the symbolic unity is a solution of the algebraic equation $x^2 = -1$. For the numbers C_3 the basis elements $1, e_2, e_3$ are roots for the cubic equation $x^3 = 1$, or with modified sign $-1, -e_2, -e_3$ they are roots for the equation $x^3 = -1$. Thus, from one side, in terms of complex numbers the equation $x^3 = 1$ has three roots

$$1, \quad -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \tag{75}$$

from which an imaginary unity can be singled out as their linear combination; from another side, the formulas (70) involve trigonometric functions, so that (in just this sense) the numbers C_3 may be regarded as a generalization of not only binary (hyperbolic) but also complex numbers for the three-dimensional case.

6. Conclusion

Up to isomorphism, two systems of hypercomplex three-dimensional numbers C_3 and H_3 are the only systems that can be selected from all the set of systems of associative-commutative hypercomplex numbers by setting forth the requirement of existence of a basis which respect to which the cube of norm of three-number (if it exists) is a non-degenerate three-form. The numbers C_3 can be juxtaposed by canonical three-form $\Omega(x_1, x_2, x_3; 1, 0, -3)$ (see Section 6 of present work), whereas the three-numbers H_3 by canonical three-forms $\Omega_3(x_1, x_2, x_3), \Omega(x_1, x_2, x_3; 0, 1, 2)$ ((see Section 4 of present work).

It is hoped that the result obtained permits entailing that also for the n-numbers with $n > 3$ the requirement of existence of a basis in term of which the n-degree of norm (provided the latter be exist) of n-number is equal to the n-form of coordinates, would select a narrow class of the hyperbolic numbers to play the role of generalization of complex and hyperbolic numbers (bi-numbers). Probably it is the hyperbolic numbers of such a type that primary find applications in mathematics and physics, being applied to the problems which involve in a sense the symmetry with respect to permutation of coordinates or some transformation "mixing" coordinates and simultaneously retaining their legitimacy.

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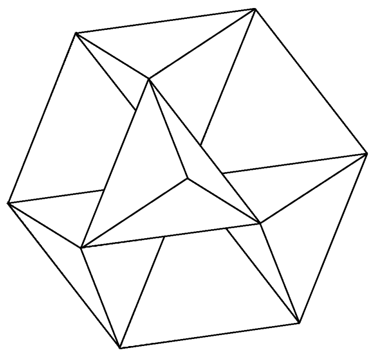

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