DYNAMICS IN $D \geq 2$-ORDER PHASE SPACE IN THE BASIS OF MULTICOMPLEX ALGEBRA

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We use commutative *algebra of multicomplex numbers*, to construct oscillator model for Hamilton-Nambu dynamics. We propose a new dynamical principle from which it follows two kind of Hamilton-Nambu equations in $D \geq 2$-dimensional phase space. The first one is formulated with $(D-1)$-evolution parameter and a single Hamiltonian. The Hamiltonian of the oscillator model in a such dynamics is given by D-degree homogeneous form. In the second formulation, vice versa, the evolution of the system along a single evolution parameter is generated by $(D-1)$ Hamiltonian. The latter is given by Nambu equations in $D \geq 3$-dimensional phase.

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1 Introduction

The algebras widely using in the mathematical physics, such as classical Clifford algebras, have their definitions from *quadratic or bilinear* relations. This is a consequence, rather, the bilinear aspects of fundamental objects such as quadratic metrics, canonical pair of the phase space variables, commutation relations etc. A real Clifford algebra is generated by the set of basis vectors $\{e_i, i = 1, ..., n\}$ and defining relations $e_ie_j + e_je_i = 2g_{ij}$ where $g_{ij} \in R$ are the coefficients of a non degenerate symmetric bilinear form. If the set is orthonormal the defining relations reduce to

$$e_ie_j = -e_je_i, \quad e_i^2 = \pm 1. \tag{1.1}$$

The classical Clifford algebra admits a Z_2- graded structure. Wide investigations of the classical Clifford algebras have been initiated by success of the Dirac equation. However, besides Clifford algebras one may build an algebraic extension, the Generalized Clifford Algebras. New algebra equipped with a metric defined by a homogeneous polynomial form of degree n naturally leads to an underlying Z_n-graded structures [1], [2]. These algebras, just as Clifford algebras, emerge from various contexts. About the problem on usefulness of the hypercomplex numbers in physics have been dedicated several papers (see, for instance, A. A. Eliovich [4] and references therein). Properties of the hypercomplex algebra of fourth order have been successfully applied to explore Berwald-Moor metric in Finslerian geometry [3]. Specially, the efforts in the developments of the multicomplex algebras are motivated by the new ideas which occur in quantum mechanics based on homogeneous metrics of degree higher than two [5]. However, besides the quantum mechanics it is also great interest to construct the classical mechanics based on high degree metrics. Generalization of Hamiltonian mechanics based on the extension of *binary* operation on classical observable to the phase space with *multiple* operation of higher order ($D \geq 3$), has received much attention in the recent literature since Nambu's contribution [6]. Y. Nambu proposed the generalization of Hamiltonian dynamics by introduce a triplet of dynamical variables which spans a tree-dimensional phase space, instead of a canonical pair. As a result, the state of a system is represented by a point in the three-dimensional phase space, and this point moves with time along a curve in the three-dimensional phase space. Since the publication of Nambu's paper, different aspects of this problem have been studied by several group of authors [7].

In the present paper we propose a dynamical principle from which it follows two kind of Hamilton-Nambu equations in $D \geq 2$-dimensional phase space. The first one is formulated with $(D-1)$ Hamiltonian and single parameter of evolution, the Nambu equations. In Ref. [8], we have proposed an extension of the Newtonian mechanics in $D \geq 3$ dimensional phase space within the Nambu's formalism. As a consequence, three-dimensional phase space oscillator model within Jacobi elliptic functions for the extended Newtonian mechanics has been constructed. The second kind of dynamics is formulated with $(D-1)$-evolution parameter and a single Hamiltonian. The Hamiltonian of the oscillator model of the latter is given by D-degree homogeneous form. This dynamics in some sense is the inverse one to the Nambu dynamics.

In Sec. 2 we recall the basic notions of the multicomplex algebra and the theory of the polytrigonometric functions.

In Sec. 3 we show that there exist two kind of the Hamiltonian equations, namely, *direct* Hamiltonian equations can be coupled by their *inverse* formulation.

In Sec. 4 we show that the Nambu-Hamilton equations admit its inverse formulation.

In Sec. 5 we derive the dynamical equations in D- dimensional phase space with $(D-1)$-evolution parameters and single Hamiltonian in terms of the co- and contra-variant coordinates defined in the space with polylinear metric form. We construct the polylinear oscillator model. The underlying algebraic structure of the polylinear oscillator model is the multicomplex algebra.

2 Commutative Part of Generalized Clifford Algebras. Polygonometric functions

The Generalized Clifford algebras (GCA) $Cl_p^{(n)}$ is generated by a set of p canonical generators $e_1, \ldots, e_p$ fulfilling:

$$e_i e_j = \omega^{sg(j-i)} e_j e_i, \quad e_i^n = \pm 1, \quad i, j = 1, \ldots, p \tag{2.1}$$

where $\omega = \exp\left(\frac{2i\pi}{n}\right)$ is a n-th primitive root of unity and $sg(x)$ the usual sign function.

In this paper we shall use the commutative part of the GCA. A commutative part of the classical Clifford algebra is generated by unique generator $\mathbf{e}$, with $e^2 = \pm 1$. When the generator e is given by $e^2 = -1$ then one has well known algebra of *Complex Numbers* . Similarly, a commutative part of GCA is the algebra of unique generator $\mathbf{e}$, satisfying to the conditions $e^n = \pm 1$. This is n-dimensional commutative algebra. A detail description of this algebra for $e^n = 1$ a reader may find in [9]. In this paper we shall consider the algebra with the unique generator defined by $e^n = -1$, we shall denominate *Algebra of Multicomplex Numbers* (MC_n). It is worth to underline that most of the results of the usual complex number analysis remain true for MC_n-number analysis. Let us sketch briefly the basic and useful properties of multicomplex algebra and its elliptic mappings which are direct extension of the cosine&sine functions. More detail description of this algebra reader may find in [10].

Any $z \in MC_n$ is defined by finite series expansion

$$z = \sum_{i=1}^{n} e^{i-1} q_i, \quad e^0 = 1. \tag{2.2}$$

Among the unitary equivalent matrix representations of the operator $\mathbf{e}$ we shall use one given by anticirculant matrix

$$(E)^l_m = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & 0 & ... & 0 \\ . & . & . & . & ... & . \\ 0 & 0 & 0 & 0 & ... & 1 \\ -1 & 0 & 0 & 0 & ... & 0 \end{pmatrix}. \tag{2.3}$$

This matrix satisfies the following condition

$$E^n = -I \tag{2.4}$$

and gives arise to convenient matrix representation of $z \in MC_n$:

$$Z^l_m = \sum_{i=1}^n q_i (E^{i-1})^l_m. \tag{2.5}$$

By using the matrix E a product of two MC_n-numbers one may represent in the following convenient way

$$\Big\{\sum_{i=1}^n e^{i-1} a_i\Big\}\Big\{\sum_{i=1}^n e^{i-1} b_i\Big\} = \sum_{i=1}^n e^{i-1} c_i,$$

with

$$c_k = \sum_{l=1}^n (E^{l-1} b_l)^m_k a_m = \sum_{l=1}^n (E^{l-1} a_l)^m_k b_m. \tag{2.6}$$

The inverse MC_n-number z^{-1} we shall define via the notion of inverse matrix Z^{-1}. In search of Z^{-1} we find n-order of pseudo-norm of Z as the determinant:

$$|Z|^n = det\{Z^l_m\}. \tag{2.7}$$

Definition: $|z|^n = |Z|^n$.

Thus $|z|^n$, as well $|Z|^n$, is an homogeneous form of n degree relatively q_i:

$$|z|^n = \eta^{i_1 \dots i_n} q_{i_1} \dots q_{i_n}, \tag{2.8}$$

where the summation convention is adopted. Obviously,

$$|z_1\ z_2|^n = |z_1|^n\ |z_2|^n \tag{2.9}$$

Any MC_n-number is coupled by its *conjugation*. The conjugation of z is given by the following

Definition:

$\bar z \in MC_n$ is conjugation of $z \in MC_n$ if $\bar z z = |z|^n$.

Let us represent $\bar z$ by the series

$$\bar z = \sum_{i=1}^n q^i e^{-(i-1)}.$$

The coordinates q_i, q^j we call as *covariant and contra-variant coordinates*, correspondingly. These are the components of the vector in n-dimensional space, while $|z|^n$ can be interpreted as a square of the length of a such vector. These coordinates satisfy the following bilinear relations

$$q^i q_i = |z|^n, \quad (E^k)^i_l q^l q_i = 0, \quad k = 1, \cdots, n-1, \tag{2.10}$$

from which one may form n- dimensional orthogonal basis defining

$$h^i{}_j = \sum_{k=1}^{n} (E^{k-1})^i_j \; q_k, \quad h_i{}^j = \sum_{k=1}^{n} (E^{-(k-1)})^j_i \; q^k. \tag{2.11}$$

These vectors are mutually orthogonal because

$$h^i{}_p h_j{}^p = \sum_{k=1}^{n} \sum_{l=1}^{n} (E^{k-1})^i_p \; q_k (E^{-(l-1)})^p_j \; q^l = |z|^n \delta^i_j, \tag{2.12}$$

where we used (2.6) and (2.10).

By using (2.8) and (2.10) one finds explicit relationships between q^i and q_i:

$$q^i = \eta^{ii_1 \cdots i_{n-1}} q_{i_1} \cdots q_{i_{n-1}}. \tag{2.13}$$

Any MC_n-number defined by the condition $|z|^n = 1$ can be given in the exponential representation:

$$z = \exp\Big(\sum_{i=1}^{n-1} \varphi_i e^i \Big). \tag{2.14}$$

Then, an expansion straightforward gives arise to the analogue of Euler formula:

$$z = \sum_{i=1}^{n} mus_i(\varphi) e^{i-1}, \quad \varphi = \{\varphi_1, \cdots, \varphi_{n-1}\}. \tag{2.15}$$

These *"mus"*-functions one may consider as extension of the usual set of cosine&sine functions. For $n = 2$ one recovers the *tri*-gonometric functions:

$$mus_1(\varphi) = \cos(\varphi), \quad mus_2(\varphi) = \sin(\varphi),$$

correspondingly, the condition $|z|^n = det(Z^l_m) = 1$ is reduced to well known identity: $\cos^2(\varphi) + \sin^2(\varphi) = 1$. We suggest to denominate the set of functions $mus_i(\varphi), \; i = 1, \cdots, n$ as *poly-gonometric functions*.

In the polar coordinates the MC_n-number is defined by

$$z = \rho \exp\Big(\sum_{i=1}^{n-1} \varphi_i e^i \Big), \tag{2.16}$$

where $\rho = |z|$.

Further, it has sense to introduce the notion of the *partially conjugated MC_n-numbers*. By using

$$\sum_{i=1}^{n} \omega^i = 0, \quad \omega = \exp\,(\frac{2i\pi}{n}),$$

we write

$$z^{(0)} z^{(1)} z^{(2)} \ldots z^{(n-1)} = \rho^n, \tag{2.17}$$

where

$$z^{(k)} = \rho \exp\Big(\sum_{i=1}^{n-1} \omega^{ki} e^i \varphi_i \Big).$$

MC_n-numbers $z^{(k)}, k = 1, 2, ..., n-1$ we shall call partially conjugated of z. Form (2.17) it follows

$$\bar{z} = z^{(1)} z^{(2)} \ldots z^{(n-1)}.$$

It is useful to keep in mind also:

$$|\bar{z}|^n = |z^{(1)}|^n |z^{(2)}|^n \dots |z^{(n-1)}|^n = \rho^{n(n-1)}, \tag{2.18}$$

because

$$|z^{(k)}| = \rho, \quad k = 1, 2, ..., n-1.$$

The following representations for $\bar{z}$ hold

$$\bar{z} = \prod_{l=1}^{n-1} \sum_{i=1}^{n} q_i \omega^{l(i-1)} e^{i-1}, \quad \bar{z} = \rho^{n-1} \exp(-\sum_{i=1}^{n-1} \varphi_i e^i). \tag{2.19}$$

We can also define conjugation of $\bar{z}$, so that,

$$\bar{z}\bar{\bar{z}} = \rho^{n(n-1)}. \tag{2.20}$$

By taking into account that for any z we have unique $\bar{z}$ we come to the conclusion that

$$\bar{\bar{z}} = \lambda z$$

where $\lambda = \rho^{n(n-2)}$.

To derive the derivatives of the poly-gonometric functions it is enough to use the series expansions (2.14), (2.15). Setting equal the expressions at any $\mathbf{e^i}$ in

$$\frac{\partial}{\partial \varphi_j} \exp\Big(\sum_{i=1}^{n-1} \varphi_i e^i\Big) = e^j \exp\Big(\sum_{i=1}^{n-1} \varphi_i e^i\Big)$$

one gets

$$\frac{\partial}{\partial \varphi_k} mus_l(\varphi) = (E^k)^m{}_l mus_m(\varphi), \quad k = 1, \dots, n-1. \tag{2.21, a}$$

For the coordinates, correspondingly, we get

$$\frac{\partial q_i}{\partial \varphi_k} = (E^k)^l{}_i q_l, \quad \frac{\partial q^i}{\partial \varphi_k} = -(E^k)^i_l q^l. \tag{2.21, b}$$

For convenience of a reader let us repeat the above formulae for the case $z \in MC_3$.

Definition:

$$z = q_1 + eq_2 + e^2 q_3, \quad e^3 = 1.$$

Conjugation:

$$\bar{z} = q^1 + e^{-1} q^2 + e^{-2} q^3.$$

Pseudo-norm:

$$|z1^3 = q_1 q^1 + q_2 q^2 + q_3 q^3.$$

Partial conjugations:

$$z^{(1)} = q_1 + \omega e q_2 + \omega^2 e^2 q_3, z^{(2)} = q_1 + \omega^2 e q_2 + \omega e^2 q_3.$$

Relationships between covariant and contra-variant coordinates:

$$q^1 = (q_1)^2 + q_2 q_3, \quad q^2 = -(q_2)^2 + q_1 q_3, \quad q^3 = (q_3)^2 + q_2 q_1.$$

Formulae of differentiation.

For the covariant vectors:

$$\frac{\partial}{\partial\varphi_1}\begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} = \begin{pmatrix} -q_3 & q_1 & q_2 \end{pmatrix}, \quad \frac{\partial}{\partial\varphi_2}\begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} = \begin{pmatrix} -q_2 & -q_3 & q_1 \end{pmatrix}$$

For the contra-variant vectors:

$$\frac{\partial}{\partial\varphi_1}\begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} = \begin{pmatrix} -q^2 \\ -q^3 \\ q^1 \end{pmatrix}, \quad \frac{\partial}{\partial\varphi_2}\begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} = \begin{pmatrix} -q^3 \\ q^1 \\ q^2 \end{pmatrix}.$$

In conclusion of this section, let us give one useful representation for the pseudo-norm of z. To give a main idea, let us begin from the case $n = 3$. According to the definition:

$$|z|^3 = det(Z) = det\begin{pmatrix} q_1 & q_2 & q_3 \\ -q_3 & q_1 & q_2 \\ -q_2 & -q_3 & q_1 \end{pmatrix}.$$

Now let us recall the definition of the determinant of the matrix:

$$det(A) = det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \epsilon^{ijk} a_i b_j c_k,$$

ϵ^{ijk} is the Levi-Civita tensor. By equating the matrix A with the matrix Z we find

$$a_l = (E^0)^i_l q_i, \quad b_l = (E^1)^i_l q_i, \quad c_l = (E^2)^i_l q_i.$$

Therefore,

$$|z|^3 = det(Z) = \epsilon^{ijk}(E^0)^p_i (E^1)^l_j (E^2)^m_k q_p q_l q_m.$$

In the general case we get

$$|z|^n = \epsilon^{l_1 l_2 \cdots l_n}(E^0)^{i_1}_{l_1}(E^1)^{i_2}_{l_2}\cdots(E^{n-1})^{i_n}_{l_n} q_{i_1}\ldots q_{i_n}. \tag{2.22}$$

3 Direct and Inverse Hamiltonian Equations

Let us recall the basic elements of Hamiltonian dynamics. One has two dimensional phase space on which the Poisson bracket structure obeying the Jacobi identity is defined. Further, one has the Hamiltonian form for the equations of motion where the evolution in time of a dynamical system is generated by a single function, the Hamiltonian. The basic canonical structure of the phase space of Hamiltonian mechanics is carried by the canonical pairs of the Cartesian coordinates.

Consider now isolated, macroscopic system consisting of N identical particles, each of which has three translational degrees of freedom. The dynamical state of the system at a given time completely specified by the $3N$ coordinates and $3N$ momenta of the particles. The values of these variables define a *phase point* in a $2n = 6N$- dimensional *phase space*. The classical phase space for Hamiltonian mechanics consists of the pair of coordinates $\{x^i, p_i\}$, $i = 1, ..., n$ and Poisson bracket:

$$\{f_1, f_2\} = \frac{\partial(f_1, f_2)}{\partial(x, p)} = \frac{\partial f_1}{\partial x^i}\frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial x^i}, \tag{3.1}$$

where f is a classical observable, smooth function on the phase space and the summation convention adopted.

A standard textbook presentation of the classical mechanics start from the first principles of the classical mechanics, such as, the principle of *least action*, according to which the world trajectories under the Hamilton phase flow are extremals of the action. Another principle is invariance of the Poincaré integral

$$P_2 = \int\int dx^i \wedge dp_i$$

over canonical mapping $(x^i(t), p_i(t)) \to (x^i(t + \delta t), p_i(t + \delta t))$ [11].

One may generalize the latter to the case of Poincaré–Cartan integral

$$PC_2 = \int\int dx^i \wedge dp_i - \int\int dh \wedge dt.$$

Instead of this we shall use the principle of the form-invariance of the integral equality

$$\int\int dx^i \wedge dp_i = \mu \int\int dh \wedge dt \tag{3.3}$$

over the mappings

$$(x^i, p_i) \to (h, t), \quad (h, t) \to (x^i, p_i). \tag{3.4}$$

Here, h is the Hamiltonian or, equal is the total energy of the system. (We suppose that an interactions explicitly no dependent of time.) The constant of motion μ can be expressed of P_2 and PC_2:

$$\mu = \frac{P_2}{P_2 - PC_2}$$

As far as our further results do not depend of μ, we shall take $\mu = 1$.

Let us start from the case of one dimensional motion $D = 2$. In that case we have two integrals of motion in the capacity of which one may choose: (1) the initial time t_0, (2) the total energy h [12]. We a priori suppose that description of a motion of the system is given by the set of two functions

$$x = X(t - t_0, h - h_0), \quad p = P(t - t_0, h - h_0),$$

where x and p to be the coordinates of trajectory and momentum, correspondingly. We also suppose that this system is invertible, namely,

$$h = H(x, p), \quad t = T(x, p).$$

Theorem 1 (§3)

Direct and Inverse Hamiltonian equations of motion are consequence of the principle of form-invariance of the integral equation (3.3) over the mapping (3.4).

Proof.

We are looking for conditions for the mappings $(h, t) \rightleftharpoons (x, p)$ over of which

$$\int\int dh \wedge dt \rightleftharpoons \int\int dx \wedge dp.$$

This condition is satisfied if the Jacobian of the mapping $(h, t) \to (x, p)$ is equal 1:

$$detJ\{(h, t) \to (x, p)\} = \frac{\partial x}{\partial h}\frac{\partial p}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial p}{\partial h} = 1,$$

where the Jacobian matrix is defined by

$$J\{(h,t) \to (x,p)\} = \begin{pmatrix} \frac{\partial x}{\partial h} & \frac{\partial x}{\partial t} \\ \frac{\partial p}{\partial h} & \frac{\partial p}{\partial t} \end{pmatrix}.$$

As far as the determinant of the matrix for $J\{(h,t) \to (x,p)\}$ is equal to one then inverse matrix is equal to adjoint matrix:

$$J^{-1}\{(h,t) \to (x,p)\} = \begin{pmatrix} -\frac{\partial p}{\partial t} & \frac{\partial x}{\partial t} \\ \frac{\partial p}{\partial h} & -\frac{\partial x}{\partial h} \end{pmatrix}.$$

According to the well known property of Jacobian inverse Jacobian matrix coincides with Jacobian matrix of inverse mapping. It gives

$$\begin{pmatrix} -\frac{\partial p}{\partial t} & \frac{\partial x}{\partial t} \\ \frac{\partial p}{\partial h} & -\frac{\partial x}{\partial h} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial x} & \frac{\partial H}{\partial p} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial p} \end{pmatrix}. \tag{3.5}$$

By equating the elements of these matrices we get two kind of Hamiltonian equations:

$$(a) \qquad \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial x}, \quad \frac{\partial x}{\partial t} = \frac{\partial H}{\partial p}, \tag{3.6, a}$$

$$(b) \qquad \frac{\partial p}{\partial h} = \frac{\partial T}{\partial x}, \quad \frac{\partial x}{\partial h} = -\frac{\partial T}{\partial p} \tag{3.6, b}$$

Of course, these two Hamiltonian systems are equivalent. From (3.6,a,b) it is easily seen, that the functions $h = H(x,p)$ and $t = T(x,p)$ are mutually exchanged. The solutions of both system are given by the same set of functions: $p = p(h,t),\ x = x(h,t)$.

Consider now the case of $2n$-dimensional phase space. The solution of the dynamical equations in that case are given by by the following set of functions

$$x^i = x^i(t - t_0, h, c_3, \ldots, c_{2n}), \quad p_i = p_i(t - t_0, h, c_3, \ldots, c_{2n}), \quad i = 1, 2, ..., n, \tag{3.7}$$

where $c_3, \ldots, c_{2n}$ are the other constants of motion.

We assume that the system (3.7) is invertible, so that

$$t = T(x_1, p_1, \ldots x_n, p_n), \quad h = H(x_1, p_1, \ldots x_n, p_n), \quad c_l = C_l(x_1, p_1, \ldots x_n, p_n), \quad l = 3, \ldots, 2n. \tag{3.8}$$

The condition of the Theorem 1 (§3) is satisfied if

$$det J\{(h,t) \to (x,p)\} = \frac{\partial x^i}{\partial h}\frac{\partial p_i}{\partial t} - \frac{\partial x^i}{\partial t}\frac{\partial p_i}{\partial h} = 1. \tag{3.9}$$

Now any element of the Jacobian matrix is n-dimensional vector:

$$J\{(h,t) \to (x,p)\} = \begin{pmatrix} \frac{\partial x^i}{\partial h} & \frac{\partial x^i}{\partial t} \\ \frac{\partial p_i}{\partial h} & \frac{\partial p_i}{\partial t} \end{pmatrix}. \tag{3.10}$$

Taking into account (3.9) we get

$$J^{-1}\{(h,t)\to(x,p)\} = \begin{pmatrix} -\dfrac{\partial p_i}{\partial t} & \dfrac{\partial x^i}{\partial t} \\ \dfrac{\partial p_i}{\partial h} & -\dfrac{\partial x^i}{\partial h} \end{pmatrix}. \tag{3.11}$$

By equating this matrix with the Jacobian matrix of the inverse mapping

$$J\{(p,x)\to(h,t)\} = \begin{pmatrix} \dfrac{\partial H}{\partial x^i} & \dfrac{\partial H}{\partial p_i} \\ \dfrac{\partial T}{\partial x^i} & \dfrac{\partial T}{\partial p_i}, \end{pmatrix} \tag{3.12}$$

we get two kind of Hamiltonian equations in $2n$-dimensional phase space

$$\frac{\partial p_i}{\partial t} = -\frac{\partial H}{\partial x^i}, \quad \frac{\partial x^i}{\partial t} = \frac{\partial H}{\partial p_i} \tag{3.13, a}$$

$$\frac{\partial p_i}{\partial h} = \frac{\partial T}{\partial x^i}, \quad \frac{\partial x^i}{\partial h} = -\frac{\partial T}{\partial p_i} \tag{3.13, b}$$

$\diamondsuit$

The validity of the above consideration can be easily demonstrated on the oscillator model. Consider direct mapping

$$x(h,t) = \sqrt{2h}\, sin(t), \quad p(h,t) = \sqrt{2h}\, cos(t)$$

and its inverse one

$$h(x,p) = \frac{1}{2}(x^2+p^2), \quad \tan(t(x,p)) = \frac{x}{p}.$$

These mappings satisfy all conditions of the Theorem 1 (§3) and gives arise oscillator equations of motion.

4 Liouville theorem. Evolution equations in $D = 2n$ phase space. Nambu dynamical equations

As it has been noted above, the solutions of two equivalent Hamilton systems (3.13,a,b) are represented by the functions

$$x^i = x^i(H, T, C_3, C_4, \ldots, C_{2n}), \quad p_i = P_i(H, T, C_3, C_4, \ldots, C_{2n}) \tag{4.1}$$

This set of functions one may consider as direct mapping. In previous section we have considered inverse mapping only with respect to the pair $\{H, T\}$. Consequently, the Hamiltonian systems of equations gave arise. Now let us put on the top the Liouville theorem, according to which the $2n$-th Poincaré integral by

$$P_{2n} = \int \ldots \int dx_1 \wedge \ldots \wedge dx_n \wedge dp_1 \ldots \wedge dp_n \tag{4.2}$$

is invariant of the motion. Define the following integral equation

$$\int \ldots \int dx_1 \wedge \ldots \wedge dx_n \wedge dp_1 \ldots \wedge dp_n = \int \ldots \int dH \wedge dT \wedge dC_3 \wedge dC_4 \ldots \wedge dC_{2n} \tag{4.3}$$

For sake of convenience let us introduce the notations

$$\{q_i\} = \{x_1, \ldots, x_n, p_n, \ldots, p_n\}, \quad \{Q_i\} = \{H, T, C_3, \ldots, C_{2n}\}, \quad i = 1, \ldots, 2n \tag{4.4}$$

and consider the complete direct-inverse mapping given by

$$\{q \rightleftharpoons Q\} \tag{4.5}$$

Theorem 2 (§4)

Direct and Inverse Hamilton-Nambu equations of motion are consequence of the principle of form-invariance of the integral equation (4.3) over the mapping (4.5).

Proof.

The mapping $\{q \to Q\}$ is mapping of $2n$ phase space coordinates onto $2n$ of constants of motion. Jacobian matrix of this mapping is defined by

$$J\{Q \to q\} = \begin{pmatrix} \dfrac{\partial q_1}{\partial Q_1} & \cdots & \dfrac{\partial q_1}{\partial Q_{2n}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial q_{2n}}{\partial Q_1} & \cdots & \dfrac{\partial q_{2n}}{\partial Q_{2n}} \end{pmatrix} \tag{4.6}$$

The principle of form-invariance of (4.3) yields the condition

$$Det J\{Q \to q\} = \epsilon_{i_1 \ldots i_{2n}} \frac{\partial q_1}{\partial Q_{i_1}} \cdots \frac{\partial q_{2n}}{\partial Q_{i_{2n}}} = 1, \tag{4.7}$$

where we used the definition of determinant of $(2n \times 2n)$ matrix. Further, as it has been done above, we shall equate any element of adjoint Jacobian matrix $J\{Q \to q\}$ with the corresponding element of Jacobian matrix of inverse mapping $J\{q \to Q\}$. As the result we get the following set of evolutionary equations

$$\frac{\partial Q_{i_k}}{\partial q_k} = \epsilon_{i_1 \ldots i_{2n}} \frac{\partial q_1}{\partial Q_{i_1}} \cdots \frac{\partial q_{k-1}}{\partial Q_{i_{k-1}}} \frac{\partial q_{k+1}}{\partial Q_{i_{k+1}}} \cdots \frac{\partial q_{2n}}{\partial Q_{i_{2n}}} \tag{4.8}$$

Now, *vice versa*, let us take the Jacobian of mapping $\{q \to Q\}$ and compare its adjoint matrix with the corresponding Jacobian matrix of inverse mapping $\{Q \to q\}$. In that case the principle of form invariance of (4.3) yields the condition

$$Det J\{q \to Q\} = \epsilon_{i_1 \ldots i_{2n}} \frac{\partial Q_1}{\partial q_{i_1}} \cdots \frac{\partial Q_{2n}}{\partial q_{i_{2n}}} = 1, \tag{4.9}$$

As the result one obtains the evolutionary equations inverse to (4.8):

$$\frac{\partial q_{i_k}}{\partial Q_k} = \epsilon_{i_1 \ldots i_{2n}} \frac{\partial Q_1}{\partial q_{i_1}} \cdots \frac{\partial Q_{k-1}}{\partial q_{i_{k-1}}} \frac{\partial Q_{k+1}}{\partial q_{i_{k+1}}} \cdots \frac{\partial Q_{2n}}{\partial q_{i_{2n}}}. \tag{4.10}$$

$\diamondsuit$

The system of equations (4.10) coincides with the Hamilton-Nambu equations in the phase space with even set of coordinates while the equations (4.8) we can consider as *inverse Hamilton-Nambu equations*. The relationships (4.7) and (4.9) we may consider as the generalization of the Lagrange and the Poisson brackets, correspondingly.

Nambu dynamical equations in $D = 3$-dimensional phase space

Now let us use Theorem 2 (§4) to obtain equations of motion in $D = 3$-dimensional phase space. Suppose that the phase space is given by the triplet of the set of variables $\{x, p, q\}$. It means, the motion of the dynamical system is described by the functions

$$x = x(t, h_1, h_2), \quad p = p(t, h_1, h_2), \quad q = q(t, h_1, h_2).$$

The variables t, h_1, h_2 in various formulations can play different role. Denote the set of functions

$$\{t = T(x, p, q), h_1 = H_1(x, p, q), h_2 = H_2(x, p, q)\}$$

by $\{Q\}$ and the set of variables $\{x, p, q\}$ by $\{q\}$.

Consider the mapping $\{q \to Q\}$ with Jacobian matrix

$$J(\{q \to Q\}) = \begin{pmatrix} \dfrac{\partial H_1}{\partial x} & \dfrac{\partial H_1}{\partial p} & \dfrac{\partial H_1}{\partial q} \\ \dfrac{\partial T}{\partial x} & \dfrac{\partial T}{\partial p} & \dfrac{\partial T}{\partial q} \\ \dfrac{\partial H_2}{\partial x} & \dfrac{\partial H_2}{\partial p} & \dfrac{\partial H_2}{\partial q} \end{pmatrix}, \tag{4.11}$$

with $det J(\{q \to Q\}) = 1$. The Jacobian of the inverse mapping is

$$J(\{Q \to q\}) = \begin{pmatrix} \dfrac{\partial x}{\partial h_1} & \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial h_2} \\ \dfrac{\partial p}{\partial h_1} & \dfrac{\partial p}{\partial t} & \dfrac{\partial p}{\partial h_2} \\ \dfrac{\partial q}{\partial h_1} & \dfrac{\partial q}{\partial t} & \dfrac{\partial q}{\partial h_2} \end{pmatrix} \tag{4.12}$$

Adjoint matrix for $J(\{q \to Q\})$ is

$$J^{-1}(\{q \to Q\}) = \begin{pmatrix} -det\begin{pmatrix} \frac{\partial T}{\partial p} & \frac{\partial T}{\partial q} \\ \frac{\partial H_2}{\partial p} & \frac{\partial H_2}{\partial q} \end{pmatrix} & det\begin{pmatrix} \frac{\partial H_1}{\partial p} & \frac{\partial H_1}{\partial q} \\ \frac{\partial H_2}{\partial p} & \frac{\partial H_2}{\partial q} \end{pmatrix} & -det\begin{pmatrix} \frac{\partial H_1}{\partial p} & \frac{\partial H_1}{\partial q} \\ \frac{\partial T}{\partial p} & \frac{\partial T}{\partial q} \end{pmatrix} \\ -det\begin{pmatrix} \frac{\partial T}{\partial q} & \frac{\partial T}{\partial x} \\ \frac{\partial H_2}{\partial q} & \frac{\partial H_2}{\partial x} \end{pmatrix} & det\begin{pmatrix} \frac{\partial H_1}{\partial q} & \frac{\partial H_1}{\partial x} \\ \frac{\partial H_2}{\partial q} & \frac{\partial H_2}{\partial x} \end{pmatrix} & -det\begin{pmatrix} \frac{\partial H_1}{\partial q} & \frac{\partial H_1}{\partial x} \\ \frac{\partial T}{\partial q} & \frac{\partial T}{\partial x} \end{pmatrix} \\ -det\begin{pmatrix} \frac{\partial T}{\partial x} & \frac{\partial T}{\partial p} \\ \frac{\partial H_2}{\partial x} & \frac{\partial H_2}{\partial p} \end{pmatrix} & det\begin{pmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_1}{\partial p} \\ \frac{\partial H_2}{\partial x} & \frac{\partial H_2}{\partial p} \end{pmatrix} & -det\begin{pmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_1}{\partial p} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial p} \end{pmatrix} \end{pmatrix} \tag{4.13}$$

By equating the matrices (4.12) and (4.13) we get

$$\frac{\partial x}{\partial t} = det\begin{pmatrix} \frac{\partial H_1}{\partial p} & \frac{\partial H_1}{\partial q} \\ \frac{\partial H_2}{\partial p} & \frac{\partial H_2}{\partial q} \end{pmatrix}, \quad \frac{\partial p}{\partial t} = det\begin{pmatrix} \frac{\partial H_1}{\partial q} & \frac{\partial H_1}{\partial x} \\ \frac{\partial H_2}{\partial q} & \frac{\partial H_2}{\partial x} \end{pmatrix}, \quad \frac{\partial q}{\partial t} = det\begin{pmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_1}{\partial p} \\ \frac{\partial H_2}{\partial x} & \frac{\partial H_2}{\partial p} \end{pmatrix} \tag{4.14}$$

Thus, we obtained the *Nambu's equations of motion* [6].

With the same way one can define:

$$
J^{-1}(\{Q \to q\}) = \begin{pmatrix} -det\begin{pmatrix} \frac{\partial p}{\partial t} & \frac{\partial p}{\partial h_2} \\ \frac{\partial q}{\partial t} & \frac{\partial q}{\partial h_2} \end{pmatrix} & det\begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial h_2} \\ \frac{\partial q}{\partial t} & \frac{\partial q}{\partial h_2} \end{pmatrix} & -det\begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial h_2} \\ \frac{\partial p}{\partial t} & \frac{\partial p}{\partial h_2} \end{pmatrix} \\ -det\begin{pmatrix} \frac{\partial p}{\partial h_1} & \frac{\partial p}{\partial h_2} \\ \frac{\partial q}{\partial h_1} & \frac{\partial q}{\partial h_2} \end{pmatrix} & det\begin{pmatrix} \frac{\partial x}{\partial h_1} & \frac{\partial x}{\partial h_2} \\ \frac{\partial q}{\partial h_1} & \frac{\partial q}{\partial h_2} \end{pmatrix} & -det\begin{pmatrix} \frac{\partial x}{\partial h_1} & \frac{\partial x}{\partial h_2} \\ \frac{\partial p}{\partial h_1} & \frac{\partial p}{\partial h_2} \end{pmatrix} \\ -det\begin{pmatrix} \frac{\partial p}{\partial h_1} & \frac{\partial p}{\partial t} \\ \frac{\partial q}{\partial h_1} & \frac{\partial q}{\partial t} \end{pmatrix} & det\begin{pmatrix} \frac{\partial x}{\partial h_1} & \frac{\partial x}{\partial t} \\ \frac{\partial q}{\partial h_1} & \frac{\partial q}{\partial t} \end{pmatrix} & -det\begin{pmatrix} \frac{\partial x}{\partial h_1} & \frac{\partial x}{\partial t} \\ \frac{\partial p}{\partial h_1} & \frac{\partial p}{\partial t} \end{pmatrix} \end{pmatrix}
$$

and equate this matrix with the Jacobian matrix $J(\{q \to Q\})$. As the result one finds

$$
\begin{aligned}
\frac{\partial p}{\partial h_2}\frac{\partial q}{\partial t} - \frac{\partial p}{\partial t}\frac{\partial q}{\partial h_2} &= \frac{\partial H_1}{\partial x} \\
\frac{\partial q}{\partial h_2}\frac{\partial x}{\partial t} - \frac{\partial q}{\partial t}\frac{\partial x}{\partial h_2} &= \frac{\partial H_1}{\partial p} \\
\frac{\partial x}{\partial h_2}\frac{\partial p}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial p}{\partial h_2} &= \frac{\partial H_1}{\partial q}
\end{aligned}
\tag{4.15}
$$

These equations we can consider as *inverse Nambu's equations.*

5 Polylinear Oscillator Model in the Basis of Multicomplex Algebra

Oscillator model is one of the oldest models of the classical mechanics. The solutions of this model are given by cosine&sine functions. The equations of motion of one dimensional oscillator may be written in two equivalent forms:
(a) In the matrix form

$$
\frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}, \tag{5.1}
$$

with the solution

$$
x = \sqrt{2h}\ sin(t), \quad p = \sqrt{2h}\ cos(t).
$$

(b) In the basis of complex algebra

$$
-i\frac{d}{dt}z = z, \quad z = p + ix, \tag{5.2}
$$

with the solution

$$
z = \sqrt{2h}\ exp(it),
$$

where the constant of motion h, energy, is defined by

$$
h = \frac{1}{2}(x^2 + p^2) = \frac{1}{2}z\bar{z}.
$$

In the Hamiltonian formalism this function plays a role of the Hamiltonian of the system: $h = H(x,p)$. Hamiltonian form of the Eqs. (5.1) and Eqs. (5.2), correspondingly, are given by

$$
\frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial p} \end{pmatrix} H(x,p), \tag{5.3, a}
$$

$$-i\frac{d}{dt}\Big(p+ix\Big)=\Big(\frac{\partial}{\partial p}+i\frac{\partial}{\partial x}\Big)H(x,p). \tag{5.3, b}$$

In the connection of complex form of equations of motion note F.Strocchi [13] gave the formulation of Schrodinger equation as Hamiltonian equations in complex coordinates where Hamiltonian function is the mean value of the corresponding quantum operator.

In this section we build one example of dynamical equations of motion in $D \geq 3$- dimensional phase space, we denominate *polylinear oscillator model.* This model is direct extension of the ordinary oscillator model in two-dimensional phase space. The solutions of the latter are given by the *tri*-gonometric functions or by the complex algebra. Correspondingly, the solutions of the polylinear oscillator model are given by the multicomplex algebra or by *poly*-gonometric functions. We show, the polylinear oscillator model is oscillator model for the Nambu's formulation of generalized Hamiltonian dynamics.

Thus, we are looking for equations for a new oscillator model the solutions of which are given by the polygonometric functions:

$$q_i=(nh)^{\frac{1}{n}}mus_i(\varphi), \quad q^i=(nh)^{\frac{n-1}{n}}mus_i(-\varphi), \quad i=1,...,n, \tag{5.4}$$

where $h = \frac{1}{n}|z|^n$ is the "energy" of the polylinear oscillator model. The polygonometric functions depend of $(n-1)$-angle. Complement this set of variables with the variable $\varphi_n = h$ to get the set of n variables: $\{\varphi_1,\ldots,\varphi_n\}$. Differentiating $q_i,\ q^i$ with respect to h we obtain

$$\frac{dq^l}{d\varphi_n}=\frac{n-1}{nh}q^l, \quad \frac{dq_l}{d\varphi_n}=\frac{1}{nh}q_l. \tag{5.5}$$

Let us consider two set of independent coordinates: $\{q^1,q^2,\ldots,q^n\}$ and $\{\varphi_1,\ldots,\varphi_{n-1},\varphi_n\}$. The Jacobian of the mapping

$$\{q^1,q^2,\ldots,q^n\}\to\{\varphi_1,\ldots,\varphi_{n-1},\varphi_n\}$$

is equal to one:

$$det\Big(\frac{\partial q_l}{\partial\varphi_k}\Big)=\frac{1}{nh}det\Big(\sum_l(E^j)^l_iq_l\Big)=1. \tag{5.6}$$

Let us evaluate the following derivatives: $\dfrac{\partial\varphi_k}{\partial q_l},\dfrac{\partial\varphi_k}{\partial q^l}$. One may find these values by using the formulae (2.21,a,b) (5.6) and (5.7). By differentiating the set of functions $\varphi_k=\varphi_k(q^1,q^2,\ldots,q^n)$, $(k=1,\ldots,n)$ with respect to the variables φ_l, $(l=1,\ldots,n)$ one obtains the linear system of the algebraic equations:

$$\delta_{ki}=\frac{\partial\varphi_k}{\partial q^l}(E^i)^l_rq^r, \quad \delta_{ki}=-\frac{\partial\varphi_k}{\partial q_l}(E^i)^r_lq_r, \quad k,i=1,2,...,n-1, \tag{5.7, a}$$

$$\delta_{kn}=\frac{n-1}{nh}\frac{\partial\varphi_k}{\partial q^l}q^l, \quad \delta_{kn}=\frac{1}{nh}\frac{\partial\varphi_k}{\partial q_l}q_l, \quad k=1,2,...,n. \tag{5.7, b}$$

The solutions of Eqs.(5.7,a,b) with respect to $\dfrac{\partial\varphi_k}{\partial q_l},\dfrac{\partial\varphi_k}{\partial q^l}$ are given by

$$\frac{\partial\varphi_k}{\partial q^l}=\frac{1}{nh}(E^{-k})^p_lq_p, \quad \frac{\partial\varphi_k}{\partial q_l}=-\frac{1}{nh}(E^{-k})^l_pq^p. \tag{5.8, a}$$

$$q_l=(n-1)\frac{\partial\varphi_n}{\partial q^l}, \quad q^l=\frac{\partial\varphi_n}{\partial q_l}. \tag{5.8, b}$$

Equations (5.8,a,b) one may consider as the extension of the oscillator equations of motion with the first integral of motion given by

$$H = \frac{1}{n}|z|^n. \tag{5.9}$$

The function $h = H(z)$ we choose as Hamiltonian of the system. By substituting (5.8,b) into right side of (2.21,b) we obtain next evolution equations for the coordinates q_i and q^i with respect to the evolution parameters $\varphi_k, k = 1, ..., n-1$:

$$\frac{\partial q_i}{\partial \varphi_k} = (n-1)(E^k)_i^l \frac{\partial H}{\partial q^l}\,, \tag{5.10, a}$$

$$\frac{\partial q^i}{\partial \varphi_k} = -(E^k)_l^i \frac{\partial H}{\partial q_l}\,. \tag{5.10, b}$$

Obviously, for $n = 2$ this system of equations is reduced to the Hamiltonian equations (5.4,a). As far as we consider H as the Hamiltonian of the conservative system, in general, the following conditions should be satisfied

$$\frac{dH}{d\varphi_k} = 0, \quad k = 1, ..., n-1. \tag{5.11}$$

By using Eqs.(2.21,b), (5.8,b) we get

$$\frac{dH}{d\varphi_k} = \frac{\partial q_l}{\partial \varphi_k}\frac{\partial H}{\partial q_l} + \frac{\partial q^l}{\partial \varphi_k}\frac{\partial H}{\partial q^l} = (n-1)(E^k)_i^l \frac{\partial H}{\partial q^l}\frac{\partial H}{\partial q_i} - (E^k)_l^i \frac{\partial H}{\partial q_l}\frac{\partial H}{\partial q^i} =$$

$$= (n-2)(E^k)_l^i \frac{\partial H}{\partial q_l}\frac{\partial H}{\partial q^i} = 0. \tag{5.12}$$

Thus the condition (5.12) is satisfied automatically for $n = 2$ case. This is the case of ordinary oscillator model. For $n \geq 3$ the condition (5.8) will be satisfied if the derivatives

$$D_m = \frac{\partial H}{\partial q^m}, \quad D^m = \frac{\partial H}{\partial q_m}$$

are related similar co- and contra- variant coordinates q_i, q^i according to our definition of the polylinear metrics given in Sec.2. It easily seen, in the case of polylinear oscillator model this condition is fulfilled because (5.8,b). In the general case the following relations between D^m and D_m hold

$$D^i = \lambda \eta^{ii_1 \cdots i_{n-1}} D_{i_1} \cdots D_{i_{n-1}}, \tag{5.13}$$

where λ for the polylinear oscillator is given by

$$\lambda = (n-1)^{(n-1)}.$$

By substituting (5.13) into Eqs.(5.10,b) we obtain

$$\frac{\partial q^i}{\partial \varphi_k} = -\lambda (E^k)_l^i \eta^{li_1 \cdots i_{n-1}} \frac{\partial H}{\partial q^{i_1}} \cdots \frac{\partial H}{\partial q^{i_{n-1}}}. \tag{5.14}$$

These equations are given in the terms of the contra-variant coordinates.

Theorem 3 (§5)

If the dynamical variables q_i of the system are coordinates of multi-oscillator model then they satisfy to the following system of equations:

$$\epsilon^{ii_1 \ldots i_{n-1}} \frac{\partial q_{i_1}}{\partial \varphi_1}\frac{\partial q_{i_2}}{\partial \varphi_2}\frac{\partial q_{i_{n-1}}}{\partial \varphi_{n-1}} = \frac{\partial H}{\partial q_i}. \tag{5.15}$$

To prove it is sufficient to transform (5.10,a) into (5.15). From Eqs.(5.10,a) we obtain the following system of equations

$$\frac{\partial H}{\partial q^{l_k}} = \frac{1}{n-1}(E^{-k})^i_{l_k}\frac{\partial q_i}{\partial \varphi_k}, \quad k = 1,\ldots,n-1; \quad l_k = 1,\ldots,n. \tag{5.16}$$

By substitute these relations into the right side of (5.13) and taking into account the formulas (2.22) we get

$$\frac{\partial H}{\partial q_i} = \eta^{ii_1\cdots i_{n-1}}(E^{-1})^{l_1}_{i_1}\frac{\partial q_{l_1}}{\partial \varphi_1}\ldots(E^{-(n-1)})^{l_{n-1}}_{i_{n-1}}\frac{\partial q_{l_{n-1}}}{\partial \varphi_{n-1}} = \epsilon_{ii_1\ldots i_{n-1}}\frac{\partial q_{i_1}}{\partial \varphi_1}\frac{\partial q_{i_2}}{\partial \varphi_2}\frac{\partial q_{i_{n-1}}}{\partial \varphi_{n-1}}.$$

$\diamondsuit$

Thus we obtained *Inverse Nambu's Equations.*

Similar the case of two-dimensional oscillator model we can elaborate inverse polylinear oscillator equations. Combining (5.8,a) with (5.5) we can write

$$\frac{\partial \varphi_k}{\partial q_i} = \frac{1}{n-1}(E^{-k})^i_l\frac{dq^l}{d\varphi_n}, \tag{5.17, a}$$

$$\frac{\partial \varphi_k}{\partial q^i} = -(E^{-k})^l_i\frac{dq_l}{d\varphi_n}. \tag{5.17, b}$$

Now let us rewrite the set of equations (5.17,a,b) in the following form

$$\frac{dq^l}{dt} = -(n-1)(E^k)^l_i\frac{\partial H_k}{\partial q_i}, \quad \frac{dq_l}{dt} = (E^k)^i_l\frac{\partial H_k}{\partial q^i}, \quad k = 1,\ldots,n-1, \tag{5.18}$$

where we denoted $H_k = \varphi_k,\ t = \varphi_n$. The condition $\dfrac{dH_k}{dt} = 0$ is fulfilled if

$$\frac{1}{n-1}(E^k)^j_i\frac{dq_j}{dt}\frac{dq^i}{dt} = 0,$$

from which we obtain

$$\frac{dq_l}{dt} = \frac{1}{\lambda}\eta_{li1\cdots i_{n-1}}\frac{dq^{i_1}}{dt}\cdots\frac{dq^{i_{n-1}}}{dt}. \tag{5.19}$$

The equations (5.18) one may consider as inversion of the system (5.10,a,b). This formulation consists of $(n-1)$- Hamiltonian $H_1 = \varphi_1, H_2 = \varphi_2,\ldots,H_n = \varphi_{n-1}$ and only one time-like parameter, $t = \varphi_n$.

The relation of obtained system equations with Nambu equations is given by following

Theorem 4 (§5)

The solutions of the equations (5.18)–(5.19) satisfy to the following system of Nambu's equations

$$\epsilon_{li_1\ldots i_{n-1}}\frac{\partial H_1}{\partial q_{i_1}}\frac{\partial H_2}{\partial q_{i_2}}\cdots\frac{\partial H_{n-1}}{\partial q_{i_{n-1}}} = \frac{dq_l}{dt}. \tag{5.20}$$

Proof

Let us form the following expression

$$Det_l(H_1,...,H_{n-1}) = \epsilon_{li_1\cdots i_{n-1}}\frac{\partial H_1}{\partial q_{i_1}}\frac{\partial H_2}{\partial q_{i_2}}\cdots\frac{\partial H_{n-1}}{\partial q_{i_{n-1}}}. \tag{5.21}$$

By using Eqs.(5.18) we get

$$Det_l(H_1, ..., H_{n-1}) = \frac{1}{(n-1)^{n-1}} \epsilon_{li_1 \cdots i_{n-1}} (E^1)^{i_1}_{l_1} \ldots (E^{n-1})^{i_{n-1}}_{l_{n-1}} \frac{dq^{l_1}}{dt} \cdots \frac{dq^{l_{n-1}}}{dt}. \qquad (5.22)$$

Now taking into account the formula (2.22) we can transform (5.22) as

$$Det_l(H_1, ..., H_{n-1}) = \eta_{li1 \cdots i_{n-1}} \frac{dq^{i_1}}{dt} \cdots \frac{dq^{i_{n-1}}}{dt}.$$

The latter just equal to $\frac{dq_l}{dt}$ because (5.19). Thus we obtain (5.20) which coincides with the well known Nambu's system of equations. ♢

Conclusion

Summarizing we come to the following conclusions.

The many particle system is described by $2n$-dimensional phase space coordinates. The solutions of the equations of motion are defined by $2n$ constants of integration or equal, constants of motion. The other form of the constants of motion one may represent via $2n$ Poincaré integrals. Let us consider even set of Poincaré integrals: $P_2, P_4, ..., P_{2n}$. We have two possibilities.

1. The particles of the system obey to form-invariance principle of P_2 (Theorem 1(§3)). Then the motion of the particles is described by Hamiltonian equations.

2. The particles of the system obey to form-invariance principle of P_{2n} (Theorem 2(§4)). Then the motion of the particles is described by Hamilton-Nambu equations.

Let us note, P_2 is the square in two-dimensional phase space while P_{2n} is the volume in $D = 2n$-dimensional phase space. This fact prompts us an idea to use the algebras based on the polylinear forms. The pseudo-norm of the multicomplex algebra exactly is defined by D-dimensional volume (see, (2.22)). Thus, it occur, the multicomplex algebra is related with Hamilton-Nambu equations similar the complex algebra is related with Hamilton equations. On the other hand, we can use multicomplex algebra as "key thread" to generalize Haniltonian equations of motion. In this way, we have constructed the oscillator model for the Hamilton-Nambu dynamical equations. We have found the dynamical equations in $D \geq 2$- dimensional phase space with $(D-1)$-evolution parameters and single Hamiltonian. The oscillator model in a such dynamics is generated by the Hamiltonian defined by D-degree homogeneous form. The solutions are represented by the set of polygonometric functions depending of $(D-1)$ angle. We have shown, these equations can be inverted. As the result we have obtained the dynamical equations in D- dimensional phase space with an single evolution parameter and $(D-1)$ Hamiltonian.

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ДИНАМИКА В ФАЗОВОМ ПРОСТРАНСТВЕ ПОРЯДКА $D \geq 2$ В БАЗИСЕ АЛГЕБРЫ МУЛЬТИКОМПЛЕКСНЫХ ЧИСЕЛ

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Мы используем коммутативную *алгебру мультикомплексных чисел*, чтобы построить модель осциллятора для динамики Гамильтона-Намбу. Мы предлагаем новый динамический принцип, из которого вытекают два вида уравнений Гамильтона-Намбу в $D \geq 2$- мерном фазовом пространстве. Первый формулируется с $(D-1)$-параметром эволюции и единственным гамильтонианом. Гамильтониан модели осциллятора в такой динамике задается однородной формой степени D. Во второй формулировке, наоборот, эволюция системы вдоль единственного параметра эволюции генерируется $(D-1)$- гамильтонианом. Последний задается уравнениями Намбу в $D \geq 3$-мерном фазовом пространстве.

Ключевые слова: Комплексные числа, обобщенная тригонометрия, дифференциальные уравнения, классическая механика, гамильтониан, фазовое пространство.

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