ON CARTAN SPACES WITH THE M-TH ROOT METRIC $K(x,p) = \sqrt[m]{a^{i_1 i_2 \ldots i_m}(x) p_{i_1} p_{i_2} \ldots p_{i_m}}$

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The aim of this paper is to expose some geometrical properties of the locally Minkowski-Cartan space with the Berwald-Moór metric of momenta $L(p) = \sqrt[n]{p_1 p_2 \ldots p_n}$. This space is regarded as a particular case of the m-th root Cartan space. Thus, Section 2 studies the v-covariant derivation components of the m-th root Cartan space. Section 3 computes the v-curvature d-tensor S^{hijk} of the m-th root Cartan space and studies conditions for $S3$-likeness. Section 4 computes the T-tensor T^{hijk} of the m-th root Cartan space. Section 5 particularizes the preceding geometrical results for the Berwald-Moór metric of momenta.

Key-words: m-th root Cartan space, $S3$-likeness, T-tensor, Berwald-Moór metric of momenta.

MSC2000: 53B40, 53C60, 53C07.

1 Introduction

Owing to the studies of E. Cartan, R. Miron [6], [7], Gh. Atanasiu [2] and many others, the geometry of Cartan spaces is today an important chapter of differential geometry, regarded as a particular case of the Hamilton geometry. By the Legendre duality of the Cartan spaces with the Finsler spaces studied by R. Miron, D. Hrimiuc, H. Shimada and S. V. Sabău [8], it was shown that the theory of Cartan spaces has the same symmetry like the Finsler geometry, giving in this way a geometrical framework for the Hamiltonian theory of Mechanics or Physical fields. In a such geometrical context we recall that a *Cartan space* is a pair $\mathcal{C}^n = (M^n, K(x,p))$ such that the following axioms hold good:

1. K is a real positive function on the cotangent bundle T^*M, differentiable on $T^*M \backslash \{0\}$ and continuous on the null section of the canonical projection $\pi^* : T^*M \to M$;
2. K is positively 1-homogenous with respect to the momenta p_i;
3. The Hessian of K^2, with the elements

$$g^{ij}(x,p) = \frac{1}{2} \frac{\partial^2 K^2}{\partial p_i \partial p_j},$$

is positive-defined on $T^*M \backslash \{0\}$.

On the other hand, in the last two decades, physical studies due to G. S. Asanov [1], D. G. Pavlov [9], [10] and their co-workers emphasize the important role played by the Berwald-Moór metric $L : TM \to \mathbb{R}$,

$$L(y) = \left(y^1 y^2 \ldots y^n\right)^{\frac{1}{n}},$$

in the theory of space-time structure and gravitation as well as in unified gauge field theories.

For such geometrical-physics reasons, following the geometrical ideas exposed by M. Matsumoto and H. Shimada in [4], [5] and [11] or by ourselves in [3], in this paper we investigate some geometrical properties of the *m-th root Cartan space* which is a natural generalization of the locally Minkowski-Cartan space with the Berwald-Moór metric of momenta.

2 The m-th root metric and v-derivation components

Let $\mathcal{C}^n = (M^n, K(x,p))$, $n \geq 4$, be an n-dimensional Cartan space with the metric

$$K(x,p) = \sqrt[m]{a^{i_1 i_2 \ldots i_m}(x) p_{i_1} p_{i_2} \ldots p_{i_m}}, \tag{2.1}$$

where $a^{i_1 i_2 \ldots i_m}(x)$, depending on the position alone, is symmetric in all the indices $i_1, i_2, \ldots, i_m$ and $m \geq 3$.

Definition 1 *The Cartan space with the metric (2.1) is called the* m***-th root Cartan space****.*

Let us consider the following notations:

$$\begin{aligned} a^i &= \left[a^{i i_2 i_3 \ldots i_m}(x) p_{i_2} p_{i_3} \ldots p_{i_m}\right] / K^{m-1}, \\ a^{ij} &= \left[a^{ij i_3 i_4 \ldots i_m}(x) p_{i_3} p_{i_4} \ldots p_{i_m}\right] / K^{m-2}, \\ a^{ijk} &= \left[a^{ijk i_4 i_5 \ldots i_m}(x) p_{i_4} p_{i_5} \ldots p_{i_m}\right] / K^{m-3}. \end{aligned} \tag{2.2}$$

The normalized supporting element $l^i (= \dot{\partial}^i K,\ \dot{\partial}^i = \partial / \partial p_i)$, the fundamental metrical d-tensor $g^{ij} (= (1/2)\, \dot{\partial}^i \dot{\partial}^j K^2)$ and the angular metrical d-tensor $h^{ij} (= K \dot{\partial}^i \dot{\partial}^j K)$ are given by

$$\begin{aligned} l^i &= a^i, \\ g^{ij} &= (m-1) a^{ij} - (m-2) a^i a^j, \\ h^{ij} &= (m-1)(a^{ij} - a^i a^j). \end{aligned} \tag{2.3}$$

Remark 1 *From the positively* 1*-homogeneity of the* m*-th root Cartan metrical function (2.1) it follows that we have*

$$K^2(x,p) = g^{ij}(x,p) p_i p_j = a^{ij}(x,p) p_i p_j.$$

Let us suppose now that the d-tensor a^{ij} is regular, that is there exists the inverse matrix $(a^{ij})^{-1} = (a_{ij})$. Obviously, we have $a_i \cdot a^i = 1$, where

$$a_i = a_{is} a^s = \frac{p_i}{K}.$$

Under these assumptions, we obtain the inverse components $g_{ij}(x,p)$ of the fundamental metrical d-tensor $g^{ij}(x,p)$, which are given by

$$g_{ij} = \frac{1}{m-1} a_{ij} + \frac{m-2}{m-1} a_i a_j. \tag{2.4}$$

The relations (2.2) and (2.3) imply that the components of the $v-$torsion d-tensor $C^{ijk} (= -(1/2) \dot{\partial}^k g^{ij})$ are given in the form

$$C^{ijk} = -\frac{(m-1)(m-2)}{2K} \left(a^{ijk} - a^{ij} a^k - a^{jk} a^i - a^{ki} a^j + 2 a^i a^j a^k \right). \tag{2.5}$$

Consequently, using the relations (2.4) and (2.5), together with the formula

$$a_s a^{sjk} = a^{jk},$$

we find the components of the $v-$derivation $C_i^{jk} (= g_{is} C^{sjk})$ in the following form:

$$C_i^{jk} = -\frac{(m-2)}{2K} \left[a_i^{jk} - \left(\delta_i^j a^k + \delta_i^k a^j \right) + a_i (2 a^j a^k - a^{jk}) \right], \tag{2.6}$$

where $a_i^{jk} = a_{is} a^{sjk}$. From (2.6) we easily find the following geometrical result:

Proposition 1 *The **torsion covector** $C^i (= C_r^{ir})$ is given by the formula*

$$C^i = -\frac{(m-2)}{2K}\left(a_r^{ir} - na^i\right),$$

where $n = \dim M$.

3 The v-curvature d-tensor S^{hijk}

Taking into account the relations (2.5) and (2.6), by calculation, we obtain

Theorem 3.1 *The v-**curvature** d-tensor $S^{hijk} (= C_r^{ij} C^{rhk} - C_r^{ik} C^{rhj})$ can be written in the form*

$$S^{hijk} = \frac{(m-1)(m-2)^2}{4K^2} \mathcal{A}_{\{j,k\}}\{a_r^{ij} a^{rhk} - a^{ij}(a^{hk} - a^h a^k) + a^i a^j a^{hk}\},$$

where $\mathcal{A}_{\{j,k\}}$ means an alternate sum.

Remark 2 *Using the relations (2.3), we underline that the v- curvature d-tensor S^{hijk} can be written as*

$$K^2 S^{hijk} = \frac{(m-2)^2}{4}\left[\left(h^{hj} h^{ik} - h^{hk} h^{ij}\right)/(m-1) + (m-1) U^{hijk}\right], \tag{3.1}$$

where

$$U^{hijk} = a_r^{ij} a^{rhk} - a_r^{ik} a^{rhj}. \tag{3.2}$$

In the sequel, let us recall the following important geometrical concept [4]:

Definition 2 *A Cartan space $\mathcal{C}^n = (M^n, K(x,p))$, $n \geq 4$, is called $S3$-**like** if there exists a positively 0-homogenous scalar function $S = S(x,p)$ such that the v-curvature d-tensor S^{hijk} to have the form*

$$K^2 S^{hijk} = S\left\{h^{hj} h^{ik} - h^{hk} h^{ij}\right\}. \tag{3.3}$$

Let $\mathcal{C}^n = (M^n, K(x,p))$, $n \geq 4$, be the m-th root Cartan space. As an immediate consequence of the above definition we have the following important result:

Theorem 3.2 *The m-th root Cartan space $\mathcal{C}^n$ is an $S3$-like Cartan space if and only if the d-tensor U^{hijk} is of the form*

$$U^{hijk} = \lambda\left\{h^{hj} h^{ik} - h^{hk} h^{ij}\right\}, \tag{3.4}$$

where $\lambda = \lambda(x,p)$ is a positively 0-homogenous scalar function.

Proof. Taking into account the formula (3.1) and the condition (3.4), we find the scalar function (see (3.3))

$$S = \frac{(m-2)^2}{4}\left[(m-1)\lambda + \frac{1}{m-1}\right]. \tag{3.5}$$

■

4 The T-tensor T^{hijk}

Let $N=(N_{ij})$ be the canonical nonlinear connection of the m-th root Cartan space with the metric (2.1), whose local coefficients are given by [8]

$$N_{ij}=-\gamma_{ij}^{0}+\frac{1}{2}\gamma_{h0}^{0}\dot{\partial}^{h}g_{ij},$$

where

$$\gamma_{jk}^{i}=\frac{g^{ir}}{2}(\partial_{k}g_{rj}+\partial_{j}g_{rk}-\partial_{r}g_{jk}),\ \ \partial_{k}=\partial/\partial x^{k},$$

$$\gamma_{ij}^{0}=\gamma_{ij}^{s}p_{s},\qquad\qquad \gamma_{h0}^{0}=\gamma_{hr}^{l}g^{rs}p_{l}p_{s}.$$

Let $C\Gamma(N)=(H_{jk}^{i},C_{i}^{jk})$ be the Cartan canonical connection of the m-th root Cartan space with the metric (2.1). The local components of the Cartan canonical connection $C\Gamma(N)$ have the expressions [8]

$$H_{jk}^{i}=\frac{g^{ir}}{2}(\delta_{k}g_{rj}+\delta_{j}g_{rk}-\delta_{r}g_{jk}),$$

$$C_{i}^{jk}=g_{is}C^{sjk}=-\frac{g_{is}}{2}\dot{\partial}^{k}g^{js},$$

where

$$\delta_{j}=\partial_{j}+N_{js}\dot{\partial}^{s}.$$

In the sequel, let us compute the T-tensor T^{hijk} of the m-th root Cartan space, which is defined as [11]

$$T^{hijk}\stackrel{def}{=}KC^{hij}|^{k}+l^{h}C^{ijk}+l^{i}C^{jkh}+l^{j}C^{khi}+l^{k}C^{hij},$$

where "$|^{k}$ "denotes the local v-covariant derivation with respect to $C\Gamma(N)$, that is we have

$$C^{hij}|^{k}=\dot{\partial}^{k}C^{hij}+C^{rij}C_{r}^{hk}+C^{hrj}C_{r}^{ik}+C^{hir}C_{r}^{jk}.$$

Using the definition of the local v-covariant derivation [8], together with the relations (2.6) and (2.3), by direct computations, we find the relations:

$$\begin{aligned}
&K|^{k}=a^{k}=l^{k},\\
&a^{i}|^{k}=\frac{(m-1)}{K}(a^{ik}-a^{i}a^{k})=\frac{h^{ik}}{K},\\
&a^{ij}|^{k}=\frac{(m-2)}{K}(a^{ik}a^{j}+a^{jk}a^{i}-2a^{i}a^{j}a^{k})=\frac{(m-2)}{(m-1)K}(h^{ik}l^{j}+h^{jk}l^{i}).
\end{aligned}\tag{4.1}$$

Suppose that we have $m\geq 4$. Then, the notation

$$a^{hijk}=\left[a^{hijki_{5}i_{6}...i_{m}}(x)p_{i_{5}}p_{i_{6}}...p_{i_{m}}\right]/K^{m-4}$$

is very useful. In this context, we can give the next geometrical results:

Lemma 1 *The v-covariant derivation of the tensor a^{hij} is given by the following formula:*

$$\begin{aligned}
a^{hij}|^{k}\ =\ &\frac{(m-3)}{K}a^{hijk}+\frac{m}{2K}a^{hij}a^{k}-\frac{(m-2)}{2K}\cdot\left\{a_{r}^{hk}a^{rij}+a_{r}^{ik}a^{rhj}+\right.\\
&+a_{r}^{jk}a^{rhi}-a^{kij}a^{h}-a^{hkj}a^{i}-a^{hik}a^{j}-a^{ij}a^{hk}-a^{hj}a^{ik}-\\
&\left.-a^{hi}a^{jk}+2\left(a^{ij}a^{h}a^{k}+\ a^{hj}a^{i}a^{k}+a^{hi}a^{j}a^{k}\right)\right\}.
\end{aligned}\tag{4.2}$$

Proof. Note that, by a direct computation, we obtain the relation

$$\frac{\partial a^{hij}}{\partial p_k} = \frac{(m-3)}{K}\left(a^{hijk} - a^{hij}a^k\right). \tag{4.3}$$

Finally, using the definition of the local v-covariant derivation, together with the formulas (4.3) and (2.6), we find the equality (4.2). ∎

Theorem 4.1 *The T-tensor T^{hijk} of the m-th root Cartan space is given by the expression*

$$\begin{aligned} T^{hijk} &= -\frac{(m-1)(m-2)(m-3)}{2K}a^{hijk} + \frac{(m-1)(m-2)^2}{4K}\cdot \\ &\cdot\left(a_r^{hk}a^{rij} + a_r^{ik}a^{rhj} + a_r^{jk}a^{rhi}\right) - \frac{m(m-1)(m-2)}{4K}\cdot \\ &\cdot\left(a^{hij}a^k + a^{hjk}a^i + a^{ijk}a^h + a^{hik}a^j - a^{ij}a^{hk} - a^{hj}a^{ik} - a^{ih}a^{jk}\right). \end{aligned} \tag{4.4}$$

Proof. It is obvious that we have the equality

$$\begin{aligned} T^{hijk} &= (KC^{hij})|^k + l^h C^{ijk} + l^i C^{jkh} + l^j C^{khi} = \\ &= (KC^{hij})|^k + a^h C^{ijk} + a^i C^{jkh} + a^j C^{khi}. \end{aligned}$$

Consequently, differentiating v-covariantly the relation (2.5) multiplied by K and using the formulas (4.1), together with the Lemma 1, by laborious computations, it follows the required result. ∎

5 The particular case of Berwald-Moór metric of momenta

Let us consider now the particular case when $m = n \geq 4$ and

$$a^{i_1 i_2 \ldots i_n}(x) = \begin{cases} 1/n!, & i_1 \neq i_2 \neq \ldots \neq i_n \\ 0, & \text{otherwise.} \end{cases}$$

In this special case, the m-th root metric (2.1) becomes the **Berwald-Moór metric of momenta** [8]

$$K(p) = \sqrt[n]{p_1 p_2 \ldots p_n}. \tag{5.1}$$

By direct computations, we deduce that the n-dimensional locally Minkowski-Cartan space $\mathcal{C}^n = (M^n, K(p))$ endowed with the Berwald-Moór metric of momenta (5.1) is characterized by the following geometrical entities and relations (E-R):

$$a^i = \frac{K}{n}\cdot\frac{1}{p_i}, \quad a_i = \frac{p_i}{K}, \quad a_i \cdot a^i = \frac{1}{n} \quad (\text{no sum by } i),$$

$$a^{ij} = \begin{cases} \dfrac{n}{n-1}\cdot a^i a^j, & i \neq j \\ 0, & i = j \end{cases}, \qquad a_{ij} = \begin{cases} n \cdot a_i a_j, & i \neq j \\ -n(n-2)\cdot (a_i)^2, & i = j, \end{cases}$$

$$a^{ijk} = \begin{cases} \dfrac{n^2}{(n-1)(n-2)}\cdot a^i a^j a^k, & i \neq j \neq k \\ 0, & \text{otherwise} \end{cases}$$

and

$$a^{hijk} = \begin{cases} \dfrac{n^3}{(n-1)(n-2)(n-3)} \cdot a^h a^i a^j a^k, & h \neq i \neq j \neq k \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, the equalities (E-R) imply that the components a_i^{jk} are given by the formulas:

$$\begin{aligned} a_i^{jk} &= -\frac{n^2}{(n-1)(n-2)} \cdot a_i a^j a^k, && i \neq j \neq k \\ a_i^{ik} &= a_i^{ki} = \frac{n}{n-1} \cdot a^k, && i \neq k \ \ \text{(no sum by } i) \\ a_i^{kk} &= 0, && \forall\ i = \overline{1,n}, \ \ \text{(no sum by } k). \end{aligned} \tag{5.2}$$

In this context, we obtain the following important geometrical result:

Theorem 5.1 *The locally Minkowski-Cartan space $\mathcal{C}^n = (M^n, K(p))$, $n \geq 4$, endowed with the Berwald-Moór metric of momenta (5.1) is characterized by the following geometrical properties:*

1. *The torsion covector C^i vanish;*
2. *$S3$-likeness with the scalar function $S = -1$;*
3. *The T-tensor T^{hijk} vanish.*

Proof. 1. It is easy to see that we have

$$\sum_r a_r^{ir} = \sum_{r,s} a_{rs} a^{sir} = n \sum_{r,s} a_r a_s a^{sir} = n \sum_r a_r a^{ir} = n a^i.$$

2. It is obvious that we have

$$h^{ij} = \begin{cases} a^i a^j, & i \neq j \\ -(n-1) \cdot (a_i)^2, & i = j. \end{cases}$$

Consequently, by computations, we obtain

$$U^{hijk} = -\frac{n^2}{(n-1)^2(n-2)^2} \left\{ h^{hj} h^{ik} - h^{hk} h^{ij} \right\},$$

where U^{hijk} is given by the relation (3.2). It follows what we were looking for (see the equalities (3.4) and (3.5)).

3. Using the relation (4.4) and the formulas (E-R) and (5.2), by laborious computations, we deduce that $T^{hijk} = 0$. ∎

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О ПРОСТРАНСТВАХ КАРТАНА С МЕТРИКОЙ M-КОРНЯ $K(x,p) = \sqrt[m]{a^{i_1 i_2 \ldots i_m}(x) p_{i_1} p_{i_2} \ldots p_{i_m}}$

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Цель статьи состоит в том, чтобы выявить некоторые геометрические свойства локальных пространств Минковского-Картана с метрикой импульсов Бервальда-Моора $L(p) = \sqrt[n]{p_1 p_2 \ldots p_n}$. Это пространство рассматривается как частный случай m-корневых пространств Картана. Раздел 2 изучает v-ковариантные дифференциальные компоненты m-корневого пространства Картана. Секция 3 вычисляет d-тензор v-кривизны S^{hijk} m-корневых пространств Картана и исследует условия $S3$-подобия. Секция 4 вычисляет T-тензор T^{hijk} m-корневого пространства Картана. Секция 5 конкретизирует предыдущие геометрические результаты для метрики моментов Бервальда-Моора.

Ключевые слова: m-корневые пространства Картана, $S3$-подобие, T-тензор, метрика импульса Бервальда-Моора.

MSC2000: 53B40, 53C60, 53C07