THE EQUATIONS OF ELECTROMAGNETISM IN SOME SPECIAL ANISOTROPIC SPACES

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We show that anisotropy of the space naturally leads to new terms in the expression of Lorentz force, as well as in the expressions of currents.

1 Introduction

Studying anisotropic spaces has an obvious meaning with regard to physical interpretations. The direction dependence of the metric could cause the appearance of motion dependent forces [1] associated with inertial forces in the accelerated frames. In case there is a physical vector field – an electromagnetic one – in the anisotropic space, this may lead to the appearance of the extra Lorentz type forces or extra currents that could reveal themselves in a special laboratory environment or even in Nature. From mathematical point of view, it is possible to treat the problem in the purely Finslerian setting when $g_{ij} = \dfrac{1}{2}\dfrac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}$ for some 2-homogeneous in y function $\mathcal{F} = \mathcal{F}(x, y)$ or introduce a more general type of anisotropic metric that could explicitly give extra terms in the equations of geodesics.

If we take into account the y-dependence of the fundamental metric tensor in anisotropic spaces, then the components of an electromagnetic-type tensor F_{ij}, $F^i_{\ j}$, F^{ij} could depend on the directional variables. In order to make sure of this, notice the following. In **isotropic** (pseudo-Riemannian) spaces with $R_{ij} = 0$, the components of the free electromagnetic potential 4-vector $A^i = A^i(x)$ obey de Rham equations:

$$A^{\mu;\nu}_{\ \ ;\nu} = 0$$

that is,

$$g^{\nu\rho}(x)\nabla_\nu\nabla_\rho(A^\mu) = 0.$$

When passing to **anisotropic** spaces with metric $g_{ij} = g_{ij}(x, y)$, the solution of such an equation would generally depend on directional variables (not to mention that the equation itself could become more complicated). So, it is meaningful to take into consideration the case when the potential 4-vector depends on the directional variables $y = (y^i)$,

$$
\begin{aligned}
A^i &= A^i(x, y), \qquad \text{and}\\
A_i &= A_i(x, y).
\end{aligned}
$$

Variational procedures applied to the above naturally lead to additional terms in Lorentz force

$$
\begin{aligned}
\frac{dy^i}{dt} + \Gamma^i_{\ jk} y^j y^k &= \frac{q}{c} F^i_{\ h} y^h + \frac{q}{c} \tilde{F}^i_{\ j} \frac{dy^j}{dt}, \qquad y^i = \dot{x}^i\\
F^i_{\ h} &= g^{ij} F_{jh}, \qquad F_{jh} = \frac{\partial A_h}{\partial x^j} - \frac{\partial A_j}{\partial x^h},\\
\tilde{F}^i_{\ j} &= g^{ih} \tilde{F}_{hj}, \qquad \tilde{F}_{hj} = -\frac{\partial A_h}{\partial y^j}
\end{aligned}
$$

as well as the appearance of a correction to the usual expression of currents:

$$D_{\frac{\partial}{\partial x^i}} F^{ki} + D_{\frac{\partial}{\partial y^a}} F^{ka} = J^k.$$

In the present paper, we investigate the case of Finslerian spaces whose metrics are obtained by a small (linearly approximable) deformation of metric tensors whose components do not depend on positional variables (*locally Minkowskian metrics*) and coordinate changes which preserve the positional independence of the undeformed metric. The construction will be generalized to arbitrary Finsler spaces in future works.

2   Weak Finslerian deformation of locally Minkowskian metrics

A *locally Minkowskian* Finsler space is a Finsler space $(M, \mathcal{F})$ with the property that there exists a local coordinate system with respect to which the components of the corresponding metric tensor do not depend on positional variables, but only on directional ones:

$$\gamma_{ij} = \gamma_{ij}(y).$$

In the following, we shall only consider coordinate changes which preserve this property.

One of the properties of locally Minkowski spaces is projective flatness, namely, their geodesics are straight lines:

This type of metrics includes as particular cases:

- Minkowski metric $\gamma = diag(1, -1, -1, -1)$;
- Berwald-Moor 4-dimensional metric, [8], [9], [10], [11].

Let us consider the space $\mathbb{R}^4$, endowed with linear coordinate changes. Let $(x, y) = (x^i, y^a)_{i,a=\overline{1,4}}$, $y^i = \dfrac{\partial x^i}{\partial t}$ (t is a parameter), $i = 1, .., 4$ be the coordinates in a local frame of $T\mathbb{R}^4 \equiv \mathbb{R}^8$.

Let g be a small (linearly approximable) deformation of a locally Minkowskian metric:

$$g_{ij}(x, y) = \gamma_{ij}(y) + \varepsilon_{ij}(x, y). \tag{1}$$

We suppose that this metric tensor is Finslerian in the sense of [3], this is,

$$g_{ij} = \frac{1}{2} \frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}$$

for some 2-homogeneous in y function $\mathcal{F} = \mathcal{F}(x, y)$, and g_{ij} is nondegenerate. We denote by $,_k$ (with commas) partial derivation w.r.t. x^k and with dots $._a$, partial derivation by the directional variable y^a. Whenever convenient – and just in order to point out the difference, we will denote the indices corresponding to y with $a, b, c, ...$ and those corresponding to x with $i, j, k, ...,$ (though, they run over the same set $\{1, 2, 3, 4\}$). Let

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{ih}(g_{hj,k} + g_{hk,j} - g_{jk,h})$$

denote the usual Christoffel symbols (with respect to x) of g.

In our case, $\Gamma^i{}_{jk}$ depend on both x and y :

$$\Gamma^i{}_{jk} = \Gamma^i{}_{jk}(x, y).$$

Let $_{|k}$ denote covariant derivation with respect to x^k :

$$X^i{}_{|k} = X^i{}_{,k} + \Gamma^i{}_{jk} X^j. \tag{2}$$

In the following, we shall also need the **Cartan tensor** C_{ijk}, [4]:

$$C_{ijk} = \frac{1}{2}(g_{ij\cdot k} + g_{ik\cdot j} - g_{jk\cdot i}) = \frac{1}{2}g_{ij\cdot k}. \tag{3}$$

Also, let

$$X^i|_a = X^i{}_{\cdot a} + C^i{}_{ja} X^j$$

denote covariant derivative with respect to y^a.

3 Lorentz force

3.1 Variational principle

The equations of electrodynamics can be obtained from the variational procedure applied to a Lagrangian. In **isotropic spaces**, the Lagrangian is, [6],

$$L(x, y) = \frac{1}{2}g_{ij}(x)y^i y^j + \frac{q}{c}A_i(x)y^i, \quad y^i = \dot{x}^i.$$

where q is the electric charge, and $A_i(x)$ are the covariant components of the 4-vector potential.

In order to obtain Lorentz force in **Finslerian spaces**, let us consider the Lagrangian

$$L = L_0 + \frac{q}{c}L_1,$$

where

$$L_0 = \frac{1}{2}g_{ij}(x, y)y^i y^j$$

(g_{ij} can be chosen as a general Finslerian metric tensor) and $L_1 = L_1(x, y)$ is a scalar function which is 1-homogeneous in the directional variables: $L_1(x, \lambda y) = \lambda L_1(x, y)$, $\forall \lambda \in \mathbb{R}$. Let

$$A_i(x, y) := \frac{\partial L_1}{\partial y^i}.$$

Then

$$L_1 = A_j(x, y)y^j$$

and our Lagrangian is written as

$$L(x, y) = \frac{1}{2}g_{ij}(x, y)y^i y^j + \frac{q}{c}A_i(x, y)y^i, \quad y^i = \dot{x}^i, \tag{4}$$

where $A_j = A_j(x, y)$ is now a **direction dependent** potential.

The components of the covector field $A_i = A_i(x, y)$ are 0-homogeneous functions in y, and possess the property

$$A_{i\cdot k}y^i = 0. \tag{5}$$

The Euler-Lagrange equations

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) = 0$$

attached to L lead to

$$g_{kh}\left(\frac{dy^h}{dt} + \Gamma^h{}_{jl}y^j y^l\right) + \frac{q}{c}\left(A_{k,h} - A_{h,k}\right)y^h + \frac{q}{c}A_{k\cdot h}\frac{dy^h}{dt} = 0. \qquad (6)$$

Let:

$$F_{kh} = A_{h,k} - A_{k,h} \qquad (7)$$

We have thus obtained

Proposition 1 *The extremal curves $t \mapsto (x^i(t)) : [0,1] \to \mathbb{R}^4$ of the Lagrangian (4) are given by*

$$\frac{dy^i}{dt} + \Gamma^i{}_{jk}y^j y^k = \frac{q}{c}F^i{}_h y^h - \frac{q}{c}g^{ik}A_{k\cdot h}\frac{dy^h}{dt}, \qquad (8)$$

Remark 1 *The term $F^i(x,y) \equiv \frac{q}{c}g^{ik}F_{kh}y^h$ is present also in the isotropic case (see [6]). But the last one,*

$$\tilde{F}^i(x,y) := -\frac{q}{c}g^{ik}(x,y)A_{k\cdot h}(x,y)\frac{dy^h}{dt}$$

can only appear in anisotropic ones.

The usual interpretation of the extremal curves is the equation of motion. Therefore, the expression in the rhs of (8) presents the Lorentz force in the anisotropic space. We see that its first term which is common with the isotropic case is proportional to velocity, while the second term is proportional to acceleration which brings to mind the idea of an "inertial force" in the accelerated reference frame.

Let us designate

$$\tilde{F}_{ia} := -A_{i\cdot a}, \quad \tilde{F}_{ai} = A_{i\cdot a}, \qquad (9)$$

where we denote by $a, b, c, d, ...$ indices corresponding to derivation by directional variables.

Then $\tilde{F}_{ia}$ is (-1)-homogeneous in the directional variables:

$$\tilde{F}_{ia}(x, \lambda y) = \frac{1}{\lambda}\tilde{F}_{ia}(x,y), \qquad \lambda \in \mathbb{R}.$$

Then the relation between $\tilde{F}_{ia}$ and the new term in (8) is

$$\tilde{F}^i = \frac{q}{c}\tilde{F}^i{}_a \frac{dy^a}{dt},$$

and we have thus obtained an antisymmetric 2-form on $T\mathbb{R}^4$:

$$F = F_{ij}dx^i \wedge dx^j + \tilde{F}_{ia}dx^i \wedge dy^a. \qquad (10)$$

The above is nothing but the exterior derivative of the 1-form $A = A_i(x,y)dx^i + 0 \cdot dy^a$ on $T\mathbb{R}^4$:

$$F = dA. \qquad (11)$$

Conclusion: Direction dependent electromagnetic potentials lead in a natural way to a correction to the expression of the electromagnetic tensor.

Example 1 *1. In the particular case when the covariant components of A do not depend on direction,*

$$A_i = A_i(x),$$

then $\tilde{F}_{ia} = 0$ and we get the regular expression of Lorentz force.

2. A simple, but nontrivial particular case is obtained when the contravariant components of the potential 4-vector do not depend on the directional variables:

$$A^i = A^i(x),$$

taking into account the y-dependence of the perturbed metric tensor g_{ij}, we get that the covariant components of A are direction dependent:

$$A_i = g_{ij}(x,y)A^j \Rightarrow \ A_i = A_i(x,y).$$

The new term to appear in Lorentz force is then

$$\tilde{F}^i = \frac{q}{c}\tilde{F}^i{}_a\frac{dy^a}{dt} = -\frac{q}{c}g^{ih}A_{h\cdot a}\frac{dy^a}{dt},$$

which leads to

$$\tilde{F}^i = -2\frac{q}{c}C^i{}_{ja}A^j\frac{dy^a}{dt}, \tag{12}$$

and

$$\tilde{F}_{ia} = -2C_{ija}A^j. \tag{13}$$

Example 2 *In particular, if $\gamma = diag(1,-1,-1,-1)$ is the Minkowski metric, and*

$$g_{ij} = \gamma_{ij} + \varepsilon_{ij}(x,y),$$

where $\varepsilon_{ij}(x,y)$ is a small Finslerian perturbation, then the above is

$$\tilde{F}_{ia} = -\varepsilon_{ij\cdot a}A^j, \tag{14}$$

hence its values are small. For other locally Minkowskian metrics, the new term $\tilde{F}_{ia}$ is not necessarily small.

Example 3 *For the case when:*

- *γ_{ij} is the Berwald-Moor Finslerian metric $\gamma_{ij} = \dfrac{1}{2}\dfrac{\partial \mathcal{F}^2}{\partial y^i \partial y^j}$, $\mathcal{F} = \sqrt[4]{y^1 y^2 y^3 y^4}$*
- *$\varepsilon_{ij} = 0$, that is,*

$$g_{ij} = \frac{1}{2}\frac{\partial \mathcal{F}^2}{\partial y^i \partial y^j}, \qquad and$$

- *$A^i = A^i(x)$, then the correction $\tilde{F}_{ia}$ is*

$$\tilde{F}_{ia} = -A_{i\cdot a} = -\frac{\partial}{\partial y^a}(g_{il}A^l) = -2C_{ila}A^l,$$

where the Cartan tensor $C_{ila} = \dfrac{1}{4}\dfrac{\partial^3 \mathcal{F}^2}{\partial y^i \partial y^l \partial y^a}$ has the explicit values

$$C_{ila} = \alpha\frac{\mathcal{F}^2}{y^i y^l y^a}, \quad \alpha = \begin{cases} \frac{3}{32} & if \ \ i = j = k \\ -\frac{1}{32} & if \ \ i = j \neq k \\ \frac{1}{32} & if \ \ i \neq j \neq k \neq i. \end{cases}$$

Here we see that C_{ila} are not necessarily small.

3.2   New term – "electromagnetic" vs. "metric"

Let us now have a look at equation (6):

$$g_{kh}\frac{Dy^h}{dt} = \frac{q}{c}F_{kh}y^h - \frac{q}{c}A_{k\cdot h}\frac{dy^h}{dt}, \quad y = \dot{x}.$$

From the mathematical point of view, we can interpret the last term in two ways:

- Since it appears multiplied by the acceleration $\frac{dy^h}{dt}$ and moreover, since $A_{k\cdot h} = A_{h\cdot k}$, we can "stick" it to the metric:

$$(g_{kh} + \frac{q}{c}A_{k\cdot h})\frac{dy^h}{dt} + \Gamma_{khl}y^h y^l = \frac{q}{c}F_{kh}y^h.$$

and get a new metric tensor

$$\tilde{g}_{kh} = g_{kh} + \frac{q}{c}A_{k\cdot h} \tag{15}$$

(if the matrix $(\tilde{g}_{kh})$ is invertible), with the property

$$\tilde{g}_{kh}y^k y^h = g_{kh}y^k y^h = \mathcal{F}^2$$

With this, we can write the equation of motion as

$$\frac{Dy^i}{dt} = \tilde{g}^{ik}\frac{q}{c}F_{kh}y^h \tag{16}$$

and the obtained expression for Lorentz force $\tilde{g}^{ik}\frac{q}{c}F_{kh}y^h$ differs from the case of isotropic perturbation $\frac{q}{c}g^{ik}F_{kh}y^h$ due to the new metric (15).

- Also, we might leave the metric as it is and move the third term in the right hand side and interpret it as a new term added to Lorentz force:

$$g_{kh}\frac{Dy^h}{dt} = \frac{q}{c}F_{kh}y^h - \frac{q}{c}A_{k\cdot h}\frac{dy^h}{dt}.$$

This would yield

$$\frac{dy^i}{dt} + \Gamma^i{}_{jk}y^j y^k = \frac{q}{c}(F^i{}_h y^h + \tilde{F}^i{}_a\frac{dy^a}{dt}). \tag{17}$$

with the influence of the anisotropy given by the second term in the rhs. Notice, that $\frac{q}{c}(F^i{}_h y^h + \tilde{F}^i{}_a\frac{dy^a}{dt})$ is equal to $\frac{q}{c}g^{ik}(F_{kh}y^h - A_{k\cdot a}\frac{dy^a}{dt})$ given by eq. (8) since $F^i{}_h = g^{ik}F_{kh}$, $\tilde{F}^i{}_a = g^{ik}A_{k\cdot a}$. In eq.(16) the term $\frac{q}{c}g^{ik}A_{k\cdot h}\ddot{x}^h$ was brought to the left hand side of the equation of motion and "swallowed" into the metric – the new "metric" was denoted by $\tilde{g}_{ik}$. This illustrates the remark concerning the equivalence principle made in [1], applied to a (curved) space with an electromagnetic field.

4   Homogeneous Maxwell equations

Let us consider again the 2-form (10)

$$F(x,y) = dA(x,y) = \frac{1}{2}F_{ij}(x,y)dx^i \wedge dx^j + \tilde{F}_{ia}(x,y)dx^i \wedge dy^a.$$

We immediately get:

Proposition 2 *There holds*

$$F_{ij,k} + F_{ki,j} + F_{jk,i} = 0; \tag{18}$$

which is just the homogeneous Maxwell equation or, in terms of covariant derivatives (2),

$$F_{ij|k} + F_{ki|j} + F_{jk|i} = 0.$$

There also hold the equalities

$$\begin{aligned} \tilde{F}_{ia,k} + \tilde{F}_{ki\cdot a} + \tilde{F}_{ak,i} &= 0; \\ \tilde{F}_{ia\cdot b} + \tilde{F}_{bi\cdot a} &= 0, \end{aligned}$$

where $F_{ia\cdot b} = \dfrac{\partial F_{ia}}{\partial y^b}$. The above two relations, together with (18) mean actually that the exterior derivative of F is 0:

$$dF = 0. \tag{19}$$

5   Currents in anisotropic spaces

In the classical Riemannian case, the inhomogeneous Maxwell equations can be obtained by means of the variational principle applied to

$$\int (\alpha F_{ij}F^{ij} - \beta j^k A_k)\sqrt{-g}\, d\Omega\,,$$

[7], where α and β are constants, $g = \det(g_{ij})$ and $\Omega = dx^1 dx^2 dx^3 dx^4$. Taking into account that in our case, at least one of the quantities F_{ij}, F^{ij} depends on y, the whole integrand depends on y, and is actually defined on some domain in $\mathbb{R}^8$.

Let

$$u^a = \frac{1}{H}y^a,$$

where H is a constant, $[H] = \dfrac{1}{\sec}$, meant to "adjust" measurement units as to have $[x^i] = [u^a]$, hence also $[F_{ij}] = [\tilde{F}_{ia}]$. So, let us consider $A_i = A_i(x,u)$ and the integral of action

$$I = \int (\alpha F_{\lambda\mu}F^{\lambda\mu} - \beta j^k A_k)\sqrt{G}d\Omega,$$

where $\lambda, \mu \in \{i, j, a, b\}$, $G = \det(G_{\alpha\beta})$ is the Sasaki lift of g to $T\mathbb{R}^4 \equiv \mathbb{R}^8$, [4]:

$$G_{\alpha\beta}(x,u) = g_{ij}(x,u)dx^i \otimes dx^j + g_{ab}(x,u)du^a \otimes du^b$$

and $\Omega = \prod_{i,a} dx^i du^a$ gives the volume form on $\mathbb{R}^8$.

Remark: The product $F_{\lambda\mu}F^{\lambda\mu}$ is in our case

$$F_{\lambda\mu}F^{\lambda\mu} = F_{ij}F^{ij} + \tilde{F}_{ia}\tilde{F}^{ia}.$$

Taking variations with respect to A_k in the above, we get, in terms of covariant derivatives with respect to the metrical connection $D\Gamma = (\Gamma^i{}_{jk}, C^i{}_{jk})$,

$$F^{ki}{}_{|i} + \tilde{F}^{ka}|_a = J^i,$$

where ${}_{|i} = D_{\frac{\partial}{\partial x^i}}$, $|_a = D_{\frac{\partial}{\partial y^a}}$.

Conclusion: In comparison to the case of isotropic spaces, there appears a new term in the expression for the current, namely,

$$\zeta^k = \tilde{F}^{ka}|_a. \tag{20}$$

This means that in an anisotropic space the measured fields would correspond to an effective current consisting of two terms: one is the current provided by the experimental environment, the other is the current corresponding to the anisotropy of space.

Examples:

1. If $A_i = A_i(x)$, then we get $\tilde{F}_{ia} = 0$ and

$$\zeta^k = 0, \quad F^{ki}{}_{|i} = J^k.$$

2. Already a nontrivial example is obtained if

$$A^i = A^i(x);$$

then we have shown above that

$$\tilde{F}_{ia} = -2C_{ija}A^j.$$

Then, $\tilde{F}^{ia} = -2C^{ia}{}_j A^j$ and

$$\zeta^k = -2(C^{ka}{}_j A^j)|_a = -2C^{ka}{}_j|_a A^j - 2C^{ka}{}_j C^j{}_{ha} A^h.$$

The presence of the last current in the experimental situation could be noticed if $\left|\tilde{F}^{ka}|_a\right| \approx \left|F^{ki}{}_{|i}\right|$.

In the case of *deformed Minkowski metric*, we get that

$$\zeta^k = -2\varepsilon^{ka}{}_{\cdot ja} A^j. \tag{21}$$

Consequently, the current $\tilde{F}^{ia}$ can be noticed when the "regular" current is small and $\left|2\varepsilon^{ka}{}_{\cdot ja} A^j\right| \approx \left|F^{ki}{}_{|i}\right|$.

If we take instead a deformed Berwald-Moor metric, then the Cartan tensor components are no longer small, and hence the correction ζ^k is generally not small, and it could be noticed even if the regular current is not small.

Conclusion: An anisotropic space with electromagnetic field possesses inherent currents that could produce observable fields.

Comparison to existent results:

R. Miron and collaborators, [7] defined electromagnetic tensors in Lagrange spaces formally, by means of nonlinear (and linear) connections on the tangent bundle TM :

$$F_{ij} = \frac{1}{2}(y_{j|i} - y_{i|j}), \quad f_{ab} = \frac{1}{2}(y_b|_a - y_a|_b)$$

where $N^i{}_j$, $L^i{}_{jk}$, $C^a{}_{bc}$ are the coefficients of the Kern nonlinear connection and respectively, the canonical metrical linear connection, [7], on TM (in the case of Finsler spaces, $(N^i{}_j$, $L^i{}_{jk}$, $C^a{}_{bc})$ provide the Cartan connection, [4]).

We notice there the appearance of new quantities f_{ab} (and additional Maxwell equations) in comparison to the Riemannian case:

$$F_{ji|k} + F_{kj|i} + F_{ik|j} = 0, \quad f_{ab|c} + f_{ca|b} + f_{bc|a} = 0,$$
$$F^{ij}{}_{|j} = J^i, \quad f^{ab}|_b = j^a.$$

Also, in the cited work is investigated the case of an electromagnetic tensor arising from a potential

$$A_i = A_i(x)$$

depending just on the positional variables x^i. In this case, the authors show that the resulting Maxwell equations and the resulting expression for Lorentz force are formally identical to the usual ones in Riemannian spaces – no additional terms appear.

In our approach, we consider **direction dependent potentials**. The components and the corrections due to anisotropy to the electromagnetic tensor appear from variational approaches. The new appearing terms are to be *added* to the currents, not regarded as separate quantities:

$$F^{ij}{}_{|j} + \tilde{F}^{ia}{}_{\cdot a} = J^i.$$

We could also relate our components of the electromagnetic tensor to linear connections. Namely, let $D\Gamma = (N^a{}_i, L^i{}_{jk}, C^i{}_{jk})$ denote the Cartan connection, [4], [7], determined by the Finslerian function g_{ij}; let us define the following tensor fields:

$$X^i{}_{jk} = \frac{2q}{c} \frac{g^{ih}}{\mathcal{F}} \frac{\partial}{\partial y^k}(\mathcal{F}A_{h|j}),$$

where the x-covariant derivative $A_{h|j}$ is taken with respect to the Cartan connection of g : $A_{h|j} = \dfrac{\delta A_h}{\delta x^j} - L^l{}_{hj}A_l$. Also, let

$$\Omega^{ih}_{rj} = \frac{1}{2}(\delta^i_r \delta^h_j - g_{rj}g^{ih})$$

denote the Obata operators, [7], of g.

Let us fix the nonlinear connection N as the Cartan one and define the linear connection $\tilde{D}\Gamma(N) = (\tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk})$ by

$$\tilde{L}^i{}_{jk} = \Gamma^i{}_{jk} + \Omega^{ih}_{rj}X^r{}_{hk}, \quad \tilde{C}^i{}_{jk} = C^i{}_{jk} - \frac{q}{c}g^{ih}A_{h\cdot jk}.$$

We denote x- and y- covariant derivatives with respect to this connection with $_{||k}$ and $_{||k}$ respectively.

Then, by direct computation, one can check the following properties:

Proposition 3 *1. A curve $t \mapsto (x^i(t), \dot{x}^i(t))$ on $T\mathbb{R}^4$ is an autoparallel curve of $\tilde{D}\Gamma(N)$ if and only if its projection $t \mapsto x^i(t)$ on the base manifold $\mathbb{R}^4$ is a solution of Lorentz equation (8).*
2. $\tilde{D}\Gamma(N)$ is h-metrical: $g_{ij||k} = 0$;
3. In the adapted basis $(\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^k})$, the local coordinates of the electromagnetic tensor are given by:

$$y_{i||j} = \frac{-1}{2}(y_{j||i} - y_{i||j}) = -\frac{q}{c}(A_{j||i} - A_{i||j}) =: -\frac{q}{c}\mathbf{F}_{ij}. \quad y_i||_k = g_{ik} - \frac{q}{c}A_{i\cdot k},$$

6   The Berwald-Moor case

In the following, we shall develop an approach for obtaining a generalization of the expression for the Lorentz force in the case of the Berwald-Moor quartic Finslerian function $\mathcal{F} = \sqrt[4]{y^1 y^2 y^3 y^4}$, [8], [9], [10], [11], [12], in terms of the 4-scalar product introduced in [8].

The 4-scalar product

$$< U, V, W, Y > = G_{ijkl} U^i V^j W^k Y^l, \qquad \forall U, V, W, Y \in \mathcal{X}(\mathbb{R}^4),$$

where the components G_{ijkl} are, [8],

$$G_{ijkl} = \begin{cases} \dfrac{1}{4!} & \text{if } i, j, k, l \text{ are all different from each other} \\ 0, & \text{elsewhere} \end{cases}$$

induces a (direction dependent) pseudo-scalar product if we specify two of the 4 vectors involved. For instance, for $W = Y = y$, we get

$$< U, V > := < U, V, y, y > = h_{ij} U^i V^j,$$

where the *flag (polynomial) metric tensor* $h = h(y)$ has the local components

$$h_{ij} = G_{ijkl} y^k y^l =: G_{ij00},$$

equivalently, [12],

$$h_{ij} = \frac{1}{12} \frac{\partial^2 \mathcal{F}^4}{\partial y^i \partial y^j}.$$

We notice that the components h_{ij} are 2-homogeneous ploynomials in the directional variables y^i and that

$$h_{ij} y^i y^j = \mathcal{F}^4$$

The Lagrangian in Section 3, providing Lorentz force can be written as

$$L = \frac{1}{2} < y, y > + \frac{q}{c} < A, y > .$$

If we use in the above, instead of the classical Finslerian metric tensor g_{ij}, the polynomial metric h_{ij}, then we get a specific (generalized) Lagrangian for the Berwald-Moor case:

$$L = \frac{1}{2} h_{ij} y^i y^j + \frac{q}{c} h_{ij} A^i y^j.$$

Remark: The first term in the above is $\frac{1}{2} h_{ij} y^i y^j = \frac{1}{2} \mathcal{F}^4$.

The second term is

$$\frac{q}{c} h_{ij} A^i y^j = \frac{q}{c} G_{ijkl} A^i y^j y^k y^l.$$

Let us denote:

$$A_{jkl} = G_{ijkl} A^i.$$

The above defined tensor is totally symmetric. Its nonvanishing components are:

$$A_{123} = \frac{1}{4!} A^4, \quad A_{124} = \frac{1}{4!} A^3, \quad A_{134} = \frac{1}{4!} A^2, \quad A_{234} = \frac{1}{4!} A^1.$$

Moreover, if $A^i = A^i(x)$ are only position dependent, then A_{jkl} depend only on x and conversely. In the following, we shall assume this, namely

$$A_{jkl} = A_{jkl}(x).$$

We get, thus, the following Lagrangian in case of the Berwald-Moor function

$$L = \frac{1}{2}\mathcal{F}^4 + \frac{q}{c}A_{ijk}(x)y^i y^j y^k. \tag{22}$$

Let us obtain the terms of the Euler-Lagrange equation

$$\frac{\partial L}{\partial x^m} = \frac{q}{c}A_{ljk,m}y^l y^j y^k;$$

$$\frac{\partial L}{\partial y^m} = \frac{1}{2}\frac{\partial \mathcal{F}^4}{\partial y^m} + 3\frac{q}{c}A_{mjk}(x)y^j y^k \;\Rightarrow$$

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial y^m}\Big) = \frac{d}{dt}\Big(\frac{1}{2}\frac{\partial \mathcal{F}^4}{\partial y^m}\Big) + 3\frac{q}{c}A_{mjk,l}y^l y^j y^k + \frac{q}{c}6A_{jmk}\frac{dy^j}{dt}y^k =$$

$$= \;\Big(\frac{12}{2}h_{mj}\frac{dy^j}{dt} + 6\frac{q}{c}A_{jmk}y^k\Big)\frac{dy^j}{dt} + \frac{q}{c}\big(A_{mjk,l} + A_{mlk,j} + A_{mjl,k}\big)y^l y^j y^k.$$

Therefore, we get

Proposition 4 *The extremal curves for the Lagrangian (22) are given by*

$$6\big(h_{mj} + \frac{q}{c}A_{mj0}\big)\frac{dy^j}{dt} + \frac{q}{c}\big(A_{mjk,l} + A_{mlk,j} + A_{mjl,k} - A_{ljk,m}\big)y^l y^j y^k = 0,$$

where $A_{mj0} = A_{mjk}y^k$.

Let us denote

$$F_{mjkl} = \frac{1}{6}(A_{mjk,l} + A_{mlk,j} + A_{mjl,k} - A_{ljk,m}). \tag{23}$$

The above relation defines a 4-covariant tensor field, which is symmetric in its 3 last indices: $F_{mjkl} = F_{mkjl} = F_{mklj}$ etc.

By contracting it twice with y^k, y^l, we get an antisymmetric tensor:

$$F_{mj00} := F_{mjkl}y^k y^l \Rightarrow \; F_{mj00} = F_{jm00}.$$

With the notation (23), the equations of extremal curves can be written as:

$$h_{mj}\frac{dy^j}{dt} + \frac{q}{c}F_{mjkl}y^j y^k y^l + \frac{1}{6}\frac{q}{c}A_{mj0}\frac{dy^j}{dt} = 0.$$

If we raise indices by means of h^{ij}, the above is equivalent to :

$$\frac{dy^i}{dt} + \frac{q}{c}F^i{}_{jkl}y^j y^k y^l + \frac{1}{6}\frac{q}{c}A^i{}_{j0}\frac{dy^j}{dt} = 0. \tag{24}$$

With this we can return to 8 and try to derive all the results obtained in what follows there in the similar manner.

Discussion

The relativity principle states that there are no means to distinguish between the inertial frames, but the equivalence principle goes even further. Since we can't distinguish between an accelerated frame and such a physical field as gravity, there is no reason to think that gravity is velocity independent. If the acceleration is not rectilinear, the inertial force acting on a body depends on the velocity of the body (e.g. Coriolis force). Therefore, measuring gravity we can't be sure to which extent we have accounted for the observable kinematics. From the

mathematical point of view it means that the space is anisotropic. Locally, it could mean the existence of a preferable direction (e.g. the axis of a rotating reference frame), but generally it demands the direction dependent metric. This leads to the anisotropic geometrodynamics [1] which appears to be able to explain some well known paradoxes observed on the cosmological scale.

In this paper the ideas formulated in [1] were developed for the case when an additional physical (electromagnetic) field is present in such an anisotropic space. Obviously, the notions of Lorentz force and electric current in an accelerated frame (in an anisotropic space) should be redefined with regard to the mentioned circumstances. This is performed for the case of the linearly approximable anisotropic perturbation of the locally Minkowski metric. We have found the expressions for the additional terms both in the Lorentz force 8 and in the current 20. It should be mentioned that it is important in which way – covariant or contravariant – do the measurable (physical) variables transform with the coordinates transformation. The calculations for the case when the unperturbed locally Minkowski metric is Minkowski one lead to the concrete expressions 14 and 21 that can be used in observations.

Alongside with the weak anisotropic perturbation of the Minkowski metric we regarded also the anisotropic Berwald-Moor metric as it was done in [8]. Working in terms of the corresponding 4-scalar product, the geodesic equations have obtained a new form and what could be called a "Lorentz force" has also transformed. This is due to the fact that the algebra chosen for the possible interpretation of experimental measurements essentially differs from the usual one. It is hard to say at once whether it is convenient or not and whether it has any new perspectives because of the lack of the corresponding language in the interpretation of physical notions.

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