AN EXTENSION OF ELECTRODYNAMICS THEORY TO THE COMPLEX LAGRANGE GEOMETRY

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In this note our purpose is to introduce the Maxwell type equations in a complex Lagrange space, particularly in a complex Finsler space.

The electromagnetic tensor fields are defined as the sum between the differential of the complex Liouville 1-form and the symplectic 2-form of the space relative to the adapted frame of Chern-Lagrange complex nonlinear connection.

Is proved that the $(1,1)$-type electromagnetic field of a complex Finsler space vanish and the differential of the $(2,0)$-type electromagnetic field yields the generalized Maxwell equations. The complex electromagnetic currents are also introduced and the conditions when they are conservative are deduced.

Finally we apply the results to the electrodynamics Lagrangian considered in [Mu] and to the case of complex Randers spaces.

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1 The basics of complex Lagrange geometry

The notion of complex Lagrange space was introduced by us ([Mu]), hankering to obtain some geometric models for quantum physics theories. It is a natural extension of complex Finsler notion, for which already exists a large reference ([A-P, Ai, Wo], ...). Here we only briefly set the basic notions needed for our purpose.

Let M be a complex manifold, $\left(z^k\right)_{k=\overline{1,n}}$ complex coordinates in a local chart, and $T'M$ be its holomorphic tangent bundle in which, as a complex manifold, we consider the complex induced coordinates $u=(z^k,\eta^k)_{k=\overline{1,n}}$.

The complexified tangent bundle $T_C(T'M)$ admits a vertical distribution $V'T'M$, locally spanned by $\{\dot{\partial}_k := \frac{\partial}{\partial\eta^k}\}_{k=\overline{1,n}}$ and its conjugate $V''T'M$, locally spanned by $\{\dot{\partial}_{\bar{k}} := \frac{\partial}{\partial\bar{\eta}^k}\}_{k=\overline{1,n}}$. A supplementary distribution in $T'(T'M)$ to $V'T'M$ is called a complex nonlinear connection, in brief (*c.n.c.*), and it is determined by a set of complex functions $N^i_j(z,\eta)$ in respect to with $\{\delta_k := \frac{\partial}{\partial z^k} - N^j_k\frac{\partial}{\partial\eta^j}\}_{k=\overline{1,n}}$ are change like vectors on the underline manifold M. The distribution spanned by $\{\delta_k\}_{k=\overline{1,n}}$ will be called horizontal adapted to the (*c.n.c.*) and will be denoted by $HT'M$. Its conjugate distribution $H''T'M$ is locally spanned by $\{\delta_{\bar{k}} := \overline{\delta_k}\}_{k=\overline{1,n}}$.

A *complex Lagrange space* is a pair (M,L), where $L: T'M \to R$ is a regular Lagrangian in sense that the Hermitian metric tensor $g_{i\bar{j}} = \partial^2 L/\partial\eta^i\partial\bar{\eta}^j$ is nondegenerated. Particularly, if L is a positive function, smooth excepting the zero sections, $(1,1)$-homogeneous, i.e. $L(z,\lambda\eta) =\mid \lambda \mid^2 L(z,\eta)$, $\forall\lambda\in\mathbb{C}$, and the quadratic form $g_{i\bar{j}}\eta^i\bar{\eta}^j$ is positive defined, then (M,L) is a complex Finsler space with fundamental function $F=\sqrt{L}$. It is obvious that the class of complex Lagrange spaces include that of complex Finsler spaces, but some properties of the last are lose in the first class.

The Lagrange function L defines a (*c.n.c.*), called by us the Chern-Lagrange (*c.n.c.*), with the following coefficients:

$$N^j_k = g^{\bar{m}j}\frac{\partial^2 L}{\partial z^k\partial\bar{\eta}^m} \tag{1.1}$$

and its adapted frames have a remarkable property concerning the brackets: $[\delta_j, \delta_k] = 0$ and the others are

$$\begin{aligned} [\delta_j, \delta_{\bar{k}}] &= (\delta_{\bar{k}} N_j^i)\dot{\partial}_i - (\delta_j N_{\bar{k}}^{\bar{i}})\dot{\partial}_{\bar{i}} \ ; \\ \left[\delta_j, \dot{\partial}_k\right] &= (\dot{\partial}_k N_j^i)\dot{\partial}_i \ ; \quad \left[\delta_j, \dot{\partial}_{\bar{k}}\right] = (\dot{\partial}_{\bar{k}} N_j^i)\dot{\partial}_i \ ; \\ \left[\dot{\partial}_j, \dot{\partial}_k\right] &= 0 \ ; \quad \left[\dot{\partial}_j, \dot{\partial}_{\bar{k}}\right] = 0. \end{aligned} \tag{1.2}$$

With respect to the adapted frames of (1.1) (*c.n.c.*) a notable derivative law of $(1,0)$-type is the so called Chern-Lagrange N-complex linear connection, which in notations from [Mu] has the coefficients

$$D\Gamma(N) = \left(L^i_{jk} = g^{\bar{m}i}\delta_k g_{j\bar{m}} \ ; \ L^i_{j\bar{k}} = 0 \ ; \ C^i_{jk} = g^{\bar{m}i}\dot{\partial}_k g_{j\bar{m}} \ ; \ C^i_{j\bar{k}} = 0\right), \tag{1.3}$$

where $D_{\delta_k}\delta_j = L^i_{jk}\delta_i \ ; \ D_{\delta_{\bar{k}}}\delta_j = L^i_{j\bar{k}}\delta_i \ ; \ D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i \ ; \ D_{\dot{\partial}_{\bar{k}}}\dot{\partial}_j = C^i_{j\bar{k}}\dot{\partial}_i$, etc.

D is a metrical connection, that is $D_{\delta_k}G = D_{\delta_{\bar{k}}}G = D_{\dot{\partial}_k}G = D_{\dot{\partial}_{\bar{k}}}G = 0$, where $G = g_{i\bar{j}}dz^i \otimes d\bar{z}^j + g_{i\bar{j}}\delta\eta^i \otimes \delta\bar{\eta}^j$ is the $N-$lift of the metric tensor $g_{i\bar{j}}$.

Also, with respect to the adapted frames of (1.1) (*c.n.c.*), two well defined forms can be considered

$$\omega = \omega' + \omega'' := \frac{\partial L}{\partial \eta^i}dz^i + \frac{\partial L}{\partial \bar{\eta}^i}d\bar{z}^i \tag{1.4}$$

$$\theta = g_{i\bar{j}}\delta\eta^i \wedge d\bar{z}^j. \tag{1.5}$$

ω is the Liouville form of the complex Lagrange space and θ is the Hermitian symplectic 2-form associated to the (M, L) space.

The complex Lagrange geometry is one Hermitian but in its applications appear sometimes non-Hermitian quantities. For instance, if we consider the non-Hermitian tensor $g_{ij} = \partial^2 L/\partial\eta^i\partial\eta^j$ (without the request of its nondegenerating) and $g_{\bar{i}\bar{j}} = \overline{g_{ij}}$, a well defined 2-form is given by

$$\varphi = g_{ij}\delta\eta^i \wedge dz^j. \tag{1.6}$$

Subsequently we shall use also this 2-form and its conjugate.

2 Maxwell equations on a complex Lagrange space

In [Mu], p. 99, we consider the following Lagrangian inspired from complex electrodynamic:

$$L_q = m_0 c\gamma_{i\bar{j}}(z)\eta^i\bar{\eta}^j - \frac{q}{c}\left(A_i(z)\eta^i + \overline{A_i(z)\eta^i}\right) \tag{2.1}$$

where $\gamma_{i\bar{j}}$ is a Hermitian metric on the complex universe M, eventually it could be constant, and the other quantities have the well-known physics meaning. $A_i(z)\eta^i$ is a 1-form which defines a complex potential.

(M, L_q) is a complex Lagrange space, with $g_{i\bar{j}} = m_0 c\gamma_{i\bar{j}}(z)$ the metric tensor, and it reduces to one complex Finsler space iff $Re A_i(z)\eta^i = 0$.

In [Mu] we contented to make a gravitational approach relative to this Lagrangian, without taking in account some electrodynamic purport of the complex potential. Actually the main difficulty in obtain a consistent theory for complex electrodynamic which we have then was the definition of complex electromagnetic fields which obey the covariance principle with respect to Chern-Lagrange complex linear connection D.

We try first time to fit at our framework one nice idea used by R. Miron ([M-A]) in real Lagrange model for electrodynamic theory. In large, it consist in define first the vertical and horizontal tensors $D^i_j = D_{\delta_j} y^i$ and $d^i_j = D_{\dot{\partial}_i} y^j$ and then the electromagnetic tensors are $\mathcal{F}_{ij} = \frac{1}{2}(D_{ij} - D_{ji})$ and $f_{ij} = \frac{1}{2}(d_{ij} - d_{ji})$, where $D_{ij} = g_{ik} D^k_j$ and $d_{ij} = g_{ik} d^k_j$. Here we use real notations with respect to canonical connection ([M-A]). The data concerning the Lagrangian expression send of to the electromagnetic tensors by means of metric tensor and connection coefficients. When we attempt to follow an analogous idea in our theory, the first remark is that from the particular case of complex Finsler spaces (M, L) the (1.3) Chern-Finsler linear connection performs, as we easy can see from the homogeneity of the fundamental function, the following conditions: $D^i_j = D_{\delta_j} \eta^i = 0$, $D^i_{\bar{j}} = D_{\delta_{\bar{j}}} \eta^i = 0$, $d^i_j = D_{\dot{\partial}_j} \eta^i = \delta^i_j - C^i_{0j}$, $d^i_{\bar{j}} = D_{\dot{\partial}_{\bar{j}}} \eta^i = 0$, and consequently the corresponding electromagnetic tensor field which could be introduced such away all vanish identically.

Neither in the more general case of complex Lagrange space such way do not offers more because D_{ij} and d_{ij} vanish and however the mixed tensors could be non zero they do not generate consistent Hermitian electromagnetic tensors. Hence this theory does not present much interest and then another approach needs follow. Such comely idea for us is also inspired by a paper of R. Miron used for an electromagnetic theory of Ingarden space ([Mi]), whose fundamental function is just of the Randers, but its geometry is not of one real Finsler space. R. Miron prove that in a Ingarden space the differential of the Liouville 1-form is the difference between its electromagnetic tensor and the symplectic 2-form of the space. This remark could be a motivation for the definitions which will come bellow.

Let (M, L) be a complex Lagrange space, ω, θ and φ the (1.4), (1.5), (1.6) forms, with respect to adapted frames of Chern-Lagrange (*c.n.c.*). The differential operator on $T_C T'M$ has the components $d = d' + d''$, with

$$d' = \delta_k dz^k + \dot{\partial}_k \delta\eta^k \quad \text{and} \quad d'' = \delta_{\bar{k}} d\bar{z}^k + \dot{\partial}_{\bar{k}} \delta\bar{\eta}^k \tag{2.2}$$

and hence for the (1.4) Liouville 1-form we have the differential

$$d\omega = d'\omega' + d'\omega'' + d''\omega' + d''\omega''.$$

Definition 2.1 *We call the complex electromagnetic fields of the* (M, L) *space, the tensors*

$$\mathcal{F}_{ij} = \frac{1}{2}\{\delta_j(\dot{\partial}_i L) - \delta_i(\dot{\partial}_j L)\}\ ; \quad \mathcal{F}_{i\bar{j}} = -\delta_i(\dot{\partial}_{\bar{j}} L) \tag{2.3}$$

and their conjugates $\mathcal{F}_{\bar{i}\bar{j}} = \overline{\mathcal{F}_{ij}}$, $\mathcal{F}_{\bar{i}j} = \overline{\mathcal{F}_{i\bar{j}}}$.

Let be $\mathcal{F}^{(2,0)} = \mathcal{F}_{ij} dz^i \wedge dz^j$ and $\mathcal{F}^{(1,1)} = \mathcal{F}_{i\bar{j}} dz^i \wedge d\bar{z}^j$.

Theorem 2.1 *We have*

$$\begin{aligned} d'\omega' &= -\mathcal{F}^{(2,0)} + \varphi\ ; \quad & d''\omega'' &= -\overline{\mathcal{F}^{(2,0)}} + \overline{\varphi} = \overline{d'\omega'} \\ d'\omega'' &= -\mathcal{F}^{(1,1)} + \theta\ ; \quad & d''\omega' &= -\overline{\mathcal{F}^{(1,1)}} - \overline{\theta} = -\overline{d'\omega''}. \end{aligned} \tag{2.4}$$

The proof follows directly from (2.2) and (2.3).

An immediate result is

Proposition 2.1 *If* (M, L) *is a complex Finsler space, then* $\mathcal{F}^{(1,1)} = 0$.

Proof. From the homogeneity condition of the Finsler function is obtain that $\dot{\partial}_i L = g_{i\bar{j}}\bar{\eta}^j$ and $\dot{\partial}_{\bar{j}} L = g_{i\bar{j}}\eta^i$. Taking in account that Chern-Finsler connection D is one metrical, we have:

$$\begin{aligned}\mathcal{F}_{i\bar{j}} &= -\delta_i(\dot{\partial}_{\bar{j}} L) = -\delta_i(g_{k\bar{j}}\eta^k) = -D_{\delta_i}(g_{k\bar{j}}\eta^k) + (g_{k\bar{m}}\eta^k)L^{\bar{m}}_{\bar{j}k} \\ &= -g_{k\bar{j}}D_{\delta_i}\eta^k = g_{k\bar{j}}(N^k_i - L^k_{hi}\eta^h) = 0.\end{aligned}$$

Here we use the fact that $L^k_{hi}\eta^h = \dot{\partial}_h(N^k_i)\eta^h = N^k_i$, in view of one property of Chern-Finsler linear connection. □

Hence, in a complex Finsler space the non zero electromagnetic tensors are only $\mathcal{F}_{ij}$ and its conjugate.

Further, from $dd\omega = 0$, it deduces that $d(\mathcal{F}^{(2,0)} + \overline{\mathcal{F}^{(2,0)}} + \mathcal{F}^{(1,1)} + \overline{\mathcal{F}^{(1,1)}}) = d(\varphi + \overline{\varphi} + \theta - \overline{\theta})$. Now writing this last formula with respect to adapted frames of Chern-Finsler (*c.n.c.*) and taking into account the (1.2) components of the Lie brackets, we have

Theorem 2.2 *In a complex Lagrange space we have the following generalized Maxwell equations*

$$\begin{aligned}&\sum\{D_{\delta_k}\mathcal{F}_{ij}\} = 0\,; \qquad \sum\{D_{\dot{\partial}_k}\mathcal{F}_{ij}\} = 0\,; \\ &\sum\{D_{\delta_{\bar{k}}}\mathcal{F}_{ij}\} = 0\,; \qquad \sum\{D_{\dot{\partial}_{\bar{k}}}\mathcal{F}_{ij}\} = 0\,; \\ &\sum\{D_{\delta_k}\mathcal{F}_{i\bar{j}}\} = \sum\{\delta_{\bar{j}}(N^h_i)g_{hk}\}\,; \quad \sum\{D_{\dot{\partial}_k}\mathcal{F}_{i\bar{j}}\} = 0\,; \\ &\sum\{D_{\delta_{\bar{k}}}\mathcal{F}_{i\bar{j}}\} = \sum\{\delta_{\bar{j}}(N^h_i)g_{h\bar{k}}\}\,; \quad \sum\{D_{\dot{\partial}_{\bar{k}}}\mathcal{F}_{i\bar{j}}\} = 0\,.\end{aligned}$$

Moreover, the following identities are fulfilled

$$\begin{aligned}&\sum\{D_{\delta_k}g_{ij} + \dot{\partial}_j(N^h_i)g_{hk}\} = 0\,; \quad \sum\{D_{\dot{\partial}_k}g_{ij}\} = 0\,; \\ &\sum\{D_{\delta_{\bar{k}}}g_{ij} + \dot{\partial}_j(N^h_i)g_{h\bar{k}}\} = 0\,; \quad \sum\{D_{\dot{\partial}_{\bar{k}}}g_{ij}\} = 0\,.\end{aligned}$$

All these sums are cyclic by (i, j, k)*, the bar indices being an abbreviation for* $\delta/\delta\bar{z}^k$ *or* $\partial/\partial\bar{\eta}^k$.

We note that these Maxwell equations become homogeneous if the complexified horizontal distribution is integrable, i.e. $\left[\delta_i, \delta_{\bar{j}}\right] = 0$. Taking into account that in a complex Finsler space $\mathcal{F}_{i\bar{j}} = 0$, another set of identities is obtain for the Chern-Finsler (*c.n.c.*), which are consequences of the Bianchi identities ([Al]).

Next, by help of the metric tensor we can lowing or raising the indices for the complex electromagnetic tensors,

$$\mathcal{F}^{ij} = g^{\bar{k}i}g^{\bar{h}j}\mathcal{F}_{\bar{k}\bar{h}} \quad \text{and} \quad \mathcal{F}^{\bar{i}j} = g^{\bar{i}k}g^{\bar{h}j}\mathcal{F}_{k\bar{h}}.$$

With these the electromagnetic currents $\overset{h}{J}, \overset{\bar{h}}{J}, \overset{v}{J}, \overset{\bar{v}}{J}$ can be given by

$$\begin{aligned}&\sum_j D_{\delta_j}\mathcal{F}^{ij} = 4\pi\,\overset{h}{J^i}\,; \quad \sum_j D_{\dot{\partial}_j}\mathcal{F}^{ij} = 4\pi\,\overset{v}{J^i}\,; \\ &\sum_j D_{\delta j}\mathcal{F}^{\bar{i}j} = 4\pi\,\overset{\bar{h}}{J^{\bar{i}}}\,; \quad \sum_j D_{\dot{\partial}_j}\mathcal{F}^{\bar{i}j} = 4\pi\,\overset{\bar{v}}{J^{\bar{i}}}\,.\end{aligned}$$

It is obvious that in complex Finsler space $\overset{\bar{h}}{J}=\overset{\bar{v}}{J}=0$. The currents will be conservative iff $DJ=0$, that is they satisfies the conditions: $D_{\delta_j}\overset{h}{J^i}=D_{\delta_{\bar{j}}}\overset{h}{J^i}=D_{\dot{\partial}_j}\overset{h}{J^i}=D_{\delta\bar{j}}\overset{h}{J^i}=0$ and analogous for others.

Further, let us come back to the (2.1) expression of the electrodynamic Lagrangian. It can be rewritten as $L_q=L_0-\frac{q}{c}\{A_i(z)\eta^i+\overline{A_i(z)\eta^i}\}$, where $L_0=m_0c\gamma_{i\bar{j}}(z)\eta^i\bar{\eta}^j$ contains specific data about the energy of the space but also about its geometry by means of $\gamma_{i\bar{j}}$. Without upper point let us denoting the classical partial derivative, $\partial_i:=\frac{\partial}{\partial z^i}$. Then we have:

$$\mathcal{F}_{ij}=-T_{ij}+\frac{q}{c}F_{ij}\quad\text{and}\quad\mathcal{F}_{i\bar{j}}=-T_{i\bar{j}}+\frac{q}{c}F_{i\bar{j}}\ , \tag{2.5}$$

where

$$T_{ij}=\frac{1}{2}m_0c\{\partial_i\gamma_{j\bar{k}}-\partial_j\gamma_{i\bar{k}}\}\bar{\eta}^k\ ;\quad T_{i\bar{j}}=m_0c\partial_i\gamma_{k\bar{j}}\eta^k$$

are the *stress-energy tensors* of the space and

$$F_{ij}=\frac{1}{2}\{\partial_iA_j-\partial_jA_i\}\ ;\quad F_{i\bar{j}}=\partial_i\overline{A_j}$$

are the *exterior electromagnetic tensors* of the space.

Let be $T^{\bar{h}\bar{k}}=g^{\bar{h}i}g^{\bar{k}j}T_{ij}$ and $T^{\bar{h}k}=g^{\bar{h}i}g^{\bar{j}k}T_{i\bar{j}}$. Since D is a metrical connection, the law of conservative energy $D_{\delta_{\bar{h}}}T^{\bar{h}\bar{k}}=D_{\delta_{\bar{j}}}T^{\bar{h}k}=D_{\dot{\partial}_{\bar{h}}}T^{\bar{h}\bar{k}}=D_{\dot{\partial}_{\bar{h}}}T^{\bar{h}k}=0$, implies

$$\sum_{\bar{h}}g^{\bar{h}i}D_{\delta_{\bar{h}}}\mathcal{F}_{ij}=\frac{q}{c}\sum_{\bar{h}}g^{\bar{h}i}\partial_{\bar{h}}F_{ij}\ ;\quad\sum_{\bar{h}}g^{\bar{h}i}D_{\dot{\partial}_{\bar{h}}}\mathcal{F}_{ij}=0\ ;$$
$$\sum_{\bar{h}}g^{\bar{h}i}D_{\delta_{\bar{h}}}\mathcal{F}_{i\bar{j}}=\frac{q}{c}\sum_{\bar{h}}g^{\bar{h}i}\partial_{\bar{h}}F_{i\bar{j}}\ ;\quad\sum_{\bar{h}}g^{\bar{h}i}D_{\dot{\partial}_{\bar{h}}}\mathcal{F}_{i\bar{j}}=0\ .$$

As we say, L_q is a complex Finsler space only in a particular case and then it reduces to one trivial with purely Hermitian metric. Recently we make with N. Aldea ([A-M]) a study of complex Randers spaces. An immediate example of such space is (M,F) with $F=\alpha+\mid\beta\mid$, where

$$\alpha^2=\gamma_{i\bar{j}}(z)\eta^i\bar{\eta}^j\quad\text{and}\quad\beta=A_i(z)\eta^i$$

and $\mid\beta\mid=\sqrt{\beta\cdot\bar{\beta}}$ is the complex norm.

Indeed (M,F) is a complex Finsler space in some smoothness assumptions, and $L=F^2$ define a complex homogeneous Lagrangian for which we can make similar reasonings like above.

The metric tensor and the Chern-Finsler ($c.n.c.$) of the complex Randers space ware determined in the general setting in [A-M]. In our notations we have:

$$g_{i\bar{j}}=\frac{F}{\alpha}h_{i\bar{j}}+\frac{F}{2|\beta|}A_iA_{\bar{j}}+\frac{1}{2L}\eta_i\eta_{\bar{j}}$$

where $h_{i\bar{j}}:=\gamma_{i\bar{j}}-\frac{1}{2\alpha^2}\gamma_{i\bar{k}}\gamma_{h\bar{j}}\eta^h\bar{\eta}^k$ and

$$\overset{CF}{N^i_j}=\overset{0}{N^i_j}+\frac{1}{\gamma}\left(\gamma_{k\bar{r}}\frac{\partial A^{\bar{r}}}{\partial z^j}\eta^k-\frac{\beta^2}{|\beta|^2}\frac{\partial A_{\bar{r}}}{\partial z^j}\bar{\eta}^r\right)\xi^i+\frac{\beta}{2|\beta|}k^{\bar{r}i}\frac{\partial A_{\bar{r}}}{\partial z^j}$$

where $\overset{0}{N^i_j}:=\gamma^{\overline{m}i}\frac{\partial\gamma_{l\overline{m}}}{\partial z^j}\eta^l$, $\xi^i:=\bar{\beta}\eta^i+\alpha^2A^i$, $A^i=\gamma^{\overline{m}i}A_{\bar{m}}$ and $k_{i\bar{j}}=\frac{1}{2\alpha}h_{i\bar{j}}+\frac{1}{4|\beta|}A_iA_{\bar{j}}$. Thus we can consider the adapted frames $\{\delta_k\}$ of $\overset{CF}{N^i_j}$ nonlinear connection.

For the complex electromagnetic fields, first we have $\mathcal{F}_{i\bar{j}} = 0$ and

$$\mathcal{F}_{ij} = \frac{1}{2}\{\delta_j(\dot{\partial}_i L) - \delta_i(\dot{\partial}_j L)\} = \frac{1}{2}\{\delta_j(g_{i\bar{k}}\bar{\eta}^k) - \delta_i(g_{j\bar{k}}\bar{\eta}^k)\}$$
$$= \frac{1}{2}\{D_{\delta_j}(g_{i\bar{k}}\bar{\eta}^k) - g_{m\bar{k}}\bar{\eta}^k L^m_{ji} - D_{\delta_i}(g_{j\bar{k}}\bar{\eta}^k) + g_{m\bar{k}}\bar{\eta}^k L^m_{ij}\}.$$

Since $D_{\delta_j} g_{i\bar{k}} = 0$ and $D_{\delta_j}\bar{\eta}^k = 0$, is obtain

$$\mathcal{F}_{ij} = \frac{1}{2} g_{m\bar{k}}\{L^m_{ij} - L^m_{ji}\}\bar{\eta}^k = \frac{1}{2}\{\delta_j g_{i\bar{k}} - \delta_i g_{j\bar{k}}\}\bar{\eta}^k.$$

In a strongly Kähler Finsler space the torsion $T^i_{jk} = L^i_{jk} - L^i_{kj} = 0$ and consequently $\mathcal{F}_{ij} = 0$. If the Finsler space is weakly Kähler (see these notions in [A-P]) then $\mathcal{F}_{ij}\eta^j = \mathcal{F}_{ij}\eta^i = 0$.

The generalized Maxwell equations are homogeneous for this example.

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