THE THEORY OF LINEAR CONNECTIONS IN THE DIFFERENTIAL GEOMETRY OF ACCELERATIONS

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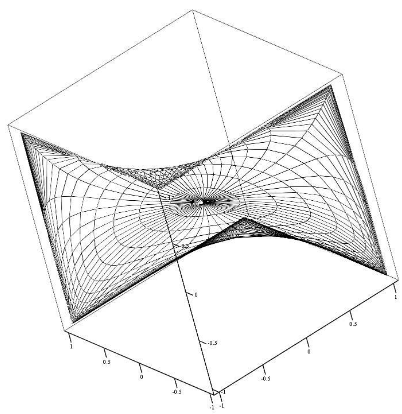

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This book presents the theory of linear connections in the differential geometry of second order. In the first part (Chapters 1 – 3) this theory is studied in the 2-jet bundle. The second part (Chapters 4 – 6) is devoted to the theory of linear connections in the differential geometry of the second order cotangent bundle.

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PREFACE

This book presents the theory of linear connections in the differential geometry of second order.

In the part one (Ch. 1 – Ch. 3) we shall study this theory in the 2-jet bundle J_0^2M.

Generally, the geometries of higher order defined as the study of the category of bundles of $k-$jet $(J_0^kM\ ,\ \pi^k,\ M)$ (see Ch. Ehresmann, [45] , [46] and References) are based on a direct approach of the properties of objects and morphisms in this category, without local coordinates.

But, many mathematical models from Lagrangian Mechanics, Theoretical Physics and Variational Calculus used multivariate Lagrangians of higher order accelerations, $L(x,\ \dfrac{dx}{dt}(t),\ ...\ ,\ \frac{1}{k!}\dfrac{dx^k}{dt^k}(t)\ )$, (see E. Cartan, [34], for $k=2$).

From here one can see the reason of construction of the geometry of the bundle of higher order accelerations (here, the tangent bundle of higher order, or the osculator bundle of higher order) in local coordinates.

Recently, this construction was achived by R. Miron and the author in the joint papers [89 – 95].

Namely, replacing the bundle of $k-$jets (J_0^kM,π^k,M) by the bundle of accelerations of order k, (or the $k-$osculator bundle (Osc^kM,π^k,M)) denoted here by (T^kM,π^k,M) one shows that the vertical distribution V_1 decomposes in k subdistributions from the sequence of inclusions $V_1\supset V_2\supset ...\supset V_k$, that there exist k independent Liouville vector fields $\overset{1}{\mathbb{C}}$, $\overset{2}{\mathbb{C}}$, ... , $\overset{k}{\mathbb{C}}$ and a natural $k-$tangent structure $\mathbb{J}$: $\mathcal{X}(T^kM)\to\mathcal{X}\left(T^kM\right)$. Then, one defines the notion of $k-$semispray S by the equation $\mathbb{J}S=\overset{k}{\mathbb{C}}$. This allows the obtaining of a nonlinear connection N from S, only. More precisely, S uniquely determines the dual coefficients of N.

Now, the nonlinear connection N gives a direct decomposition

$$T_u(T^kM)=N_0\,(u)\otimes N_1\,(u)\otimes ...\otimes N_{k-1}\,(u)\otimes V_k,\forall u\in T^kM, \tag{0.1}$$

to which all geometric objects on T^kM are described.

Thus N leads to define of the simplest linear connection D on T^kM, that which preserves by parallelism the distribution defining N and the $2-$tangent structure $\mathbb{J}$ is absolute parallel with respect to D, i.e. $D_X\mathbb{J}=0$, $\forall X\in\mathcal{X}\left(T^kM\right)$, (a $\mathbb{J}N$- linear connection in this book, cf. with Def. 6.2, Ch. 2). It comes out that $\mathbb{J}N$ preserves by parallelism the distributions from the decomposition (0.1), too. The local coefficients characterising N are in the smallest possible number: $\mathbb{J}D\Gamma(N) = (L^a_{bc}\ ,\ \underset{(\beta)}{C}{}^a_{bc})$, $(\beta = 1, ..., k)$. The whole Miron-Atanasiu's theory ($[89] - [95]$) is based on the decomposition (0.1) and the linear connection $\mathbb{J}D\Gamma(N)$. These are exposed unitary in R.Miron's monograph [84] .

In this book, for $k = 2$, also, we consider the decomposition (0.1):

$$T_u(T^2M) = N_0\,(u) \oplus N_1\,(u) \oplus V_2\,(u)\,, \forall u \in T^kM, \tag{0.2}$$

but we use a linear connection D on T^2M which preserves by parallelism the horizontal and verticals distributions N_0, N_1 and V_2 on T^2M, only. This linear connection is characterized by the local coefficients

$$D\Gamma(N) = (\underset{(\alpha 0)}{L}{}^a_{bc},\ \underset{(\alpha 1)}{C}{}^a_{bc},\ \underset{(\alpha 2)}{C}{}^a_{bc}), (\alpha = 0, 1, 2).$$

Moreover, there exist the natural almost contact structures $\underset{\alpha}{\mathbb{F}} : \mathcal{X}\left(\widetilde{T^2M}\right) \to \mathcal{X}\left(\widetilde{T^2M}\right), (\alpha = 0, 1, 2)$ on $\widetilde{T^2M}$ (cf. with Section 1.8, Ch. 1) and we can define the $\underset{\alpha}{\mathbb{F}}N$-linear connections as an $N-$linear connection D for which $D_X\underset{\alpha}{\mathbb{F}} = 0$, $\forall X \in T^2M$, $(\alpha = 0, 1, 2)$. We have the inclusion:

$$\mathbb{J}D\Gamma\,(N) \subset \underset{\alpha}{\mathbb{F}}D\Gamma\,(N) \subset D\Gamma\,(N)\,, (\alpha = 0, 1, 2)\,.$$

A detailed study of an $N-$linear connection $D\Gamma(N)$ allows the introduction of $h_\alpha-$ and $v_{\beta\alpha}$- covariant derivatives, $(\alpha = 0, 1, 2; \beta = 1, 2)$. With these one studies the parallelism of vector fields (Sect. 2.9 Ch. 2), the torsion, the curvature, the structure equations (Sections 2.3, 2.7, 2.10, Ch. 2), the Ricci and Bianchi identities (Sections 2.8 and 2.11, Ch. 2), etc.

Finally of the part one, in the Chapter 3, we study the metric structures on T^2M and some remarkable metrics on T^2M, especially an $(h_1, v_1, v_2)\,-$metric $\mathbb{G}, h-$Riemannian, v_1-locally Minkowski and v_2-locally accelerate, which can be use to the geometrical model on tangent bundle on T^2M.

In the part two of the book (Ch. 4 – Ch. 6) we shall study the theory of linear connections in the differential geometry of second order cotangent bundle.

The differential geometry of the second order cotangent bundle $\left(T^{*2}M, \pi^{*2}, M\right)$ introduced and studied, recently, by Acad. R. Miron [85], [86] and Acad. R. Miron with his partners, [97], is based on the differential geometries of the tangent bundle (TM, π, M) and the cotangent bundle (T^*M, π^*, M), [134] (see also, Gh. Atanasiu [7]-[13], S. Ianuş [55]-[57], R. Miron [78]-[83], V. Oproiu [110], [111], etc.), namely

$$T^{*2}M = TM \times_M T^*M. \tag{0.3}$$

In this way, the point $x \in M$, the velocity $y \in TM$ and the momenta $p \in T^*M$ there exist, intrinsec in $T^{*2}M$: $(x, y, p) = (u) \in T^{*2}M$. They are fasten in the Liouville $1-$form ω, in the $2-$form of presymplectic structure θ, in the natural tangent structure $\mathbb{J} : \chi\left(T^{*2}M\right) \longrightarrow \chi\left(T^{*2}M\right)$, etc., by

$$\omega = p_a dx^a, \tag{0.4}$$

$$\theta = d\omega = dp_a \wedge dx^a, \tag{0.5}$$

$$\mathbb{J}\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial y^a},\ \mathbb{J}\left(\frac{\partial}{\partial y^a}\right) = 0,\ \mathbb{J}\left(\frac{\partial}{\partial p_a}\right) = 0, \tag{0.6}$$
$$(a = 1, 2, ..., n;\ n = \dim M).$$

But, in the tangent bundle of differentiable manifold $T^{*2}M$, $\left(TT^*M, \tau^{*2}, M\right)$, where τ^{*2} is the canonical projection, there exist the natural subdistributions $V_1(u) = \left\{\frac{\partial}{\partial y^a}\ |_u\right\}$ and $W_2(u) = \left\{\frac{\partial}{\partial p_a}\ |_u\right\}$ and we have a direct decomposition of vector spaces

$$T_u T^{*2}M = N(u) \oplus V_1(u) \oplus W_2(u),\ \forall u \in T^{*2}M, \tag{0.7}$$

where N is a nonlinear connection.

The main geometrical objects on $T^{*2}M$ can be reported to the direct sum (0.7) and by use the natural tensors $\omega, \theta, \mathbb{J}$, etc., expressed in the formulae (0.4), (0.5), (0.6), etc.

For example, a linear connection in the R.Miron's theory on the $2-$cotangent bundle $T^{*2}M$ have the smallest possible number:

$$MD\Gamma(N) = \left(H^a{}_{bc}, C^a{}_{bc}, C_a{}^{bc}\right), (a, b, c = 1, ..., n),$$

(in this book, it is called a Miron $N-$linear connection).

Generally, in the monograph [97], the whole calculus: the parallelism of vector fields, the torsion, the curvature, the Ricci identities, etc., are based on the set of these coefficients.

In this book, a linear connection in the differential geometry of the $2-$cotangent bundle $T^{*2}M$ will be more rich. It will have a set of nine coefficients:

$$D\Gamma(N) = \left(\underset{(00)}{H}{}^a{}_{bc}, \underset{(10)}{H}{}^a{}_{bc}, \underset{(20)}{H}{}^a{}_{bc}, \underset{(01)}{C}{}^a{}_{bc}, \underset{(11)}{C}{}^a{}_{bc}, \underset{(21)}{C}{}^a{}_{bc}, \underset{(02)}{C}{}_a{}^{bc}, \underset{(12)}{C}{}_a{}^{bc}, \underset{(22)}{C}{}_a{}^{bc}\right)$$

(here, it is called an $N-$linear connection)

This is an advantage in the physical applications in electrodynamics [103], [104], elasticity [105], quantum field theories [40], [109], [119], in the deviations of geodesics [29], [30], [133], etc., because the torsion, the curvature, remarkable identities, etc., are much more substantials.

The calculus with these $N-$linear connections is not difficult and we develop him in the Chapters 4,5 and 6.

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Chapter 1

The 2-tangent bundle (T^2M, π^2, M)

1.1 The manifold T^2M

Let M be a real differentiable manifold of dimension n. A point of M will be denoted by x and its local coordinate system by (U,φ), $\varphi(x) = (x^a)$. The indices a,b,... run over set {1,2,...,n} and Einstein convention of summarizing is adopted all over this work.

Let us consider two curves $\rho, \sigma : I \to M$, having images in a domain of local chart $U \subset M$. We say that ρ and σ have a "contact or order 2" or the "same tangent line and the same curvature" in the point $x_0 \in U$ if: $\rho(0) = \sigma(0) = x_0, (0 \in I)$, and for any function $f \in \mathcal{F}(U)$:

$$\frac{d^\beta}{dt^\beta}(f \circ \rho)(t)|_{t=0} = \frac{d^\beta}{dt^\beta}(f \circ \sigma)(t)|_{t=0}, (\beta = 1, 2). \tag{1.1}$$

The relation "contact of order 2" is an equivalence on the set of smooth curves in M, which pas through the point x_0. Let $[\rho]_{x_0}$be a class of equivalence. It will be called a "2-osculator space" in the point $x_0 \in M$. The set of 2-osculator spaces in the point $x_0 \in M$ will be denoted by Osc^2M, and we put

$$Osc^2M = \bigcup_{x_0 \in M} Osc^2_{x_0}.$$

One considers the mapping $\pi^2 : Osc^2M \to M$ define by $\pi^2([\rho]_{x_0}) = x_0$. Clearly, π^2 is a surjection.

The set Osc^2M is endowed with a natural differentiable structure, induced by that of the manifold M, so that π^2 is a differentiable mapping. It will be described below.

The curve $\rho : I \to M, (\operatorname{Im}\rho \subset U)$ is analytically represented in the local chart (U, φ) by $x^a = x^a(t), t \in I$, $x_0 = x_o^a (= x^a(0))$. Taking the function f from (1.1), successively equal to the coordinate functions x^a, then a representative of the class $[\rho]_{x_0}$ is given by

$$x^{*a}(t) = x^a(0) + t\frac{dx^a}{dt}(0) + \frac{1}{2}t^2\frac{d^2x^a}{dt^2}(0), t \in (-\epsilon, \epsilon) \subset I.$$

The previous polynomials are determined by the coefficients

$$x_0^a = x^a(0), y^{(1)a} = \frac{dx^a}{dt}(0), y^{(2)a} = \frac{1}{2}\frac{d^2x^a}{dt^2}(0). \tag{1.2}$$

Hence, the pair $\left(\left(\pi^2\right)^{-1}(U), \Phi\right)$, with $\Phi\left([\rho]_{x_0}\right) = \left(x_0^a, y^{(1)a}, y^{(2)a}\right) \in R^{3n}$, $\forall [\rho]_{x_0} \in \left(\pi^2\right)^{-1}(U)$ is a local chart on Osc^2M. Thus a differentiable atlas $\mathcal{A}_M$ of the differentiable structure on the manifold M determines a differentiable atlas $\mathcal{A}_{Osc^2M}$ on Osc^2M and therefore the triple $\left(Osc^2M, \pi^2, M\right)$ is a differentiable bundle. We will denote the 2-osculator bundle $\left(Osc^2M, \pi^2, M\right)$ also with $\left(T^2M, \pi^2, M\right)$.

By (1.2), a transformation of local coordinates $\left(x^a, y^{(1)a}, y^{(2)a}\right) \to (\widetilde{x}^a, \widetilde{y}^{(1)a}, \widetilde{y}^{(2)a})$ on the manifold T^2M is given by

$$\begin{cases} \widetilde{x}^a &= \widetilde{x}^a\left(x^1, ..., x^n\right), \det(\dfrac{\partial\widetilde{x}^a}{\partial x^b}) \neq 0, \\ \widetilde{y}^{(1)a} &= \dfrac{\partial\widetilde{x}^a}{\partial x^b}y^{(1)b}, \\ 2\widetilde{y}^{(1)a} &= \dfrac{\partial\widetilde{y}^{(1)a}}{\partial x^b}y^{(1)b} + 2\dfrac{\partial\widetilde{y}^{(1)a}}{\partial y^{(1)b}}y^{(2)b}. \end{cases} \tag{1.3}$$

One can see that T^2M is of dimension 3n.

Moreover, if M is a paracompact manifold, then T^2M is paracompact, too.

Let us present here some notations. A point $u \in T^2M$ whose projection by π^2 is x, i.e. $\pi^2(u) = x$, will be denoted by $\left(x, y^{(1)}, y^{(2)}\right)$, its local coordinates being $\left(x^a, y^{(1)a}, y^{(2)a}\right)$.

The null section $0 : M \to T^2M$ of the projection π^2 is defined by $0 : (x) \in M \to (x, 0, 0) \in T^2M$ we denote by $\widetilde{T^2M} = T^2M \backslash \{0\}$.

The coordinate transformation (1.3) determines the transformation of the natural basis $\left(\dfrac{\partial}{\partial x^a}|_u, \dfrac{\partial}{\partial y^{(1)a}}|_u, \dfrac{\partial}{\partial y^{(2)a}}|_u\right)$, $(a = 1, .., n)$, of the tangent space TT^2M at the point $u \in T^2M$ the following:

$$\begin{aligned} \frac{\partial}{\partial x^a}|_u &= \frac{\partial\widetilde{x}^b}{\partial x^a}\frac{\partial}{\partial\widetilde{x}^b}|_u + \frac{\partial\widetilde{y}^{(1)b}}{\partial x^a}\frac{\partial}{\partial\widetilde{y}^{(1)b}}|_u + \frac{\partial\widetilde{y}^{(2)b}}{\partial x^a}\frac{\partial}{\partial\widetilde{y}^{(2)b}}|_u, \\ \frac{\partial}{\partial y^{(1)a}}|_u &= \frac{\partial\widetilde{y}^{(1)b}}{\partial y^{(1)a}}\frac{\partial}{\partial\widetilde{y}^{(1)b}}|_u + \frac{\partial\widetilde{y}^{(2)b}}{\partial y^{(1)a}}\frac{\partial}{\partial\widetilde{y}^{(2)b}}|_u, \\ \frac{\partial}{\partial y^{(2)a}}|_u &= \frac{\partial\widetilde{y}^{(2)b}}{\partial y^{(2)a}}\frac{\partial}{\partial\widetilde{y}^{(2)b}}|_u. \end{aligned} \tag{1.4}$$

By means of (1.3) we obtain

$$\frac{\partial \widetilde{x}^a}{\partial x^b} = \frac{\partial \widetilde{y}^{(1)a}}{\partial y^{(1)b}} = \frac{\partial \widetilde{y}^{(2)a}}{\partial y^{(2)b}}, \quad \frac{\partial \widetilde{y}^{(1)a}}{\partial x^b} = \frac{\partial y^{(2)a}}{\partial y^{(1)b}}. \tag{1.4'}$$

Looking at the formula (1.4) we remark the existence of some natural object fields on T^2M.

First of all, the tangent space V_{1u} to the fibre $(\pi^2)^{-1}(x)$ in the point $u \in T^2M$ is locally spanned by $\left\{ \frac{\partial}{\partial y^{(1)1}}, ..., \frac{\partial}{\partial y^{(1)n}}, \frac{\partial}{\partial y^{(2)1}}, ..., \frac{\partial}{\partial y^{(2)n}} \right\}$. Therefore, the mapping $V_1 : u \in T^2M \to V_{1u} \subset T_uT^2M$ provides a regular distribution which is generated by $\left\{ \frac{\partial}{\partial y^{(1)a}}, \frac{\partial}{\partial y^{(2)a}} \right\}, (a = 1, .., n)$. Consequently, V_1 is an integrable distribution on T^2M of local dimension 2n, called the vertical distribution on T^2M. Similarly, the tangent space V_{2u} to the fibre $\left(\pi_1^2\right)^{-1}(x, y^1)$ in the point $u \in T^2M$, where $\pi_1^2 : (x, y^{(1)}, y^{(2)}) \in T^2M \to (x, y^{(1)}) \in Osc^1M = TM$, is locally spanned by $\left\{ \frac{\partial}{\partial y^{(2)1}}, ..., \frac{\partial}{\partial y^{(2)n}} \right\}$. Therefore, the mapping $V_2 : u \in T^2M \to V_{2u} \subset T_uT^2M$ provides a new regular vertical distribution which is generated by $\left\{ \frac{\partial}{\partial y^{(2)a}} \right\}, (a = 1, ..., n)$. So, V_2 is an integrable distribution on T^2M of local dimension n and it is a subdistribution of V_1.

Therefore, in every point $u \in T^2M$, we have the vector space $V_2(u), V_1(u)$, T_uT^2M of dimensions n, 2n, 3n, respectively and satisfying the inclusions

$$V_2(u) \subset V_1(u) \subset T_uT^2M, \quad \forall u \in T^2M.$$

We denote

$$V(u) = V_1(u) \oplus V_2(u), \quad \forall u \in T^2M. \tag{1.5}$$

and we call V the **vertical distribution** on T^2M.

Taking in account (1.3), (1.4), it follow that

$$\overset{1}{\mathbb{C}} \ = \ y^{(1)a}\frac{\partial}{\partial y^{(2)a}}, \tag{1.6}$$

$$\overset{2}{\mathbb{C}} \ = \ y^{(1)a}\frac{\partial}{\partial y^{(1)a}} + 2y^{(2)a}\frac{\partial}{\partial y^{(2)a}}, \tag{1.6'}$$

are two vertical vector fields, global define on T^2M and linear independent on $\widetilde{T^2M}$. $\overset{1}{\mathbb{C}}$ belongs to the distribution V_2 and $\overset{2}{\mathbb{C}}$ belongs to the distribution V_1. They are called the **Liouville vector fields**. The existence of the Liouville vector field is very important in the study of geometry of the manifold T^2M.

Let us consider the $\mathcal{F}(\mathrm{T}^2M)$ - linear mapping J:$\mathcal{X}(T^2M) \to \mathcal{X}(T^2M)$,

$$J\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial y^{(1)a}}, \quad J\left(\frac{\partial}{\partial y^{(1)a}}\right) = \frac{\partial}{\partial y^{(2)a}}, \quad J\left(\frac{\partial}{\partial y^{(2)a}}\right) = 0. \tag{1.7}$$

Theorem 1.1. *The following properties hold:*
1°. J is globally defined on T^2M.
2°. $J \circ J \circ J = 0, \quad rank\, \|J\| = 2n$.
3°. Im $J = V_1$, Ker $J = V_2$, $J(V_1) = V_2$.
4°. J is an integrable structure on T^2M.
5°. $J\overset{2}{\mathbb{C}} = \overset{1}{\mathbb{C}}$, $J\overset{1}{\mathbb{C}} = 0$.

The proof can be found in [89], [90].

We say that J is the 2-**tangent structure** on T^2M.
The previous geometrical notions are useful in the next sections of this work.

1.2 Homogeneity

The notion of homogeneity, (see, De Léon M. and Vasquez E., [76]), of function $f \in \mathcal{F}(T^2M)$ with respect to the variables $y^{(1)a}, y^{(2)a}$ is necessary in our considerations because some fundamental object fields on T^2M have the homogeneous components.

In the osculator manifold $Osc^2M = T^2M$, a point $[\rho]_{x_0}$ has a geometrical meaning, i.e. changing of parametrization of the curve $\rho : I \to M$ does not change the space $[\rho]_{x_0}$. Taking into account the affine transformations of parameter

$$t = a\bar{t} + b, \bar{t} \in I, a \in R_+^* \tag{*}$$

we obtain the transformation of coordinate of $[\rho]_{x_0}$ in the form

$$\overline{x}^c = x^c, \overline{y}^{(1)c} = ay^{(1)c}, \overline{y}^{(2)c} = a^2y^{(2)c}, (c = 1, 2, ..., n). \tag{**}$$

Therefore, the transformation of coordinates (1.3) on the manifold T^2M preserve the transformations (**).

Let $H = \left\{h_t : R \to R, t \in R_+^*\right\}$ be the group of homotheties of real numbers field R. H acts as an uniparameter group of transformations on T^2M as follows

$$H \times T^2M \to T^2M, \{(h_t, u) \to h_t(u)\},$$

where

$$h_t(x, y^{(1)}, y^{(2)}) = \left(x, ty^{(1)}, t^2y^{(2)}\right).$$

Consequently, H acts as a group of transformations on T^2M, with preserving of the fibres.

The transformation group H of the homotethies is invariant under the transformation of local coordinates on T^2M.

The orbit of a point $u_0 = \left(x_0, y_0^{(1)}, y_0^{(2)}\right) \in T^2M$ is given by

$$\begin{aligned} x^a &= x_0^a \\ y^{(1)a} &= t y_0^{(1)a} \\ y^{(2)a} &= t^2 y_0^{(2)a}, t \in R_+^*. \end{aligned}$$

The tangent vector field to orbit in the point $u_0 = h_1(u_0)$ is given by

$$\overset{2}{\mathbb{C}}_{u_0} = y_0^{(1)a} \left(\frac{\partial}{\partial y^{(1)a}}\right)_{u_0} + 2y_0^{(2)a} \left(\frac{\partial}{\partial y^{(2)a}}\right)_{u_0}.$$

This is the Liouville vector field $\overset{2}{\mathbb{C}}$, (1.6') in the point u_0. For this vector field $(h_t)_{t\in R_+^*}$ is his uniparameter group. Let us consider the vector field $\overset{1}{\mathbb{C}} = J\overset{2}{\mathbb{C}}$, where $\overset{1}{\mathbb{C}}$ is the Liouville vector field given by (1.6).

Now, we can formulate:

Definition 2.1. *A function $f : T^2M \to R$ differentiable on $\widetilde{T^2M}$ and continuos on the null section $0 : M \to T^2M$ is called homogeneous of degree $r, (r \in Z)$, on the fibres on T^2M, (briefly r-homogeneous) if*

$$f \circ h_t = t^r f, \forall t \in R_+^*. \tag{2.1}$$

The following theorem of Euler type holds, [76]:

Theorem 2.1. *A function $f \in F(T^2M)$ differentiable on $\widetilde{T^2M}$ and continuous on the null sections is homogeneous of degree r on the fibres of T^2M if and only if we have*

$$\mathcal{L}_{\overset{2}{\mathbb{C}}} f = rf, \tag{2.2}$$

$L_{\overset{2}{\mathbb{C}}}$ being the Lie derivative with respect to the Liouville vector field $\overset{2}{\mathbb{C}}$.

The equality (2.2) is equivalent to

$$y^{(1)a}\frac{\partial f}{\partial y^{(1)a}} + 2y^{(2)a}\frac{\partial f}{\partial y^{(2)a}} = rf \tag{2.2'}$$

The following properties hold:

1. f_1, f_2 r-homogeneous $\Longrightarrow \lambda_1 f_1 + \lambda_2 f_2, \lambda_1, \lambda_2 \in R$ is r-homogeneous,

2. f_1 r-homogeneous, f_2 s-homogeneous $\Longrightarrow f_1 \cdot f_2$ is $(r+s)$-homogeneous,

3. f_1 r-homogeneous, $f_2 \neq 0$ s-homogeneous $\Longrightarrow \dfrac{f_1}{f_2}$ is $(r-s)$-homogeneous

By extension we can define the homogeneity of vector fields and 1-forms on T^2M.

Definition 2.2. *A vector field $X \in X(T^2M)$ is r-homogeneous if*

$$X \circ h_t = t^{r-1} h_t^* \circ X, \forall t \in R_+^*.$$

It follows:

Theorem 2.2. *A vector field $X \in X(\widetilde{T^2M})$ is r-homogeneous if and only if we have*

$$\mathcal{L}_{\overset{2}{\mathbb{C}}} X = (r-1)X. \tag{2.3}$$

Of course, $\mathcal{L}_{\overset{2}{\mathbb{C}}} X = [\overset{2}{\mathbb{C}}, X]$ is the Lie derivative of X with respect to $\overset{2}{\mathbb{C}}$.

Consequently, we can prove:

Corollary 2.1.

1. *The vector fields $\dfrac{\partial}{\partial x^a}, \dfrac{\partial}{\partial y^{(1)a}}, \dfrac{\partial}{\partial y^{(2)a}}$ are 1,0 and -1-homogeneous, respectively.*
2. *If $f \in F(\widetilde{T^2M})$ is s-homogeneous and $X \in X(\widetilde{T^2M})$ is r-homogeneous then fX is $(r+s)$-homogeneous.*
3. *The Liouville vector field $\overset{1}{\mathbb{C}}$ is 0-homogeneous.*
4. *The Liouville vector field $\overset{2}{\mathbb{C}}$ is 1-homogeneous.*

Corollary 2.2.

1°. A vector field on $\widetilde{T^2M}$:

$$X = X^{(0)a}\frac{\partial}{\partial x^a} + X^{(1)a}\frac{\partial}{\partial y^{(1)a}} + X^{(2)a}\frac{\partial}{\partial y^{(2)a}}$$

is r-homogeneous if and only if $X^{(0)a}$ are functions $(r-1)$-homogeneous, $X^{(1)a}$ are functions r-homogeneous and $X^{(2)a}$ are functions $(r+1)$-homogeneous.

2°. If $X \in X(\widetilde{T^2M})$ is r-homogeneous and $f \in F(\widetilde{T^2M})$ is s-homogeneous, then $Xf \in F(\widetilde{T^2M})$ is a $(r+s-1)$-homogeneous function

3°. If $f \in F(\widetilde{T^2M})$ is an arbitrary s-homogeneous function, then $\dfrac{\partial f}{\partial y^{(2)a}}$ is a $(s-2)$-homogeneous function and $\dfrac{\partial^2 f}{\partial y^{(2)a}\partial y^{(2)b}}$ is $(s-4)$-homogeneous function.

Proposition 2.1. *If $X_1 \in X(\widetilde{T^2M})$ and $X_2 \in X(\widetilde{T^2M})$ are vector fields r_1- and r_2- homogeneous, respectively then the bracket $[X_1, X_2]$ are $(r_1 + r_2 - 1)$-homogeneous vector field.*

Proof. Indeed, we have

$$\begin{aligned}\mathcal{L}_{\overset{2}{\mathbb{C}}}[X_1, X_2] &= [\overset{2}{\mathbb{C}}, [X_1, X_2]] = [X_1, [\overset{2}{\mathbb{C}}, X_2]] - [X_2, [\overset{2}{\mathbb{C}}, X_1]] \\ &= [X_1, (r_2 - 1) X_2] - [X_2, (r_1 - 1) X_1] = (r_1 + r_2 - 2)[X_1, X_2].\end{aligned}$$

q.e.d.

In the case of q-form we can give:

Definition 2.3. *A q-form $\omega \in \Lambda^q\left(\widetilde{T^2M}\right)$ is s-homogeneous if*

$$\omega \circ h_t^* = t^s \omega, \forall t \in R_+^*.$$

It follows:

Theorem 2.3. *A q-form $\omega \in \Lambda^q\left(\widetilde{T^2M}\right)$ is s-homogeneous if and only if*

$$\mathcal{L}_{\overset{2}{\mathbb{C}}}\omega = s\omega. \tag{2.4}$$

Corollary 2.3.

1°. If $\omega \in \Lambda^q\left(\widetilde{T^2M}\right)$ is s-homogeneous and $\omega' \in \Lambda^{q'}\left(\widetilde{T^2M}\right)$ is s'-homogeneous $\Longrightarrow \omega \wedge \omega'$ is $(s + s')$-homogeneous

2°. If $\omega \in \Lambda^q\left(\widetilde{T^2M}\right)$ is s-homogeneous and $\underset{1}{X}, ..., \underset{q}{X}$ are each r-homogeneous $\Longrightarrow \omega\left(\underset{1}{X}, ..., \underset{q}{X}\right)$ is $(r + s - 1)$-homogeneous.

3°. $dx^a\ (a = 1, ..., n)$ are 0-homogeneous 1-forms.

The applications of those properties in the problems of homogeneous lifts of the Riemannian, Finslerian and Lagrangian structures on T^2M and in the geometry of Finsler space of order two are numberless, [17], [18].

1.3 Second order semispray

In applications we shall use the operator

$$C = y^{(1)a}\frac{\partial}{\partial x^a} + 2y^{(2)a}\frac{\partial}{\partial y^{(1)a}}. \tag{3.1}$$

This operator is not a vector field on T^2M. By a direct calculation one checks the following:

Lemma 3.1.

1. *Under a coordinate transformation (1.3) on T^2M, C changes as follows*

$$C = \widetilde{C} + \left(y^{(1)b} \frac{\partial \widetilde{y}^{(2)c}}{\partial x^b} + 2y^{(2)b} \frac{\partial \widetilde{y}^{(2)c}}{\partial y^{(1)b}} \right) \frac{\partial}{\partial \widetilde{y}^{(2)c}}. \tag{3.1'}$$

2. *For any function $f \in F(T^2M)$ having the property* $\dfrac{\partial f}{\partial y^{(2)a}} = 0$, *with respect to (1.3), we have*

$$Cf = \widetilde{C}f. \tag{3.1"}$$

Now we can introduce the following definition:

Definition 3.1. *A second order semispray S on $\widetilde{T^2M}$ (briefly, a **2-semispray**) is a vector field $S \in x(\widetilde{T^2M})$ with the property:*

$$JS = \overset{2}{\mathbb{C}}. \tag{3.2}$$

If S is 2-homogeneous, then S will be called a **2-spray.**

Not always there exists a vector field S with the property (3.2). Therefore, the notion of local 2-semispray must be formulated taking $S \in \mathcal{X}(\widetilde{U})$, $\widetilde{U}$ being an open set in the manifold T^2M.

Theorem 3.1.

1°. A 2-semispray can be uniquely written in the form

$$S = y^{(1)a} \frac{\partial}{\partial x^a} + 2y^{(2)a} \frac{\partial}{\partial y^{(1)a}} - 3G^a \left(x, y^{(1)}, y^{(2)}\right) \frac{\partial}{\partial y^{(2)a}}. \tag{3.3}$$

2°. The set of functions $G^a\left(x, y^{(1)}, y^{(2)}\right)$, $(a = 1, ..., n)$, are changed with respect to (1.3) as follows:

$$3\widetilde{G}^a = 3\frac{\partial \widetilde{x}^a}{\partial x^b} G^b - \left(y^{(1)b} \frac{\partial \widetilde{y}^{(2)a}}{\partial x^b} + 2y^{(2)b} \frac{\partial \widetilde{y}^{(2)a}}{\partial y^{(1)b}} \right). \tag{3.4}$$

3°. If the set of functions G^a are a apriori given on every domain of a local chart in $\widetilde{T^2M}$, so that (3.4) holds, then S from (3.3) is a 2-semispray.

Proof.

1°. If a vector field

$$S = f_0^a \left(x, y^{(1)}, y^{(2)}\right) \frac{\partial}{\partial x^a} + f_1^a \left(x, y^{(1)}, y^{(2)}\right) \frac{\partial}{\partial y^{(1)a}} + f_2^a \left(x, y^{(1)}, y^{(2)}\right) \frac{\partial}{\partial y^{(2)a}}$$

is a 2-semispray S, then $JS = \overset{2}{\mathbb{C}}$ implies $f_0^a = y^{(1)a}, f_1^a = 2y^{(2)a}$ and $f_2^a(x,\, y^{(1)},\, y^{(2)}) = -3G^a(x,\, y^{(1)},\, y^{(2)})$. So that G^a are uniquely determined and (3.2) holds.

2°. The formula (3.4) follows from (1.3), (1.4) and the fact that S is a vector field on $\widetilde{T^2M}$, i.e.

$$S = C - 3G^a\left(x, y^{(1)}, y^{(2)}\right)\frac{\partial}{\partial y^{(2)a}} = \widetilde{C} - 3\widetilde{G}^a\left(\widetilde{x}, \widetilde{y}^{(1)}, \widetilde{y}^{(2)}\right)\frac{\partial}{\partial \widetilde{y}^{(2)a}} = \widetilde{S}.$$

3°. Using the rule of transformation (3.4) of the set of functions G^a it follows that S is a vector field which satisfies $JS = \overset{2}{\mathbb{C}}$.

q.e.d.

From the previous theorem, it results that S is uniquely determined by $G^a\left(x, y^{(1)}, y^{(2)}\right)$ and conversely. Because of this reason, G^a are called **the coefficients of the 2-semispray.**

Theorem 3.2. *A 2-semispray S on $\widetilde{T^2M}$ is a 2-spray if and only if its coefficients G^a are 3-homogeneous functions with respect to $y^{(2)a}$.*

Proof. By means of 1° and 2°, Corollary 2.1 it follows that $y^{(1)a}\dfrac{\partial}{\partial x^a}$ is 2-homogeneous, $y^{(2)a}\dfrac{\partial}{\partial y^{(1)a}}$ is 2-homogeneous, $\dfrac{\partial}{\partial y^{(2)a}}$ is (-1)-homogeneous vector fields. Hence, S is 2-homogeneous if and only if G^a are 3-homogeneous functions with respect to $y^{(2)a}$.

q.e.d.

The integral curves of the 2-semispray S from (3.3) are given by

$$\frac{dx^a}{dt} = y^{(1)a}, \frac{dy^{(1)a}}{dt} = 2y^{(2)a}, \frac{dy^{(2)a}}{dt} = -3G^a\left(x, y^{(1)}, y^{(2)}\right). \tag{3.5}$$

It follows that, on M, these curves are expressed as solutions of the following differential equations

$$\frac{d^3x^a}{dt^3} + 3!G^a\left(x, \frac{dx}{dt}, \frac{1}{2}\frac{d^2x}{dt^2}\right) = 0. \tag{3.6}$$

The curves $c : t \in I \to (x^a(t)) \subset U \subset M$, solutions of (3.6), are called the paths of the 2-semispray S. The differential equation (3.6) has geometrical meaning. Conversely, if the differential equation (3.6) is given on a domain of a local chart U of the manifold M, and this equation is preserved by the transformations of local coordinates on M, then coefficients $G^a(x,\, y^{(1)},\, y^{(2)})$, $\left(y^{(1)a} = \dfrac{dx^a}{dt}, y^{(2)a} = \frac{1}{2}\dfrac{d^2x^a}{dt^2}\right)$, obey the transformations (3.4). Hence $G^a(x,\, y^{(1)},\, y^{(2)})$ are the coefficients of a 2-semispray. Consequently, we obtain:

Theorem 3.3. *A 2-semispray S on $\widetilde{T^2M}$, with the coefficients $G^a(x,\, y^{(1)},\, y^{(2)})$ is characterized by a system of differential equations (3.6), which has a geometrical meaning.*

Using the previous theorem we prove:

Theorem 3.4. *If the base manifold M is paracompact, then on T^2M there exist 2-semisprays.*

Proof. M being paracompact, there exist a Riemannian metric g on M with local coefficients $\gamma_{ab}(x)$. Consider $\gamma^a_{bc}(x)$ the Christoffel symbols of g. It is easy to prove that

$$z^{(2)a} = y^{(2)a} + \frac{1}{2}\gamma^a_{bc}(x)\, y^{(1)b} y^{(1)c} \tag{3.7}$$

is a distinguished vector field, i.e., with respect to (1.3), we have $\widetilde{z}^{(2)a} = \dfrac{\partial \widetilde{x}^a}{\partial x^b} z^{(2)b}$. It follows, that the function

$$L\left(x, y^{(1)}, y^{(2)}\right) = \gamma_{ab}(x)\, z^{(2)a} z^{(2)b} \tag{3.8}$$

does not depend on the transformations of coordinates (1.3). Then the set of functions

$$\begin{aligned} G^a\left(x, y^{(1)}, y^{(2)}\right) = \tfrac{1}{3!}\gamma^{ab}(x) \Bigg\{ y^{(1)c} \frac{\partial}{\partial x^c}\left(\frac{\partial L}{\partial y^{(2)b}}\right) + \\ + 2y^{(2)c} \frac{\partial}{\partial y^{(1)c}}\left(\frac{\partial L}{\partial y^{(2)b}}\right) - \frac{\partial L}{\partial y^{(1)b}} \Bigg\} \end{aligned} \tag{3.9}$$

is transformed, by means of a transformation (1.3), like in formula (3.4). Theorem 3.1 may be applied. It follow that the set of functions G^a are the coefficients of a 2-semispray S.

q.e.d.

Finally, in this section, we consider the function determined by a 2-semispray S:

$$\underset{1}{N}{}^a_b = \frac{\partial G^a}{\partial y^{(2)b}}. \tag{3.10}$$

Using the rule of transformation (3.4) of the coefficients G^a we can prove, without difficulties:

Theorem 3.5. *If $G^a\left(x, y^{(1)}, y^{(2)}\right)$ are the coefficients of a 2-semispray S then the set of functions $N^a_b\left(x, y^{(1)}, y^{(2)}\right)$ from (3.10) has the following rule of transformation with respect to (1.3):*

$$\underset{1}{\widetilde{N}}{}^a_c \frac{\partial \widetilde{x}^c}{\partial x^b} = \underset{1}{N}{}^c_b \frac{\partial \widetilde{x}^a}{\partial x^c} - \frac{\partial \widetilde{y}^{(1)a}}{\partial x^b}. \tag{3.11}$$

The system of functions $\underset{1}{N}{}^a_b\left(x, y^{(1)}, y^{(2)}\right)$ is important to define the notion of nonlinear connection on T^2M.

1.4 Nonlinear connection

We extend the classical definition of the nonlinear connection, [134] , on the total space bundle $\left(T^2M, \pi^2, M\right)$.

Definition 4.1. *A nonlinear connection on the manifold T^2M is a regular distribution N on T^2M supplementary to the vertical distribution V, i.e.*

$$T_uT^2M = N(u) \oplus V(u), \quad \forall u \in T^2M. \tag{4.1}$$

Taking into account Proposition 1.1 it follow that the distribution N has the property

$$T_uT^2M = N(u) \oplus V_1(u) \oplus V_2(u) \tag{4.1'}$$

Generally, we consider a nonlinear connection on T^2M from the point of view of Definition 4.1. We denote it by N and call it a **horizontal distribution.** According to (4.1) we deduce that the local dimension of N is n=dim M.

Proposition 4.1. *If the manifold M is paracompact, then there exists non-linear connections on T^2M.*

Indeed, the manifold M being paracompact, it result that T^2M is a paracompact manifold. There exists at least a Riemannian metric G on T^2M. Considering N as the orthogonal distribution to the vertical distribution V with respect to G, the relation (4.1) is true. Thus, N is a nonlinear connection on T^2M.

q.e.d.

Let h and v be the horizontal and vertical projectors determined by the distributions N and V. We have

$$h + v = I, \quad h^2 = h, \quad v^2 = v, \quad hv = vh = 0. \tag{4.2}$$

For simplicity, we denote

$$X^H = hX, X^V = vX, \quad \forall X \in \mathcal{X}(T^2M). \tag{4.3}$$

Therefore, we have

$$X = X^H + X^V, \quad \forall X \in \mathcal{X}(T^2M). \tag{4.4}$$

We call an **horizontal lift** an $\mathcal{F}$(M)-linear map $l_h : \mathcal{X}(M) \to \mathcal{X}(T^2M)$ with the properties

$$v \circ l_h = 0, \quad d\pi^2 \circ l_h = Id,$$

where dπ^2 is the differential mapping of the projection $\pi^2, d\pi^2 : TT^2M \to TM$. Consequently, locally, for any vector field X$\in \mathcal{X}(M)$ it follows that l_hX is a uniquely determined vector field in the horizontal distribution N.

Then, we obtain a unique local basis $\left\{\frac{\delta}{\delta x^a}\right\}$, adapted to the horizontal distribution N which is projected by dπ^2 on to the natural basis $\left\{\frac{\partial}{\partial x^a}\right\}$. It is given by

$$\frac{\delta}{\delta x^a} = l_h\left(\frac{\partial}{\partial x^a}\right). \tag{4.6}$$

This can uniquely written in the form:

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - \underset{1}{N}{}^b{}_a \frac{\partial}{\partial y^{(1)b}} - \underset{2}{N}{}^b{}_a \frac{\partial}{\partial y^{(2)b}}. \tag{4.7}$$

The system of differential functions $(\underset{1}{N}{}^b{}_a(x, y^{(1)}, y^{(2)}),\ \underset{2}{N}{}^b{}_a(x, y^{(1)}, y^{(2)}))$, $a, b \in \{1, 2, ..., n\}$, defined on the domain of local chart on T^2M, are called **the coefficients** of the nonlinear connection N and $\left\{\frac{\delta}{\delta x^a}\right\}$ is called **the adapted basis** to N.

We can see that, with respect to (1.3), we have

$$\frac{\delta}{\delta x^a} = \frac{\partial \widetilde{x}^b}{\partial x^a} \frac{\delta}{\delta \widetilde{x}^b}. \tag{4.8}$$

It is not difficult to prove the following property, [92]:

Theorem 4.1. *With respect to a changing of local coordinates (1.3) on T^2M the coefficients $\left(\underset{1}{N}{}^b{}_a, \underset{2}{N}{}^b{}_a\right)$ of the nonlinear connection N on T^2M obey the rule of transformation*

$$\begin{aligned} \underset{1}{\widetilde{N}}{}^a{}_f \frac{\partial \widetilde{x}^f}{\partial x^b} &= \frac{\partial \widetilde{x}^a}{\partial x^f} \underset{1}{N}{}^f{}_b - \frac{\partial \widetilde{y}^{(1)a}}{\partial x^b}, \\ \underset{2}{\widetilde{N}}{}^a{}_f \frac{\partial \widetilde{x}^f}{\partial x^b} &= \frac{\partial \widetilde{x}^a}{\partial x^f} \underset{2}{N}{}^f{}_b + \frac{\partial \widetilde{y}^{(1)a}}{\partial x^f} \underset{1}{N}{}^f{}_b - \frac{\partial \widetilde{y}^{(2)a}}{\partial x^b}. \end{aligned} \tag{4.9}$$

Conversely:

Theorem 4.2. *If the systems of functions $\left(\underset{1}{N}{}^a{}_b, \underset{2}{N}{}^a{}_b\right)$ are given on every domain of local chart of the manifold T^2M such that the equations (4.9) hold, then $\left(\underset{1}{N}{}^a{}_b, \underset{2}{N}{}^a{}_b\right)$ are the coefficients of a nonlinear connection on T^2M.*

Let N be a nonlinear connection on T^2M. The 2-tangent structure J, defined by (1.7), applies the horizontal distribution N in a vertical subdistribution N_1 from V_1 of local dimension n, supplementary to the subdistribution V_2. Setting $\mathrm{N}_0 = N, J(N_0) = N_1$, we obtain from Theorem 1.1:

Theorem 4.3. *The following direct decomposition of linear spaces holds*

$$T_uT^2M = N_0(u) \oplus N_1(u) \oplus V_2(u), \forall u \in T^2M. \tag{4.10}$$

N_1 is called the **J-vertical distribution**.

Theorem 4.4. *The adapted basis to the distribution N_0, N_1, V_2 are given, respectively, by*

$$\begin{aligned} \frac{\delta}{\delta x^a} &= \frac{\partial}{\partial x^a} - \underset{1}{N}{}^b{}_a \frac{\partial}{\partial y^{(1)b}} - \underset{2}{N}{}^b{}_a \frac{\partial}{\partial y^{(2)b}}, \\ \frac{\delta}{\delta y^{(1)a}} &= \frac{\partial}{\partial y^{(1)a}} - \underset{1}{N}{}^b{}_a \frac{\partial}{\partial y^{(2)b}}, \\ \frac{\delta}{\delta y^{(2)a}} &= \frac{\partial}{\partial y^{(2)a}}. \end{aligned} \tag{4.11}$$

Consequently,

$$\left\{\frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^{(1)a}}, \frac{\delta}{\delta y^{(2)a}}\right\} = \left\{\frac{\delta}{\delta y^{(\alpha)a}}\right\}, \quad x^a = y^{(0)a}, (\alpha = 0,1,2)\,, \tag{4.12}$$

is a local basis adapted to the direct decomposition (4.10) and we have

$$\frac{\delta}{\delta y^{(\beta)a}} = \frac{\partial \widetilde{x}^b}{\partial x^a}\frac{\delta}{\delta \widetilde{y}^{(\beta)b}}, (\beta = 1,2). \tag{4.13}$$

Indeed, (4.8) is transformed by J in to (4.13).
As usually, let we denote

$$\partial_a = \frac{\partial}{\partial x^a}, \dot{\partial}_{1a} = \frac{\partial}{\partial y^{(1)a}}, \dot{\partial}_{2a} = \frac{\partial}{\partial y^{(2)a}},$$

and from now on we denote the basis (4.12) by

$$\left(\delta_a, \delta_{1a}, \dot{\partial}_{2a}\right). \tag{4.12'}$$

1.5 The dual coefficients of a nonlinear connection. Determination of a nonlinear connection from a 2-semispray

The dual basis (or adapted cobasis) of the adapted basis (4.12) will be denoted by

$$\left(dx^a, \delta y^{(1)a}, \delta y^{(2)a}\right), (a = 1,...,n)\,. \tag{5.1}$$

The scalar product of the covector fields (5.1) and vector fields (4.12) are expressed as follows:

$$\begin{array}{llllll} \delta_b\rfloor dx^a &= \delta^a_b, & \delta_b\rfloor \delta y^{(1)a} &= 0, & \delta_b\rfloor \delta y^{(2)a} &= 0, \\ \delta_{1b}\rfloor dx^a &= 0, & \delta_{1b}\rfloor \delta y^{(1)a} &= \delta^a_b, & \delta_{1b}\rfloor \delta y^{(2)a} &= 0, \\ \delta_{2b}\rfloor dx^a &= 0, & \delta_{2b}\rfloor \delta y^{(1)a} &= 0, & \delta_{2b}\rfloor \delta y^{(2)a} &= \delta^a_b, \end{array} \tag{5.2}$$

$$\left(\delta_{2b} = \dot{\partial}_{2b}\right).$$

By a straightforward calculus we obtain:
Theorem 5.1.
1°. The dual basis (5.1) of the adapted basis (4.12) is given by

$$\begin{aligned} \delta x^a &= dx^b, \\ \delta y^{(1)a} &= dy^{(1)a} + \underset{1}{M}{}^a{}_b dx^b, \\ \delta y^{(2)a} &= dy^{(2)a} + \underset{1}{M}{}^a{}_b dy^{(1)b} + \underset{2}{M}{}^a{}_b dx^b, \end{aligned} \tag{5.3}$$

where

$$\underset{1}{M}{}^a{}_b = \underset{1}{N}{}^a{}_b, \underset{2}{M}{}^a{}_b = \underset{2}{N}{}^a{}_b + \underset{1}{N}{}^a{}_f \underset{1}{N}{}^f{}_b. \tag{5.4}$$

2°. *Conversely, if the adapted cobasis (5.1) is given in the form (5.3), then the adapted basis (4.12) is expressed in the form (4.11), where*

$$\underset{1}{N}{}^a{}_b = \underset{1}{M}{}^a{}_b, \underset{2}{N}{}^a{}_b = \underset{2}{M}{}^a{}_b - \underset{1}{M}{}^a{}_f \underset{1}{M}{}^f{}_b. \tag{5.4'}$$

These new coefficients $\underset{1}{M}{}^a{}_b, \underset{2}{M}{}^a{}_b$ will be called the **dual coefficients** of the nonlinear connection N.

With respect to (1.3) the covector fields of the adapted cobasis (5.1) transform as follows

$$\delta\widetilde{y}^{(\alpha)a} = \frac{\partial\widetilde{x}^a}{\partial x^b}\delta y^{(\alpha)b} \qquad (\alpha = 0, 1, 2; \delta y^{(0)} = dx). \tag{5.5}$$

By a straightforward calculus the rule of transformations of a dual coefficients $\underset{1}{M}{}^a{}_b, \underset{2}{M}{}^a{}_b$ with respect to (1.3), it is not difficult to obtain, [92]:

Theorem 5.2.

1°. *A transformation of coordinates (1.3) on the differentiable manifold T^2M implies the following transformation of the dual coefficients*

$$\begin{aligned} \frac{\partial\widetilde{x}^a}{\partial x^c}\underset{1}{M}{}^c{}_b &= \underset{1}{\widetilde{M}}{}^a{}_c\frac{\partial\widetilde{x}^c}{\partial x^b} + \frac{\partial\widetilde{y}^{(1)a}}{\partial x^b} \\ \frac{\partial\widetilde{x}^a}{\partial x^c}\underset{2}{M}{}^c{}_b &= \underset{2}{\widetilde{M}}{}^a{}_c\frac{\partial\widetilde{x}^c}{\partial x^b} + \underset{1}{\widetilde{M}}{}^a{}_c\frac{\partial\widetilde{y}^{(1)c}}{\partial y^b} + \frac{\partial\widetilde{y}^{(2)a}}{\partial x^b}. \end{aligned} \tag{5.6}$$

2°. *If on each domain of local chart on T^2M a set of function $\left(\underset{1}{M}{}^a{}_b, \underset{2}{M}{}^a{}_b\right)$ is given, such that, with respect to (1.3), the equations (5.6) hold, then there exists on T^2M an unique nonlinear connection N which has as dual coefficients just the given set of functions.*

One of the important problems concerning the notion of nonlinear connection consists in its determinations from a 2-semispray.

Let us consider S a 2-semispray with the coefficients $G^a\left(x, y^{(1)}, y^{(2)}\right)$:

$$S = y^{(1)a}\frac{\partial}{\partial x^a} + 2y^{(2)a}\frac{\partial}{\partial y^{(1)a}} - 3G^a\left(x, y^{(1)}, y^{(2)}\right)\frac{\partial}{\partial y^{(2)a}}. \tag{5.7}$$

We have, [91]:

Theorem 5.3. *The set of functions*

$$\underset{1}{M}{}^{a}{}_{b} = \frac{\partial G^{a}}{\partial y^{(2)b}}, \underset{2}{M}{}^{a}{}_{b} = \frac{1}{2}\left(S\underset{1}{M}{}^{a}{}_{b} + \underset{1}{M}{}^{a}{}_{c}\underset{1}{M}{}^{c}{}_{b}\right) \tag{5.8}$$

gives the dual coefficients of a nonlinear connection N, determined by the 2-semispray S only, with the coefficients $G^{a}\left(x, y^{(1)}, y^{(2)}\right)$.

By a straightforward calculus we obtain:
Corollary 5.1. ([33]) *The following functions*

$$\underset{1}{M}{}^{a}{}_{b} = \frac{\partial G^{a}}{\partial y^{(2)b}}, \underset{2}{M}{}^{a}{}_{b} = \frac{\partial G^{a}}{\partial y^{(1)b}} \tag{5.8'}$$

are geometrical object fields on $T^{2}M$, having the rules of transformations (5.6) with respect to the changing of local coordinates (1.3).

These results are very important for the construction of the canonical nonlinear connections in the various geometries of second order.

1.6 Distinguished vector and covector fields

Let N be a nonlinear connection. Then, it given rise to the object decomposition (4.10). Let h, v_1, v_2 be the projectors defined by the distributions N_0, N_1, V_2. They have the following properties:

$$\begin{array}{l} h + v_1 + v_2 = I, h^2 = h, v_1^2 = v_1, v_2^2 = v_2, \\ h \circ v_1 = v_1 \circ h = 0, h \circ v_2 = v_2 \circ h = 0, v_1 \circ v_2 = v_2 \circ v_1 = 0. \end{array} \tag{6.1}$$

If X$\in \mathcal{X}(\widetilde{T^{2}M})$ we denote:

$$X^{H} = hX, X^{V_1} = v_1 X, X^{V_2} = v_2 X. \tag{6.2}$$

Therefore we have the unique decomposition:

$$X = X^{H} + X^{V_1} + X^{V_2}. \tag{6.3}$$

Each of the components X^{H}, X^{V_1}, X^{V_2} is called a **d-vector** field on $T^{2}M$.
In the adapted basis (4.12) we get

$$X^{H} = X^{(0)a}\delta_{a}, X^{V_1} = X^{(1)a}\delta_{1a}, X^{V_2} = X^{(2)a}\dot{\partial}_{2a}. \tag{6.3'}$$

By means of (4.8) we have

$$\tilde{X}^{(0)a} = \frac{\partial \tilde{x}^{a}}{\partial x^{b}} X^{(0)b}, \tilde{X}^{(1)a} = \frac{\partial \tilde{x}^{a}}{\partial x^{b}} X^{(1)b}, \tilde{X}^{(2)a} = \frac{\partial \tilde{x}^{a}}{\partial x^{b}} X^{(2)b}. \tag{6.4}$$

But, these are the classical rule of the transformations of the local coordinates of vector fields on the base manifold M. Therefore, $X^{(o)a}, X^{(1)a}, X^{(2)a}$ are called **d-vector** fields.

For instance, the Liouville vector fields $\overset{1}{\mathbb{C}}$ and $\overset{2}{\mathbb{C}}$ have the properties

$$\begin{array}{llll} \overset{1}{\mathbb{C}}{}^H = 0, & \overset{1}{\mathbb{C}}{}^{V_1} = 0, & \overset{1}{\mathbb{C}} & = \overset{1}{\mathbb{C}}{}^{V_2} = y^{(1)a}\dot{\partial}_{2a}, \\ \overset{2}{\mathbb{C}}{}^H = 0, & \overset{2}{\mathbb{C}}{}^{V_1} = y^{(1)a}\delta_{1a}, & \overset{2}{\mathbb{C}}{}^{V_2} & = (2y^{(2)a} + \underset{1}{N}{}^a{}_b y^{(1)a})\dot{\partial}_{2a}. \end{array}$$

A similar theory can be done for distinguished 1-forms.

With respect to the direct decomposition (4.12) a 1-form $\omega \in \mathcal{X}^*(\widetilde{T^2M})$ can be written in the form

$$\omega = \omega^H + \omega^{V_1} + \omega^{V_2}, \tag{6.5}$$

where

$$\omega^H = \omega_0 h, \omega^{V_1} = \omega_0 v_1, \omega^{V_2} = \omega_0 v_2. \tag{6.5'}$$

In the adapted cobasis (5.1), we have

$$\omega = \underset{(0)}{\omega}{}_a dx^a + \underset{(1)}{\omega}{}_a \delta y^{(1)a} + \underset{(2)}{\omega}{}_a \delta y^{(2)a}. \tag{6.6}$$

The quantities $\omega^H, \omega^{V_1}, \omega^{V_2}$ are called **d-1-forms.**

The coefficients $\underset{(0)}{\omega}{}_a, \underset{(1)}{\omega}{}_a, \underset{(2)}{\omega}{}_a$ are transformed by (1.3) as follows:

$$\underset{(0)}{\omega}{}_a = \frac{\partial \tilde{x}^b}{\partial x^a} \underset{(0)}{\widetilde{\omega}}{}_b, \underset{(1)}{\omega}{}_a = \frac{\partial \tilde{x}^b}{\partial x^a} \underset{(1)}{\widetilde{\omega}}{}_b, \underset{(2)}{\omega}{}_a = \frac{\partial \tilde{x}^b}{\partial x^a} \underset{(2)}{\widetilde{\omega}}{}_b.$$

Hence $\underset{(0)}{\omega}{}_a, \underset{(1)}{\omega}{}_a, \underset{(2)}{\omega}{}_a$ are called **d-covector fields.**

Particularly, we remark that the differential of a function $f \in \mathcal{F}(\widetilde{T^2M})$ can be written in the form

$$df = \frac{\delta f}{\delta x^a} dx^a + \frac{\delta f}{\delta y^{(1)a}} \delta y^{(1)a} + \frac{\partial f}{\partial y^{(2)a}} \dot{\partial}^{2a}. \tag{6.7}$$

Therefore

$$\begin{array}{c} df = (df)^H + (df)^{V_1} + (df)^{V_2} \\ where \\ (df)^H = \delta_a f dx^a, (df)^{V_1} = \delta_{1a} f \delta y^{(1)a}, (df)^{V_2} = \dot{\partial}_{2a} f \delta y^{(2)a}. \end{array} \tag{6.7'}$$

Let us consider a smooth parametrized curve $\gamma : I \subset R \to \widetilde{T^2M}$ such that $\operatorname{Im}\gamma \subset (\pi^2)^{-1}(U)$. It can be analytical represented by:

$$x^a = x^a(t), y^{(1)a} = y^{(1)a}(t), y^{(2)a} = y^{(2)a}(t), t \in I \tag{6.8}$$

The tangent vector $\dfrac{d\gamma}{dt}$, in a point of the curve γ, can be written in the form:

$$\frac{d\gamma}{dt} = \left(\frac{d\gamma}{dt}\right)^H + \left(\frac{d\gamma}{dt}\right)^{V_1} + \left(\frac{d\gamma}{dt}\right)^{V_2} = \frac{dx^a}{dt}\delta_a + \frac{\delta y^{(1)a}}{dt}\delta_{1a} + \frac{\delta y^{(2)a}}{dt}\dot{\partial}_{2a}, \quad (6.9)$$

where

$$\frac{\delta y^{(1)a}}{dt} = \frac{dy^{(1)a}}{dt} + \underset{1}{M}{}^a_b \frac{dx^b}{dt}, \frac{\delta y^{(2)a}}{\delta t} = \frac{dy^{(2)a}}{dt} + \underset{1}{M}{}^a_b \frac{dy^{(1)b}}{dt} + \underset{2}{M}{}^a_b \frac{dy^{(2)b}}{dt}. \quad (6.10)$$

The curve (6.8) is called **horizontal** if $\dfrac{d\gamma}{dt} = \left(\dfrac{d\gamma}{dt}\right)^H$ in every point of the curve γ.

Proposition 6.1. *An horizontal curve on $\widetilde{T^2M}$ is characterized by the following system of differential equations:*

$$x^a = x^a(t), \frac{\delta y^{(1)a}}{dt} = 0, \frac{\delta y^{(2)a}}{dt} = 0, t \in I. \quad (6.11)$$

Clearly, the system of differential equations (6.11) has local solutions, if the initial point $x_0^a = x^a(t_0), y_0^{(1)a}, y_0^{(2)a}$ on T^2M are given, $t_0 \in I$.

Let c:I$\to$ M be a parametrized curve on the base manifold M, given by $x^a = x^a(t), t \in I$. Let us, also, consider to extension $\widetilde{c}(=\gamma)$ to T^2M of the curve c. The curve c:I$\to$ M on the base manifold M is called an **autoparallel** curve of the nonlinear connection N if its extension $\widetilde{c}(=\gamma)$ to T^2M is an horizontal curve.

Theorem 6.1. *The autoparallel curve of the nonlinear connection N with the dual coefficients $\left(\underset{1}{M}{}^a_b, \underset{2}{M}{}^a_b\right)$ are characterized by the system of differential equations*

$$\begin{aligned} y^{(1)a} &= \frac{dx^a}{dt}, y^{(2)a} = \tfrac{1}{2}\frac{d^2x^a}{dt^2}, \\ \frac{\delta y^{(1)a}}{dt} &= \frac{dy^{(1)a}}{dt} + \underset{1}{M}{}^a_b \frac{dx^b}{dt} = 0, \\ \frac{\delta y^{(2)a}}{dt} &= \frac{dy^{(2)a}}{dt} + \underset{1}{M}{}^a_b \frac{dy^{(1)b}}{dt} + \underset{2}{M}{}^a_b \frac{dx^b}{dt} = 0. \end{aligned} \quad (6.12)$$

A theorem of existence and uniqueness of the autoparallel curves of a non-linear connection can now be easy formulated.

Finally, we can represent the Liouville vector fields $\overset{1}{\mathbb{C}}$ and $\overset{2}{\mathbb{C}}$ from (1.6), (1.6') in the adapted basis (4.12). We get

$$\overset{1}{\mathbb{C}} = z^{(1)a}\dot{\partial}_{2a}, \overset{2}{\mathbb{C}} = z^{(1)a}\delta_{1a} + 2z^{(2)a}\dot{\partial}_{2a}, \quad (6.13)$$

where

$$z^{(1)a} = y^{(1)a}, z^{(2)a} = y^{(2)a} + \frac{1}{2}\underset{1}{M}{}^a_b y^{(1)b}. \quad (6.14)$$

Therefore, $z^{(1)a}$ and $z^{(2)a}$ are d-vector fields. They are called the **Liouville d-vector fields**.

1.7 Lie brackets. Exterior differentials

In applications, the Lie brackets of the vector fields $\left\{\delta_a, \delta_{1a}, \dot{\partial}_{2a}\right\}$ from the adapted basis to the direct decomposition (4.10), are important.

Proposition 7.1. *The Lie brackets of the vector fields of the adapted basis are given by*

$$\begin{array}{lcl} [\delta_b, \delta_c] &=& \underset{(01)}{R}{}^{a}{}_{bc}\delta_{1a} + \underset{(02)}{R}{}^{a}{}_{bc}\dot{\partial}_{2a}, \\ [\delta_b, \delta_{1c}] &=& \underset{(11)}{B}{}^{a}{}_{bc}\delta_{1a} + \underset{(12)}{B}{}^{a}{}_{bc}\dot{\partial}_{2a}, \\ \left[\delta_b, \dot{\partial}_{2c}\right] &=& \underset{(21)}{B}{}^{a}{}_{bc}\delta_{1a} + \underset{(22)}{B}{}^{a}{}_{bc}\dot{\partial}_{2a}, \\ [\delta_{1b}, \delta_{1c}] &=& \underset{(12)}{R}{}^{a}{}_{bc}\dot{\partial}_{2a}, \\ \left[\delta_{1b}, \dot{\partial}_{2c}\right] &=& \underset{(21)}{B}{}^{a}{}_{bc}\dot{\partial}_{2a}, \end{array} \tag{7.1}$$

where

$$\begin{array}{lcl} \underset{(01)}{R}{}^{a}{}_{bc} &=& \delta_c \underset{1}{N}{}^{a}{}_{b} - \delta_b \underset{1}{N}{}^{a}{}_{c}, \\ \underset{(02)}{R}{}^{a}{}_{bc} &=& \delta_c \underset{2}{N}{}^{a}{}_{b} - \delta_b \underset{2}{N}{}^{a}{}_{c} + \underset{1}{N}{}^{a}{}_{f} \underset{(01)}{R}{}^{f}{}_{bc}, \\ \underset{(11)}{B}{}^{a}{}_{bc} &=& \delta_{1c} \underset{1}{N}{}^{a}{}_{b},\ \underset{(12)}{B}{}^{a}{}_{bc} = \delta_{1c} \underset{2}{N}{}^{a}{}_{c} - \delta_b \underset{1}{N}{}^{a}{}_{c} + \underset{1}{N}{}^{a}{}_{f} \underset{(11)}{B}{}^{f}{}_{bc}, \\ \underset{(21)}{B}{}^{a}{}_{bc} &=& \dot{\partial}_{2c} \underset{1}{N}{}^{a}{}_{b},\ \underset{(22)}{B}{}^{a}{}_{bc} = \dot{\partial}_{2c} \underset{2}{N}{}^{a}{}_{b} + \underset{1}{N}{}^{a}{}_{f} \underset{(21)}{B}{}^{f}{}_{bc}, \\ \underset{(12)}{R}{}^{a}{}_{bc} &=& \delta_{1c} \underset{1}{N}{}^{a}{}_{b} - \delta_{1b} \underset{1}{N}{}^{a}{}_{c}. \end{array} \tag{7.2}$$

The proof of this relations can be done by a direct calculus.

Now we can establish, [14]:

Proposition 7.2. *The exterior differentials of the 1-forms (dx^a, $\delta y^{(1)a}$, $\delta y^{(2)a}$), which determine the adapted cobasis (5.1), are given by*

$$\begin{array}{lcl} d(dx^a) &=& 0, \\ d(\delta y^{(1)a}) &=& \left\{\frac{1}{2}\underset{(01)}{R}{}^{a}{}_{bc}dx^c + \underset{(11)}{B}{}^{a}{}_{bc}\delta y^{(1)c} + \underset{(21)}{B}{}^{a}{}_{bc}\delta y^{(2)c}\right\} \wedge dx^b, \\ d(\delta y^{(2)a}) &=& \left\{\frac{1}{2}\underset{(02)}{R}{}^{a}{}_{bc}dx^c + \underset{(12)}{B}{}^{a}{}_{bc}\delta y^{(1)c} + \underset{(22)}{B}{}^{a}{}_{bc}\delta y^{(2)c}\right\} \wedge dx^b + \\ && + \left\{\frac{1}{2}\underset{(12)}{R}{}^{a}{}_{bc}\delta y^{(1)c} + \underset{(21)}{B}{}^{a}{}_{bc}\delta y^{(2)c}\right\} \wedge \delta y^{(1)b}. \end{array} \tag{7.3}$$

Indeed, from (5.3) we deduce

$$d(\delta y^{(1)a}) = d\underset{1}{M}{}^{a}{}_{b} \wedge dx^b, d(\delta y^{(2)a}) = d\underset{1}{M}{}^{a}{}_{b} \wedge dy^{(1)b} + d\underset{2}{M}{}^{a}{}_{b} \wedge dx^b,$$

where we substitute $\mathrm{d}y^{(1)a}$ and $\mathrm{d}y^{(2)a}$ from (5.3) and take into account the relations (5.4) and the formulae (7.2).

Let us consider the following coefficients from (7.1):

$$\underset{(11)}{B}{}^{a}{}_{bc} = \delta_{1c}\underset{1}{N}{}^{a}{}_{b}, \underset{(22)}{B}{}^{a}{}_{bc} = \dot{\partial}_{2c}\underset{2}{N}{}^{a}{}_{b} + \underset{1}{N}{}^{a}{}_{f}\dot{\partial}_{2c}\underset{1}{N}{}^{f}{}_{b}. \tag{7.4}$$

By means of (4.9) it follows:

Proposition 7.3. *The coefficients* $\underset{(11)}{B}{}^{a}{}_{bc}, \underset{(22)}{B}{}^{a}{}_{bc}$ *have the same rule of transformation with respect to the local changing of coordinates (1.3) on* T^2M. *This is*

$$\underset{(\beta\beta)}{\widetilde{B}}{}^{a}{}_{df}\frac{\partial\widetilde{x}^d}{\partial x^b}\frac{\partial\widetilde{x}^f}{\partial x^c} = \frac{\partial\widetilde{x}^a}{\partial x^d}\underset{(\beta\beta)}{B}{}^{d}{}_{bc} - \frac{\partial^2\widetilde{x}^a}{\partial x^b\partial x^c}, (\beta = 1, 2). \tag{7.5}$$

We will be see that these coefficients are the horizontal coefficients of an N-linear connection on T^2M.

We obtain also:

Proposition 7.4. *The coefficients:* $\underset{(01)}{R}{}^{a}{}_{bc}, \underset{(02)}{R}{}^{a}{}_{bc}, \underset{(12)}{R}{}^{a}{}_{bc}$ *and*

$$\begin{aligned} \underset{(12)}{B}{}^{a}{}_{bc} &= \delta_{1c}\underset{2}{N}{}^{a}{}_{b} - \delta_b\underset{1}{N}{}^{a}{}_{c} + \underset{1}{N}{}^{a}{}_{f}\delta_{1c}\underset{1}{N}{}^{f}{}_{b}, \\ \underset{(21)}{B}{}^{a}{}_{bc} &= \dot{\partial}_{2c}\underset{1}{N}{}^{a}{}_{b}, \end{aligned} \tag{7.6}$$

are d-tensor fields on T^2M.

We get:

Theorem 7.1. *The horizontal distribution N is integrable if and only if for any vector fields* $X, Y \in X\left(T^2M\right)$ *we have*

$$[X^H, Y^H]^{V_1} = [X^H, Y^H]^{V_2} = 0.$$

Indeed, the Lie bracket of any two horizontal vector fields X^H, Y^H belongs to the horizontal distribution N if and only if the last two equations hold.

Also, we get:

Theorem 7.2. *The J-vertical distribution* N_1 *is integrable if and only if for any vector field* $X, Y \in X\left(T^2M\right)$ *we have:*

$$[X^{V_1}, Y^{V_1}]^H = [X^{V_1}, Y^{V_1}]^{V_2} = 0.$$

Taking into account (7.1) we can formulate:

Theorem 7.3.

1° *The horizontal distribution N is integrable if and only if the following d-tensor fields vanish:*

$$\underset{(01)}{R}{}^{a}{}_{bc} = 0, \underset{(02)}{R}{}^{a}{}_{bc} = 0. \tag{7.7}$$

2° *The J-vertical distribution* N_1 *is integrable if and only if we have:*

$$\underset{(12)}{R}{}^{a}{}_{bc} = 0. \tag{7.8}$$

1.8 The almost product structure P. The almost (α) −contact structure $\underset{\alpha}{\mathbb{F}}, (\alpha = 0, 1, 2)$

Assuming that a nonlinear connection N is given, we define a $\mathcal{F}\left(T^2(M)\right)$-linear mapping

$$\mathbb{P} : \mathcal{X}(T^2M) \to \mathcal{X}(T^2M),$$

by defined

$$\mathbb{P}(X^H) = X^H, \mathbb{P}(X^{V_1}) = -X^{V_1}, \mathbb{P}(X^{V_2}) = -X^{V_2}, \forall X \in \mathcal{X}(T^2M) \tag{8.1}$$

We have, also

$$\begin{cases} \mathbb{P} \circ \mathbb{P} & = I, \\ \mathbb{P} & = I - 2(v_1 + v_2) = 2h - I, \\ rank\mathbb{P} & = 3n. \end{cases} \tag{8.2}$$

We can prove, without difficulties, [90], [92]:

Theorem 8.1. *A nonlinear connection N on T^2M is characterized by the existence of an almost product structure P on T^2M whose eigenspaces corresponding to the eigenvalue -1 coincide with the linear spaces of the vertical distribution V on T^2M.*

The nonlinear connection N being fixed we have the direct decomposition (4.1), (4.10) and the corresponding adapted basis (4.11).

Let us consider the $\mathcal{F}(\widetilde{T^2M})$-linear mapping:

$$\underset{0}{\mathbb{F}}(\delta_a) = 0, \underset{0}{\mathbb{F}}(\delta_{1a}) = -\dot{\partial}_{2a}, \underset{0}{\mathbb{F}}(\dot{\partial}_{2a}) = \delta_{1a}. \tag{8.3}$$

Then, we deduce:

Theorem 8.2. *The mapping $\underset{0}{\mathbb{F}}$ has the following properties:*

1°. It is globally defined on T^2M.

2°. $\underset{0}{\mathbb{F}}$ is a tensor field of type (1,1).

3°. $Ker\underset{0}{\mathbb{F}}=N_0$, $Im\underset{0}{\mathbb{F}}=N_1 \oplus V_2$.

4°. rank $\underset{0}{\mathbb{F}}$=2n.

5°. $\underset{0}{\mathbb{F}}^3 + \underset{0}{\mathbb{F}} = 0$.

Proof.

1°. Taking into account (4.13) we have $\frac{\partial x^a}{\partial \widetilde{x}^b}\underset{0}{\mathbb{F}}(\frac{\delta}{\delta x^a}) = 0$, implies $\underset{0}{\mathbb{F}}(\frac{\delta}{\delta \widetilde{x}^a}) = 0$. Also, $\frac{\partial x^a}{\partial \widetilde{x}^b}\underset{0}{\mathbb{F}}(\frac{\delta}{\delta y^{(1)a}}) = -\frac{\partial x^a}{\partial \widetilde{x}^b}\frac{\partial}{\partial y^{(2)a}}$, $\frac{\partial x^a}{\partial \widetilde{x}^b}\underset{0}{\mathbb{F}}(\frac{\partial}{\partial y^{(2)a}}) = \frac{\partial x^a}{\partial \widetilde{x}^b}\frac{\delta}{\delta y^{(1)a}}$, lead to $\underset{0}{\mathbb{F}}(\frac{\delta}{\delta \widetilde{y}^{(1)b}}) = -\frac{\partial}{\partial \widetilde{y}^{(2)b}}$ and $\underset{0}{\mathbb{F}}(\frac{\partial}{\partial \widetilde{y}^{(2)b}}) = \frac{\delta}{\delta \widetilde{y}^{(1)b}}$.

2°. $\underset{0}{\mathbb{F}}$ is $\mathcal{F}(\mathrm{T}^2M)$-linear mapping from $\mathcal{X}(\mathrm{T}^2M)$ to $\mathcal{X}(\mathrm{T}^2M)$.

3°. $\underset{0}{\mathbb{F}}(\frac{\delta}{\delta x^a})$=0 implies $\underset{0}{\mathbb{F}}|_{N_0}$ is trivial and $\underset{0}{\mathbb{F}}(N_0 \oplus N_1 \oplus V_2) = N_1 \oplus V_2$.
4°. Evidently, by means of 3°.
5°. $\underset{0}{\mathbb{F}}^2(\mathrm{X}^{V_1})$=$\underset{0}{\mathbb{F}}(-X^{V_2})$=$-X^{V_1}$; $\underset{0}{\mathbb{F}}^3(\mathrm{X}^{V_1})$=$\mathrm{X}^{V_2}$ and $\underset{0}{\mathbb{F}}(\mathrm{X}^{V_1})$=$-X^{V_2}$;

$\underset{0}{\mathbb{F}}^2(\mathrm{X}^{V_2})$=$\underset{0}{\mathbb{F}}(\mathrm{X}^{V_1})$=$-X^{V_2}$; $\underset{0}{\mathbb{F}}^3(\mathrm{X}^{V_2})$=$-X^{V_1}$ and $\underset{0}{\mathbb{F}}(\mathrm{X}^{V_2})$=$\mathrm{X}^{V_1}$. So,
$\left(\underset{0}{\mathbb{F}}^3 + \underset{0}{\mathbb{F}}\right)(X^H) = 0, \left(\underset{0}{\mathbb{F}}^3 + \underset{0}{\mathbb{F}}\right)(X^{V_1}) = 0, \left(\underset{0}{\mathbb{F}}^3 + \underset{0}{\mathbb{F}}\right)(X^{V_2}) = 0,$
$\forall X^H \in N_0, \forall X^{V_1} \in N_1, \forall X^{V_2} \in V_2.$

Therefore 5° holds.

q.e.d.

Thus, $\underset{0}{\mathbb{F}}$ is a $\underset{0}{\mathbb{F}}$(3,1)-structure. We can say that $\underset{0}{\mathbb{F}}$ is a **natural almost (0)-contact structure** determined by the nonlinear connection N.

Indeed, the dimension of T^2M is 3n=n+2n. Let us consider a local basis $\left\{\underset{1a}{\xi}\right\}$ of the distribution N_0 and $\left\{\overset{1a}{\eta}\right\}$ its dual. Then the set $\left(\underset{0}{\mathbb{F}}, \underset{1a}{\xi}, \overset{1a}{\eta}\right)$, determine an almost n-contact structure. Namely, we have

$$\begin{aligned} \underset{0}{\mathbb{F}}(\underset{1a}{\xi}) &= 0, \overset{1a}{\eta}(\underset{1b}{\xi}) = \delta^a_b, \\ \underset{0}{\mathbb{F}}^2(X) &= -X + \sum_{a=1}^{n} \overset{1a}{\eta}(X) \underset{1a}{\xi}, \forall X \in \mathcal{X}(\widetilde{T^2M}). \end{aligned}$$

Let us consider the $\mathcal{F}(\widetilde{T^2M})$-linear mapping

$$\underset{1}{\mathbb{F}}(\delta_a) = -\dot{\partial}_{2a}, \underset{1}{\mathbb{F}}(\delta_{1a}) = 0, \underset{1}{\mathbb{F}}(\dot{\partial}_{2a}) = \delta_a. \tag{8.4}$$

We have

Theorem 8.3. *The mapping $\underset{1}{\mathbb{F}}$ has the following properties*
1°. It is globally defined on T^2M.
2°.$\underset{1}{\mathbb{F}}$ is a tensor field of type (1,1).
3°. Ker$\underset{1}{\mathbb{F}}$=N_1, Im$\underset{1}{\mathbb{F}}$=$N_0 \oplus V_2$.
4°. rank $\underset{1}{\mathbb{F}}$=2n.
5°. $\underset{1}{\mathbb{F}}^3 + \underset{1}{\mathbb{F}} = 0$.

The proof fallow the same manner.

We can say that $\underset{1}{\mathbb{F}}$ is a **natural almost (1)-contact structure** determined by the nonlinear connection N.

Analogous, let us consider the $\mathcal{F}(\widetilde{T^2M})$ - linear mapping

$$\underset{2}{\mathbb{F}}(\delta_a) = -\delta_{1a}, \underset{2}{\mathbb{F}}(\delta_{1a}) = \delta_a, \underset{2}{\mathbb{F}}(\dot{\partial}_{2a}) = 0. \tag{8.5}$$

We have

Theorem 8.4. *The mapping* $\underset{2}{\mathbb{F}}$ *has the following properties*

1°. It is globally defined on T^2M.

2°. $\underset{2}{\mathbb{F}}$ *is a tensor field of type (1,1).*

3°. $Ker\underset{2}{\mathbb{F}}=V_2$, $Im\underset{2}{\mathbb{F}}=N_0 \oplus N_1$.

4°. rank $\underset{2}{\mathbb{F}}=2n$.

5°. $\underset{2}{\mathbb{F}}^3 + \underset{2}{\mathbb{F}} = 0$.

We can say that $\underset{2}{\mathbb{F}}$ is a **natural almost (2)-contact structure** determined by the nonlinear connection N.

The Nijenhuis tensor of the structures $\underset{\alpha}{\mathbb{F}}$, $(\alpha = 0,1,2)$, is given by:

$$\mathcal{N}_{\mathbb{F}}(X,Y) = \mathbb{F}^2[X,Y] + [\mathbb{F}X, \mathbb{F}Y] - \mathbb{F}[\mathbb{F}X, Y] - \mathbb{F}[X, \mathbb{F}Y], (\mathbb{F} = \underset{\alpha}{\mathbb{F}}),$$

and the normality condition of reads as follow:

$$\begin{aligned} \mathcal{N}_{\mathbb{F}}(X,Y) + \sum_{a=1}^{n} d\left(\delta y^{(2)a}\right)(X,Y) &= 0,\ \forall X,Y \in \mathcal{X}(T^2M), \\ (\mathbb{F} &= \underset{\alpha}{\mathbb{F}},\ \alpha = 0,1,2). \end{aligned} \tag{8.6}$$

Of course, in the adapted basis, using the formula (7.3) we can obtain the explicit form of the equations (8.6), $(\alpha = 0,1,2)$.

The structures $\left(\underset{\alpha}{\mathbb{F}}, \underset{1a}{\overset{\alpha}{\xi}}, \underset{\alpha}{\overset{1a}{\eta}}\right)$ will be used in the case when we have a Riemann structure $\mathbb{G}$ on T^2M, so that the set $\left(\underset{\alpha}{\mathbb{F}}, \underset{1a}{\overset{\alpha}{\xi}}, \underset{\alpha}{\overset{1a}{\eta}}, \mathbb{G}\right)$ will be the almost (α)-contact Riemannian structures on $\widetilde{T^2M}$, $(\alpha = 0,1,2)$. The manifold T^2M endowed with this structures gives us the geometrical models, $\underset{(\alpha)}{H}^{3n} = \left\{T^2M, \mathbb{G}, \underset{\alpha}{\mathbb{F}}\right\}$, $(\alpha = 0,1,2)$, for these spaces.

Taking into account (1.7) and (4.11), we obtain:

Proposition 8.1. *The following equalities hold:*

$$J(\delta_a) = \delta_{1a}, J(\delta_{1a}) = \dot{\partial}_{2a}, J(\dot{\partial}_{2a}) = 0. \tag{8.7}$$

1.9 The Riemann structure on $\widetilde{T^2M}$

Let us consider a Riemannian structure $\mathbb{G}$ on the manifold $\widetilde{T^2M}$.

The following problem is arises: Can the Riemannian structure $\mathbb{G}$ determine a nonlinear connection N on $\widetilde{T^2M}$?

In order to determine a nonlinear connection on $\widetilde{T^2M}$ by means of $\mathbb{G}$, it is sufficient to determine the distribution vertical V_2 orthogonal to the distributions

N_1 and N_0. The solution is immediate. Namely, it is important to determine the coefficients $\underset{1}{N}{}^a_{\ b}, \underset{2}{N}{}^a_{\ b}$ on N.

In the natural basis, $\mathbb{G}$ is given locally by

$$\mathbb{G} = \underset{00}{g}{}_{ab}dx^a \otimes dx^b + \underset{01}{g}{}_{ab}dx^a \otimes dy^{(1)b} + \underset{02}{g}{}_{ab}dx^a \otimes dy^{(2)b} + ... + \underset{22}{g}{}_{ab}dy^{(2)a} \otimes dy^{(2)b}, \tag{9.1}$$

where the matrix $||\underset{\alpha\beta}{g}{}_{ab}||, (\alpha, \beta = 0, 1, 2)$, is positively defined.

Let $\left(\delta_a, \delta_{1a}, \dot{\partial}_{2a}\right)$ be the adapted basis of N given by (4.11).

The following conditions of orthogonality between N_1 and V_2, respectively, N_0 and V_2:

$$\mathbb{G}(\delta_{1a}, \dot{\partial}_{2b}) = 0, \mathbb{G}(\delta_a, \dot{\partial}_{2b}) = 0 \tag{9.2}$$

give us the following system of equations for determining the coefficients $\underset{1}{N}{}^a_{\ b}$ and $\underset{2}{N}{}^a_{\ b}$:

$$\begin{aligned} \underset{12}{g}{}_{ab} - \underset{1}{N}{}^c_{\ a}\underset{22}{g}{}_{bc} &= 0, \\ \underset{02}{g}{}_{ab} - \underset{1}{N}{}^c_{\ a}\underset{12}{g}{}_{bc} - \underset{2}{N}{}^c_{\ a}\underset{22}{g}{}_{bc} &= 0. \end{aligned} \tag{9.3}$$

The restriction of a Riemannian structure $\mathbb{G}$ on T^2M to the vertical distribution V_2 on T^2M is completely determined by

$$\underset{22}{g}{}_{ab} = \mathbb{G}(\dot{\partial}_{2a}, \dot{\partial}_{2b}), (a, b = 1, ..., n). \tag{9.4}$$

Performing a change of coordinates on T^2M it comes out that the functions $\underset{22}{g}{}_{ab}$ may be viewed as the components of a tensor field on M. Assuming that rank($\underset{22}{g}{}_{ab}$)=n, let ($\underset{22}{g}{}^{cd}$) be the inverse of matrix ($\underset{22}{g}{}_{ab}$).

Consequently, we have:

Theorem 9.1. *A Riemannian structure G on T^2M determines uniquely a nonlinear connection N, if the distribution V_2 is orthogonal to distributions N_1 and N_0. The coefficients $\underset{1}{N}{}^a_{\ b}$, $\underset{2}{N}{}^a_{\ b}$ of N are given by*

$$\begin{aligned} \underset{1}{N}{}^a_{\ b} &= \underset{12}{g}{}_{bc}\underset{22}{g}{}^{ca}, \\ \underset{2}{N}{}^a_{\ b} &= \underset{02}{g}{}_{bc}\underset{22}{g}{}^{ca} - \underset{1}{N}{}^c_{\ b}\underset{1}{N}{}^a_{\ c}. \end{aligned} \tag{9.5}$$

Corollary 9.1. *If the distribution V_2 is orthogonal to distributions N_1 and N_0, then a Riemannian structure $\mathbb{G}$ on T^2M determines uniquely the dual coefficients $\underset{1}{M}{}^a_{\ b}$, $\underset{2}{M}{}^a_{\ b}$ of a nonlinear connection N by*

$$\underset{1}{M}{}^a_{\ b} = \underset{12}{g}{}_{bc} \underset{22}{g}{}^{ca}, \underset{2}{M}{}^a_{\ b} = \underset{02}{g}{}_{bc} \underset{22}{g}{}^{ca}. \tag{9.6}$$

Let $\underset{\alpha}{\mathbb{F}}$ be the natural almost (α)-contact structures, $(\alpha = 0, 1, 2)$, determined by the previous nonlinear connection N.

The following problem arises: When will the pair $(\mathbb{G}, \underset{\alpha}{\mathbb{F}})$ is a Riemannian almost (α)-contact structure?

Of course, it is necessary to have:

$$\mathbb{G}(\underset{\alpha}{\mathbb{F}}X, Y) = -\mathbb{G}(X, \underset{\alpha}{\mathbb{F}}Y), \forall X, Y \in \mathcal{X}(\widetilde{T^2M}), (\alpha = 0, 1, 2).$$

Consequently, we get:

Theorem 9.2. *The pair* $(\mathbb{G}, \underset{0}{\mathbb{F}})$ *is a Riemannian almost (0)-contact structure on* T^2M *if and only if in the adapted basis determined by* N_0*,* N_1 *and* V_2 *the tensor* $\mathbb{G}$ *has the form*

$$\mathbb{G}(X, Y) = g_{ab} dx^a \otimes dx^b + h_{ab} \delta y^{(1)a} \otimes \delta y^{(1)b} + h_{ab} \delta y^{(2)a} \otimes \delta y^{(2)b}. \tag{9.7}$$

Theorem 9.3. *The pair* $(\mathbb{G}, \underset{1}{\mathbb{F}})$ *is a Riemannian almost (1)-contact structure on* T^2M *if and only if in the adapted basis determined by* N_0*,* N_1 *and* V_2 *the tensor* $\mathbb{G}$ *has the form*

$$\mathbb{G}(X, Y) = g_{ab} dx^a \otimes dx^b + h_{ab} \delta y^{(1)a} \otimes \delta y^{(1)b} + g_{ab} \delta y^{(2)a} \otimes \delta y^{(2)b}. \tag{9.8}$$

Theorem 9.4. *The pair* $(\mathbb{G}, \underset{2}{\mathbb{F}})$ *is a Riemannian almost (2)-contact structure on* T^2M *if and only if in the adapted basis determined by* N_0*,* N_1 *and* V_2 *the tensor* $\mathbb{G}$ *has the form*

$$\mathbb{G}(X, Y) = g_{ab} dx^a \otimes dx^b + g_{ab} \delta y^{(1)a} \otimes \delta y^{(1)b} + h_{ab} \delta y^{(2)a} \otimes \delta y^{(2)b}. \tag{9.9}$$

Corollary 9.2. *With respect to each the Riemannian structures (9.7), (9.8), (9.9) the distributions* N_0*,* N_1*,* V_2 *are orthogonal respectively.*

Remark The forms (9.7), (9.8), (9.9) will be used to define a lift to T^2M of a metric structure given only by a nonsingular and symmetric d-tensor field g_{ab}. Namely, we have

$$\mathbb{G}(X, Y) = g_{ab} dx^a \otimes dx^b + g_{ab} \delta y^{(1)a} \otimes \delta y^{(1)b} + g_{ab} \delta y^{(2)a} \otimes \delta y^{(2)b}. \tag{9.10}$$

We can prove:

Theorem 9.5. *If the Riemannian structure* $\mathbb{G}$ *given by (9.1) satisfy:*

$$\mathbb{G}\left(\partial_a, \dot{\partial}_{2b}\right) = 0, \mathbb{G}\left(\dot{\partial}_{1a}, \dot{\partial}_{2b}\right) = 0 \textit{ and } \mathbb{G}\left(\dot{\partial}_{2a}, \dot{\partial}_{2b}\right) \neq 0, \tag{9.11}$$

then we have

$$\underset{1}{N}{}^a{}_b = 0, \underset{2}{N}{}^a{}_b = 0 \left(equiv. \underset{1}{M}{}^a{}_b = 0, \underset{2}{M}{}^a{}_b = 0 \right), \tag{9.12}$$

$$\delta_a = \partial_a, \delta_{1a} = \dot{\partial}_{1a}, \delta y^{(1)a} = dy^{(1)a}, \delta y^{(2)a} = dy^{(2)a}. \tag{9.13}$$

Proof. By (9.11) and (9.3) we obtain (9.12). The (4.11) and (5.3) given us (9.13).

q.e.d.

Corollary 9.3. *If the Riemannian structure (9.1) satisfy the equations (9.11) then* $\mathbb{G}$ *has the following expression:*

$$\mathbb{G} = \underset{(00)}{g}{}_{ab} \left(x, y^{(1)}, y^{(2)} \right) dx^a \otimes dx^b + \sum_{\beta=1}^{2} \underset{(\beta\beta)}{g}{}_{ab} \left(x, y^{(1)}, y^{(2)} \right) dy^{(\beta)a} \otimes dy^{(\beta)b} \tag{9.14}$$

if and only if we have

$$\mathbb{G} \left(\partial_a, \dot{\partial}_{1b} \right) = 0, \tag{9.15}$$

where

$$\det(\underset{(00)}{g}{}_{ab}). \det(\underset{(11)}{g}{}_{ab}). \det(\underset{(22)}{g}{}_{ab}) \neq 0. \tag{9.16}$$

Chapter 2

Linear connections on the manifold T^2M

The main topics of this chapter showing that there are linear connection compatible to the direct decomposition (4.10) determined by a nonlinear connection N, on the total space of the bundle (T^2M, π^2, M).

We are going to study the distinguished Tensor Algebra (or d-Tensor Algebra), N-linear connections, torsions and curvatures, parallelism, structures equations, etc.

2.1 The $d-$tensor algebra

Let N be a nonlinear connection on T^2M. Then N determines the direct decomposition (4.10), Ch. 1. With respect to (4.10), Ch.1, a vector field X and one form ω can be uniquely written in the form (6.3) and (6.5), Ch. 1, respectively, i.e.

$$\begin{aligned} X &= X^H + X^{V_1} + X^{V_2}, \\ \omega &= \omega^H + \omega^{V_1} + \omega^{V_2}. \end{aligned} \tag{1.1}$$

Definition 1.1. *A distinguished tensor field (briefly: d-tensor field) on T^2M of type (r,s) is a tensor field T of type (r,s) on T^2M with the property:*

$$T(\overset{1}{\omega}, ..., \overset{r}{\omega}, \underset{1}{X}, ..., \underset{s}{X}) = T(\overset{1}{\omega}{}^H, ..., \overset{r}{\omega}{}^{V_2}, \underset{1}{X}{}^H, ..., \underset{s}{X}{}^{V_2}), \tag{1.2}$$

for any $(\overset{1}{\omega}, ..., \overset{r}{\omega}) \in \mathcal{X}^*(T^2M)$ *and for any* $(\underset{1}{X}, ..., \underset{s}{X}) \in \mathcal{X}(T^2M)$.

For instance, every component X^H, X^{V_1} and X^{V_2} of a vector field $X \in \mathcal{X}(T^2M)$ is a d-vector field.

Also, every component ω^H, ω^{V_1} and ω^{V_2} of the form $\omega \in \mathcal{X}(T^2M)$ is a d-1-form.

If T$\in \mathcal{T}_s^r(T^2M)$ is not a d-tensor, then using (1.1) in $T(\overset{1}{\omega}, ..., \overset{r}{\omega}, \underset{1}{X}, ..., \underset{s}{X})$ we get

$$T(\overset{1}{\omega}^H + \overset{1}{\omega}^{V_1} + \overset{1}{\omega}^{V_2}, ..., \underset{s}{X}^H + \underset{s}{X}^{V_1} + \underset{s}{X}^{V_2}) = T(\overset{1}{\omega}^H, ..., \underset{s}{X}^H) +$$
$$+T(\overset{1}{\omega}^H, ..., \underset{s}{X}^{V_1}) + ... + T(\overset{1}{\omega}^{V_2}, ..., \underset{s}{X}^{V_2}).$$

Then, every term in the second member is a d-tensor field.

Let us consider the coordinates of a d-tensor field with respect to the adapted basis $(\delta_a, \delta_{1a}, \dot{\partial}_{2a})$ and cobasis $(\mathrm{dx}^a, \delta y^{(1)a}, \delta y^{(2)a})$

$$T^{a_1...a_r}_{b_1...b_s}(x, y^{(1)}, y^{(2)}) = T(dx^{a_1}, ..., \delta y^{(2)a_r}, \delta_{b_1}, ..., \delta_{(2)b_s}).$$

It follows that T of type (r,s) can be locally written in the form

$$T = T^{a_1...a_r}_{b_1...b_s}(x, y^{(1)}, y^{(2)})\delta_{a_1} \otimes ... \otimes \dot{\partial}_{2a_r} \otimes dx^{b_1} \otimes ... \otimes \delta y^{(2)b_s}. \tag{1.3}$$

Hence, the set $\{1, \delta_a, \delta_{1a}, \dot{\partial}_{2a}\}$ generates the algebra of the d-tensor fields over the ring of functions $\mathcal{F}(\mathrm{T}^2M)$.

Examples

1°. If f$\in \mathcal{F}(\mathrm{T}^2M)$, then $\dfrac{\delta f}{\delta x^a} = \delta_a f$, $\dfrac{\delta f}{\delta y^{(1)a}} = \delta_{1a} f$, $\dfrac{\partial f}{\partial y^{(2)a}} = \dot{\partial}_{2a} f$ are d-1-covectors.

2°. Let us consider a Riemannian structure G on T^2M and assume that the distributions N_0, N_1, V_2 are orthogonal in pairs, with respect to G:

$$G(X^H, Y^{V_1}) = G(X^H, Y^{V_2}) = G(X^{V_1}, Y^{V_2}) = 0, \forall X, Y \in \mathcal{X}(T^2M). \tag{1.4}$$

In this case G can be uniquely written as a sum of d-tensors:

$$G = G^H + G^{V_1} + G^{V_2}, \tag{1.4'}$$

where, for any X,Y$\in \mathcal{X}(T^2M)$, we have

$$\begin{aligned} G^H(X,Y) &= G(X^H, Y^H), \\ G^{V_1}(X,Y) &= G(X^{V_1}, Y^{V_1}), G^{V_2}(X,Y) = G(X^{V_2}, Y^{V_2}). \end{aligned} \tag{1.4''}$$

Consequently, in the adapted cobasis, G can be uniquely written as

$$G = \underset{(0)}{g}_{ab} dx^a \otimes dx^b + \underset{(1)}{g}_{ab} \delta y^{(1)a} \otimes \delta y^{(1)b} + \underset{(2)}{g}_{ab} \delta y^{(2)a} \otimes \delta y^{(2)b}, \tag{1.5}$$

where

$$\underset{(\alpha)}{g}_{ab}(x, y^{(1)}, y^{(2)}) = \underset{(\alpha)}{g}_{ba}(x, y^{(1)}, y^{(2)}), (\alpha = 0, 1, 2), \tag{1.5'}$$

$$rank||\underset{(\alpha)}{g}_{ab}|| = n, (\alpha = 0, 1, 2) \tag{1.5''}$$

The quantities $\underset{(\alpha)}{g}_{ab}, (\alpha = 0, 1, 2)$ are d-tensors of type (0,2) on T^2M.

2.2 $N-$linear connection

On the total space T^2M of the 2-osculator bundle (T^2M,π^2,M) there are linear connections compatible with the direct decomposition (4.10), Ch.1. The advantage of considering these linear connections is that in the adapted basis they have as coefficients some geometrical objects, possibly to find in usual cases.

Definition 2.1. *A linear connection D on T^2M is called a N-linear connection if it preserves by parallelism the horizontal and vertical distributions N_0, N_1 and V_2 on T^2M.*

By a general theory of connections on manifolds, the horizontal and vertical distributions are preserves by parallelism if for any X$\in \mathcal{X}(T^2M), D_X$ carries the horizontal vector fields to the horizontal vector fields and the vertical vector fields to the vertical vector fields. Thus D$_XY^H$ is always an horizontal vector field and D$_XY^{V_\beta}$ are verticals, $(\beta=1,2)$.

Theorem 2.1. *For any N-linear connection D we have*

$$D_Xh=0, D_Xv_1=0, D_Xv_2=0, \tag{2.1}$$

$$D_X\mathbb{P}=0, \qquad \forall X\in \mathcal{X}\left(T^2M\right). \tag{2.2}$$

Indeed, from (D$_Xh$)(Y)=D$_X(hY)-h(D_XY)$ if Y=Y^H, and Y=Y$^{V_\beta}$,$(\beta=1,2)$ we obtain D$_Xh=0$. Similarly, we get D$_Xv_1=0$,D$_Xv_2=0$.
Now, taking into account the expression (8.2), Ch. 1, of $\mathbb{P}$ it follows D$_X\mathbb{P}=0$.

Theorem 2.2. *A linear connection D on T^2M is a N-linear connection if and only if the following properties are verified*

$$\begin{array}{rcl}\left(D_XY^H\right)^{V_\beta} &=& 0, \left(D_XY^{V_\beta}\right)^H=0, (\beta=1,2),\\ \left(D_XY^{V_1}\right)^{V_2} &=& 0, \left(D_XY^{V_2}\right)^{V_1}=0, \forall X,Y\in\mathcal{X}(T^2M).\end{array} \tag{2.3}$$

Let us consider a vector field X$\in$ $\mathcal{X}(T^2M)$, written in the form (1.1). It follows from the property of an N-linear connection that

$$D_XY=D_{X^H}Y+D_{X^{V_1}}Y+D_{X^{V_2}}Y, \forall X,Y\in\mathcal{X}(T^2M). \tag{2.4}$$

Also, let us consider Y on T^{2}M, written in the form (1.1).Since D$_XY$ is $\mathcal{F}$(T^{2}M)-linear with respect to X, we have

$$\begin{array}{rcl} D_XY &=& \sum\limits_{\alpha=0}^{2}\left(D_{X^H}Y^{V_\alpha}+D_{X^{V_1}}Y^{V_\alpha}+D_{X^{V_2}}Y^{V_\alpha}\right),\\ && \forall X,Y\in\mathcal{X}(T^2M), (V_0=H)\end{array} \tag{2.5}$$

We find here new operators in the algebra of d-tensor fields: $\underset{\alpha}{D}_{X^H}$, $\underset{\alpha}{D}_{X^{V_1}}$, $\underset{\alpha}{D}_{X^{V_2}}$, $(\alpha=0,1,2)$, denoted by

$$\underset{\alpha}{D}_{X^H}=\underset{\alpha}{D}{}^H_X, \underset{\alpha}{D}_{X^{V_1}}=\underset{\alpha}{D}{}^{V_1}_X, \underset{\alpha}{D}_{X^{V_2}}=\underset{\alpha}{D}{}^{V_2}_X, (\alpha=0,1,2). \tag{2.6}$$

We have

$$\left\{\begin{array}{lll} \underset{0}{D}{}^H_X Y = D_{X^H} Y^H, & \underset{0}{D}{}^{V_1}_X Y = D_{X^{V_1}} Y^H, & \underset{0}{D}{}^{V_2}_X Y = D_{X^{V_2}} Y^H, \\ \underset{\beta}{D}{}^H_X Y = D_{X^H} Y^{V_\beta}, & \underset{\beta}{D}{}^{V_1}_X Y = D_{X^{V_1}} Y^{V_\beta}, & \underset{\beta}{D}{}^{V_2}_X Y = D_{X^{V_2}} Y^{V_\beta}, \\ & & (\beta = 1,2). \end{array}\right. \tag{2.7}$$

These operators are not covariant derivations in the algebra of d-tensor fields, since $\underset{\alpha}{D}{}^H_X f = X^H f \neq Xf$, etc. But they have similar properties with the covariant derivations.

Theorem 2.3. *The operators* $\underset{\alpha}{D}{}^H, \underset{\alpha}{D}{}^{V_1}, \underset{\alpha}{D}{}^{V_2}, (\alpha = 0,1,2)$, *have the properties:*

1°. All equalities (2.3) are verified for X=$X^H, X = X^{V_1}, X = X^{V_2}$ *and* $\underset{\alpha}{D}{}^H_X f = X^H f, \underset{\alpha}{D}{}^{V_1}_X f = X^{V_1} f, \underset{\alpha}{D}{}^{V_2}_X f = X^{V_2} f,$

2°. $\underset{\alpha}{D}{}^H_X (fY) = X^H(fY) + f\underset{\alpha}{D}{}^H_X Y, \underset{\alpha}{D}{}^{V_\beta}_X (fY) = X^{V_\beta}(fY^{V_\beta}) + f\underset{\alpha}{D}{}^{V_\beta}_X Y,$

3°. $\left(\underset{\alpha}{D}{}^H_X Y\right)_{|U} = \underset{\alpha}{D}{}^H_{X|U} Y_{|U}, \left(\underset{\alpha}{D}{}^{V_\beta}_X Y\right)_{|U} = \underset{\alpha}{D}{}^{V_\beta}_{X|U} Y_{|U}$ *for any open set* $U \subset T^2M$,

4°. $\underset{\alpha}{D}{}^H_{X+Y} = \underset{\alpha}{D}{}^H_X + \underset{\alpha}{D}{}^H_Y, \underset{\alpha}{D}{}^{V_\beta}_{X+Y} = \underset{\alpha}{D}{}^{V_\beta}_X + \underset{\alpha}{D}{}^{V_\beta}_Y,$

5°. $\underset{\alpha}{D}{}^H_{fX} = f\underset{\alpha}{D}{}^H_X, \underset{\alpha}{D}{}^{V_\beta}_{fX} = f\underset{\alpha}{D}{}^{V_\beta}_X,$

for any f∈ $F(T^2M)$ *and any vector fields X,Y on* T^2M, *(*$\alpha = 0,1,2; \beta = 1,2$*).*

$\underset{\alpha}{D}{}^H, \underset{\alpha}{D}{}^{V_1}, \underset{\alpha}{D}{}^{V_2}$ are called **$\mathbf{h}_\alpha$-, $\mathbf{v}_{1\alpha}$-, and $\mathbf{v}_{2\alpha}$-covariant derivatives** respectively, $(\alpha = 0,1,2)$.

We can extend the action of the h_α-, $v_{1\alpha}$-, and $v_{2\alpha}$-derivatives to any tensor field on T^2M, particularly to any d-tensor fields. So, for any $\omega \in \mathcal{X}^*(T^2M)$ and for any X,Y∈ $\mathcal{X}(T^2M)$ we have

$$\begin{array}{rcl} \left(\underset{\alpha}{D}{}^H_X \omega\right)(Y) & = & X^H \omega(Y) - \omega\left(\underset{\alpha}{D}{}^H_X Y\right), \\ \left(\underset{\alpha}{D}{}^{V_\beta}_X \omega\right)(Y) & = & X^{V_\beta} \omega(Y) - \omega\left(\underset{\alpha}{D}{}^{V_\beta}_X Y\right), (\alpha = 0,1,2; \beta = 1,2). \end{array} \tag{2.8}$$

If T∈ $\mathcal{T}^r_s(T^2M)$, taking in $\underset{\alpha}{D}_X T, X = X^H$ or X=X^{V_β}, we have

$$\begin{array}{l} \left(\underset{\alpha}{D}{}^H_X T\right)\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right) = X^H T\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right) - \\ -T\left(\underset{\alpha}{D}{}^H_X \overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right) - ... - T\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{\alpha}{D}{}^H_X \underset{s}{X}\right), \\ \left(\underset{\alpha}{D}{}^{V_\beta}_X T\right)\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right) = X^{V_\beta} T\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right) - \\ -T\left(\underset{\alpha}{D}{}^{V_\beta}_X \overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right) - ... - T\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{\alpha}{D}{}^{V_\beta}_X \underset{s}{X}\right), \\ (\alpha = 0,1,2; \beta = 1,2) \end{array} \tag{2.9}$$

Now, let us consider a parametrized smooth curve $\gamma : t \subset I \to \gamma(t) \in \widetilde{T^2M}$, having the image in a domain of a local chart.

Its tangent vector field $\dot{\gamma}=\dfrac{d\gamma}{dt}$ can be uniquely written in the form

$$\dot{\gamma} = \dot{\gamma}^H + \dot{\gamma}^{V_1} + \dot{\gamma}^{V_2}. \tag{2.10}$$

In the case when γ is analytically given by the equation (6.8), Ch. 1, then $\dot{\gamma}^H, \dot{\gamma}^{V_1}, \dot{\gamma}^{V_2}$ are given by (6.9), Ch. 1. And we can define the horizontal curve.

A vector field Y defined along the curve γ has the covariant derivative

$$D_{\dot{\gamma}}Y = D^H_{\dot{\gamma}}Y + D^{V_1}_{\dot{\gamma}}Y + D^{V_2}_{\dot{\gamma}}Y.$$

The vector field Y(u(γ)) is called **parallel** along the curve γ if

$$D_{\dot{\gamma}}Y = 0.$$

In particular, the curve γ is **autoparallel** with respect to an N-linear connection D if $D_{\dot{\gamma}}\dot{\gamma}=0$.

In a next section we will study these notions by means of adapted basis.

2.3 Torsion and curvature

The torsion $\mathbb{T}$ of an N-linear connection D is expressed, as usually, by

$$\mathbb{T}(X, Y) = D_X Y - D_Y X - [X, Y]. \tag{3.1}$$

It can be evaluated for the pairs of d-vector fields (X^H, Y^H), (X^H, Y^{V_β}), $(X^{V_\beta}, Y^{V_\gamma})$, $(\beta, \gamma = 1, 2)$. We obtain the vector fields

$$\mathbb{T}(X^H, Y^H), \mathbb{T}(X^H, Y^{V_\beta}), \mathbb{T}(X^{V_\beta}, Y^{V_\gamma}), (\beta, \gamma = 1, 2).$$

Since D preserves by parallelism the distributions N_0, N_1, V_2 and the distributions N_1, V_2 are integrable it follows

Proposition 3.1. *The following property of the torsion $\mathbb{T}$ holds:*

$$h\mathbb{T}(X^{V_\beta}, X^{V_\gamma}) = 0, (\beta, \gamma = 1, 2). \tag{3.2}$$

Now, we deduce

Proposition 3.2. *The tensor of torsion $\mathbb{T}$ of an N-linear connection D is well determined by the following components, where in the right hand we have d-tensor fields of type (1,2):*

$$
\begin{array}{rcllll}
\mathbb{T}(X^H,Y^H) & = & h\mathbb{T}(X^H,Y^H) & +v_1\mathbb{T}(X^H,Y^H) & +v_2\mathbb{T}(X^H,Y^H), \\
\mathbb{T}(X^H,Y^{V_\beta}) & = & h\mathbb{T}(X^H,Y^{V_\beta}) & +v_1\mathbb{T}(X^H,Y^{V_\beta}) & +v_2\mathbb{T}(X^H,Y^{V_\beta}), \\
 & & & & (\beta=1,2), \\
\mathbb{T}(X^{V_1},Y^{V_\beta}) & = & & v_1\mathbb{T}(X^{V_1},Y^{V_\beta}) & +v_2\mathbb{T}(X^{V_1},Y^{V_\beta}), \\
 & & & & (\beta=1,2), \\
\mathbb{T}(X^{V_2},Y^{V_2}) & = & & v_1\mathbb{T}(X^{V_2},Y^{V_2}) & +v_2\mathbb{T}(X^{V_2},Y^{V_2}).
\end{array}
\tag{3.3}
$$

The d-tensor fields from the right hand of (3.3) are called the **d-tensors of torsion** of the N-linear connection D.

For instance, we have

$$\mathbb{T}(X^H,Y^{V_\alpha}) = \underset{\alpha}{D}{}^H_X Y^{V_\alpha} - \underset{0}{D}{}^{V_\alpha}_Y X^H - [X^H,Y^{V_\alpha}], (\alpha=0,1,2), V^0 = H.$$

We shall say that h$\mathbb{T}(X^H,Y^H)$ is h(hh)-tensor of torsion of D, $v_1\mathbb{T}(X^H,Y^H)$ is v$_1$(hh)-tensor of torsion of D and so on.

The curvature tensor $\mathbb{R}$ of D is given by

$$\mathbb{R}(X,Y)Z = (D_XD_Y - D_YD_X)Z - D_{[X,Y]}Z, \forall X,Y,Z \in \mathcal{X}(T^2M). \tag{3.4}$$

We will express $\mathbb{R}$ by means of the components (2.4), taking into account the decomposition (1.1) for the vector fields on T^2M.

We prove

Theorem 3.1. *The curvature tensor $\mathbb{R}$ of the N-linear connection D has the properties*

$$
\begin{array}{rcl}
v_\beta\mathbb{R}(X,Y)Z^H & = & 0, \qquad h\mathbb{R}(X,Y)Z^{V_\beta} = 0, (\beta=1,2), \\
\mathbb{R}(X,Y)Z & = & h\mathbb{R}(X,Y)Z^H + v_1\mathbb{R}(X,Y)Z^{V_1} + v_2\mathbb{R}(X,Y)Z^{V_2}.
\end{array}
\tag{3.5}
$$

Proof. Since D preserves by parallelism the verticals and horizontal distributions, by (3.4) the operator $\mathbb{R}$(X,Y) carries horizontal vector fields to horizontals and verticals vector fields to verticals. Thus the first four equations from (3.5) hold. The next one is an easy consequence of the first four.

q.e.d.

By Theorem 3.1 and the equation $\mathbb{R}(X,Y) = -\mathbb{R}(Y,X), \forall X,Y \in \mathcal{X}(T^2M)$ we get

Theorem 3.2. *The curvature tensor of a d-linear connection D on the total space T^2M of a 2-osculator bundle (T^2M,π^2,M) is completely determined by the following d-tensor fields:*

$$
\begin{array}{rcl}
\mathbb{R}(X^H,Y^H)Z^H & = & \underset{0}{D}{}^H_X \underset{0}{D}{}^H_Y Z^H - \underset{0}{D}{}^H_Y \underset{0}{D}{}^H_X Z^H - \sum\limits_{\alpha=0}^{2} \underset{0}{D}{}^{V_\alpha}_{[X^H,Y^H]} Z^H, \\
\mathbb{R}(X^H,Y^H)Z^{V_\gamma} & = & \underset{\gamma}{D}{}^H_X \underset{\gamma}{D}{}^H_Y Z^{V_\gamma} - \underset{\gamma}{D}{}^H_Y \underset{\gamma}{D}{}^H_X Z^{V_\gamma} - \sum\limits_{\alpha=0}^{2} \underset{\gamma}{D}{}^{V_\alpha}_{[X^H,Y^H]} Z^{V_\gamma}, \\
 & & (\gamma=1,2),
\end{array}
$$

$$\begin{aligned}
\mathbb{R}(X^{V_\beta},Y^H)Z^H &= \underset{0}{D}{}^{V_\beta}_X \underset{0}{D}{}^H_Y Z^H - \underset{0}{D}{}^H_Y \underset{0}{D}{}^{V_\beta}_X Z^H - \sum_{\alpha=0}^{2} \underset{0}{D}{}^{V_\alpha}_{[X^H,Y^H]} Z^H, \\
&\qquad (\beta=1,2), \\
\mathbb{R}(X^{V_\beta},Y^H)Z^{V_\gamma} &= \underset{\gamma}{D}{}^{V_\beta}_X \underset{\gamma}{D}{}^H_Y Z^{V_\gamma} - \underset{\gamma}{D}{}^H_Y \underset{\gamma}{D}{}^{V_\beta}_X Z^{V_\gamma} - \sum_{\alpha=0}^{2} \underset{\gamma}{D}{}^{V_\alpha}_{[X^{V_\beta},Y^H]} Z^{V_\gamma}, \\
&\qquad (\beta,\gamma=1,2),
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\mathbb{R}(X^{V_\beta},Y^{V_\gamma})Z^H &= \underset{0}{D}{}^{V_\beta}_X \underset{0}{D}{}^{V_\gamma}_Y Z^H - \underset{0}{D}{}^{V_\gamma}_Y \underset{0}{D}{}^{V_\beta}_X Z^H - \sum_{\varepsilon=1}^{2} \underset{0}{D}{}^{V_\varepsilon}_{[X^{V_\beta},Y^{V_\gamma}]} Z^H, \\
&\qquad (\beta,\gamma=1,2,\beta\le\gamma), \\
\mathbb{R}(X^{V_\beta},Y^{V_\gamma})Z^{\mathbb{V}_\delta} &= \underset{\delta}{D}{}^{V_\beta}_X \underset{\delta}{D}{}^{V_\gamma}_Y Z^{V_\delta} - \underset{\delta}{D}{}^{V_\gamma}_Y \underset{\delta}{D}{}^{V_\beta}_X Z^{V_\delta} - \sum_{\varepsilon=1}^{2} \underset{\delta}{D}{}^{V_\varepsilon}_{[X^{V_\beta},Y^{V_\gamma}]} Z^{V_\delta}, \\
&\qquad (\beta,\gamma,\delta=1,2,\beta\le\gamma),
\end{aligned}$$

where $V^0 = H$.

The d-tensors (3.6) are called **d-tensors of curvature** of the N-linear connection D.

In applications it is suitable to consider the equalities (3.6) as Ricci identities. We shall establish such identities for vector fields only, although these may be written for every tensor fields. A simple aranjament of (3.6) gives us

Theorem 3.3. *For any N-linear connection D the following Ricci identities hold:*

$$\begin{aligned}
&\underset{0}{D}{}^H_X \underset{0}{D}{}^H_Y Z^H - \underset{0}{D}{}^H_Y \underset{0}{D}{}^H_X Z^H = \mathbb{R}(X^H,Y^H)Z^H + \underset{0}{D}{}^H_{[X^H,Y^H]} Z^H + \sum_{\varepsilon=1}^{2} \underset{0}{D}{}^{V_\varepsilon}_{[X^H,Y^H]} Z^H, \\
&\underset{0}{D}{}^{V_\beta}_X \underset{0}{D}{}^H_Y Z^H - \underset{0}{D}{}^H_Y \underset{0}{D}{}^{V_\beta}_X Z^H = \mathbb{R}(X^{V_\beta},Y^H)Z^H + \underset{0}{D}{}^H_{[X^{V_\beta},Y^H]} Z^H + \sum_{\varepsilon=1}^{2} \underset{0}{D}{}^{V_\varepsilon}_{[X^{V_\beta},Y^H]} Z^H, \\
&\qquad (\beta=1,2), \\
&\underset{0}{D}{}^{V_\beta}_X \underset{0}{D}{}^{V_\gamma}_Y Z^H - \underset{0}{D}{}^{V_\gamma}_Y \underset{0}{D}{}^{V_\beta}_X Z^H = \mathbb{R}(X^{V_\beta},Y^{V_\gamma})Z^H + \sum_{\varepsilon=1}^{2} \underset{0}{D}{}^{V_\varepsilon}_{[X^{V_\beta},Y^{V_\gamma}]} Z^H, \\
&\qquad (\beta,\gamma=1,2,\beta\le\gamma),
\end{aligned} \tag{3.7$_1$}$$

$$\begin{aligned}
&\underset{\gamma}{D}{}^H_X \underset{\gamma}{D}{}^H_Y Z^{V_\gamma} - \underset{\gamma}{D}{}^H_Y \underset{\gamma}{D}{}^H_X Z^{V_\gamma} = \mathbb{R}(X^H,Y^H)Z^{V_\gamma} + \underset{\gamma}{D}{}^H_{[X^H,Y^H]} Z^{V_\gamma} + \sum_{\varepsilon=1}^{2} \underset{\gamma}{D}{}^{V_\varepsilon}_{[X^H,Y^H]} Z^{V_\gamma}, \\
&\qquad (\gamma=1,2), \\
&\underset{\gamma}{D}{}^{V_\beta}_X \underset{\gamma}{D}{}^H_Y Z^{V_\gamma} - \underset{\gamma}{D}{}^H_Y \underset{\gamma}{D}{}^{V_\beta}_X Z^{V_\gamma} = \mathbb{R}(X^{V_\beta},Y^H)Z^{V_\gamma} + \underset{\gamma}{D}{}^H_{[X^{V_\beta},Y^H]} Z^{V_\gamma} + \sum_{\varepsilon=1}^{2} \underset{\gamma}{D}{}^{V_\varepsilon}_{[X^{V_\beta},Y^H]} Z^{V_\gamma}, \\
&\qquad (\beta,\gamma=1,2), \\
&\underset{\delta}{D}{}^{V_\beta}_X \underset{\delta}{D}{}^{V_\gamma}_Y Z^{V_\delta} - \underset{\delta}{D}{}^{V_\gamma}_Y \underset{\delta}{D}{}^{V_\beta}_X Z^{\delta} = \mathbb{R}(X^{V_\beta},Y^{V_\gamma})Z^{V_\delta} + \sum_{\varepsilon=1}^{2} \underset{\delta}{D}{}^{V_\varepsilon}_{[X^{V_\beta},Y^{V_\gamma}]} Z^{V_\delta}, \\
&\qquad (\beta,\gamma,\delta=1,2,\beta\le\gamma),
\end{aligned} \tag{3.7$_2$}$$

where $V^0 = H$

As a consequence, we obtain

Theorem 3.4. *For any N-linear connection D the following identities hold*

$$[D_X, D_Y] \overset{\beta}{\mathbb{C}} = \mathbb{R}(X,Y)\overset{\beta}{\mathbb{C}} - D_{[X,Y]}\overset{\beta}{\mathbb{C}}, \qquad (\beta = 1,2), \tag{3.8}$$

where $\overset{1}{\mathbb{C}}, \overset{2}{\mathbb{C}}$ *are the Liouville vector fields on* T^2M.

The d-tensors of torsion and curvature of a d-linear connection D are not independent. As it is well knows the torsion $\mathbb{T}$ and curvature $\mathbb{R}$ of every linear connection D on $\mathrm{T}^2\mathrm{M}$ satisfies the following classical Bianchi identities:

$$\sum \left[\left(D_X\mathbb{T}\right)(Y,Z) - \mathbb{R}\left(X,Y\right)Z + \mathbb{T}\left(\mathbb{T}\left(X,Y\right),Z\right)\right] = 0, \tag{3.9}$$

$$\sum \left[\left(D_X\mathbb{R}\right)(U,Y,Z) + \mathbb{R}(\mathbb{T}\left(X,Y\right),Z)U\right] = 0, \tag{3.10}$$

where $\sum$ means cyclic sum over X, Y, Z.

If D is a N-linear connection on T^2M,then by the Theorem 3.1 and

$$\begin{array}{l} v_\beta \left(D_X\mathbb{R}\right)\left(U^H, Y, Z\right) = 0 \;\;,\;\; h\left(D_X\mathbb{R}\right)\left(U^{V_\beta}, Y, Z\right) = 0, \\ v_\beta \left(D_X\mathbb{R}\right)\left(U^{V_\gamma}, Y, Z\right) = 0 \;\;,\;\; \left(\beta, \gamma = 1, 2, \beta \neq \gamma\right), \end{array}$$

the identities (3.9) and (3.10) become:

$$\begin{array}{rl} \sum[h\left(D_X\mathbb{T}\right)(Y,Z) & - \; h\mathbb{R}\left(X,Y\right)Z + \\ & + \; h\mathbb{T}\left(h\mathbb{T}\left(X,Y\right),Z\right) + \sum\limits_{\gamma=1}^{2} h\mathbb{T}\left(v_\gamma\mathbb{T}\left(X,Y\right),Z\right)] = 0, \\ \sum[v_\beta\left(D_X\mathbb{T}\right)(Y,Z) & - \; v_\beta\mathbb{R}\left(X,Y\right)Z + \\ & + \; v_\beta\mathbb{T}\left(h\mathbb{T}\left(X,Y\right),Z\right) + \sum\limits_{\gamma=1}^{2} v_\beta\mathbb{T}\left(v_\gamma\mathbb{T}\left(X,Y\right),Z\right)] = 0, \\ & \qquad (\beta = 1,2) \end{array} \tag{3.11}$$

$$\begin{array}{rl} \sum[h\left(D_X\mathbb{R}\right)(U,Y,Z) & + \; h\mathbb{R}\left(h\mathbb{T}\left(X,Y\right),Z\right)U + \\ & + \; \sum\limits_{\gamma=1}^{2} h\mathbb{R}\left(v_\gamma\mathbb{T}\left(X,Y\right),Z\right)U] = 0, \\ \sum[v_\beta\left(D_X\mathbb{R}\right)(U,Y,Z) & + \; v_\beta\mathbb{R}\left(h\mathbb{T}\left(X,Y\right),Z\right)U + \\ & + \; \sum\limits_{\gamma=1}^{2} v_\beta\mathbb{R}\left(v_\gamma\mathbb{T}\left(X,Y\right),Z\right)U] = 0, (\beta = 1,2). \end{array} \tag{3.12}$$

2.4 The coefficients of an $N-$linear connection

An N-linear connection is characterized by its coefficients in the adapted basis

$$\delta_a = \frac{\delta}{\delta x^a}, \delta_{1a} = \frac{\delta}{\delta y^{(1)a}}, \dot{\partial}_{2a} = \frac{\partial}{\partial y^{(2)a}}.$$

These coefficients obey particular rules of transformation with respect to the changes of local coordinates on manifold T^2M.

We can prove

Theorem 4.1.

1°. *An N-linear connection D can be uniquely represented, in the adapted basis* $\left(\delta_a, \delta_{1a}, \dot{\partial}_{2a}\right)$ *in the form*

$$\begin{cases} D_{\delta_c}\delta_b &= \underset{(00)}{L}{}^a{}_{bc}\delta_a, \ D_{\delta_c}\delta_{1b} &= \underset{(10)}{L}{}^a{}_{bc}\delta_{1a}, \ D_{\delta_c}\dot{\partial}_{2b} &= \underset{(20)}{L}{}^a{}_{bc}\dot{\partial}_{2a}, \\ D_{\delta_{1c}}\delta_b &= \underset{(01)}{C}{}^a{}_{bc}\delta_a, \ D_{\delta_{1c}}\delta_{1b} &= \underset{(11)}{C}{}^a{}_{bc}\delta_{1a}, \ D_{\delta_{1c}}\dot{\partial}_{2b} &= \underset{(21)}{C}{}^a{}_{bc}\dot{\partial}_{2a}, \\ D_{\dot{\partial}_{2c}}\delta_b &= \underset{(02)}{C}{}^a{}_{bc}\delta_a, \ D_{\dot{\partial}_{2c}}\delta_{1b} &= \underset{(12)}{C}{}^a{}_{bc}\delta_{1a}, \ D_{\dot{\partial}_{2c}}\dot{\partial}_{2b} &= \underset{(22)}{C}{}^a{}_{bc}\dot{\partial}_{2a}. \end{cases} \tag{4.1}$$

2°. *With respect to the coordinate transformations (1.3), Ch.1, the coefficients* $\underset{(\alpha 0)}{L}{}^a{}_{bc}\left(x, y^{(1)}, y^{(2)}\right)$ *obey the rule of transformation:*

$$\underset{(\alpha 0)}{\widetilde{L}}{}^a{}_{df}\frac{\partial\widetilde{x}^d}{\partial x^b}\frac{\partial\widetilde{x}^f}{\partial x^c} = \frac{\partial\widetilde{x}^a}{\partial x^f}\underset{(\alpha 0)}{L}{}^f{}_{bc} - \frac{\partial^2\widetilde{x}^a}{\partial x^b\partial x^c} \qquad (\alpha = 0,1,2). \tag{4.2}$$

3°. *The coefficients* $\underset{(\alpha\beta)}{C}{}^a{}_{bc}, (\alpha = 0,1,2; \beta = 1,2)$ *are d-tensor fields of type (1,2):*

$$\underset{(\alpha\beta)}{\widetilde{C}}{}^a{}_{bc} = \frac{\partial\widetilde{x}^a}{\partial x^d}\frac{\partial x^e}{\partial\widetilde{x}^b}\frac{\partial x^f}{\partial\widetilde{x}^c}\underset{(\alpha\beta)}{C}{}^d{}_{ef}, \qquad (\alpha = 0,1,2, \beta = 1,2). \tag{4.2'}$$

Indeed, we can uniquely write

$$D_{\delta_c}\delta_b = \underset{(00)}{L}{}^a{}_{bc}\delta_a + \underset{(00)}{\overset{1}{L}}{}^a{}_{bc}\delta_{1a} + \underset{(00)}{\overset{2}{L}}{}^a{}_{bc}\dot{\partial}_{2a},$$

and taking into account that $D_X\delta_b$ belongs to the horizontal distribution N_0, we get $\underset{(00)}{\overset{1}{L}}{}^a{}_{bc} = 0$, $\underset{(00)}{\overset{2}{L}}{}^a{}_{bc} = 0$. Hence, we have the first equality (4.2) for $\alpha = 0$. Similarly, we prove the following equalities of (4.2). The statements 2° and 3° can be proved by a direct calculus, taking into account the rule of transformations (4.13), Ch. 1, for $\delta_a, \delta_{1a}, \dot{\partial}_{2a}$.

q.e.d.

The system of functions:

$$D\Gamma(N) = \left(\underset{(00)}{L}{}^a{}_{bc}, \underset{(10)}{L}{}^a{}_{bc}, \underset{(20)}{L}{}^a{}_{bc}, \underset{(01)}{C}{}^a{}_{bc}, \underset{(11)}{C}{}^a{}_{bc}, \underset{(21)}{C}{}^a{}_{bc}, \underset{(02)}{C}{}^a{}_{bc}, \underset{(12)}{C}{}^a{}_{bc}, \underset{(22)}{C}{}^a{}_{bc}\right) \tag{4.3}$$

are called the **coefficients of the N-linear connection** D.

The inverse statement of Theorem 4.1 holds also.

Theorem 4.2. *If the systems of functions (4.3) are a priori given over every domain of local chart on the manifold T^2M, having the rule of transformation* *mentioned in Theorem 4.1, then there exists an unique N-linear connection D whose coefficients are just the system of given functions.*

Corollary 4.1. *The following formula hold*

$$\begin{aligned} D_{\delta_c} dx^a &= -\underset{(00)}{L}{}^a{}_{bc} dx^b, D_{\delta_c}\delta y^{(1)a} = -\underset{(10)}{L}{}^a{}_{bc}\delta y^{(1)b}, D_{\delta_c}\delta y^{(2)a} = -\underset{(20)}{L}{}^a{}_{bc}\delta y^{(2)b}, \\ D_{\delta_{1c}} dx^a &= -\underset{(01)}{C}{}^a{}_{bc} dx^b, D_{\delta_{1c}}\delta y^{(1)a} = -\underset{(11)}{C}{}^a{}_{bc}\delta y^{(1)b}, D_{\delta_{1c}}\delta y^{(2)a} = -\underset{(21)}{C}{}^a{}_{bc}\delta y^{(2)b}, \\ D_{\dot\partial_{2c}} dx^a &= -\underset{(02)}{C}{}^a{}_{bc} dx^b, D_{\dot\partial_{2c}}\delta y^{(1)a} = -\underset{(12)}{C}{}^a{}_{bc}\delta y^{(1)b}, D_{\dot\partial_{2c}}\delta y^{(2)a} = -\underset{(22)}{C}{}^a{}_{bc}\delta y^{(2)b}. \end{aligned} \tag{4.4}$$

Indeed, the formula (4.1), the condition of duality between $\left(\delta_a, \delta_{1a}, \dot\partial_{2a}\right)$ and $\left(dx^a, \delta y^{(1)a}, \delta^{(2)a}\right)$ lead to formula (4.4).

2.5 The h_α-, $v_{1\alpha}$- and $v_{2\alpha}$-covariant derivatives in local adapted basis

Let us consider a d-tensor field T, of type (r,s) in the adapted basis $\left(\delta_a, \delta_{1a}, \dot\partial_{2a}\right)$ and its dual $\left(dx^a, \delta y^{(1)}, \delta y^{(2)a}\right)$, (1.3), Ch. 1:

$$T = T^{a_1..a_r}_{b_1...b_s}\delta_{a_1}\otimes ...\otimes \dot\partial_{2a_r}\otimes dx^{b_1}\otimes ...\otimes \delta y^{(2)b_s}. \tag{5.1}$$

For $X = X^H = \overset{(0)}{X}{}^a\delta_a$, applying (4.1), (4.4) and using the properties of the operators $\underset{\alpha}{D}{}^H_X$ we deduce:

$$\underset{\alpha}{D}{}^H_X T = \overset{(0)}{X}{}^d T^{a_1...a_r}_{b_1...b_s\,|\alpha d}\delta_{a_1}\otimes ...\otimes \dot\partial_{2a_r}\otimes dx^{b_1}\otimes ...\otimes \delta y^{(2)b_s}, (\alpha = 0,1,2), \tag{5.2}$$

where

$$\begin{gathered} T^{a_1...a_r}_{b_1...b_s\,|\alpha d} = \delta_d T^{a_1...a_r}_{b_1...b_s} + \underset{(\alpha 0)}{L}{}^{a_1}{}_{cd} T^{ca_2...a_r}_{b_1...b_s} + ...+ \\ + \underset{(\alpha 0)}{L}{}^{a_r}{}_{cd} T^{a_1...a_{r-1}c}_{b_1...b_s} - \underset{(\alpha 0)}{L}{}^{c}{}_{b_1 d} T^{a_1...a_r}_{cb_2...b_s} - ... - \underset{(\alpha 0)}{L}{}^{c}{}_{b_s d} T^{a_1...a_r}_{b_1...b_{s-1}c}, \\ (\alpha = 0,1,2). \end{gathered} \tag{5.2'}$$

The operators "$_{|\alpha d}$" are called **h$_\alpha$-covariant derivatives** with respect to $D\Gamma(N), (\alpha = 0,1,2)$.

Let us consider the operators $\underset{\alpha}{D}{}^{V_1}_X$, for the vector fields $X^{V_1} = \overset{(1)}{X}{}^a\delta_{1a}$, $(\alpha = 0,1,2)$, we obtain for the d-tensor field T from (5.1) the formula:

$$\underset{\alpha}{D}{}^{V_1}_X T = \overset{(1)}{X}{}^d T^{a_1...a_r}_{b_1...b_s}\overset{(1)}{|}_{\alpha d}\delta_{a_1}\otimes ...\otimes \dot\partial_{2a_r}\otimes dx^{b_1}\otimes ...\otimes \delta y^{(2)b_s}, (\alpha = 0,1,2), \tag{5.3}$$

where

$$\begin{gathered}T^{a_1\ldots a_r}_{b_1\ldots b_s}\overset{(1)}{|}_{\alpha d}=\delta_{1d}T^{a_1\ldots a_r}_{b_1\ldots b_s}+\underset{(\alpha 1)}{C}{}^{a_1}{}_{cd}T^{ca_2\ldots a_r}_{b_1\ldots b_s}+\ldots+\\+\underset{(\alpha 1)}{C}{}^{a_r}{}_{cd}T^{a_1\ldots a_{r-1}c}_{b_1\ldots b_s}-\underset{(\alpha 1)}{C}{}^{c}{}_{b_1 d}T^{a_1\ldots a_r}_{cb_2\ldots b_s}-\ldots-\underset{(\alpha 1)}{C}{}^{c}{}_{b_s d}T^{a_1\ldots a_r}_{b_1\ldots b_{s-1}c},\\(\alpha=0,1,2)\,.\end{gathered}\tag{5.3'}$$

The operators " $\overset{(1)}{|}_{\alpha d}$" are called **$\mathbf{v}_{1\alpha}$-covariant derivatives** with respect to $D\Gamma\left(N\right),\left(\alpha=0,1,2\right).$

Finally, taking , $X=X^{V_2}=\overset{(2)}{X}{}^{a}\dot{\partial}_{2a}$, then for $\underset{\alpha}{D}{}^{V_2}_{X}T$ we get

$$\underset{\alpha}{D}{}^{V_2}_{X}T=\overset{(2)}{X}{}^{d}T^{a_1\ldots a_r}_{b_1\ldots b_s}\overset{(2)}{|}_{\alpha d}\delta_{a_1}\otimes\ldots\otimes\dot{\partial}_{2a_r}\otimes dx^{b_1}\otimes\ldots\otimes\delta y^{(2)b_s},\left(\alpha=0,1,2\right),\tag{5.4}$$

where

$$\begin{gathered}T^{a_1\ldots a_r}_{b_1\ldots b_s}\overset{(2)}{|}_{\alpha d}=\dot{\partial}_{2d}T^{a_1\ldots a_r}_{b_1\ldots b_s}+\underset{(\alpha 2)}{C}{}^{a_1}{}_{cd}T^{ca_2\ldots a_r}_{b_1\ldots b_s}+\ldots+\\+\underset{(\alpha 2)}{C}{}^{a_r}{}_{cd}T^{a_1\ldots a_{r-1}c}_{b_1\ldots b_s}-\underset{(\alpha 2)}{C}{}^{c}{}_{b_1 d}T^{a_1\ldots a_r}_{cb_2\ldots b_s}-\ldots-\underset{(\alpha 2)}{C}{}^{c}{}_{b_s d}T^{a_1\ldots a_r}_{b_1\ldots b_{s-1}c},\\(\alpha=0,1,2)\,.\end{gathered}\tag{5.4'}$$

The operators " $\overset{(2)}{|}_{\alpha d}$" are called **$\mathbf{v}_{2\alpha}$-covariant derivatives** with respect to $D\Gamma\left(N\right),\left(\alpha=0,1,2\right)$.

It is not difficult to prove:

Proposition 5.1. *The quantities:*

$$T^{a_1\ldots a_r}_{b_1\ldots b_s|\alpha d},T^{a_1\ldots a_r}_{b_1\ldots b_s}\overset{(1)}{|}_{\alpha d},T^{a_1\ldots a_r}_{b_1\ldots b_s}\overset{(2)}{|}_{\alpha d},\left(\alpha=0,1,2\right),$$

are d-tensor fields of type (r,s+1).

Proposition 5.2. *The operators* "$|_{\alpha d}$", " $\overset{(1)}{|}_{\alpha d}$" *and* " $\overset{(2)}{|}_{\alpha d}$", $(\alpha=0,1,2)$, *have the properties:*

1°. $f_{|\alpha d}=\delta_d f,f\overset{(1)}{|}_{\alpha d}=\delta_{1d}f,f\overset{(2)}{|}_{\alpha d}=\dot{\partial}_{2d}f,\forall f\in F\left(T^2M\right).$

2°. They are distributive with respect to the adition of the d-tensor of the same type.

3°. They commute with the operation of contraction.

4°. They verify the Leibnitz rule with respect to the tensor product.

As an application, let us consider **"the $\left(z^{(1)}\right)$- and $\left(z^{(2)}\right)$-deflection tensor fields"**, where $z^{(1)a}$ and $z^{(2)a}$ are the Liouville d-vector fields of the N linear connection D (see (6.14), Ch.1):

$$\begin{aligned}
\overset{(1)}{\underset{\alpha}{D}}{}^a{}_b &= z^{(1)a}{}_{|\alpha b}, \quad \overset{(11)}{\underset{\alpha}{d}}{}^a{}_b = z^{(1)a}\overset{(1)}{|}{}_{\alpha b}, \quad \overset{(12)}{\underset{\alpha}{d}}{}^a{}_b = z^{(1)a}\overset{(2)}{|}{}_{\alpha b},\\
\overset{(2)}{\underset{\alpha}{D}}{}^a{}_b &= z^{(2)a}{}_{|\alpha b}, \quad \overset{(21)}{\underset{\alpha}{d}}{}^a{}_b = z^{(2)a}\overset{(1)}{|}{}_{\alpha b}, \quad \overset{(22)}{\underset{\alpha}{d}}{}^a{}_b = z^{(2)a}\overset{(2)}{|}{}_{\alpha b},
\end{aligned} \tag{5.5}$$

$$(\alpha = 0, 1, 2).$$

Proposition 5.3. *The ($z^{(1)}$)-deflection tensor fields have the expression*

$$\overset{(1)}{\underset{\alpha}{D}}{}^a{}_b = -\underset{1}{N}{}^a{}_b + z^{(1)c}\underset{(\alpha 0)}{L}{}^a{}_{cb}, \quad \overset{(11)}{\underset{\alpha}{d}}{}^a{}_b = \delta^a_b + z^{(1)c}\underset{(\alpha 1)}{C}{}^a{}_{cb}, \quad \overset{(12)}{\underset{\alpha}{d}}{}^a{}_b = z^{(1)c}\underset{(\alpha 2)}{C}{}^a{}_{cb} \tag{5.5'}$$

$$(\alpha = 0, 1, 2).$$

These equalities are easy to prove, if one notice

$$z^{(1)a}{}_{|\alpha b} = \delta_b z^{(1)a} + z^{(1)c}\underset{(\alpha 0)}{L}{}^a{}_{cb}, \quad z^{(1)a}\overset{(\beta)}{|}{}_{\alpha b} = \delta_{\beta b} z^{(1)a} + z^{(1)c}\underset{(\alpha\beta)}{C}{}^a{}_{cb},$$

$$\left(\alpha = 0, 1, 2, \beta = 1, 2, \delta_{2b} = \dot{\partial}_{2b}\right).$$

Also, we have:

Proposition 5.4. *The ($z^{(2)}$)-deflection tensor fields are given by*

$$\begin{aligned}
\overset{(2)}{\underset{\alpha}{D}}{}^a{}_b &= -\tfrac{1}{2}\left(\underset{2}{N}{}^a{}_b + \underset{2}{M}{}^a{}_b\right) + \tfrac{1}{2}z^{(1)c}\delta_b \underset{1}{N}{}^a{}_c + z^{(2)c}\underset{(\alpha 0)}{L}{}^a{}_{cb},\\
\overset{(21)}{\underset{\alpha}{d}}{}^a{}_b &= \tfrac{1}{2}\underset{1}{N}{}^a{}_b + \tfrac{1}{2}z^{(1)c}\underset{(11)}{B}{}^a{}_{cb} + z^{(2)c}\underset{(\alpha 1)}{C}{}^a{}_{cb},\\
\overset{(22)}{\underset{\alpha}{d}}{}^a{}_b &= \delta^a_b + \tfrac{1}{2}z^{(1)c}\underset{(21)}{B}{}^a{}_{cb} + z^{(2)c}\underset{(\alpha 2)}{C}{}^a{}_{cb},
\end{aligned} \tag{5.5''}$$

$$(\alpha = 0, 1, 2).$$

We conclude this section with the following theorem of existence of N-linear connection on T^2M.

Theorem 5.1. *If the manifold M is paracompact and N is a nonlinear connection on T^2M, with the coefficients $\underset{1}{N}{}^a{}_b, \underset{2}{N}{}^a{}_b$, then there exists a N-linear connection on T^2M.*

Proof. Since M is paracompact, there exists a linear connection on M of local coefficients, say $\Gamma^a{}_{bc}(x)$. Let $\underset{1}{N}{}^a{}_b\left(x, y^{(1)}, y^2\right)$ and $\underset{2}{N}{}^a{}_b\left(x, y^{(1)}, y^2\right)$ be the local coefficients of the nonlinear connection N. We set $\underset{(00)}{L}{}^a{}_{bc} = \Gamma^a{}_{bc}(x)$, $\underset{(10)}{L}{}^a{}_{bc} =$

$\delta_{1b}\underset{1}{N}{}^{a}{}_{c}$, $\underset{(20)}{L}{}^{a}{}_{bc} = \dot{\partial}_{2b}\underset{2}{N}{}^{a}{}_{c} + \underset{1}{N}{}^{a}{}_{d}\dot{\partial}_{2c}\underset{1}{N}{}^{d}{}_{b}$. Thus, taking into account the Proposition 7.3, Ch. 1, we obtain three set of functions which transform, with respect to (1.3),Ch.1, by (4.2), Ch.1. It result that $D\Gamma(N)$ given by

$$D\Gamma(N) = \left(\Gamma^{a}_{bc}(x)\,,\ \underset{(11)}{B}{}^{a}{}_{cb},\ \underset{(22)}{B}{}^{a}{}_{cb}, 0, 0, 0, 0, 0, 0\right)$$

defines an N-linear connection on T^2M.

Definition 5.1. *An N-linear connection D on T^2M with coefficients*

$$D\Gamma(N) = \left(L^{a}_{bc},\ \underset{(11)}{B}{}^{a}{}_{cb},\ \underset{(22)}{B}{}^{a}{}_{cb}, 0,\ \underset{(11)}{C}{}^{a}{}_{bc}, 0, 0, 0,\ \underset{(22)}{C}{}^{a}{}_{bc}\right)$$

is called an N-linear connection of Berwald type on T^2M.

2.6 $\underset{\alpha}{\mathbb{F}}N-$and $JN-$linear connections, $(\alpha = 0,1,2)$

Generally, an N-linear connection D on T^2M is not compatible with the natural (α)-contact structures $\underset{\alpha}{\mathbb{F}}\,(\alpha = 0,1,2)$, determined by the nonlinear connection, given by (8.3), (8.4), (8.5), Ch. 1.

Definition 6.1. *An N-linear connection D on T^2M is called $\underset{\alpha}{\mathbb{F}}N$- linear connection ($\alpha = 0,1,2$) if $\underset{\alpha}{\mathbb{F}}$ is absolutely parallel with respect to D:*

$$D_X\underset{\alpha}{\mathbb{F}} = 0, \qquad \forall X \in \mathcal{X}\left(T^2M\right), (\alpha = 0,1,2)\,. \tag{6.1}$$

By direct calculus we prove:

Theorem 6.1.

1°. *An $\underset{0}{\mathbb{F}}N$-linear connection on T^2M is characterized by the coefficients $\underset{0}{\mathbb{F}}D\Gamma(N)$ given by (4.3) where*

$$\underset{(20)}{L}{}^{a}{}_{bc} = \underset{(10)}{L}{}^{a}{}_{bc},\ \underset{(21)}{C}{}^{a}{}_{bc} = \underset{(11)}{C}{}^{a}{}_{bc},\ \underset{(22)}{C}{}^{a}{}_{bc} = \underset{(12)}{C}{}^{a}{}_{bc}. \tag{6.2}$$

2°. *An $\underset{1}{\mathbb{F}}N$-linear connection on T^2M is characterized by the coefficients $\underset{1}{\mathbb{F}}D\Gamma(N)$ given by (4.3) where*

$$\underset{(20)}{L}{}^{a}{}_{bc} = \underset{(00)}{L}{}^{a}{}_{bc},\ \underset{(21)}{C}{}^{a}{}_{bc} = \underset{(01)}{C}{}^{a}{}_{bc},\ \underset{(22)}{C}{}^{a}{}_{bc} = \underset{(02)}{C}{}^{a}{}_{bc}. \tag{6.3}$$

3°. *An $\underset{2}{\mathbb{F}}N$-linear connection on T^2M is characterized by the coefficients $\underset{2}{\mathbb{F}}D\Gamma(N)$ given by (4.3) where*

$$\underset{(10)}{L}{}^{a}{}_{bc} = \underset{(00)}{L}{}^{a}{}_{bc},\ \underset{(11)}{C}{}^{a}{}_{bc} = \underset{(01)}{C}{}^{a}{}_{bc},\ \underset{(12)}{C}{}^{a}{}_{bc} = \underset{(02)}{C}{}^{a}{}_{bc}, \tag{6.4}$$

i.e.,

$$\underset{2}{\mathbb{F}}D\Gamma\left(N\right)=\left(\underset{(00)}{L}{}^{a}{}_{bc},\underset{(20)}{L}{}^{a}{}_{bc},\underset{(11)}{C}{}^{a}{}_{bc},\underset{(21)}{C}{}^{a}{}_{bc},\underset{(02)}{C}{}^{a}{}_{bc},\underset{(22)}{C}{}^{a}{}_{bc}\right). \qquad (6.4')$$

Remarks

A. $\underset{0}{F}D\left(\Gamma\right)\left(N\right)=\underset{1}{F}D\left(\Gamma\right)\left(N\right)=\left(\underset{(00)}{L}{}^{a}{}_{bc},\underset{(10)}{L}{}^{a}{}_{bc},\underset{(01)}{C}{}^{a}{}_{bc},\underset{(11)}{C}{}^{a}{}_{bc},\underset{(02)}{C}{}^{a}{}_{bc},\underset{(12)}{C}{}^{a}{}_{bc}\right).$

B. The essential lifts to T^2M a one pairs of metric structures given by non-singular and symmetric d-tensor fields (g_{ab}, h_{ab}) are (9.7) and (9.9), Ch. 1.

Also, an N-linear connection D on T^2M is not compatible with the natural 2-tangent structure J, given by (1.7), Ch. 1.

Definition 6.2. *An N-linear connection D on T^2M is called JN-linear connection if J is absolutely parallel with respect to D:*

$$D_XJ=0, \qquad \forall X\in\mathcal{X}\left(T^2M\right). \qquad (6.5)$$

Theorem 6.2. *A JN-linear connection on T^2M is characterized by the coefficients $JD\Gamma\left(N\right)$ given by (4.3), where*

$$\begin{array}{lllllll}\underset{(00)}{L}{}^{a}{}_{bc} & = & \underset{(10)}{L}{}^{a}{}_{bc} & = & \underset{(20)}{L}{}^{a}{}_{bc} & (= & L^{a}_{bc}),\\ \underset{(01)}{C}{}^{a}{}_{bc} & = & \underset{(11)}{C}{}^{a}{}_{bc} & = & \underset{(21)}{C}{}^{a}{}_{bc} & (= & \underset{(1)}{C}{}^{a}_{bc}),\\ \underset{(02)}{C}{}^{a}{}_{bc} & = & \underset{(12)}{C}{}^{a}{}_{bc} & = & \underset{(22)}{C}{}^{a}{}_{bc} & (= & \underset{(2)}{C}{}^{a}_{bc}).\end{array} \qquad (6.6)$$

Proof. Indeed, by (1.7), Ch.1, we can write

$$\left(D_{\delta_c}J\right)\left(\delta_b\right)=D_{\delta_c}J\left(\delta_b\right)-J\left(D_{\delta_c}\delta_b\right)=D_{\delta_c}\delta_{1b}-J\left(\underset{(00)}{L}{}^{a}{}_{bc}\delta_a\right)=\left(\underset{(10)}{L}{}^{a}{}_{bc}-\underset{(00)}{L}{}^{a}{}_{bc}\right)\delta_{1b},$$

$$\left(D_{\delta_c}J\right)\left(\delta_{1b}\right)=D_{\delta_c}J\left(\delta_{1b}\right)-J\left(D_{\delta_c}\delta_{1b}\right)=D_{\delta_c}\dot{\partial}_{2b}-J\left(\underset{(10)}{L}{}^{a}{}_{bc}\delta_{1a}\right)=\left(\underset{(20)}{L}{}^{a}{}_{bc}-\underset{(10)}{L}{}^{a}{}_{bc}\right)\dot{\partial}_{2a}.$$

Hence, (6.5) gives us the first equalities (6.6). Similarly, we prove the others.

q.e.d.

Remarks

1°. We have

$$\left\{JD\Gamma\left(N\right)\right\}\subset\left\{\underset{\alpha}{\mathbb{F}}D\Gamma\left(N\right)\right\}\subset\left\{D\Gamma\left(N\right)\right\},\left(\alpha=0,1,2\right).$$

2°. For any JN-linear connection, the h_α- and $\mathrm{v}_{\beta\alpha}$-covariant derivatives, $(\alpha=0,1,2,\ \beta=1,2)$, one reduce to h-, v_1- and v_2-covariant derivatives. Also, " $|_{\alpha c}$", $(\alpha=0,1,2)$, one reduce to " $|_c$", only and " $\overset{(\beta)}{|}_{\alpha c}$", $(\alpha=0,1,2,\beta=1,2)$, one reduce to " $\overset{(1)}{|}_{c}$" and " $\overset{(2)}{|}_{c}$", respectively.

3°. For any JN-linear connection with the coefficients

$$JD\Gamma\left(N\right) = \left(L^a_{bc}, \underset{(1)}{C}{}^a{}_{bc}, \underset{(2)}{C}{}^a{}_{bc}\right) \tag{6.7}$$

the deflection d-tensor fields have the expression

$$\overset{(\beta)}{D}{}^a{}_b = z^{(\beta)a}{}_{|b}, \ \overset{(\beta1)}{d}{}^a{}_b = z^{(\beta)a}\overset{(1)}{|}{}_b, \ \overset{(\beta2)}{d}{}^a{}_b = z^{(\beta)a}\overset{(2)}{|}{}_b, (\beta = 1, 2)\,. \tag{6.8}$$

All these correspond to Miron-Atanasiu's theory on $Osc^2M = T^2M$ [89–94].

2.7 The local expression of torsion and curvature

In order to determine the local expressions of d-tensors of torsion and curvature of an N-linear connection we use the covariant derivatives in the adapted basis.

Theorem 7.1. *The d-tensors of torsion of an N-linear connection D with coefficients* $D\Gamma\left(N\right) = \left(\underset{(\alpha0)}{L}{}^a{}_{bc}, \underset{(\alpha1)}{C}{}^a{}_{bc}, \underset{(\alpha2)}{C}{}^a{}_{bc}\right)$, $(\alpha = 0, 1, 2)$*, in the adapted basis (4.12), Ch.1., have the following expressions:*

$$\begin{aligned}
h\mathbb{T}\left(\delta_c, \delta_b\right) &= \underset{(00)}{T}{}^a{}_{bc}\delta_a, \ v_\gamma\mathbb{T}\left(\delta_c, \delta_b\right) = \underset{(0\gamma)}{T}{}^a{}_{bc}\delta_{\gamma a}, \\
&\qquad (\gamma = 1, 2)\,, \\
h\mathbb{T}\left(\delta_{\beta c}, \delta_b\right) &= \underset{(\beta0)}{P}{}^a{}_{bc}\delta_a, \ v_\gamma\mathbb{T}\left(\delta_{\beta c}, \delta_b\right) = \underset{(\beta\gamma)}{P}{}^a{}_{bc}\delta_{\gamma a}, \\
&\qquad (\beta, \gamma = 1, 2, \delta_{2a} = \dot{\partial}_{2a}), \\
v_\gamma\mathbb{T}\left(\dot{\partial}_{2c}, \delta_{1b}\right) &= \underset{(2\gamma)}{Q}{}^a{}_{bc}\delta_{\gamma a}, \\
&\qquad (\gamma = 1, 2, \delta_{2a} = \dot{\partial}_{2a}), \\
v_\gamma\mathbb{T}\left(\delta_{\beta c}, \delta_{\beta b}\right) &= \underset{(\beta\gamma)}{S}{}^a{}_{bc}\delta_{\gamma a}, \\
&\qquad (\beta, \gamma = 1, 2, \delta_{2a} = \dot{\partial}_{2a}),
\end{aligned} \tag{7.1}$$

where

$$\begin{array}{llll}
\underset{(00)}{T}{}^a{}_{bc} = \underset{(00)}{L}{}^a{}_{bc} - \underset{(00)}{L}{}^a{}_{cb}, & \underset{(01)}{T}{}^a{}_{bc} = \underset{(01)}{R}{}^a{}_{bc}, & \underset{(02)}{T}{}^a{}_{bc} = \underset{(02)}{R}{}^a{}_{bc}, \\
\underset{(10)}{P}{}^a{}_{bc} = \underset{(01)}{C}{}^a{}_{bc}, & \underset{(11)}{P}{}^a{}_{bc} = \underset{(11)}{B}{}^a{}_{bc} - \underset{(10)}{L}{}^a{}_{cb}, & \underset{(12)}{P}{}^a{}_{bc} = \underset{(12)}{B}{}^a{}_{bc}, \\
\underset{(20)}{P}{}^a{}_{bc} = \underset{(02)}{C}{}^a{}_{bc}, & \underset{(21)}{P}{}^a{}_{bc} = \underset{(21)}{B}{}^a{}_{bc}, & \underset{(22)}{P}{}^a{}_{bc} = \underset{(22)}{B}{}^a{}_{bc} - \underset{(20)}{L}{}^a{}_{cb}, \\
& \underset{(11)}{S}{}^a{}_{bc} = \underset{(11)}{C}{}^a{}_{bc} - \underset{(11)}{C}{}^a{}_{cb}, & \underset{(12)}{S}{}^a{}_{bc} = \underset{(12)}{R}{}^a{}_{bc}, \\
& \underset{(21)}{Q}{}^a{}_{bc} = \underset{(12)}{C}{}^a{}_{bc}, & \underset{(22)}{Q}{}^a{}_{bc} = \underset{(21)}{B}{}^a{}_{bc} - \underset{(21)}{C}{}^a{}_{cb}, \\
& \underset{(21)}{S}{}^a{}_{bc} = 0, & \underset{(22)}{S}{}^a{}_{bc} = \underset{(22)}{C}{}^a{}_{bc} - \underset{(22)}{C}{}^a{}_{cb}.
\end{array} \tag{7.2}$$

Indeed, (7.1),Ch. 1 and(4.1), Ch.2, imply (7.1) and (7.2).

Especially, we have

$$\underset{(00)}{T}{}^{a}{}_{bc}=\underset{(00)}{L}{}^{a}{}_{bc}-\underset{(00)}{L}{}^{a}{}_{cb},\ \underset{(11)}{S}{}^{a}{}_{bc}=\underset{(11)}{C}{}^{a}{}_{bc}-\underset{(11)}{C}{}^{a}{}_{cb},\ \underset{(22)}{S}{}^{a}{}_{bc}=\underset{(22)}{C}{}^{a}{}_{bc}-\underset{(22)}{C}{}^{a}{}_{cb}. \tag{7.3}$$

Therefore,

Proposition 7.1. *The following statements are equivalent*

1. $\underset{(00)}{T}{}^{a}{}_{bc}=\underset{(11)}{S}{}^{a}{}_{bc}=\underset{(22)}{S}{}^{a}{}_{bc}=0,$
2. $\underset{(00)}{L}{}^{a}{}_{bc}=\underset{(00)}{L}{}^{a}{}_{cb},\ \underset{(11)}{C}{}^{a}{}_{bc}=\underset{(11)}{C}{}^{a}{}_{cb},\ \underset{(22)}{C}{}^{a}{}_{bc}=\underset{(22)}{C}{}^{a}{}_{cb}.$

We pay attention to the N-linear connection given in following definition:

Definition 7.1. *An N-linear connection on T^2M is called semisymmetric if*

$$\underset{(00)}{T}{}^{a}{}_{bc}=\frac{1}{2}\left(\delta^a_b\sigma_c-\delta^a_c\sigma_b\right),\ \underset{(\beta\beta)}{S}{}^{a}{}_{bc}=\frac{1}{2}\left(\delta^a_b\underset{(\beta)}{\tau}{}_c-\delta^a_c\underset{(\beta)}{\tau}{}_b\right),(\beta=1,2)\,, \tag{7.4}$$

where $\sigma,\ \underset{(1)}{\tau},\ \underset{(2)}{\tau}\in\mathcal{X}^*\left(T^2M\right).$

In the next calculus we have need of the following d-tensor fields:

$$\begin{aligned}
&\overset{\alpha}{\underset{(0)}{T}}{}^{a}{}_{bc}=\underset{(\alpha 0)}{L}{}^{a}{}_{bc}-\underset{(\alpha 0)}{L}{}^{a}{}_{cb},\quad \overset{\alpha}{\underset{(\beta\beta)}{P}}{}^{a}{}_{bc}=\underset{(\beta\beta)}{B}{}^{a}{}_{bc}-\underset{(\alpha 0)}{L}{}^{a}{}_{cb},\\
&\overset{\alpha}{\underset{(22)}{Q}}{}^{a}{}_{bc}=\underset{(21)}{B}{}^{a}{}_{bc}-\underset{(\alpha 1)}{C}{}^{a}{}_{cb},\quad \overset{\alpha}{\underset{(\beta)}{S}}{}^{a}{}_{bc}=\underset{(\alpha\beta)}{C}{}^{a}{}_{bc}-\underset{(\alpha\beta)}{C}{}^{a}{}_{cb},\\
&\qquad\qquad(\alpha=0,1,2,\beta=1,2)\,.
\end{aligned} \tag{7.5}$$

We remark that we have

$$\begin{aligned}
&\overset{0}{\underset{(0)}{T}}{}^{a}{}_{bc}=\underset{(00)}{T}{}^{a}{}_{bc},\quad \overset{\beta}{\underset{(\beta\beta)}{P}}{}^{a}{}_{bc}=\underset{(\beta\beta)}{P}{}^{a}{}_{bc},\\
&\overset{2}{\underset{(22)}{Q}}{}^{a}{}_{bc}=\underset{(22)}{Q}{}^{a}{}_{bc},\quad \overset{\beta}{\underset{(\beta)}{S}}{}^{a}{}_{bc}=\underset{(\beta\beta)}{S}{}^{a}{}_{bc},\quad (\beta=1,2)\,.
\end{aligned} \tag{7.6}$$

Proposition 7.2. *For any JN-linear connection with coefficients $JD\Gamma(N)=\left(L^a{}_{bc},\underset{(1)}{C}{}^{a}{}_{bc},\underset{(2)}{C}{}^{a}{}_{bc}\right)$ we get* $(\beta=1,2)$

$$\begin{aligned}
\overset{0}{\underset{(0)}{T}}{}^{a}{}_{bc}&=\overset{1}{\underset{(0)}{T}}{}^{a}{}_{bc}=\overset{2}{\underset{(0)}{T}}{}^{a}{}_{bc}=L^a{}_{bc}-L^a{}_{cb}\quad\left(=\underset{(0)}{T}{}^{a}{}_{bc}\right),\\
\overset{0}{\underset{(\beta\beta)}{P}}{}^{a}{}_{bc}&=\overset{1}{\underset{(\beta\beta)}{P}}{}^{a}{}_{bc}=\overset{2}{\underset{(\beta\beta)}{P}}{}^{a}{}_{bc}=\underset{(\beta\beta)}{B}{}^{a}{}_{bc}-L^a{}_{cb}\quad\left(=\underset{(\beta\beta)}{P}{}^{a}{}_{bc}\right),\\
\overset{0}{\underset{(22)}{Q}}{}^{a}{}_{bc}&=\overset{1}{\underset{(22)}{Q}}{}^{a}{}_{bc}=\overset{2}{\underset{(22)}{Q}}{}^{a}{}_{bc}=\underset{(21)}{B}{}^{a}{}_{bc}-\underset{(1)}{C}{}^{a}{}_{cb}\quad\left(=\underset{(22)}{Q}{}^{a}{}_{bc}\right),\\
\overset{0}{\underset{(\beta)}{S}}{}^{a}{}_{bc}&=\overset{1}{\underset{(\beta)}{S}}{}^{a}{}_{bc}=\overset{2}{\underset{(\beta)}{S}}{}^{a}{}_{bc}=\underset{(\beta)}{C}{}^{a}{}_{bc}-\underset{(\beta)}{C}{}^{a}{}_{cb}\quad\left(=\underset{(\beta)}{S}{}^{a}{}_{bc}\right),
\end{aligned} \tag{7.7}$$

Now, it is easy to write the d-tensors of torsion for the $\underset{\alpha}{\mathbb{F}}N$-linear connection $\underset{(\alpha)}{\mathbb{F}} D\Gamma(N), (\alpha = 0,1,2)$, given in Theorem 6.1, and for the JN-linear connection $JD\Gamma(N)$, given by (6.7). For instance, we have

Corollary 7.1. ([91]) *The d-tensors of torsion of a JN-linear connection with the coefficients* $JD\Gamma(N) = \left(L^a_{bc}, \underset{(1)}{C}{}^a_{bc}, \underset{(2)}{C}{}^a_{bc}\right)$ *are the following:*

$$\begin{array}{llllll} \underset{(0)}{T}{}^a_{bc}, & \underset{(01)}{R}{}^a_{bc}, & \underset{(02)}{R}{}^a_{bc}, & & & \\ \underset{(1)}{C}{}^a_{bc}, & \underset{(11)}{P}{}^a_{bc}, & \underset{(12)}{B}{}^a_{bc}, & \underset{(2)}{C}{}^a_{bc}, & \underset{(21)}{B}{}^a_{bc}, & \underset{(22)}{P}{}^a_{bc}, \\ \underset{(1)}{S}{}^a_{bc}, & \underset{(12)}{R}{}^a_{bc}, & \underset{(22)}{Q}{}^a_{bc}, & \underset{(2)}{S}{}^a_{bc}. & & \end{array} \tag{7.8}$$

Indeed, (6.6), (7.2) and (7.7), imply (7.8).

The local expressions of the d-tensors of curvature of an N-linear connection D with the coefficients $D\Gamma(N) = \left(\underset{(\alpha 0)}{L}{}^a_{bc}, \underset{(\alpha 1)}{C}{}^a_{bc}, \underset{(\alpha 2)}{C}{}^a_{bc}\right)$ $(\alpha = 0,1,2)$, in the adapted basis (4.12), Ch.1, can be found from the formulae (3.4).

If X,Y$\in \left\{\delta_a, \delta_{1a}, \dot{\partial}_{2a}\right\}$ we denote $\mathbb{R}(X,Y)$ by

$$\begin{array}{ll} \mathbb{R}(\delta_b, \delta_c) = \underset{(0)}{\mathbb{R}}{}_{bc}, & \mathbb{R}(\delta_{\beta b}, \delta_c) = \underset{(\beta)}{\mathbb{P}}{}_{bc}, \quad \left(\beta = 1,2, \delta_{2a} = \dot{\partial}_{2a}\right), \\ \mathbb{R}\left(\dot{\partial}_{2b}, \delta_{1c}\right) = \underset{(2)}{\mathbb{Q}}{}_{bc}, & \mathbb{R}(\delta_{\beta b}, \delta_{\beta c}) = \underset{(\beta)}{\mathbb{S}}{}_{bc}, \quad \left(\beta = 1,2, \delta_{2a} = \dot{\partial}_{2a}\right), \end{array} \tag{7.9}$$

and the action of $\mathbb{R}(X,Y)$ on Z$\in \left\{\delta_a, \delta_{1a}, \dot{\partial}_{2a}\right\}$ we denote by

$$\begin{array}{ll} \underset{(0)}{\mathbb{R}}{}_{dc}\delta_{\alpha b} = \underset{(0\alpha)}{R}{}_b{}^a{}_{cd}\delta_{\alpha a}, & \underset{(\beta)}{\mathbb{P}}{}_{dc}\delta_{\alpha b} = \underset{(\beta\alpha)}{P}{}_b{}^a{}_{cd}\delta_{\alpha a}, \\ \underset{(2)}{\mathbb{Q}}{}_{dc}\delta_{\alpha b} = \underset{(2\alpha)}{Q}{}_b{}^a{}_{cd}\delta_{\alpha a}, & \underset{(\beta)}{\mathbb{S}}{}_{dc}\delta_{\alpha b} = \underset{(\beta\alpha)}{S}{}_b{}^a{}_{cd}\delta_{\alpha a}, \end{array} \tag{7.10}$$

$$(\alpha = 0,1,2, y^{(0)} = x, \delta_{0a} = \delta_a, \delta_{2a} = \dot{\partial}_{2a}; \beta = 1,2).$$

By a direct computation, taking into account the Lie brackets (7.1), Ch.1, we get

Theorem 7.2. *An N-linear connection D with the coefficients* $D\Gamma(N) = \left(\underset{(\alpha 0)}{L}{}^a_{bc}, \underset{(\alpha 1)}{C}{}^a_{bc}, \underset{(\alpha 2)}{C}{}^a_{bc}\right)$ $(\alpha = 0,1,2)$ *has the d-tensors of curvature (7.10) expressed by the following formulae:*

$$\begin{aligned} \underset{(0\alpha)}{R}{}_b{}^a{}_{cd} = {} & \delta_d \underset{(\alpha 0)}{L}{}^a_{bc} - \delta_c \underset{(\alpha 0)}{L}{}^a_{bd} + \underset{(\alpha 0)}{L}{}^f_{bc} \underset{(\alpha 0)}{L}{}^a_{fd} - \underset{(\alpha 0)}{L}{}^f_{bd} \underset{(\alpha 0)}{L}{}^a_{fc} + \\ & + \underset{(\alpha 1)}{C}{}^a_{bf} \underset{(01)}{R}{}^f_{cd} + \underset{(\alpha 2)}{C}{}^a_{bf} \underset{(02)}{R}{}^f_{cd}, \end{aligned}$$

$$\underset{(\beta\alpha)}{P}{}_{b\ cd}^{\ a} = \delta_{\beta d}\underset{(\alpha 0)}{L}{}^{a}_{\ bc} - \underset{(\alpha\beta)}{C}{}^{a}_{\ bd|\alpha c} + \underset{(\alpha 1)}{C}{}^{a}_{\ bf}\overset{\alpha}{\underset{(\beta 1)}{P}}{}^{f}_{\ cd} + \underset{(\alpha 2)}{C}{}^{a}_{\ bf}\overset{\alpha}{\underset{(\beta 2)}{P}}{}^{f}_{\ cd}, \tag{7.11}$$

$$\underset{(2\alpha)}{Q}{}_{b\ cd}^{\ a} = \dot{\partial}_{2d}\underset{(\alpha 1)}{C}{}^{a}_{\ bc} - \delta_{1c}\underset{(\alpha 2)}{C}{}^{a}_{\ bd} + \underset{(\alpha 1)}{C}{}^{f}_{\ bc}\underset{(\alpha 2)}{C}{}^{a}_{\ fd} - \underset{(\alpha 2)}{C}{}^{f}_{\ bd}\underset{(\alpha 1)}{C}{}^{a}_{\ fc} + \underset{(\alpha 2)}{C}{}^{a}_{\ bf}\underset{(21)}{P}{}^{f}_{\ cd},$$

$$\underset{(\beta\alpha)}{S}{}_{b\ cd}^{\ a} = \delta_{\beta d}\underset{(\alpha\beta)}{C}{}^{a}_{\ bc} - \delta_{\beta c}\underset{(\alpha\beta)}{C}{}^{a}_{\ bd} + \underset{(\alpha\beta)}{C}{}^{f}_{\ bc}\underset{(\alpha\beta)}{C}{}^{a}_{\ fd} - \underset{(\alpha\beta)}{C}{}^{f}_{\ bd}\underset{(\alpha\beta)}{C}{}^{a}_{\ fc} + \underset{(\alpha 2)}{C}{}^{a}_{\ bf}\underset{(\beta 2)}{R}{}^{f}_{\ cd},$$

$$\left(\alpha = 0,1,2, \beta = 1,2,\ \overset{\alpha}{\underset{(12)}{P}}{}^{a}_{\ bc} = \underset{(12)}{P}{}^{a}_{\ bc},\ \overset{\alpha}{\underset{(21)}{P}}{}^{a}_{\ bc} = \underset{(21)}{P}{}^{a}_{\ bc},\ \underset{(22)}{R}{}^{a}_{\ bc} = 0, y^{(0)} = x, \delta_{2a} = \dot{\partial}_{2a}\right).$$

Taking into account (6.6), (7.8) and (7.11), we obtain

Corollary 7.2. *The essential d-tensors of curvature of a JN-linear connection with the coefficients* $D\Gamma(N) = \left(L^a_{bc}, \underset{(1)}{C}{}^{a}_{\ bc}, \underset{(2)}{C}{}^{a}_{\ bc}\right)$ *are as follows*

$$\begin{aligned}
&R^{\ a}_{b\ cd} &&\left(= \underset{(00)}{R}{}_{b\ cd}^{\ a} = \underset{(01)}{R}{}_{b\ cd}^{\ a} = \underset{(02)}{R}{}_{b\ cd}^{\ a}\right),\\
&\underset{(\beta)}{P}{}_{b\ cd}^{\ a} &&\left(= \underset{(\beta 0)}{P}{}_{b\ cd}^{\ a} = \underset{(\beta 1)}{P}{}_{b\ cd}^{\ a} = \underset{(\beta 2)}{P}{}_{b\ cd}^{\ a}\right),\ (\beta = 1,2),\\
&Q^{\ a}_{b\ cd} &&\left(= \underset{(20)}{Q}{}_{b\ cd}^{\ a} = \underset{(21)}{Q}{}_{b\ cd}^{\ a} = \underset{(22)}{Q}{}_{b\ cd}^{\ a}\right),\\
&\underset{(\beta)}{S}{}_{b\ cd}^{\ a} &&\left(= \underset{(\beta 0)}{S}{}_{b\ cd}^{\ a} = \underset{(\beta 1)}{S}{}_{b\ cd}^{\ a} = \underset{(\beta)}{S}{}_{b\ cd}^{\ a}\right),\ (\beta = 1,2).
\end{aligned} \tag{7.12}$$

From (7.11), the expression of these d-tensors of curvature are easy to write, [14], [91].

2.8 The Ricci identites in the adapted basis

Theorem 8.1. *For any N-linear connection D with the coefficients* $D\Gamma(N) = \left(\underset{(\alpha 0)}{L}{}^{a}_{\ bc}, \underset{(\alpha 1)}{C}{}^{a}_{\ bc}, \underset{(\alpha 2)}{C}{}^{a}_{\ bc}\right)$, $(\alpha = 0,1,2)$, *the following Ricci identities hold:*

$$\begin{aligned}
X^a{}_{|\alpha b|\alpha c} - X^a{}_{|\alpha c|\alpha b} &= X^f\underset{(0\alpha)}{R}{}_{f\ bc}^{\ a} - \overset{\alpha}{\underset{(0)}{T}}{}^{f}_{\ bc}X^a{}_{|\alpha f} - \\
&- \underset{(01)}{R}{}^{f}_{\ bc}X^a\overset{(1)}{|}_{\alpha f} - \underset{(02)}{R}{}^{f}_{\ bc}X^a\overset{(2)}{|}_{\alpha f},\\
X^a{}_{|\alpha b}\overset{(1)}{|}_{\alpha c} - X^a\overset{(1)}{|}_{\alpha c|\alpha b} &= X^f\underset{(1\alpha)}{P}{}_{f\ bc}^{\ a} - \underset{(\alpha 1)}{C}{}^{f}_{\ bc}X^a{}_{|\alpha f} - \\
&- \overset{\alpha}{\underset{(11)}{P}}{}^{f}_{\ bc}X^a\overset{(1)}{|}_{\alpha f} - \underset{(12)}{P}{}^{f}_{\ bc}X^a\overset{(2)}{|}_{\alpha f},
\end{aligned}$$

$$
\begin{aligned}
X^a{}_{|\alpha b}\overset{(2)}{|}_{\alpha c} - X^a\overset{(2)}{|}_{\alpha c|\alpha b} &= X^f \underset{(2\alpha)}{P}{}_f{}^a{}_{bc} - \underset{(\alpha 2)}{C}{}^f{}_{bc} X^a{}_{|\alpha f} - \\
&\quad - \underset{(21)}{P}{}^f{}_{bc} X^a \overset{(1)}{|}_{\alpha f} - \underset{(22)}{\overset{\alpha}{P}}{}^f{}_{bc} X^a \overset{(2)}{|}_{\alpha f}, \\
X^a \overset{(1)}{|}_{\alpha b} \overset{(2)}{|}_{\alpha c} - X^a \overset{(2)}{|}_{\alpha c} \overset{(1)}{|}_{\alpha b} &= X^f \underset{(2\alpha)}{Q}{}_f{}^a{}_{bc} - \underset{(\alpha 2)}{C}{}^f{}_{bc} X^a \overset{(1)}{|}_{\alpha f} - \\
&\quad - \underset{(22)}{\overset{\alpha}{Q}}{}^f{}_{bc} X^a \overset{(2)}{|}_{\alpha f}, \\
X^a \overset{(\beta)}{|}_{\alpha b} \overset{(\beta)}{|}_{\alpha c} - X^a \overset{(\beta)}{|}_{\alpha c} \overset{(\beta)}{|}_{\alpha b} &= X^f \underset{(\beta\alpha)}{S}{}_f{}^a{}_{bc} - \underset{(\beta)}{\overset{\alpha}{S}}{}^f{}_{bc} X^a \overset{(1)}{|}_{\alpha f} - \\
&\quad - \underset{(\beta 2)}{R}{}^f{}_{bc} X^a \overset{(2)}{|}_{\alpha f}, \\
&(\alpha = 0,1,2, \beta = 1,2),
\end{aligned}
\tag{8.1}
$$

where $\underset{(22)}{R}{}^a{}_{bc} = 0$ and $\underset{(0)}{\overset{\alpha}{T}}{}^a{}_{bc}$, $\underset{(\beta\beta)}{\overset{\alpha}{P}}{}^a{}_{bc}$, $\underset{(22)}{\overset{\alpha}{Q}}{}^a{}_{bc}$, $\underset{(\beta)}{\overset{\alpha}{S}}{}^a{}_{bc}$, $(\alpha = 0,1,2, \beta = 1,2)$ are given by (7.6).

Remark

Using the previous considerations we can express the Ricci identities for any JN-linear connections with coefficients $D\Gamma(N) = (L^a_{bc}, \underset{(1)}{C}{}^a{}_{bc}, \underset{(2)}{C}{}^a{}_{bc})$, [91], [92].

As usually, we extend the Ricci identities for any d-tensor field, given by (5.1).

As a first application let us consider a Riemannian metric G on T^2M in the form

$$
G = \underset{(0)}{g}{}_{ab} dx^a \otimes dx^b + \underset{(1)}{g}{}_{ab} \delta y^{(1)a} \otimes \delta y^{(1)b} + \underset{(2)}{g}{}_{ab} \delta y^{(2)a} \otimes \delta y^{(2)b}, \tag{8.2}
$$

having the properties

$$
\underset{(\alpha)}{g}{}_{ab|\alpha c} = 0,\ \underset{(\alpha)}{g}{}_{ab} \overset{(1)}{|}_{\alpha c} = 0,\ \underset{(\alpha)}{g}{}_{ab} \overset{(2)}{|}_{\alpha c} = 0, (\alpha = 0,1,2), \tag{8.3}
$$

with respect to an N-linear connection D.

Then we have:

Theorem 8.2. *If the Riemannian metric G, (8.2), verifies the conditions (8.3), then the following d-tensors*

$$
\begin{cases}
\underset{(0\alpha)}{R}{}_{abcd} = \underset{(\alpha)}{g}{}_{fb} \underset{(0\alpha)}{R}{}_a{}^f{}_{cd}, & \underset{(\beta\alpha)}{P}{}_{abcd} = \underset{(\alpha)}{g}{}_{fb} \underset{(\beta\alpha)}{P}{}_a{}^f{}_{cd}, \\
\underset{(2\alpha)}{Q}{}_{abcd} = \underset{(\alpha)}{g}{}_{fb} \underset{(2\alpha)}{Q}{}_a{}^f{}_{cd}, & \underset{(\beta\alpha)}{S}{}_{abcd} = \underset{(\alpha)}{g}{}_{fb} \underset{(\beta\alpha)}{S}{}_a{}^f{}_{cd},\ (\alpha = 0,1,2),
\end{cases}
\tag{8.3'}
$$

are skew-symmetric in the first two indices (ab).

Indeed, writting the Ricci identities for d-tensor fields $\underset{(\alpha)}{g}_{ab}$ and taking into account by the equations (8.3) we deduce

$$\underset{(\alpha)}{g}_{af}\underset{(0\alpha)}{R}{}^{f}{}_{b\,cd}+\underset{(\alpha)}{g}_{bf}\underset{(0\alpha)}{R}{}^{f}{}_{a\,cd}=0,...$$

And using (8.3), we get $\underset{(0\alpha)}{R}{}_{bacd}+\underset{(0\alpha)}{R}{}_{abcd}=0,(\alpha=0,1,2)$,etc.

The Ricci identities (8.1) applied to the Liouville d-vector fields $z^{(1)a},z^{(2)a}$ lead to the some fundamental identities.

Theorem 8.3. *For any N-linear connection D with the coefficients* $D\Gamma\left(N\right)=\left(\underset{(\alpha0)}{L}{}^{a}{}_{bc},\underset{(\alpha1)}{C}{}^{a}{}_{bc},\underset{(\alpha2)}{C}{}^{a}{}_{bc}\right)$, $(\alpha=0,1,2)$ *the deflection tensor fields satisfy the following identities:*

$$\begin{aligned}
\overset{(\beta)}{\underset{\alpha}{D}}{}^{a}{}_{b|\alpha c}-\overset{(\beta)}{\underset{\alpha}{D}}{}^{a}{}_{c|\alpha b} &= z^{(\beta)f}\underset{(0\alpha)}{R}{}_{f}{}^{a}{}_{bc}-\overset{(\beta)}{\underset{\alpha}{D}}{}^{a}{}_{f}\overset{\alpha}{\underset{(0)}{T}}{}^{f}{}_{bc}-\\
&\quad-\overset{(\beta1)}{\underset{\alpha}{d}}{}^{a}{}_{f}\underset{(01)}{R}{}^{f}{}_{bc}-\overset{(\beta2)}{\underset{\alpha}{d}}{}^{a}{}_{f}\underset{(02)}{R}{}^{f}{}_{bc},\\
\overset{(\beta)}{\underset{\alpha}{D}}{}^{a}{}_{b}\overset{(1)}{|}{}_{\alpha c}-\overset{(\beta1)}{\underset{\alpha}{d}}{}^{a}{}_{c|\alpha b} &= z^{(\beta)f}\underset{(1\alpha)}{P}{}_{f}{}^{a}{}_{bc}-\overset{(\beta)}{\underset{\alpha}{D}}{}^{a}{}_{f}\underset{(\alpha1)}{C}{}^{f}{}_{bc}-\\
&\quad-\overset{(\beta1)}{\underset{\alpha}{d}}{}^{a}{}_{f}\overset{(\alpha)}{\underset{(11)}{P}}{}^{f}{}_{bc}-\overset{(\beta2)}{\underset{\alpha}{d}}{}^{a}{}_{f}\underset{(12)}{P}{}^{f}{}_{bc},\\
\overset{(\beta)}{\underset{\alpha}{D}}{}^{a}{}_{b}\overset{(2)}{|}{}_{\alpha c}-\overset{(\beta2)}{\underset{\alpha}{d}}{}^{a}{}_{c|\alpha b} &= z^{(\beta)f}\underset{(2\alpha)}{P}{}_{f}{}^{a}{}_{bc}-\overset{(\beta)}{\underset{\alpha}{D}}{}^{a}{}_{f}\underset{(\alpha2)}{C}{}^{f}{}_{bc}-\\
&\quad-\overset{(\beta1)}{\underset{\alpha}{d}}{}^{a}{}_{f}\underset{(21)}{P}{}^{f}{}_{bc}-\overset{(\beta2)}{\underset{\alpha}{d}}{}^{a}{}_{f}\overset{(\alpha)}{\underset{(22)}{P}}{}^{f}{}_{bc},\\
\overset{(\beta1)}{\underset{\alpha}{d}}{}^{a}{}_{b}\overset{(2)}{|}{}_{\alpha c}-\overset{(\beta2)}{\underset{\alpha}{d}}{}^{a}{}_{c}\overset{(1)}{|}{}_{\alpha b} &= z^{(\beta)f}\underset{(2\alpha)}{Q}{}_{f}{}^{a}{}_{bc}-\\
&\quad-\overset{(\beta1)}{\underset{\alpha}{d}}{}^{a}{}_{f}\underset{(\alpha2)}{C}{}^{f}{}_{bc}-\overset{(\beta2)}{\underset{\alpha}{d}}{}^{a}{}_{f}\overset{(\alpha)}{\underset{(22)}{Q}}{}^{f}{}_{bc},\\
\overset{(\beta\gamma)}{\underset{\alpha}{d}}{}^{a}{}_{b}\overset{(\gamma)}{|}{}_{\alpha c}-\overset{(\beta\gamma)}{\underset{\alpha}{d}}{}^{a}{}_{c}\overset{(\gamma)}{|}{}_{\alpha b} &= z^{(\beta)f}\underset{(\gamma\alpha)}{S}{}_{f}{}^{a}{}_{bc}-\\
&\quad-\overset{(\beta\gamma)}{\underset{\alpha}{d}}{}^{a}{}_{f}\overset{\gamma}{\underset{(\alpha)}{S}}{}^{f}{}_{bc}-\overset{(\beta2)}{\underset{\alpha}{d}}{}^{a}{}_{f}\underset{(\gamma2)}{R}{}^{f}{}_{bc},
\end{aligned}\tag{8.4}$$

$$(\alpha=0,1,2,\beta,\gamma=1,2,\ \underset{(22)}{R}{}^{a}{}_{bc}=0).$$

Also, if the $\left(z^{(1)}\right)$-deflection tensors and $\left(z^{(2)}\right)$-deflection tensors have the following particular form

$$\begin{cases} \overset{(1)}{\underset{\alpha}{D}}{}^a{}_b = 0, & \overset{(11)}{\underset{\alpha}{d}}{}^a{}_b = \delta^a_b, \quad \overset{(12)}{\underset{\alpha}{d}}{}^a{}_b = 0, \\ \overset{(2)}{\underset{\alpha}{D}}{}^a{}_b = 0, & \overset{(21)}{\underset{\alpha}{d}}{}^a{}_b = 0, \quad \overset{(22)}{\underset{\alpha}{d}}{}^a{}_b = \delta^a_b, \end{cases} \tag{8.5}$$

then, the fundamental identities from (8.4) are very important, especially for applications.

Proposition 8.1. *If the deflection tensors are given by (8.5), then the following identities hold:*

$$\begin{gathered} z^{(\beta)f} \underset{(0\alpha)}{R}{}_f{}^a{}_{bc} = \underset{(0\beta)}{R}{}^a{}_{bc}, \quad z^{(1)f} \underset{(2\alpha)}{P}{}_f{}^a{}_{bc} = \underset{(21)}{P}{}^a{}_{bc}, \quad z^{(2)f} \underset{(1\alpha)}{P}{}_f{}^a{}_{bc} = \underset{(12)}{P}{}^a{}_{bc}, \\ z^{(\beta)f} \underset{(\beta\alpha)}{P}{}_f{}^a{}_{bc} = \overset{\alpha}{\underset{(\beta\beta)}{P}}{}^a{}_{bc}, \quad z^{(1)f} \underset{(2\alpha)}{Q}{}_f{}^a{}_{bc} = \underset{(\alpha 2)}{C}{}^a{}_{bc}, \quad z^{(2)f} \underset{(2\alpha)}{Q}{}_f{}^a{}_{bc} = \overset{\alpha}{\underset{(22)}{Q}}{}^a{}_{bc}, \\ z^{(\beta)f} \underset{(\beta\alpha)}{S}{}_f{}^a{}_{bc} = \overset{\beta}{\underset{(\alpha)}{S}}{}^a{}_{bc}, \quad z^{(1)f} \underset{(2\alpha)}{S}{}_f{}^a{}_{bc} = 0, \qquad z^{(2)f} \underset{(1\alpha)}{S}{}_f{}^a{}_{bc} = \underset{(12)}{R}{}^a{}_{bc}, \\ (\alpha = 0,1,2; \beta = 1,2)\,. \end{gathered} \tag{8.6}$$

Let us consider the covariant h_α- and $\mathrm{v}_{\beta\alpha}$-deflection tensors of $\mathrm{D}\Gamma\,(N)$

$$\overset{(\beta)}{\underset{\alpha}{D}}{}_{ab} = \underset{(\alpha)}{g}{}_{ac} \overset{(\beta)}{\underset{\alpha}{D}}{}^c{}_b, \ \overset{(\beta\gamma)}{\underset{\alpha}{d}}{}_{ab} = \underset{(\alpha)}{g}{}_{ac} \overset{(\beta\gamma)}{\underset{\alpha}{d}}{}^c{}_b (\alpha = 0,1,2; \beta,\gamma = 1,2).$$

Theorem 8.4. *If the Riemann metric G, (8.2), verifies the conditions (8.3), then the covariant deflection tensors satisfy the following identities:*

$$\begin{aligned} \overset{(\beta)}{\underset{\alpha}{D}}{}_{bc|\alpha d} - \overset{(\beta)}{\underset{\alpha}{D}}{}_{bd|\alpha c} &= z^{(\beta)f} \underset{(0\alpha)}{R}{}_{fbcd} - \overset{(\beta)}{\underset{\alpha}{D}}{}_{bf} \overset{\alpha}{\underset{(0)}{T}}{}^f{}_{cd} - \\ &\quad - \overset{(\beta 1)}{\underset{\alpha}{d}}{}_{bf} \underset{(01)}{R}{}^f{}_{cd} - \overset{(\beta 2)}{\underset{\alpha}{d}}{}_{bf} \underset{(02)}{R}{}^f{}_{cd}, \\ \overset{(\beta)}{\underset{\alpha}{D}}{}_{bc} \overset{(1)}{|}{}_{\alpha d} - \overset{(\beta 1)}{\underset{\alpha}{d}}{}_{bd|\alpha c} &= z^{(\beta)f} \underset{(1\alpha)}{P}{}_{fbcd} - \overset{(\beta)}{\underset{\alpha}{D}}{}_{bf} \underset{(\alpha 1)}{C}{}^f{}_{cd} - \\ &\quad - \overset{(\beta 1)}{\underset{\alpha}{d}}{}_{bf} \overset{\alpha}{\underset{(11)}{P}}{}^f{}_{cd} - \overset{(\beta 2)}{\underset{\alpha}{d}}{}_{bf} \underset{(12)}{P}{}^f{}_{cd}, \\ \overset{(\beta)}{\underset{\alpha}{D}}{}_{bc} \overset{(2)}{|}{}_{\alpha d} - \overset{(\beta 2)}{\underset{\alpha}{d}}{}_{bd|\alpha c} &= z^{(\beta)f} \underset{(2\alpha)}{P}{}_{fbcd} - \overset{(\beta)}{\underset{\alpha}{D}}{}_{bf} \underset{(\alpha 2)}{C}{}^f{}_{cd} - \\ &\quad - \overset{(\beta 1)}{\underset{\alpha}{d}}{}_{bf} \underset{(21)}{P}{}^f{}_{cd} - \overset{(\beta 2)}{\underset{\alpha}{d}}{}_{bf} \overset{\alpha}{\underset{(22)}{P}}{}^f{}_{cd}, \\ \overset{(\beta 1)}{\underset{\alpha}{d}}{}_{bc} \overset{(2)}{|}{}_{\alpha d} - \overset{(\beta 2)}{\underset{\alpha}{d}}{}_{bd} \overset{(1)}{|}{}_{\alpha c} &= z^{(\beta)f} \underset{(2\alpha)}{Q}{}_{fbcd} - \\ &\quad - \overset{(\beta 1)}{\underset{\alpha}{d}}{}_{bf} \underset{(\alpha 2)}{C}{}^f{}_{cd} - \overset{(\beta 2)}{\underset{\alpha}{d}}{}_{bf} \underset{(22)}{Q}{}^f{}_{cd}, \end{aligned} \tag{8.7}$$

$$\overset{(\beta\gamma)}{\underset{\alpha}{d}}{}_{bc}\overset{(\gamma)}{|}_{\alpha d} - \overset{(\beta\gamma)}{\underset{\alpha}{d}}{}_{bd}\overset{(\gamma)}{|}_{\alpha c} = z^{(\beta)f}\underset{(\gamma\alpha)}{S}{}_{fbcd} - \\ - \overset{(\beta\gamma)}{\underset{\alpha}{d}}{}_{bf}\overset{\gamma}{\underset{(\alpha)}{S}}{}^{f}{}_{cd} - \overset{(\beta 2)}{\underset{\alpha}{d}}{}_{bf}\underset{(\gamma 2)}{R}{}^{f}{}_{bc},$$

$$\left(\alpha = 0,1,2; \beta,\gamma = 1,2;\ \underset{(22)}{R}{}^{a}{}_{bc} = 0\right).$$

2.9 Parallelism of the vector fields on the manifold T^2M

Let D be an N-linear connection with the coefficients $D\Gamma(N) = (\underset{(\alpha 0)}{L}{}^{a}{}_{bc}, \underset{(\alpha 1)}{C}{}^{a}{}_{bc}, \underset{(\alpha 2)}{C}{}^{a}{}_{bc})$, $(\alpha = 0,1,2)$, in the adapted basis $\left(\delta_a, \delta_{1a}, \dot{\partial}_{2a}\right)$.

Let us consider a smooth parametrized curve $\gamma : I \to T^2M$ having the image in a domain of a chart of T^2M.

Thus, γ has an analytical expression of the form

$$x^a = x^a(t), y^{(1)a} = y^{(1)a}(t), y^{(2)a} = y^{(2)a}(t), t \in I \tag{9.1}$$

The tangent vector field $\dot{\gamma} = \frac{d\gamma}{dt}$, by means of (6.9), Ch. 1, can be written as follows:

$$\dot{\gamma} = \frac{dx^a}{dt}\delta_a + \frac{\delta y^{(1)a}}{dt}\delta_{1a} + \frac{\delta y^{(2)a}}{dt}\dot{\partial}_{2a}, \tag{9.2}$$

where

$$\frac{\delta y^{(1)a}}{dt} = \frac{dy^{(1)a}}{dt} + \underset{1}{M}{}^{a}_{b}\frac{dx^b}{dt}, \frac{\delta y^{(2)a}}{dt} = \frac{dy^{(2)a}}{dt} + \underset{1}{M}{}^{a}_{b}\frac{dy^{(1)b}}{dt} + \underset{2}{M}{}^{a}_{b}\frac{dx^b}{dt}. \tag{9.3}$$

Let us denote

$$D_{\dot{\gamma}}X = \frac{DX}{dt}, DX = \frac{DX}{dt}dt, \forall X \in \mathcal{X}\left(T^2M\right). \tag{9.4}$$

The quantity DX is the **covariant differential** of the vector X, and $\dfrac{DX}{dt}$ is the **covariant differential** along the curve γ.

If X is written in the form

$$X = X^H + X^{V_1} + X^{V_2} = X^{(0)a}\delta_a + X^{(1)a}\delta_{1a} + X^{(2)a}\dot{\partial}_{2a}$$

and we put

$$D_{\dot{\gamma}} = D_{\dot{\gamma}^H} + D_{\dot{\gamma}^{V_1}} + D_{\dot{\gamma}^{V_2}} = D^H_{\dot{\gamma}} + D^{V_1}_{\dot{\gamma}} + D^{V_2}_{\dot{\gamma}} = \frac{dx^a}{dt}D_{\delta_a} + \frac{\delta y^{(1)a}}{dt}D_{\delta_{1a}} + \frac{\delta y^{(2)a}}{dt}D_{\dot{\partial}_{2a}},$$

then, after a straightforward calculus, we have

$$\begin{aligned} DX = &\left(dX^{(0)a} + X^{(0)f} \underset{(0)}{\omega}{}^{a}{}_{f}\right)\delta_a + \\ &+\left(dX^{(1)a} + X^{(1)f} \underset{(1)}{\omega}{}^{a}{}_{f}\right)\delta_{1a} + \left(dX^{(2)a} + X^{(2)f} \underset{(2)}{\omega}{}^{a}{}_{f}\right)\dot{\partial}_{2a}. \end{aligned} \tag{9.5}$$

where

$$\underset{(\alpha)}{\omega}{}^{a}{}_{b} = \underset{(\alpha 0)}{L}{}^{a}{}_{bc} dx^c + \underset{(\alpha 1)}{C}{}^{a}{}_{bc}\delta y^{(1)c} + \underset{(\alpha 2)}{C}{}^{a}{}_{bc}\delta y^{(2)c}, (\alpha = 0, 1, 2). \tag{9.6}$$

The 1-forms $\underset{(0)}{\omega}{}^{a}{}_{b}, \underset{(1)}{\omega}{}^{a}{}_{b}, \underset{(2)}{\omega}{}^{a}{}_{b}$ from (9.6) are called **1-forms of connections** of D.

Putting

$$\frac{\underset{(a)}{\omega}{}^{a}{}_{b}}{dt} = \underset{(\alpha 0)}{L}{}^{a}{}_{bc}\frac{dx^c}{dt} + \underset{(\alpha 1)}{C}{}^{a}{}_{bc}\frac{\delta y^{(1)c}}{dt} + \underset{(\alpha 2)}{C}{}^{a}{}_{bc}\frac{\delta y^{(2)c}}{dt}, \tag{9.6'}$$

the covariant differential along the curve γ is given by

$$\begin{aligned} \frac{DX}{dt} = &\left(\frac{dX^{(0)a}}{dt} + X^{(0)f}\frac{\underset{(0)}{\omega}{}^{a}{}_{f}}{dt}\right)\delta_a + \\ +&\left(\frac{dX^{(1)a}}{dt} + X^{(1)f}\frac{\underset{(1)}{\omega}{}^{a}{}_{f}}{dt}\right)\delta_{1a} + \left(\frac{dX^{(2)a}}{dt} + X^{(2)f}\frac{\underset{(2)}{\omega}{}^{a}{}_{f}}{dt}\right)\dot{\partial}_{2a}. \end{aligned} \tag{9.7}$$

The theory of the parallelism of the vector fields along a curve γ presented in Section 2 of this chapter can be applied here. We obtain:

Theorem 9.1. *The vector field* $X = X^{(0)a}\delta_a + X^{(1)a}\delta_{1a} + X^{(2)a}\dot{\partial}_{2a}$ *is parallel along the parametrized curve* γ*, with respect to* D *if and only if its coordinates* $X^{(0)a}, X^{(1)a}, X^{(2)a}$ *are solutions of the differential equations*

$$\frac{dX^{(\alpha)a}}{dt} + X^{(\alpha)f}\frac{\underset{(\alpha)}{\omega}{}^{a}{}_{f}}{dt} = 0, (\alpha = 0, 1, 2). \tag{9.8}$$

The proof is immediate, by means of the expression (9.7) for $\dfrac{DX}{dt}$.

A theorem of existence and uniqueness for the parallel vector fields along a given parametrized curve in T^2M can be formulated in the classical manner.

The vector field X$\in \mathcal{X}\left(T^2M\right)$ is called **absolutely parallel** with respect to the N-linear connection D$\Gamma\left(N\right)$, if DX=0 for any curve γ. It is equivalent to the integrability of the following system of Pfaff equations

$$dX^{(\alpha)a} + X^{(\alpha)f}\underset{(\alpha)}{\omega}{}^{a}{}_{f} = 0, (\alpha = 0, 1, 2). \tag{9.9}$$

The system (9.9) is equivalent to the system

$$X^{(\alpha)a}{}_{|\alpha b} = X^{(\alpha)}\overset{(1)}{|}{}_{\alpha b} = X^{(\alpha)}\overset{(2)}{|}{}_{\alpha b} = 0, (\alpha = 0,1,2) \tag{9.9'}$$

which must be integrable.

Using the Ricci identities, the system (9.9') is integrable if and only if the coordinates $X^{(\alpha)a}, (\alpha = 0,1,2)$, of the vector X satisfy the following equations

$$X^{(\alpha)f}\underset{(0\alpha)}{R}{}_{f\ bc}^{\ a} = 0, X^{(\alpha)f}\underset{(\beta\alpha)}{P}{}_{f\ bc}^{\ a} = 0, X^{(\alpha)f}\underset{(2\alpha)}{Q}{}_{f\ bc}^{\ a} = 0, X^{(\alpha)f}\underset{(\beta\alpha)}{S}{}_{f\ bc}^{\ a} = 0$$
$$(\alpha = 0,1,2; \beta = 1,2)\,. \tag{9.10}$$

The manifold T^2M is called with **absolutely parallelism** of vectors with respect to D, if any vector field on T^2M is absolutely parallel.

In this case the system (9.10) are verified for any vector field X. It follows:

Theorem 9.2. *The manifold T^2M is with absolutely parallelism of vectors, with respect to the N-linear connection D if, and only if, all d-tensors of curvatures of D vanish.*

The curve γ is autoparallel with respect to D if $D_{\dot\gamma}\dot\gamma = 0$.

By means of (9.2) and (9.7) we deduce

$$\begin{aligned}\frac{D\dot\gamma}{dt} &= \left(\frac{d^2x^a}{dt^2} + \frac{dx^b}{dt}\frac{\underset{(0)}{\omega}{}^a{}_b}{dt}\right)\delta_a + \\ &+ \left(\frac{d}{dt}\frac{\delta y^{(1)a}}{dt} + \frac{\delta y^{(1)b}}{dt}\frac{\underset{(1)}{\omega}{}^a{}_b}{dt}\right)\delta_{1a} + \left(\frac{d}{dt}\frac{\delta y^{(2)a}}{dt} + \frac{\delta y^{(2)b}}{dt}\frac{\underset{(2)}{\omega}{}^a{}_b}{dt}\right)\dot\partial_{2a}.\end{aligned} \tag{9.11}$$

Theorem 9.3. *A smooth parametrized curve (9.1) is an autoparallel curve with respect to the N-linear connection D if and only if the functions $x^a(t)$, $y^{(1)a}(t)$, $y^{(2)a}(t)$, $t \in I$, verify the following system of differential equations*

$$\begin{aligned}\frac{d^2x^a}{dt^2} &+ \frac{dx^b}{dt}\frac{\underset{(0)}{\omega}{}^a{}_b}{dt} &= 0,\\ \frac{d}{dt}\frac{\delta y^{(1)a}}{dt} &+ \frac{\delta y^{(1)a}}{dt}\frac{\underset{(1)}{\omega}{}^a{}_b}{dt} &= 0,\\ \frac{d}{dt}\frac{\delta y^{(2)a}}{dt} &+ \frac{\delta y^{(2)a}}{dt}\frac{\underset{(2)}{\omega}{}^a{}_b}{dt} &= 0.\end{aligned} \tag{9.12}$$

Evidently, the theorem of existence and uniqueness for the autoparallel curve can be easily formulated.

We recall that γ is an horizontal curve if $\dot{\gamma} = \dot{\gamma}^H$. The horizontal curve are characterized by

$$x^a = x^a(t), \frac{\delta y^{(1)a}}{dt} = 0, \frac{\delta y^{(2)a}}{dt} = 0. \tag{9.13}$$

Definition 9.1. *The horizontal path of an N-linear connection D is an horizontal autoparallel curve with respect to D.*

Theorem 9.4. *The horizontal paths of an N-linear connection D are characterized by the system of differential equations:*

$$\frac{d^2x^a}{dt^2} + \underset{(\alpha 0)}{L}{}^{a}{}_{bc}\left(x, y^{(1)}, y^{(2)}\right)\frac{dx^b}{dt}\frac{dx^c}{dt} = 0, \frac{\delta y^{(1)a}}{dt} = 0, \frac{\delta y^{(2)a}}{dt} = 0, \tag{9.14}$$

$$(\alpha = 0, 1, 2).$$

Indeed, the equations (9.13), (9.6') and (9.12) imply (9.14).

A parametrized curve γ is v_β-vertical curve in the point $\mathrm{x}_0 \in M$ if its tangent vector field $\dot{\gamma}$ belongs to the distributions $\mathrm{N}_1$and V_2, respectively, $(\beta = 1, 2)$.

Of course, a v_1-vertical curve γ in the point $\mathrm{x}_0 \in M$ is analytically represented by the equations of the form

$$x^a = x_0^a, y^{(1)a} = y^{(1)a}(t), y^{(2)a} = 0, t \in I, \tag{9.15}$$

and a v_2-vertical curve γ in the point $\mathrm{x}_0 \in M$ is analytically represented by the equations of the form

$$x^a = x_0^a, y^{(1)a} = 0, y^{(2)a} = y^{(2)a}(t), t \in I. \tag{9.15'}$$

We define a v_β-path $(\beta = 1, 2)$ in the point $\mathrm{x}_0 \in M$ with respect to D to be a v_β-vertical curve in the point $\mathrm{x}_0 \in M$, which is an autoparellel curve with respect to D.

By means of (9.15), (9.15') and (9.11) we can prove:

Theorem 9.5.

1°. *The v_1-vertical paths in the point $x_0 \in M$ are characterized by the system of differential equations*

$$x^a = x_0^a, \frac{d^2y^{(1)a}}{dt^2} + \underset{(\alpha 1)}{C}{}^{a}{}_{bc}\left(x_0, y^{(1)}, 0\right)\frac{dy^{(1)b}}{dt}\frac{dy^{(1)c}}{dt} = 0, y^{(2)a} = 0, (\alpha = 0, 1, 2).$$

2°. *The v_2-vertical paths in the point $x_0 \in M$ are characterized by the system of differential equations*

$$x^a = x_0^a, y^{(1)a} = 0, \frac{d^2y^{(2)a}}{dt^2} + \underset{(\alpha 2)}{C}{}^{a}{}_{bc}\left(x_0, 0, y^{(2)}\right)\frac{dy^{(2)b}}{dt}\frac{dy^{(2)c}}{dt} = 0, (\alpha = 0, 1, 2).$$

Remarks

1. We assume that there exists the coefficients $\underset{(\alpha 1)}{C}{}^{a}{}_{bc}(x_0, y^{(1)}, 0)$ and $\underset{(\alpha 2)}{C}{}^{a}{}_{bc}(x_0, 0, y^{(2)a})$.
2. By Theorem 6.2, we can write the results of this section with respect to the JN-linear connection JD on T^2M, [91], [92].

2.10 Structure equations of $N-$linear connection

For an N-linear connection D, with the coefficients $D\Gamma(N) = (\underset{(\alpha 0)}{L}{}^{a}{}_{bc}, \underset{(\alpha 1)}{C}{}^{a}{}_{bc}, \underset{(\alpha 2)}{C}{}^{a}{}_{bc})$, $(\alpha = 0, 1, 2)$, in the adapted basis $\left(\delta_a, \delta_{1a}, \dot{\partial}_{2a}\right)$ we can prove:

Lemma 10.1.

1°. *Each of following geometrical object fields*

$$d\left(dx^a\right) - dx^b \wedge \underset{(\alpha)}{\omega}{}^{a}{}_{b}, d\left(\delta y^{(\beta)a}\right) - \delta y^{(\beta)b} \wedge \underset{(\alpha)}{\omega}{}^{a}{}_{b}, \qquad (\alpha = 0, 1, 2, \beta = 1, 2),$$

are d-vector fields.

2°. *The geometrical object fields*

$$d\underset{(\alpha)}{\omega}{}^{a}{}_{b} - \underset{(\alpha)}{\omega}{}^{c}{}_{b} \wedge \underset{(\alpha)}{\omega}{}^{a}{}_{c}, \qquad (\alpha = 0, 1, 2),$$

and d-tensor fields, with respect to indices a and b.

Using the previous Lemma we can prove, by a straightforward calculus, a fundamental result in the geometry of T^2M.

Theorem 10.1. *For any N-linear connection D, with the coefficients* $D\Gamma(N) = \left(\underset{(\alpha 0)}{L}{}^{a}{}_{bc}, \underset{(\alpha 1)}{C}{}^{a}{}_{bc}, \underset{(\alpha 2)}{C}{}^{a}{}_{bc}\right)$, $(\alpha = 0, 1, 2)$, *the following structure equations hold good:*

$$\begin{aligned} d\left(dx^a\right) - dx^b \wedge \underset{(\alpha)}{\omega}{}^{a}{}_{b} &= -\underset{(\alpha)}{\overset{(0)}{\Omega}}{}^{a}, \\ d\left(\delta y^{(1)a}\right) - \delta y^{(1)b} \wedge \underset{(\alpha)}{\omega}{}^{a}{}_{b} &= -\underset{(\alpha)}{\overset{(1)}{\Omega}}{}^{a}, \\ d\left(\delta y^{(2)a}\right) - \delta y^{(2)b} \wedge \underset{(\alpha)}{\omega}{}^{a}{}_{b} &= -\underset{(\alpha)}{\overset{(2)}{\Omega}}{}^{a}, \qquad (\alpha = 0, 1, 2), \end{aligned} \tag{10.1}$$

and

$$d\underset{(\alpha)}{\omega}{}^{a}{}_{b} - \underset{(\alpha)}{\omega}{}^{f}{}_{b} \wedge \underset{(\alpha)}{\omega}{}^{a}{}_{f} = -\underset{(\alpha)}{\Omega}{}^{a}{}_{b}, \qquad (\alpha = 0, 1, 2), \tag{10.2}$$

where $\underset{(\alpha)}{\overset{(0)}{\Omega}}{}^a, \underset{(\alpha)}{\overset{(1)}{\Omega}}{}^a, \underset{(\alpha)}{\overset{(2)}{\Omega}}{}^a, (\alpha = 0,1,2)$ are the 2-forms of torsion

$$
\begin{aligned}
\underset{(\alpha)}{\overset{(0)}{\Omega}}{}^a &= \tfrac{1}{2}\underset{(0)}{\overset{\alpha}{T}}{}^a{}_{bc}dx^b \wedge dx^c + \\
&\quad + \underset{(\alpha 1)}{C}{}^a{}_{bc}dx^b \wedge \delta y^{(1)c} + \underset{(\alpha 2)}{C}{}^a{}_{bc}dx^b \wedge \delta y^{(2)c},
\end{aligned}
$$

$$
\begin{aligned}
\underset{(\alpha)}{\overset{(1)}{\Omega}}{}^a &= \tfrac{1}{2}\underset{(01)}{R}{}^a{}_{bc}dx^b \wedge dx^c + \\
&\quad + \underset{(11)}{\overset{\alpha}{P}}{}^a{}_{bc}dx^b \wedge \delta y^{(1)c} + \underset{(21)}{P}{}^a{}_{bc}dx^b \wedge \delta y^{(2)c} + \\
&\quad + \tfrac{1}{2}\underset{(1)}{\overset{\alpha}{S}}{}^a{}_{bc}\delta y^{(1)b} \wedge \delta y^{(1)c} + \underset{(\alpha 2)}{C}{}^a{}_{bc}\delta y^{(1)b} \wedge \delta y^{(2)c},
\end{aligned}
\tag{10.3}
$$

$$
\begin{aligned}
\underset{(\alpha)}{\overset{(2)}{\Omega}}{}^a &= \tfrac{1}{2}\underset{(02)}{R}{}^a{}_{bc}dx^b \wedge dx^c + \\
&\quad + \underset{(12)}{P}{}^a{}_{bc}dx^b \wedge \delta y^{(1)c} + \underset{(22)}{\overset{\alpha}{P}}{}^a{}_{bc}dx^b \wedge \delta y^{(2)c} + \\
+\frac{1}{2}\underset{(12)}{R}{}^a{}_{bc}\delta y^{(1)b} \wedge \delta y^{(1)c} &+ \underset{(22)}{\overset{\alpha}{Q}}{}^a{}_{bc}\delta y^{(1)b} \wedge \delta y^{(2)c} + \frac{1}{2}\underset{(2)}{\overset{\alpha}{S}}{}^a{}_{bc}\delta y^{(2)b} \wedge \delta y^{(2)c},
\end{aligned}
$$

and where $\underset{(\alpha)}{\Omega}{}^a{}_b, (\alpha = 0,1,2)$,are the 2-forms of curvature

$$
\begin{aligned}
\underset{(\alpha)}{\Omega}{}^a{}_b &= \tfrac{1}{2}\underset{(0\alpha)}{R}{}_b{}^a{}_{cd}dx^c \wedge dx^d + \\
&\quad + \underset{(1\alpha)}{P}{}_b{}^a{}_{cd}dx^c \wedge \delta y^{(1)d} + \underset{(2\alpha)}{P}{}_b{}^a{}_{cd}dx^c \wedge \delta y^{(2)d} + \\
+\frac{1}{2}\underset{(1\alpha)}{S}{}_b{}^a{}_{cd}\delta y^{(1)c} \wedge \delta y^{(1)d} &+ \underset{(2\alpha)}{Q}{}_b{}^a{}_{cd}\delta y^{(1)c} \wedge \delta y^{(2)d} + \frac{1}{2}\underset{(2\alpha)}{S}{}_b{}^a{}_{cd}\delta y^{(2)c} \wedge \delta y^{(2)d}.
\end{aligned}
\tag{10.4}
$$

Remarks

1°. The theorem 10.1 is extremely important in a theory of submanifold embedding in the total space T^2M of the bundle $\left(T^2M, \pi^2, M\right)$.

2°. For any JN-linear connection JD with coefficients $JD\Gamma(N) = (L^a{}_{bc}, \underset{(1)}{C}{}^a{}_{bc}, \underset{(2)}{C}{}^a{}_{bc})$ we have

$$
\begin{aligned}
&\underset{(0)}{\overset{(0)}{\Omega}}{}^a = \underset{(1)}{\overset{(0)}{\Omega}}{}^a = \underset{(2)}{\overset{(0)}{\Omega}}{}^a = \overset{(0)}{\Omega}{}^a, \underset{(0)}{\overset{(1)}{\Omega}}{}^a = \underset{(1)}{\overset{(1)}{\Omega}}{}^a = \underset{(2)}{\overset{(1)}{\Omega}}{}^a = \overset{(1)}{\Omega}{}^a, \\
&\underset{(0)}{\overset{(2)}{\Omega}}{}^a = \underset{(1)}{\overset{(2)}{\Omega}}{}^a = \underset{(2)}{\overset{(2)}{\Omega}}{}^a = \overset{(2)}{\Omega}{}^a, \underset{(0)}{\Omega}{}^a{}_b = \underset{(1)}{\Omega}{}^a{}_b = \underset{(2)}{\Omega}{}^a{}_b = \Omega^a{}_b.
\end{aligned}
\tag{10.5}
$$

Then, by Theorem 6.2 the structure equations for the JN-linear connection are easy to write, [14].

2.11 The Bianchi identities in the adapted basis

For applications, the form of the Bianchi identities (3.11) and (3.12) in the adapted basis $\left(\delta_a, \delta_{1a}, \dot{\partial}_{2a}\right)$ is needed. In order to get it we shall insert in (3.11) and (3.12) the vector fields X, Y, Z as in the following table:

X	δ_d	δ_{1d}	$\dot{\partial}_{2d}$	δ_{1d}	δ_{1d}	$\dot{\partial}_{2d}$	δ_{1d}	$\dot{\partial}_{2d}$	$\dot{\partial}_{2d}$	$\dot{\partial}_{2d}$
Y	δ_c	δ_c	δ_c	δ_{1c}	$\dot{\partial}_{2c}$	$\dot{\partial}_{2c}$	δ_{1c}	δ_{1c}	$\dot{\partial}_{2c}$	$\dot{\partial}_{2c}$
Z	δ_b	δ_b	δ_b	δ_b	δ_b	δ_b	δ_{1b}	δ_{1b}	δ_{1b}	$\dot{\partial}_{2b}$

and U will be taken successively equal to $\delta_a, \delta_{1a}, \dot{\partial}_{2a}$. Then taking into account (7.1), (7.2), (7.5) and (7.10) we prove

Theorem 11.1. *For any N-linear connection D with the coefficients* $D\Gamma(N) = \left(\underset{(\alpha 0)}{L}{}^{a}{}_{bc}, \underset{(\alpha 1)}{C}{}^{a}{}_{bc}, \underset{(\alpha 2)}{C}{}^{a}{}_{bc}\right)$, $(\alpha = 0,1,2)$, *the following Bianchi identities hold:*

$$\sum^{0}\left[\underset{(0\alpha)}{T}{}^{a}{}_{bc|\alpha d} + \overset{\alpha}{\underset{(0)}{T}}{}^{f}{}_{bc}\underset{(0\alpha)}{T}{}^{a}{}_{df} + \underset{(01)}{T}{}^{f}{}_{bc}\underset{(1\alpha)}{P}{}^{a}{}_{df} + \underset{(02)}{T}{}^{f}{}_{bc}\underset{(2\alpha)}{P}{}^{a}{}_{df} - \overset{\alpha}{\underset{(00)}{R}}{}_{b}{}^{a}{}_{cd}\right] = 0, \quad (\alpha = 0,1,2), \tag{11.1$_1$}$$

where

$$\overset{0}{\underset{(00)}{R}}{}_{b}{}^{a}{}_{cd} = \underset{(00)}{R}{}_{b}{}^{a}{}_{cd}, \quad \overset{\beta}{\underset{(00)}{R}}{}_{b}{}^{a}{}_{cd} = 0, (\beta = 1,2),$$

$$\begin{aligned}
&\underset{(0\alpha)}{T}{}^{a}{}_{bc}\overset{(1)}{|}{}_{\alpha d} - \underset{(1\alpha)}{P}{}^{a}{}_{bd|\alpha c} + \underset{(1\alpha)}{P}{}^{a}{}_{cd|\alpha b} - \\
&- \overset{\alpha}{\underset{(0)}{T}}{}^{f}{}_{bc}\underset{(1\alpha)}{P}{}^{a}{}_{fd} - \underset{(1\alpha)}{C}{}^{f}{}_{bd}\underset{(0\alpha)}{T}{}^{a}{}_{cf} + \underset{(1\alpha)}{C}{}^{f}{}_{cd}\underset{(0\alpha)}{T}{}^{a}{}_{bf} + \\
&+ \underset{(01)}{T}{}^{f}{}_{bc}\underset{(1\alpha)}{S}{}^{a}{}_{df} - \overset{\alpha}{\underset{(11)}{P}}{}^{f}{}_{bd}\underset{(1\alpha)}{P}{}^{a}{}_{cf} + \overset{\alpha}{\underset{(11)}{P}}{}^{f}{}_{cd}\underset{(1\alpha)}{P}{}^{a}{}_{bf} + \\
&+ \underset{(02)}{T}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}^{a}{}_{df} - \underset{(12)}{P}{}^{f}{}_{bd}\underset{(2\alpha)}{P}{}^{a}{}_{cf} + \underset{(12)}{P}{}^{f}{}_{cd}\underset{(2\alpha)}{P}{}^{a}{}_{bf} - \overset{\alpha}{\underset{(10)}{A}}{}_{b}{}^{a}{}_{cd} = 0, \\
&\qquad (\alpha = 0,1,2),
\end{aligned} \tag{11.2$_1$}$$

where

$$\overset{0}{\underset{(10)}{A}}{}_{b}{}^{a}{}_{cd} = \underset{(10)}{P}{}_{b}{}^{a}{}_{cd} - \underset{(10)}{P}{}_{c}{}^{a}{}_{bd}, \quad \overset{1}{\underset{(10)}{A}}{}_{b}{}^{a}{}_{cd} = \underset{(01)}{R}{}_{d}{}^{a}{}_{bc}, \quad \overset{2}{\underset{(10)}{A}}{}_{b}{}^{a}{}_{cd} = 0,$$

$$
\begin{aligned}
&\underset{(0\alpha)}{T}{}^{a}{}_{bc}\overset{(2)}{|}{}_{\alpha d} - \underset{(2\alpha)}{P}{}^{a}{}_{bd|\alpha c} + \underset{(2\alpha)}{P}{}^{a}{}_{cd|\alpha b} - \\
&- \underset{(0)}{\overset{\alpha}{T}}{}^{f}{}_{bc}\underset{(2\alpha)}{P}{}^{a}{}_{fd} - \underset{(20)}{P}{}^{f}{}_{bd}\underset{(0\alpha)}{T}{}^{a}{}_{cf} + \underset{(20)}{P}{}^{f}{}_{cd}\underset{(0\alpha)}{T}{}^{a}{}_{bf} - \\
&- \underset{(01)}{T}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}^{a}{}_{fd} - \underset{(21)}{P}{}^{f}{}_{bd}\underset{(1\alpha)}{P}{}^{a}{}_{cf} + \underset{(21)}{P}{}^{f}{}_{cd}\underset{(1\alpha)}{P}{}^{a}{}_{bf} - \\
&- \underset{(02)}{T}{}^{f}{}_{bc}\underset{(2\alpha)}{S}{}^{a}{}_{fd} - \underset{(22)}{\overset{\alpha}{P}}{}^{f}{}_{bd}\underset{(2\alpha)}{P}{}^{a}{}_{cf} + \underset{(22)}{\overset{\alpha}{P}}{}^{f}{}_{cd}\underset{(2\alpha)}{P}{}^{a}{}_{bf} - \underset{(20)}{\overset{\alpha}{A}}{}_{b}{}^{a}{}_{cd} = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\alpha = 0, 1, 2)\,,
\end{aligned}
\tag{11.3$_1$}
$$

where

$$
\underset{(20)}{\overset{0}{A}}{}_{b}{}^{a}{}_{cd} = \underset{(20)}{P}{}_{b}{}^{a}{}_{cd} - \underset{(20)}{P}{}_{c}{}^{a}{}_{bd},\ \underset{(20)}{\overset{1}{A}}{}_{b}{}^{a}{}_{cd} = 0,\ \underset{(20)}{\overset{2}{A}}{}_{b}{}^{a}{}_{cd} = \underset{(02)}{R}{}_{d}{}^{a}{}_{bc},
$$

$$
\begin{aligned}
&\underset{(1\alpha)}{P}{}^{a}{}_{bc}\overset{(1)}{|}{}_{\alpha d} - \underset{(1\alpha)}{P}{}^{a}{}_{bd}\overset{(1)}{|}{}_{\alpha c} + \underset{(1\alpha)}{S}{}^{a}{}_{cd|\alpha b} - \\
&- \underset{(\alpha 1)}{C}{}^{f}{}_{bc}\underset{(1\alpha)}{P}{}^{a}{}_{fd} + \underset{(\alpha 1)}{C}{}^{f}{}_{bd}\underset{(1\alpha)}{P}{}^{a}{}_{fc} + \\
&+ \underset{(11)}{\overset{\alpha}{P}}{}^{f}{}_{bc}\underset{(1\alpha)}{S}{}^{a}{}_{df} - \underset{(11)}{\overset{\alpha}{P}}{}^{f}{}_{bd}\underset{(1\alpha)}{S}{}^{a}{}_{cf} + \underset{(1)}{\overset{\alpha}{S}}{}^{f}{}_{cd}\underset{(1\alpha)}{P}{}^{a}{}_{bf} + \\
&+ \underset{(12)}{P}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}^{a}{}_{df} - \underset{(12)}{P}{}^{f}{}_{bd}\underset{(2\alpha)}{Q}{}^{a}{}_{cf} + \underset{(12)}{S}{}^{f}{}_{cd}\underset{(2\alpha)}{P}{}^{a}{}_{bf} - \underset{(11)}{\overset{\alpha}{A}}{}_{b}{}^{a}{}_{cd} = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\alpha = 0, 1, 2)\,,
\end{aligned}
\tag{11.4$_1$}
$$

where

$$
\underset{(11)}{\overset{0}{A}}{}_{b}{}^{a}{}_{cd} = \underset{(10)}{S}{}_{b}{}^{a}{}_{cd},\ \underset{(11)}{\overset{1}{A}}{}_{b}{}^{a}{}_{cd} = \underset{(11)}{P}{}_{d}{}^{a}{}_{bc} - \underset{(11)}{P}{}_{c}{}^{a}{}_{bd},\ \underset{(11)}{\overset{2}{A}}{}_{b}{}^{a}{}_{cd} = 0,
$$

$$
\begin{aligned}
&\underset{(2\alpha)}{P}{}^{a}{}_{bc}\overset{(1)}{|}{}_{\alpha d} - \underset{(1\alpha)}{P}{}^{a}{}_{bd}\overset{(2)}{|}{}_{\alpha c} - \underset{(2\alpha)}{Q}{}^{a}{}_{dc|\alpha b} - \\
&- \underset{(\alpha 2)}{C}{}^{f}{}_{bc}\underset{(1\alpha)}{P}{}^{a}{}_{fd} + \underset{(\alpha 1)}{C}{}^{f}{}_{bd}\underset{(2\alpha)}{P}{}^{a}{}_{fc} - \underset{(\alpha 2)}{C}{}^{f}{}_{dc}\underset{(1\alpha)}{P}{}^{a}{}_{bf} + \\
&+ \underset{(21)}{P}{}^{f}{}_{bc}\underset{(1\alpha)}{S}{}^{a}{}_{df} + \underset{(11)}{\overset{\alpha}{P}}{}^{f}{}_{bd}\underset{(2\alpha)}{Q}{}^{a}{}_{fc} - \underset{(22)}{\overset{\alpha}{Q}}{}^{f}{}_{dc}\underset{(2\alpha)}{P}{}^{a}{}_{bf} + \\
&+ \underset{(22)}{\overset{\alpha}{P}}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}^{a}{}_{df} + \underset{(12)}{P}{}^{f}{}_{bd}\underset{(2\alpha)}{S}{}^{a}{}_{fc} - \underset{(12)}{\overset{\alpha}{A}}{}_{b}{}^{a}{}_{cd} = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\alpha = 0, 1, 2)\,,
\end{aligned}
\tag{11.5$_1$}
$$

where

$$
\underset{(12)}{\overset{0}{A}}{}_{b}{}^{a}{}_{cd} = 0,\ \underset{(12)}{\overset{1}{A}}{}_{b}{}^{a}{}_{cd} = \underset{(21)}{P}{}_{d}{}^{a}{}_{bc},\ \underset{(12)}{\overset{2}{A}}{}_{b}{}^{a}{}_{cd} = -\underset{(12)}{P}{}_{c}{}^{a}{}_{bd},
$$

$$\begin{aligned}
&\underset{(2\alpha)}{P}{}^{a}{}_{bc}\overset{(2)}{|}{}_{\alpha d}-\underset{(2\alpha)}{P}{}^{a}{}_{bd}\overset{(2)}{|}{}_{\alpha c}+\underset{(2\alpha)}{S}{}^{a}{}_{cd|\alpha b}-\\
&-\underset{(\alpha 2)}{C}{}^{f}{}_{bc}\underset{(2\alpha)}{P}{}^{a}{}_{fd}+\underset{(\alpha 2)}{C}{}^{f}{}_{bd}\underset{(2\alpha)}{P}{}^{a}{}_{fc}+\underset{(2)}{\overset{\alpha}{S}}{}^{f}{}_{cd}\underset{(2\alpha)}{P}{}^{a}{}_{bf}-\\
&-\underset{(21)}{P}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}^{a}{}_{fd}+\underset{(21)}{P}{}^{f}{}_{bd}\underset{(2\alpha)}{Q}{}^{a}{}_{fc}-\\
&-\underset{(22)}{P}{}^{f}{}_{bc}\underset{(2\alpha)}{S}{}^{a}{}_{fd}+\underset{(22)}{P}{}^{f}{}_{bd}\underset{(2\alpha)}{S}{}^{a}{}_{fc}-\underset{(22)}{\overset{\alpha}{A}}{}^{a}{}_{b\,cd}=0,\\
&\qquad\qquad(\alpha=0,1,2)\,,
\end{aligned}\tag{11.6$_1$}$$

where

$$\underset{(22)}{\overset{0}{A}}{}^{a}{}_{b\,cd}=\underset{(20)}{S}{}^{a}{}_{b\,cd},\ \underset{(22)}{\overset{1}{A}}{}^{a}{}_{b\,cd}=0,\ \underset{(22)}{\overset{2}{A}}{}^{a}{}_{b\,cd}=\underset{(22)}{P}{}^{a}{}_{d\,bc}-\underset{(22)}{P}{}^{a}{}_{c\,bd},$$

$$\begin{aligned}
&\sum^{0}\left[\underset{(1\beta)}{S}{}^{a}{}_{bc}\overset{(1)}{|}{}_{\beta d}+\underset{(1)}{\overset{\beta}{S}}{}^{f}{}_{bc}\underset{(1\beta)}{S}{}^{a}{}_{df}+\underset{(12)}{S}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}^{a}{}_{df}-\underset{(11)}{\overset{\beta}{S}}{}^{a}{}_{b\,cd}\right]\ =\ 0,\\
&(\beta=1,2)\,,
\end{aligned}\tag{11.7$_1$}$$

where

$$\underset{(11)}{\overset{1}{S}}{}^{a}{}_{b\,cd}=\underset{(11)}{S}{}^{a}{}_{b\,cd},\ \underset{(11)}{\overset{2}{S}}{}^{a}{}_{b\,cd}=0,$$

$$\begin{aligned}
&\underset{(1\beta)}{S}{}^{a}{}_{bc}\overset{(2)}{|}{}_{\beta d}-\underset{(2\beta)}{Q}{}^{a}{}_{bd}\overset{(1)}{|}{}_{\beta c}+\underset{(2\beta)}{Q}{}^{a}{}_{cd}\overset{(1)}{|}{}_{\beta b}-\\
&-\underset{(1)}{\overset{\beta}{S}}{}^{f}{}_{bc}\underset{(2\beta)}{Q}{}^{a}{}_{fd}-\underset{(\beta 2)}{C}{}^{f}{}_{bd}\underset{(1\beta)}{S}{}^{a}{}_{cf}+\underset{(\beta 2)}{C}{}^{f}{}_{cd}\underset{(1\beta)}{S}{}^{a}{}_{bf}-\\
&-\underset{(12)}{S}{}^{f}{}_{bc}\underset{(2\beta)}{S}{}^{a}{}_{fd}-\underset{(22)}{\overset{\beta}{Q}}{}^{f}{}_{bd}\underset{(2\beta)}{Q}{}^{a}{}_{cf}+\underset{(22)}{\overset{\beta}{Q}}{}^{f}{}_{cd}\underset{(2\beta)}{Q}{}^{a}{}_{bf}-\underset{(21)}{\overset{\beta}{B}}{}^{a}{}_{b\,cd}=0,\\
&\qquad\qquad(\beta=1,2)\,,
\end{aligned}\tag{11.8$_1$}$$

where

$$\underset{(21)}{\overset{1}{B}}{}^{a}{}_{b\,cd}=\underset{(21)}{Q}{}^{a}{}_{b\,cd}-\underset{(21)}{Q}{}^{a}{}_{c\,bd},\ \underset{(21)}{\overset{2}{B}}{}^{a}{}_{b\,cd}=\underset{(12)}{S}{}^{a}{}_{d\,bc},$$

$$\begin{aligned}
&\underset{(2\beta)}{Q}{}^{a}{}_{bc}\overset{(2)}{|}{}_{\beta d}-\underset{(2\beta)}{Q}{}^{a}{}_{bd}\overset{(2)}{|}{}_{\beta c}+\underset{(2\beta)}{S}{}^{a}{}_{cd}\overset{(1)}{|}{}_{\beta b}-\\
&-\underset{(\beta 2)}{C}{}^{f}{}_{bc}\underset{(2\beta)}{Q}{}^{a}{}_{fd}+\underset{(\beta 2)}{C}{}^{f}{}_{bd}\underset{(2\beta)}{Q}{}^{a}{}_{fc}+\underset{(2)}{\overset{\beta}{S}}{}^{f}{}_{cd}\underset{(2\beta)}{Q}{}^{a}{}_{bf}-\\
&-\underset{(22)}{Q}{}^{f}{}_{bc}\underset{(2\beta)}{S}{}^{a}{}_{fd}+\underset{(22)}{Q}{}^{f}{}_{bd}\underset{(2\beta)}{S}{}^{a}{}_{fc}-\underset{(22)}{\overset{\beta}{B}}{}^{a}{}_{b\,cd}=0,\\
&\qquad\qquad(\beta=1,2)\,,
\end{aligned}\tag{11.9$_1$}$$

where

$$\underset{(22)}{\overset{1}{B}}{}^{a}{}_{b\,cd}=\underset{(21)}{S}{}^{a}{}_{b\,cd},\ \underset{(22)}{\overset{2}{B}}{}^{a}{}_{b\,cd}=\underset{(22)}{Q}{}^{a}{}_{d\,bc}-\underset{(22)}{Q}{}^{a}{}_{c\,bd},$$

$$\sum^{0}\left[\underset{(22)}{S}{}^{a}{}_{bc}\overset{(2)}{|}{}_{2d}+\underset{(22)}{S}{}^{f}{}_{bc}\underset{(22)}{S}{}^{a}{}_{df}-\underset{(22)}{S}{}^{a}{}_{b\,cd}\right]=0, \tag{11.10$_1$}$$

and

$$\begin{aligned}\sum^{0}\Big[&\underset{(0\alpha)}{R}{}_{a}{}^{e}{}_{bc|\alpha d}+\underset{(o\alpha)}{R}{}_{a}{}^{e}{}_{bf}\overset{\alpha}{\underset{(0)}{T}}{}^{f}{}_{cd}+\\&+\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{bf}\underset{(01)}{R}{}^{f}{}_{cd}+\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{bf}\underset{(02)}{R}{}^{f}{}_{cd}\Big]=0,\ (\alpha=0,1,2)\,,\end{aligned} \tag{11.1$_2$}$$

$$\begin{aligned}&\underset{(0\alpha)}{R}{}_{a}{}^{e}{}_{bc}\overset{(1)}{|}{}_{\alpha d}-\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{bd|\alpha c}+\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{cd|\alpha b}-\\&-\overset{\alpha}{\underset{(0)}{T}}{}^{f}{}_{bc}\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{df}-\underset{(\alpha 1)}{C}{}^{f}{}_{bd}\underset{(0\alpha)}{R}{}_{a}{}^{e}{}_{cf}+\underset{(\alpha 1)}{C}{}^{f}{}_{cd}\underset{(0\alpha)}{R}{}_{a}{}^{e}{}_{bf}+\\&+\underset{(01)}{R}{}^{f}{}_{bc}\underset{(1\alpha)}{S}{}_{a}{}^{e}{}_{df}-\overset{\alpha}{\underset{(11)}{P}}{}^{f}{}_{bd}\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{cf}+\overset{\alpha}{\underset{(11)}{P}}{}^{f}{}_{cd}\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{bf}+\\&+\underset{(02)}{R}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{df}-\underset{(12)}{P}{}^{f}{}_{bd}\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{cf}+\underset{(12)}{P}{}^{f}{}_{cd}\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{bf}=0,\\&\qquad(\alpha=0,1,2)\,,\end{aligned} \tag{11.2$_2$}$$

$$\begin{aligned}&\underset{(0\alpha)}{R}{}_{a}{}^{e}{}_{bc}\overset{(2)}{|}{}_{\alpha d}-\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{bd|\alpha c}+\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{cd|\alpha b}-\\&-\overset{\alpha}{\underset{(0)}{T}}{}^{f}{}_{bc}\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{fd}-\underset{(\alpha 2)}{C}{}^{f}{}_{bd}\underset{(0\alpha)}{R}{}_{a}{}^{e}{}_{cf}+\underset{(\alpha 2)}{C}{}^{f}{}_{cd}\underset{(0\alpha)}{R}{}_{a}{}^{e}{}_{bf}-\\&-\underset{(01)}{R}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{fd}-\underset{(21)}{P}{}^{f}{}_{bd}\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{cf}+\underset{(21)}{P}{}^{f}{}_{cd}\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{bf}+\\&+\underset{(02)}{R}{}^{f}{}_{bc}\underset{(2\alpha)}{S}{}_{a}{}^{e}{}_{df}-\overset{\alpha}{\underset{(22)}{P}}{}^{f}{}_{bd}\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{cf}+\overset{\alpha}{\underset{(22)}{P}}{}^{f}{}_{cd}\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{bf}=0,\\&\qquad(\alpha=0,1,2)\,,\end{aligned} \tag{11.3$_2$}$$

$$\begin{aligned}&\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{bc}\overset{(1)}{|}{}_{\alpha d}-\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{bd}\overset{(1)}{|}{}_{\alpha c}+\underset{(1\alpha)}{S}{}_{a}{}^{e}{}_{cd}\overset{(1)}{|}{}_{\alpha b}-\\&-\underset{(\alpha 1)}{C}{}^{f}{}_{bc}\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{fd}+\underset{(\alpha 1)}{C}{}^{f}{}_{bd}\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{fc}+\\&+\overset{\alpha}{\underset{(11)}{P}}{}^{f}{}_{bc}\underset{(1\alpha)}{S}{}_{a}{}^{e}{}_{df}+\overset{\alpha}{\underset{(11)}{P}}{}^{f}{}_{bd}\underset{(1\alpha)}{S}{}_{a}{}^{e}{}_{fc}+\overset{\alpha}{\underset{(1)}{S}}{}^{f}{}_{cd}\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{bf}+\\&+\underset{(12)}{P}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{df}-\underset{(12)}{P}{}^{f}{}_{bd}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{cf}+\underset{(12)}{S}{}^{f}{}_{cd}\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{bf}=0,\\&\qquad(\alpha=0,1,2)\,,\end{aligned} \tag{11.4$_2$}$$

$$\begin{aligned}&\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{bc}\overset{(1)}{|}{}_{\alpha d}-\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{bd}\overset{(2)}{|}{}_{\alpha c}-\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{dc|\alpha b}-\\&-\underset{(\alpha 2)}{C}{}^{f}{}_{bc}\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{fd}+\underset{(\alpha 1)}{C}{}^{f}{}_{bd}\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{fc}+\end{aligned}$$

$$\begin{aligned}
&+ \underset{(21)}{P}{}^{f}{}_{bc}\underset{(1\alpha)}{S}{}_{a}{}^{e}{}_{df} + \overset{\alpha}{\underset{(11)}{P}}{}^{f}{}_{bd}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{fc} - \underset{(\alpha 2)}{C}{}^{f}{}_{dc}\underset{(1\alpha)}{P}{}_{a}{}^{e}{}_{bf} + \\
&+ \overset{\alpha}{\underset{(22)}{P}}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{df} + \underset{(12)}{P}{}^{f}{}_{bd}\underset{(2\alpha)}{S}{}_{a}{}^{e}{}_{fc} - \overset{\alpha}{\underset{(22)}{Q}}{}^{f}{}_{dc}\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{bf} = 0,
\end{aligned} \tag{11.5$_2$}$$

$$(\alpha = 0,1,2),$$

$$\begin{aligned}
&\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{bc}\overset{(2)}{|}{}_{\alpha d} - \underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{bd}\overset{(2)}{|}{}_{\alpha c} + \underset{(2\alpha)}{S}{}_{a}{}^{e}{}_{cd|\alpha b} - \\
&- \underset{(\alpha 2)}{C}{}^{f}{}_{bc}\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{fd} + \underset{(\alpha 2)}{C}{}^{f}{}_{bd}\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{fc} - \\
&- \underset{(21)}{P}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{fd} + \underset{(21)}{P}{}^{f}{}_{bd}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{fc} - \\
&- \overset{\alpha}{\underset{(22)}{P}}{}^{f}{}_{bc}\underset{(2\alpha)}{S}{}_{a}{}^{e}{}_{fd} + \overset{\alpha}{\underset{(12)}{P}}{}^{f}{}_{bd}\underset{(2\alpha)}{S}{}_{a}{}^{e}{}_{fc} - \overset{\alpha}{\underset{(2)}{S}}{}^{f}{}_{cd}\underset{(2\alpha)}{P}{}_{a}{}^{e}{}_{bf} = 0,
\end{aligned} \tag{11.6$_2$}$$

$$(\alpha = 0,1,2),$$

$$\overset{0}{\sum}\left[\underset{(1\alpha)}{S}{}_{a}{}^{e}{}_{bc}\overset{(1)}{|}{}_{\alpha d} + \overset{\alpha}{\underset{(1)}{S}}{}^{f}{}_{bc}\underset{(1\alpha)}{S}{}_{a}{}^{e}{}_{df} + \underset{(12)}{S}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{df}\right] = 0, \tag{11.7$_2$}$$

$$(\alpha = 0,1,2),$$

$$\begin{aligned}
&\underset{(1\alpha)}{S}{}_{a}{}^{e}{}_{bc}\overset{(2)}{|}{}_{\alpha d} - \underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{bd}\overset{(1)}{|}{}_{\alpha c} + \underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{cd}\overset{(1)}{|}{}_{\alpha b} - \\
&- \overset{\alpha}{\underset{(1)}{S}}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{fd} - \underset{(\alpha 2)}{C}{}^{f}{}_{bd}\underset{(1\alpha)}{S}{}_{a}{}^{e}{}_{cf} + \underset{(\alpha 2)}{C}{}^{f}{}_{cd}\underset{(1\alpha)}{S}{}_{a}{}^{e}{}_{bf} - \\
&- \underset{(12)}{S}{}^{f}{}_{bc}\underset{(2\alpha)}{S}{}_{a}{}^{e}{}_{fd} - \overset{\alpha}{\underset{(22)}{Q}}{}^{f}{}_{bd}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{cf} + \overset{\alpha}{\underset{(22)}{Q}}{}^{f}{}_{cd}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{bf} = 0,
\end{aligned} \tag{11.8$_2$}$$

$$(\alpha = 0,1,2),$$

$$\begin{aligned}
&\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{bc}\overset{(2)}{|}{}_{\alpha d} - \underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{bd}\overset{(2)}{|}{}_{\alpha c} + \underset{(2\alpha)}{S}{}_{a}{}^{e}{}_{cd}\overset{(1)}{|}{}_{\alpha b} - \\
&- \underset{(\alpha 2)}{C}{}^{f}{}_{bc}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{fd} + \underset{(\alpha 2)}{C}{}^{f}{}_{bd}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{fc} - \overset{\alpha}{\underset{(2)}{S}}{}^{f}{}_{cd}\underset{(2\alpha)}{Q}{}_{a}{}^{e}{}_{bf} - \\
&- \overset{\alpha}{\underset{(22)}{Q}}{}^{f}{}_{bc}\underset{(2\alpha)}{S}{}_{a}{}^{e}{}_{fd} + \overset{\alpha}{\underset{(22)}{Q}}{}^{f}{}_{bd}\underset{(22)}{S}{}_{a}{}^{e}{}_{fc} = 0,
\end{aligned} \tag{11.9$_2$}$$

$$(\alpha = 0,1,2),$$

$$\overset{0}{\sum}\left[\underset{(2\alpha)}{S}{}_{a}{}^{e}{}_{bc}\overset{(2)}{|}{}_{\alpha d} + \overset{\alpha}{\underset{(22)}{S}}{}^{f}{}_{bc}\underset{(22)}{S}{}_{a}{}^{e}{}_{df}\right] = 0, \tag{11.10$_2$}$$

$$(\alpha = 0,1,2).$$

Here, everywhere, $\overset{0}{\sum}$ means cyclic sum over (d,c,b).

Remarks.

1°. These identities become simpler if

$$\underset{(00)}{T}{}^{a}{}_{bc} = 0, \ \underset{(11)}{S}{}^{a}{}_{bc} = 0, \ \underset{(22)}{S}{}^{a}{}_{bc} = 0.$$

2°. By Theorem 6.2, Proposition 7.2 and Corolarlly 7.2, the Bianchi identities for the JN-linear connection, with the coefficients $JD\Gamma(N) = (L^{a}_{\ bc}, \underset{(1)}{C}{}^{a}{}_{bc}, \underset{(2)}{C}{}^{a}{}_{bc})$ are not difficult to write (see [27]).

Chapter 3

Metric structures on the manifold T^2M

3.1 Metric $N-$linear connections on T^2M

Definition 1.1. *A metric structure on the manifold T^2M is a symmetric covariant tensor field G of the type (0,2) which is non degenerate at each point $u \in T^2M$ and of constant signature on T^2M. If G is positive definite we say it defines a Riemannian structure on T^2M.*

As in the Section 9, Ch. 1, we can prove that there is uniquely a nonlinear connection such that the distribution V_2 is orthogonal to distribution N_1 and N_0 , with respect to G, i.e.:

$$G\left(X^{V_1}, Y^{V_2}\right) = 0, G\left(X^H, Y^{V_2}\right) = 0. \tag{1.1}$$

Proposition 1.1. *A metric structure G on T^2M determines uniquely a nonlinear connection N, if the distribution V_2 is orthogonal to distributions N_1 and N_0. The coefficients $\underset{1}{N}{}^a{}_b$, $\underset{2}{N}{}^a{}_b$ of N are given by*

$$\underset{1}{N}{}^a{}_b = \underset{12}{g}{}_{bc}\underset{22}{g}{}^{ca}, \underset{2}{N}{}^a{}_b = \underset{02}{g}{}_{bc}\underset{22}{g}{}^{ca} - \underset{1}{N}{}^a{}_c\underset{1}{N}{}^c{}_b, \tag{1.2}$$

where

$$\underset{\alpha\beta}{g}{}_{ab} = G\left(\dot{\partial}_{\alpha a}, \dot{\partial}_{\beta b}\right), (\alpha, \beta = 0, 1, 2), \dot{\partial}_{0a} = \dot{\partial}_a, ||\underset{22}{g}{}^{ab}|| = ||\underset{22}{g}{}_{ab}||^{-1}. \tag{1.3}$$

In this Section we shall use only this nonlinear connection.

Also, we suppose that the distribution N_0 is orthogonal to distribution N_1, with respect to G, namely:

$$G\left(X^H, Y^{V_1}\right) = 0. \tag{1.4}$$

Proposition 1.2 *If the distribution* $\underset{0}{N}$ *is orthogonal to distribution* $\underset{1}{N}$ *with respect to G, is necessary that between the components* $\underset{\alpha\beta}{g}{}_{ab}$, $(\alpha,\beta=0,1,2)$ *of G there exists the following relation*

$$\begin{aligned} \underset{01}{g}{}_{ab} - \underset{1}{N}{}^c{}_a \underset{11}{g}{}_{cb} \ - \ \underset{2}{N}{}^c{}_a \underset{12}{g}{}_{cb} - \underset{02}{g}{}_{ac} \underset{1}{N}{}^c{}_b + \\ + \ \underset{1}{N}{}^c{}_a \underset{1}{N}{}^d{}_b \underset{12}{g}{}_{cd} + \underset{2}{N}{}^c{}_a \underset{1}{N}{}^d{}_b \underset{22}{g}{}_{cd} = 0. \end{aligned} \tag{1.5}$$

Proof. By (4.11), Ch. 1, (1.3) and (1.4) we get (1.5).

q.e.d.

Corollary 1.1. *If the distributions* $\underset{0}{N}, \underset{1}{N}, V_2$ *are orthogonals in pairs with respect to a metric structure G and rank(* $\underset{22}{g}{}_{ab}$*)=n, then between the components* $\underset{\alpha\beta}{g}{}_{ab}$ *of G, (1.3), necessary exists the following relation*

$$\underset{01}{g}{}_{ab} - \left(\underset{12}{g}{}_{ac} \underset{11}{g}{}_{db} + \underset{02}{g}{}_{ac} \underset{12}{g}{}_{db} \right) \underset{22}{g}{}^{cd} + \underset{12}{g}{}_{ac} \underset{12}{g}{}_{db} \underset{12}{g}{}_{fg} \underset{22}{g}{}^{cf} \underset{22}{g}{}^{dg} = 0. \tag{1.5'}$$

Let us consider a metric structure G on T^2M and the distributions $\underset{0}{N}, \underset{1}{N}, V_2$ are orthogonals in pairs with respect to a metric structure G. By (1.1) and (1.4) we have the following decomposition of G:

$$G(X,Y) = G(X^H,Y^H) + G(X^{V_1},Y^{V_1}) + G(X^{V_2},Y^{V_2}), X,Y \in \mathcal{X}(T^2M). \tag{1.6}$$

With the other words, G decomposes into sum of three d-tensor fields: G^H of type (0,2) defined by $G^H(X,Y) = G(X^H,Y^H)$, G^{V_1} of type (0,2) defined by $G^{V_1}(X,Y) = G(X^{V_1},Y^{V_1})$, G^{V_2} of type (0,2) defined by $G^{V_2}(X,Y) = G(X^{V_2},Y^{V_2})$.

Locally, these d-tensor fields can be written as

$$G^H = \underset{(0)}{g}{}_{ab} dx^a \otimes dx^b, G^{V_1} = \underset{(1)}{g}{}_{ab} \delta y^{(1)a} \otimes \delta y^{(1)a}, G^{V_2} = \underset{(2)}{g}{}_{ab} \delta y^{(2)a} \otimes \delta y^{(2)b}, \tag{1.7}$$

where

$$\underset{(0)}{g}{}_{ab} = G\left(\delta_a, \delta_b\right), \underset{(1)}{g}{}_{ab} = G\left(\delta_{1a}, \delta_{1b}\right), \underset{(2)}{g}{}_{ab} = G\left(\dot{\partial}_{2a}, \dot{\partial}_{2b}\right), \tag{1.8}$$

$$rank|| \underset{(\alpha)}{g}{}_{ab}|| = n, (\alpha = 0,1,2)\,. \tag{1.9}$$

Thus the decomposition (1.6) looks locally as follows:

$$G = \underset{(0)}{g}{}_{ab} dx^a \otimes dx^b + \underset{(1)}{g}{}_{ab} \delta y^{(1)a} \otimes \delta y^{(1)b} + \underset{(2)}{g}{}_{ab} \delta y^{(2)a} \otimes \delta y^{(2)b}. \tag{1.10}$$

Definition 1.2. *An N-linear connection D on* T^2M *endowed with a metric structure G is said to be a metric N-linear connection if* $D_X G = 0$ *for every* $X \in X\left(T^2M\right)$.

Proposition 1.3. *If a linear connection D on T^2M has the properties:*
a) D preserves by parallelism the vertical distributions N_1 and V_2,
b) $D_X G = 0, X \in X(T^2M)$
then it is a metric N-linear connection.

Proof. It suffices to prove that D preserves by parallelism the horizontal distribution $u \to \underset{0}{N}(u)$. Using a), (1.1) and (1.4) in

$$(D_X G)\left(Y^H, Z^{V_\beta}\right) = XG\left(Y^H, Z^{V_\beta}\right) - G\left(D_X Y^H, Z^{V_\beta}\right) - G\left(Y^H, D_X Z^{V_\beta}\right) = 0,$$
$$(\beta = 1, 2),$$

one gets $G\left(D_X Y^H, Z^{V_\beta}\right) = 0, (\beta = 1, 2)$,for every $Z \in \mathcal{X}\left(T^2M\right)$. Thus, by (1.1) and (1.4), $D_X Y^H$ is an horizontal vector field,

q.e.d.

Proposition 1.4. *An N-linear connection D on T^2M endowed with a metric structure G is a metric N-linear connection if and only if*

$$\begin{array}{l} \underset{0}{D}{}^H_X G^H = 0, \;\; \underset{0}{D}{}^{V_1}_X G^H = 0, \;\; \underset{0}{D}{}^{V_2}_X G^H = 0, \\ \underset{\beta}{D}{}^H_X G^{V_\beta} = 0, \;\; \underset{\beta}{D}{}^{V_1}_X G^{V_\beta} = 0, \;\; \underset{\beta}{D}{}^{V_2}_X G^{V_\beta} = 0, \;\; (\beta = 1, 2) . \end{array} \tag{1.11}$$

Proof. The equation $D_X G = 0$ implies $D^H_X G = 0$, $D^{V_1}_X G = 0$ and $D^{V_2}_X G = 0$. By (1.6) we have

$$(\underset{0}{D}{}^H_X G^H)(Y, Z) + (\underset{1}{D}{}^H_X G^{V_1})(Y, Z) + (\underset{2}{D}{}^H_X G^{V_2})(Y, Z) = (D^H_X G)(Y, Z) = 0, \quad (*)$$

$$(\underset{0}{D}{}^{V_1}_X G^H)(Y, Z) + (\underset{1}{D}{}^{V_1}_X G^{V_1})(Y, Z) + (\underset{2}{D}{}^{V_1}_X G^{V_2})(Y, Z) = (D^{V_1}_X G)(Y, Z) = 0, \quad (**)$$

$$(\underset{0}{D}{}^{V_2}_X G^H)(Y, Z) + (\underset{1}{D}{}^{V_2}_X G^{V_1})(Y, Z) + (\underset{2}{D}{}^{V_2}_X G^{V_2})(Y, Z) = (D^{V_2}_X G)(Y, Z) = 0. \quad (***)$$

Taking in (*) Y=Y^H, Z=Z^H one gets $\underset{0}{D}{}^H_X G^H = 0$, taking Y=$Y^{V_1}$, Z=$Z^{V_1}$ one gets $\underset{1}{D}{}^H_X G^{V_1} = 0$, taking Y=$Y^{V_2}$, Z=$Z^{V_2}$ one obtains $\underset{2}{D}{}^H_X G^{V_2} = 0$. Now, putting in (**) Y=Y^H, Z=Z^H one obtains $\underset{0}{D}{}^{V_1}_X G^H = 0$, putting Y=$Y^{V_1}$, Z=$Z^{V_1}$ one gets $\underset{1}{D}{}^{V_1}_X G^{V_1} = 0$ and then Y=Y^{V_2}, Z=Z^{V_2} one gets $\underset{2}{D}{}^{V_1}_X G^{V_2} = 0$. Similarly, taking in (***) Y=Y^H, Z=Z^H one obtains $\underset{0}{D}{}^{V_2}_X G^H = 0$, putting Y=$Y^{V_1}$, Z=$Z^{V_1}$ one gets $\underset{1}{D}{}^{V_2}_X G^{V_1} = 0$, and then Y=$Y^{V_2}$, Z=$Z^{V_2}$ one obtains $\underset{2}{D}{}^{V_2}_X G^{V_2} = 0$. Conversely, using (1.11) in (*) one results $D^H_X G = 0$, using (1.11) in (**) one results $D^{V_1}_X G = 0$, and then by (***) one deduces $D^{V_2}_X G = 0$. From these three equations it follows $D_X G = 0$.

q.e.d.

We shall now discuss the existence of metric N-linear connection on T^2M.

First we prove

Theorem 1.1. *If $\overset{*}{D}$ is a fixed N-linear connection on T^2M, then the N-linear connection given by the following formulae is metric with respect to G:*

$$
\begin{aligned}
2G^H(\underset{0}{D}{}^H_X Y, Z) &= X^H(G^H)(Y,Z) + Y^H(G^H)(Z,X) - Z^H(G^H)(X,Y) - \\
&- G^H(X,[Y^H,Z^H]) + G^H(Y,[Z^H,X^H]) + G^H(Z,[X^H,Y^H]),
\end{aligned}
$$

$$
\begin{aligned}
\underset{\beta}{D}{}^H_X Y &= \underset{\beta}{\overset{*}{D}}{}^H_X Y + \underset{(\beta 0)}{A}\left(Y^{V_\beta}, X^H\right), \textit{such that} \\
& 2\left(G^{V_\beta}\right)\left(\underset{(\beta 0)}{A}(Y^{V_\beta}, X^H), Z\right) = \left(\underset{\beta}{\overset{*}{D}}{}^H_X G^{V_\beta}\right)(Y^{V_\beta}, Z^{V_\beta}), (\beta = 1,2),
\end{aligned}
$$

$$
\begin{aligned}
\underset{\delta}{D}{}^{V_1}_X Y &= \underset{\delta}{\overset{*}{D}}{}^{V_1}_X Y + \underset{(\delta 1)}{A}\left(Y^{V_\delta}, X^{V_1}\right), \textit{such that} \\
& 2\left(G^{V_\delta}\right)\left(\underset{(\delta 1)}{A}(Y^{V_\delta}, X^{V_1}), Z\right) = \left(\underset{\delta}{\overset{*}{D}}{}^{V_1}_X G^{V_\delta}\right)(Y^{V_\delta}, Z^{V_\delta}), (\delta = 0,2), V_0 = H,
\end{aligned}
$$

$$
\begin{aligned}
2G^{V_1}(\underset{1}{D}{}^{V_1}_X Y, Z) &= X^{V_1}(G^{V_1})(Y,Z) + Y^{V_1}(G^{V_1})(Z,X) - Z^{V_1}(G^{V_1})(X,Y) - \\
&- G^{V_1}(X,[Y^{V_1},Z^{V_1}]) + G^{V_1}(Y,[Z^{V_1},X^{V_1}]) + G^{V_1}(Z,[X^{V_1},Y^{V_1}]),
\end{aligned} \tag{1.12}
$$

$$
\begin{aligned}
\underset{\varepsilon}{D}{}^{V_2}_X Y &= \underset{\varepsilon}{\overset{*}{D}}{}^{V_2}_X Y + \underset{(\varepsilon 2)}{A}\left(Y^{V_\varepsilon}, X^{V_2}\right), \textit{such that} \\
& 2\left(G^{V_\varepsilon}\right)\left(\underset{(\varepsilon 2)}{A}(Y^{V_\varepsilon}, X^{V_2}), Z\right) = \left(\underset{\varepsilon}{\overset{*}{D}}{}^{V_2}_X G^{\varepsilon}\right)(Y^{V_\varepsilon}, Z^{V_\varepsilon}), (\varepsilon = 0,1), V_0 = H,
\end{aligned}
$$

$$
\begin{aligned}
2G^{V_2}(\underset{2}{D}{}^{V_2}_X Y, Z) &= X^{V_2}(G^{V_2})(Y,Z) + Y^{V_2}(G^{V_2})(Z,X) - Z^{V_2}(G^{V_2})(X,Y) - \\
&- G^{V_2}(X,[Y^{V_2},Z^{V_2}]) + G^{V_2}(Y,[Z^{V_2},X^{V_2}]) + G^{V_2}(Z,[X^{V_2},Y^{V_2}]).
\end{aligned}
$$

Proof. It is obvious that the formulae (1.12) uniquely determine D^H_X, $\mathrm{D}^{V_1}_X$ and $\mathrm{D}^{V_2}_X$, hence they uniquely determine an N-linear connection on T^2M. By a direct computation one checks that D^H_X, $\mathrm{D}^{V_1}_X$ and $\mathrm{D}^{V_2}_X$ verify (1.11). Thus D is a metric N-linear connection. We note that the h(hh)-, $v_1(v_1v_1)$- and $v_2(v_2v_2)$-tensors of torsion of D vanish.

q.e.d.

Next, we have:

Theorem 1.2. *Let G be a metric structure on T^2M. There exist metric N-linear connections on T^2M which depend on G, only. One of them is given by (1.12), in which*

$$\begin{aligned} \overset{*}{\underset{\beta}{D}}{}^{H}_{X}Y &= [X^H, Y^{V_\beta}]^{V_\beta}, (\beta = 1,2), \\ \overset{*}{\underset{\delta}{D}}{}^{V_1}_{X}Y &= [X^{V_1}, Y^{V_\delta}]^{H}, (\delta = 0,2), V_0 = H, \\ \overset{*}{\underset{\varepsilon}{D}}{}^{V_2}_{X}Y &= [X^{V_2}, Y^{V_\varepsilon}]^{V_\varepsilon}, (\varepsilon = 0,1), V_0 = H. \end{aligned} \tag{1.13}$$

Proof. It is evident that $\underset{0}{D}{}^{H}_{X}Y = D_{X^H}Y^H$, $\underset{1}{D}{}^{V_1}_{X}Y = D_{X^{V_1}}Y^{V_1}$ and $\underset{2}{D}{}^{V_2}_{X}Y = D_{X^{V_2}}Y^{V_2}$ given by the first, the fourth and the six equations from (1.12) depend on G only. If we chose the N-linear connection $\overset{*}{D}_X$such that $v_\beta \overset{*}{T}\left(X^H, Y^{V_\beta}\right) = 0$, $v_\beta \overset{*}{T}\left(Y^{V_\beta}, X^{V_1}\right) = [X^{V_1}, Y^{V_\beta}]^{V_\beta}$, $h\overset{*}{T}\left(X^H, Y^{V_\beta}\right) = 0$, $(\beta = 1,2)$, then the equations (1.13) hold and by the second, the third and the five equations from (1.12), $\underset{1}{D}{}^{H}_{X}Y$, $\underset{2}{D}{}^{H}_{X}Y$, $\underset{0}{D}{}^{V_1}_{X}Y$, $\underset{2}{D}{}^{V_1}_{X}Y$ and $\underset{0}{D}{}^{V_2}_{X}Y$, $\underset{1}{D}{}^{V_2}_{X}Y$, respectively, depend on G, only.

q.e.d.

Now, we shall express a metric N-linear connection and related results in terms of local coordinate systems.

As we have seen, a metric structure G uniquely determines a nonlinear connection N and if this metric satisfies (1.4), then G takes the local form (1.10), where the basis $(dx^a, \delta y^{(1)a}, \delta y^{(2)a})$, was used.

Traslating the Proposition 1.4 in local coordinates one obtains:

Proposition 1.5. *An N-linear connection on T^2M is a metric N-linear connection if and only if*

$$\underset{(\alpha)}{g}{}_{ab|\alpha c} = 0,\ \underset{(\alpha)}{g}{}_{ab} \overset{(1)}{|}{}_{\alpha c} = 0,\ \underset{(\alpha)}{g}{}_{ab} \overset{(2)}{|}{}_{\alpha c} = 0, (\alpha = 0,1,2). \tag{1.14}$$

If we proceed similarly with the Theorem 1.2 we deduce:

Theorem 1.3. *If the manifold T^2M is endowed with the metric structure G given by (1.10) then there exists on T^2M a metric N-linear connection, depending only on G, whose h(hh)-, $v_1(v_1v_1)$- and $v_2(v_2v_2)$- tensors of torsion vanish. Its local coefficients* $\overset{c}{D}\Gamma(N) = (\underset{(\alpha 0)}{\overset{c}{L}}{}^{a}{}_{bc}.\ \underset{(\alpha 1)}{\overset{c}{C}}{}^{a}{}_{bc},\ \underset{(\alpha 2)}{\overset{c}{C}}{}^{a}{}_{bc})$, $(\alpha = 0,1,2)$, *are as follows:*

$$
\begin{aligned}
\underset{(00)}{\overset{c}{L}}{}^{a}{}_{bc} &= \tfrac{1}{2}\underset{(0)}{g}{}^{ad}(\delta_c \underset{(0)}{g}{}_{bd} + \delta_b \underset{(0)}{g}{}_{dc} - \delta_d \underset{(0)}{g}{}_{bc}),\\
\underset{(\beta 0)}{\overset{c}{L}}{}^{a}{}_{bc} &= \underset{(\beta\beta)}{B}{}^{a}{}_{cb} + \tfrac{1}{2}\underset{(\beta)}{g}{}^{ad}(\delta_c \underset{(\beta)}{g}{}_{bd} - \underset{(\beta\beta)}{B}{}^{f}{}_{cb}\underset{(\beta)}{g}{}_{fd} - \underset{(\beta\beta)}{B}{}^{f}{}_{cd}\underset{(\beta)}{g}{}_{bf}), (\beta = 1,2)\,,\\
\underset{(\delta 1)}{\overset{c}{C}}{}^{a}{}_{bc} &= \tfrac{1}{2}\underset{(\delta)}{g}{}^{ad}\delta_{1c}\underset{(\delta)}{g}{}_{bd}, (\delta = 0,2)\,;\ \underset{(\varepsilon 2)}{\overset{c}{C}}{}^{a}{}_{bc} = \tfrac{1}{2}\underset{(\varepsilon)}{g}{}^{ad}\dot{\partial}_{2c}\underset{(\varepsilon)}{g}{}_{bd}, (\varepsilon = 0,1)\,,\\
\underset{(\beta\beta)}{\overset{c}{C}}{}^{a}{}_{bc} &= \tfrac{1}{2}\underset{(\beta)}{g}{}^{ad}(\delta_{\beta c}\underset{(\beta)}{g}{}_{bd} + \delta_{\beta b}\underset{(\beta)}{g}{}_{dc} - \delta_{\beta d}\underset{(\beta)}{g}{}_{bc}), (\beta = 1,2)\,, \delta_{2a} = \dot{\partial}_{2a}.
\end{aligned}
\tag{1.15}
$$

Definition 1.3. *The metric N-linear connection given by (1.15) will be called the canonical N-linear connection associated with G.*

Let $D\overset{*}{\Gamma}(N) = (\underset{(\alpha 0)}{\overset{*}{L}}{}^{a}{}_{bc}, \underset{(\alpha 1)}{\overset{*}{C}}{}^{a}{}_{bc}, \underset{(\alpha 2)}{\overset{*}{C}}{}^{a}{}_{bc})$, $(\alpha = 0,1,2)$, be an N-linear connection on T^2M which is endowed with a metric structure G.

If we denote by $\overset{*}{|}_{\alpha c}$, $\overset{(1)}{\overset{*}{|}}{}_{\alpha c}$, $\overset{(2)}{\overset{*}{|}}{}_{\alpha c}$ the h_α- and $v_{\beta\alpha}$- covariant derivations $(\alpha = 0,1,2; \beta = 1,2)$, with respect to $D\overset{*}{\Gamma}(N)$, then by a direct calculation one checks that the N-linear connection whose local coefficients are given by

$$
\begin{aligned}
\underset{(\alpha 0)}{L}{}^{a}{}_{bc} &= \underset{(\alpha 0)}{\overset{*}{L}}{}^{a}{}_{bc} + \tfrac{1}{2}\underset{(\alpha)}{g}{}^{ad}\underset{(\alpha)}{g}{}_{db\overset{*}{|}\alpha c},\\
\underset{(\alpha 1)}{C}{}^{a}{}_{bc} &= \underset{(\alpha 1)}{\overset{*}{C}}{}^{a}{}_{bc} + \tfrac{1}{2}\underset{(\alpha)}{g}{}^{ad}\underset{(\alpha)}{g}{}_{db}\overset{(1)}{\overset{*}{|}}{}_{\alpha c},\\
\underset{(\alpha 2)}{C}{}^{a}{}_{bc} &= \underset{(\alpha 2)}{\overset{*}{C}}{}^{a}{}_{bc} + \tfrac{1}{2}\underset{(\alpha)}{g}{}^{ad}\underset{(\alpha)}{g}{}_{db}\overset{(2)}{\overset{*}{|}}{}_{\alpha c}, (\alpha = 0,1,2)\,,
\end{aligned}
\tag{1.16}
$$

is a metrical N-linear connection.

This method of metrisation of an N-linear connection is called the Kawaguchi metrisation process, [44].

Let us associate to G the following operators of Obata type:

$$
\underset{1}{\overset{\alpha}{O}}{}^{cd}_{ab} = \frac{1}{2}(\delta^c_a\delta^d_b - \underset{(\alpha)}{g}{}_{ab}\underset{(\alpha)}{g}{}^{cd}), \underset{2}{\overset{\alpha}{O}}{}^{cd}_{ab} = \frac{1}{2}(\delta^c_a\delta^d_b + \underset{(\alpha)}{g}{}_{ab}\underset{(\alpha)}{g}{}^{cd}), (\alpha = 0,1,2)\,. \tag{1.17}
$$

Theorem 1.4. *The set of all metric N-linear connections with respect to G on the manifold T^2M is given by*

$$
\begin{aligned}
\underset{(\alpha 0)}{L}{}^{a}{}_{bc} &= \underset{(\alpha 0)}{\overset{c}{L}}{}^{a}{}_{bc} + \underset{1}{\overset{\alpha}{O}}{}^{fa}_{bd}\overset{\alpha}{X}{}^{d}{}_{fc},\\
\underset{(\alpha 1)}{C}{}^{a}{}_{bc} &= \underset{(\alpha 1)}{\overset{c}{C}}{}^{a}{}_{bc} + \underset{1}{\overset{\alpha}{O}}{}^{fa}_{bd}\overset{\alpha}{Y}{}^{d}{}_{fc},\\
\underset{(\alpha 2)}{C}{}^{a}{}_{bc} &= \underset{(\alpha 2)}{\overset{c}{C}}{}^{a}{}_{bc} + \underset{1}{\overset{\alpha}{O}}{}^{fa}_{bd}\overset{\alpha}{Z}{}^{d}{}_{fc}, (\alpha = 0,1,2)\,,
\end{aligned}
\tag{1.18}
$$

where $\left(\underset{(\alpha 0)}{\overset{c}{L}}{}^{a}{}_{bc}, \underset{(\alpha 1)}{\overset{c}{C}}{}^{a}{}_{bc}, \underset{(\alpha 2)}{\overset{c}{C}}{}^{a}{}_{bc}\right)$ is the canonical N-linear connection (1.15) and $\overset{\alpha}{X}{}^{a}{}_{bc}, \overset{\alpha}{Y}{}^{a}{}_{bc}, \overset{\alpha}{Z}{}^{a}{}_{bc}$, $(\alpha = 0, 1, 2)$, are arbitrary d-tensor fields.

Proof. See V. Cruceanu, R. Miron [44], V. Oproiu [111].

3.2 Metric $N-$linear connections with the torsion prescribed

We have proved above the existence of metric N-linear connections whose h(hh)-, $v_1(v_1v_1)$- and $v_2(v_2v_2)$- tensors of torsion vanish. But there are certain problems, especially related to the theory of relativity, in which metrical N-linear connections with h(hh)-, $v_1(v_1v_1)$- and $v_2(v_2v_2)$- tensors of torsion prescribed are needed. In the following we show that such metric N-linear connections do exist.

Definition 2.1. *An N-linear connection D on T^2M is $h_0v_{11}v_{22}$-metric with respect to a metric structure G if*

$$\underset{0}{D}{}^{H}_{X}G^{H} = 0, \underset{1}{D}{}^{V_1}_{X}G^{V_1} = 0, \underset{2}{D}{}^{V_2}_{X}G^{V_2} = 0, X \in \mathcal{X}\left(T^2M\right). \tag{2.1}$$

An easy computation in local coordinates leads to

Proposition 2.1. *An N-linear connection $D\overset{c}{\Gamma}(N) = (\underset{(\alpha 0)}{L}{}^{a}{}_{bc}, \underset{(\alpha 1)}{C}{}^{a}{}_{bc}, \underset{(\alpha 2)}{C}{}^{a}{}_{bc})$ is $h_0v_{11}v_{22}$-metric with respect to $G = \underset{(0)}{g}{}_{ab}dx^a \otimes dx^b + \underset{(1)}{g}{}_{ab}\delta y^{(1)a} \otimes \delta y^{(1)b} + \underset{(2)}{g}{}_{ab}\delta y^{(2)a} \otimes \delta y^{(2)b}$ if and only if*

$$\underset{(0)}{g}{}_{ab|0c} = 0, \ \underset{(1)}{g}{}_{ab} \overset{(1)}{|}{}_{1c} = 0, \ \underset{(2)}{g}{}_{ab} \overset{(2)}{|}{}_{2c} = 0. \tag{2.1'}$$

Let us consider an N-linear connection of the Berwald type

$$B\overset{c}{\Gamma}(N) = (\underset{(00)}{\overset{c}{L}}{}^{a}{}_{bc}, \underset{(11)}{B}{}^{a}{}_{cb}, \underset{(22)}{B}{}^{a}{}_{cb}, 0, \underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc}, 0, 0, 0, \underset{(22)}{\overset{c}{C}}{}^{a}{}_{bc}), \tag{2.2}$$

where

$$\begin{aligned} \underset{(00)}{\overset{c}{L}}{}^{a}{}_{bc} &= \frac{1}{2}\underset{(0)}{g}{}^{ad}(\delta_c \underset{(0)}{g}{}_{bd} + \delta_b \underset{(0)}{g}{}_{dc} - \delta_d \underset{(0)}{g}{}_{bc}), \\ \underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc} &= \frac{1}{2}\underset{(1)}{g}{}^{ad}(\delta_{1c} \underset{(1)}{g}{}_{bd} + \delta_{1b} \underset{(1)}{g}{}_{dc} - \delta_{1d} \underset{(1)}{g}{}_{bc}), \\ \underset{(22)}{\overset{c}{C}}{}^{a}{}_{bc} &= \frac{1}{2}\underset{(2)}{g}{}^{ad}(\dot{\partial}_{2c} \underset{(2)}{g}{}_{bd} + \dot{\partial}_{2b} \underset{(2)}{g}{}_{dc} - \dot{\partial}_{2d} \underset{(2)}{g}{}_{bc}). \end{aligned} \tag{2.3}$$

We have:

Proposition 2.2.

1°. The N-linear connection of the Berwald type (2.2) is $h_0v_{11}v_{22}$-metric. It depends on the metric structure G, only.

2°. The d-tensors of torsions of $B\overset{c}{\Gamma}(N)$ are given by

$$
\begin{aligned}
\underset{(00)}{T}{}^{a}{}_{bc} &= 0,\ \underset{(01)}{R}{}^{a}{}_{bc},\ \underset{(02)}{R}{}^{a}{}_{bc},\\
\underset{(\beta 0)}{P}{}^{a}{}_{bc} &= 0,\ \underset{(\beta\beta)}{P}{}^{a}{}_{bc} = 0, (\beta = 1,2)\,,\ \underset{(12)}{P}{}^{a}{}_{bc},\ \underset{(21)}{P}{}^{a}{}_{bc}\\
\underset{(21)}{Q}{}^{a}{}_{bc} &= 0,\ \underset{(22)}{Q}{}^{a}{}_{bc} = \underset{(21)}{P}{}^{a}{}_{bc},\ \underset{(12)}{S}{}^{a}{}_{bc} = \underset{(12)}{R}{}^{a}{}_{bc},\ \underset{(\beta\beta)}{S}{}^{a}{}_{bc} = 0, (\beta = 1,2)\,.
\end{aligned}
$$

3°. The d-tensors of curvature of $B\overset{c}{\Gamma}(N)$ have the following expressions:

$$
\begin{aligned}
\underset{(00)}{R}{}_{b}{}^{a}{}_{cd} &= \delta_d \underset{(00)}{\overset{c}{L}}{}^{a}{}_{bc} - \delta_c \underset{(00)}{\overset{c}{L}}{}^{a}{}_{bd} + \underset{(00)}{\overset{c}{L}}{}^{f}{}_{bc}\underset{(00)}{\overset{c}{L}}{}^{a}{}_{fd} - \underset{(00)}{\overset{c}{L}}{}^{f}{}_{bd}\underset{(00)}{\overset{c}{L}}{}^{a}{}_{fc},\\
\underset{(0\beta)}{R}{}_{b}{}^{a}{}_{cd} &= \delta_d \underset{(\beta\beta)}{B}{}^{a}{}_{cb} - \delta_c \underset{(\beta\beta)}{B}{}^{a}{}_{db} + \underset{(\beta\beta)}{B}{}^{f}{}_{cb}\underset{(\beta\beta)}{B}{}^{a}{}_{df} - \underset{(\beta\beta)}{B}{}^{f}{}_{db}\underset{(\beta\beta)}{B}{}^{a}{}_{cf} +\\
&\quad + \underset{(\beta\beta)}{\overset{c}{C}}{}^{a}{}_{bf}\underset{(0\beta)}{R}{}^{f}{}_{cd}, (\beta = 1,2)\,,
\end{aligned}
$$

$$
\begin{aligned}
\underset{(\beta 0)}{P}{}_{b}{}^{a}{}_{cd} &= \delta_{\beta d} \underset{(00)}{\overset{c}{L}}{}^{a}{}_{bc}, (\beta = 1,2, \delta_{2a} = \dot{\partial}_{2a}),\\
\underset{(\beta\beta)}{P}{}_{b}{}^{a}{}_{cd} &= \delta_{\beta d} \underset{(\beta\beta)}{B}{}^{a}{}_{cb} - \underset{(\beta\beta)}{\overset{c}{C}}{}^{a}{}_{bd|\beta c}, (\beta = 1,2, \delta_{2a} = \dot{\partial}_{2a}),
\end{aligned}
$$

$$
\begin{aligned}
\underset{(12)}{P}{}_{b}{}^{a}{}_{cd} &= \delta_{1d} \underset{(22)}{B}{}^{a}{}_{cb} + \underset{(22)}{\overset{c}{C}}{}^{a}{}_{bf}\underset{(12)}{B}{}^{f}{}_{cd},\\
\underset{(21)}{P}{}_{b}{}^{a}{}_{cd} &= \dot{\partial}_{2d} \underset{(11)}{B}{}^{a}{}_{cb} + \underset{(11)}{\overset{c}{C}}{}^{a}{}_{bf}\underset{(21)}{B}{}^{f}{}_{cd},
\end{aligned}
$$

$$
\begin{aligned}
\underset{(20)}{Q}{}_{b}{}^{a}{}_{cd} &= 0,\ \underset{(21)}{Q}{}_{b}{}^{a}{}_{cd} = \dot{\partial}_{2d} \underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc},\\
\underset{(22)}{Q}{}_{b}{}^{a}{}_{cd} &= -\delta_{1c} \underset{(22)}{\overset{c}{C}}{}^{a}{}_{bd} + \underset{(22)}{\overset{c}{C}}{}^{a}{}_{bf}\underset{(21)}{B}{}^{f}{}_{cd},
\end{aligned}
$$

$$
\begin{aligned}
\underset{(10)}{S}{}_{b}{}^{a}{}_{cd} &= 0,\ \underset{(12)}{S}{}_{b}{}^{a}{}_{cd} = \underset{(22)}{C}{}^{a}{}_{bf}\underset{(12)}{R}{}^{f}{}_{cd},\\
\underset{(\beta\beta)}{S}{}_{b}{}^{a}{}_{cd} &= \delta_{\beta d} \underset{(\beta\beta)}{\overset{c}{C}}{}^{a}{}_{bc} - \delta_{\beta c} \underset{(\beta\beta)}{\overset{c}{C}}{}^{a}{}_{bd} + \underset{(\beta\beta)}{\overset{c}{C}}{}^{f}{}_{bc}\underset{(\beta\beta)}{\overset{c}{C}}{}^{a}{}_{fd} - \underset{(\beta\beta)}{\overset{c}{C}}{}^{f}{}_{bd}\underset{(\beta\beta)}{\overset{c}{C}}{}^{a}{}_{fc},\\
&\qquad\qquad (\beta = 1,2, \delta_{2a} = \dot{\partial}_{2a}).
\end{aligned}
$$

The N-linear connection (2.2) will be called **the canonical Berwald type connection** on T^2M.

Now, we shall prove:

Theorem 2.1. *There exists an unique $h_0v_{11}v_{22}$-metric N-linear connection of the Berwald type with prescribed h(hh)-, $v_1(v_1v_1)$- and $v_2(v_2v_2)$- tensors of torsion .*

Proof. Let us fix the Berwald connection $B\overset{c}{\Gamma}(N)$ introduced in the above. Then, by the general theory of connections, every other N-linear connection of the Berwald type is of the form

$$(\underset{(00)}{L}{}^a{}_{bc} + \underset{(0)}{\tau}{}^a{}_{bc}, \underset{(11)}{B}{}^a{}_{bc}, \underset{(22)}{B}{}^a{}_{cb}, 0, \underset{(11)}{C}{}^a{}_{bc} + \underset{(1)}{\tau}{}^a{}_{bc}, 0, 0, 0, \underset{(22)}{C}{}^a{}_{bc} + \underset{(2)}{\tau}{}^a{}_{bc})$$

where $\underset{(0)}{\tau}{}^a{}_{bc}$, $\underset{(1)}{\tau}{}^a{}_{bc}$ and $\underset{(2)}{\tau}{}^a{}_{bc}$ are arbitrary d-tensor fields. Let $\underset{(00)}{T}{}^a{}_{bc}$, $\underset{(11)}{S}{}^a{}_{bc}$ and $\underset{(22)}{S}{}^a{}_{bc}$ be three d-tensor fields which are skew-symmetric with respect to the covariance indices. We shall determine the d-tensor fields $\underset{(0)}{\tau}{}^a{}_{bc}$, $\underset{(1)}{\tau}{}^a{}_{bc}$ and $\underset{(2)}{\tau}{}^a{}_{bc}$ such that the Berwald type connection of general form given above to be $h_0v_{11}v_{22}$-metric and to have $\underset{(00)}{T}{}^a{}_{bc}$, $\underset{(11)}{S}{}^a{}_{bc}$ and $\underset{(22)}{S}{}^a{}_{bc}$ as h(hh)-, $v_1(v_1v_1)$- and $v_2(v_2v_2)$-tensors of torsion, respectively. These conditions show us that $\underset{(0)}{\tau}{}^a{}_{bc}$, $\underset{(1)}{\tau}{}^a{}_{bc}$ and $\underset{(2)}{\tau}{}^a{}_{bc}$ must satisfy the following systems of equations:

$$\begin{cases} \underset{(0)}{\tau}{}^a{}_{bc} - \underset{(0)}{\tau}{}^a{}_{cb} = \underset{(00)}{T}{}^a{}_{bc} \\ \underset{(0)}{\tau}{}^d{}_{ac}\, \underset{(0)}{g}{}_{db} + \underset{(0)}{\tau}{}^d{}_{bc}\, \underset{(0)}{g}{}_{ad} = 0, \end{cases} \quad (2.4) \quad \begin{cases} \underset{(\beta)}{\tau}{}^a{}_{bc} - \underset{(\beta)}{\tau}{}^a{}_{cb} = \underset{(\beta\beta)}{S}{}^a{}_{bc} \\ \underset{(\beta)}{\tau}{}^d{}_{ac}\, \underset{(\beta)}{g}{}_{db} + \underset{(\beta)}{\tau}{}^d{}_{bc}\, \underset{(\beta)}{g}{}_{ad} = 0, (\beta = 1, 2)\,. \end{cases} \quad (2.5)$$

If in the second equation (2.4) we cyclicly permute the indices a, b, c then we add the equations such obtained and take into account the first equation (2.4) we obtain

$$\underset{(0)}{\tau}{}^a{}_{bc} = \frac{1}{2}\underset{(0)}{g}{}^{ad}(\underset{(0)}{g}{}_{df}\,\underset{(00)}{T}{}^f{}_{bc} - \underset{(0)}{g}{}_{bf}\,\underset{(00)}{T}{}^f{}_{dc} + \underset{(0)}{g}{}_{fc}\,\underset{(00)}{T}{}^f{}_{bd}). \tag{2.6}$$

If we similarly proceed with the equations (2.5) we deduce

$$\underset{(\beta)}{\tau}{}^a{}_{bc} = \frac{1}{2}\underset{(\beta)}{g}{}^{ad}(\underset{(\beta)}{g}{}_{df}\,\underset{(\beta\beta)}{S}{}^f{}_{bc} - \underset{(\beta)}{g}{}_{bf}\,\underset{(\beta\beta)}{S}{}^f{}_{dc} + \underset{(\beta)}{g}{}_{fc}\,\underset{(\beta\beta)}{S}{}^f{}_{bd}), (\beta = 1, 2)\,. \tag{2.7}$$

Consequently $\underset{(0)}{\tau}{}^a{}_{bc}$ and $\underset{(\beta)}{\tau}{}^a{}_{bc}$, $(\beta = 1, 2)$, are uniquely determined,

q.e.d.

From (1.16) we see directly that the Kawaguchi metrisation process leaves unchanged the h(hh)-, $v_1(v_1v_1)$- and $v_2(v_2v_2)$- tensors of torsion. Thus we have:

Theorem 2.2. *Let T^2M be endowed with a metric structure G given by (1.10). There exists on T^2M a metric N-linear connection completely determined by G whose h(hh)-, $v_1(v_1v_1)$- and $v_2(v_2v_2)$- tensors of torsion are prescribed. It is obtained from the $h_0v_{11}v_{22}$- metric Berwald type connection given by Theorem 2.1 via the Kawaguchi metrization process and has the following local coefficients*

$$\begin{aligned} \underset{(00)}{L}{}^a{}_{bc} &= \tfrac{1}{2}\underset{(0)}{g}{}^{ad}(\delta_c\,\underset{(0)}{g}{}_{bd} + \delta_b\,\underset{(0)}{g}{}_{dc} - \delta_d\,\underset{(0)}{g}{}_{bc}) + \underset{(0)}{\tau}{}^a{}_{bc}, \\ \underset{(\beta 0)}{L}{}^a{}_{bc} &= \underset{(\beta 0)}{\overset{c}{L}}{}^a{}_{bc}, (\beta = 1, 2)\,, \underset{(01)}{C}{}^a{}_{bc} = \underset{(01)}{\overset{c}{C}}{}^a{}_{bc}, \end{aligned}$$

$$\underset{(11)}{C}{}^{a}{}_{bc} = \frac{1}{2}\underset{(1)}{g}{}^{ad}(\delta_{1c}\underset{(1)}{g}{}_{bd} + \delta_{1b}\underset{(1)}{g}{}_{dc} - \delta_{1d}\underset{(1)}{g}{}_{bc}) + \underset{(1)}{\tau}{}^{a}{}_{bc}, \tag{2.8}$$

$$\begin{aligned} \underset{(\varepsilon 2)}{C}{}^{a}{}_{bc} &= \underset{(\varepsilon 2)}{\overset{c}{C}}{}^{a}{}_{bc}, (\varepsilon = 0,1), \underset{(21)}{C}{}^{a}{}_{bc} = \underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc}, \\ \underset{(22)}{C}{}^{a}{}_{bc} &= \frac{1}{2}\underset{(2)}{g}{}^{ad}(\dot{\partial}_{2c}\underset{(2)}{g}{}_{bd} + \dot{\partial}_{2b}\underset{(2)}{g}{}_{dc} - \dot{\partial}_{2d}\underset{(2)}{g}{}_{bc}) + \underset{(2)}{\tau}{}^{a}{}_{bc}, \end{aligned}$$

where $\underset{(0)}{\tau}{}^{a}{}_{bc}$ *and* $\underset{(\beta)}{\tau}{}^{a}{}_{bc}$, $(\beta = 1,2)$, *are given by (2.6) and (2.7), respectively.*

3.3 The Levi-Civita connection on T^2M

It is well known that there exists an unique linear connection on T^2M which is metric with respect to G and symmetric, that is, it has no tensor of torsion. This is called the Levi-Civita connection of G. Note that it is not an N-linear connection of G on T^2M.

We shall give the local coefficients of the Levi-Civita connection G in the adapted basis $(\delta_a, \delta_{1a}, \dot{\partial}_{2a})$. These coefficients will be expressed by using the local coefficients of the canonical metrical N-linear connection $D\overset{c}{\Gamma}(N)$ from (1.15).

If we denote by ∇ the Levi-Civita connection of G, then by a well known fact about the difference of two linear connections, we can write:

$$\nabla_X = \overset{c}{D}_X + \tau_X, X \in \mathcal{X}\left(T^2M\right), \tag{3.1}$$

where τ_X is a tensor field of the type (1,1) on T^2M. Taking into account that the linear connections ∇ and $\overset{c}{D}$ are metric with respect to G and ∇ is without torsion the following system of equations for the determination of τ_X is obtained:

$$\begin{aligned} &G(\tau(Y,X),Z) + G(Y,\tau(Z,X)) = 0 \\ &\tau(X,Y) - \tau(Y,X) = \overset{c}{T}(X,Y), X, Y \in \mathcal{X}(T^2M), \end{aligned} \tag{3.2}$$

where we have set $\tau_X(Y) = \tau(Y,X)$ and $\overset{c}{T}$ is the torsion of $\overset{c}{D}$.

In the adapted basis $(\delta_a, \delta_{1a}, \dot{\partial}_{2a})$ the Levi-Civita connection looks as follows:

$$\begin{aligned} \nabla_{\delta_c}\delta_b &= \underset{(0)}{L}{}^{a}{}_{bc}\delta_a + \underset{(1)}{L}{}^{a}{}_{bc}\delta_{1a} + \underset{(2)}{L}{}^{a}{}_{bc}\dot{\partial}_{2a}, \\ \nabla_{\delta_c}\delta_{\beta b} &= \underset{(\beta 0)}{K}{}^{a}{}_{bc}\delta_a + \underset{(\beta 1)}{K}{}^{a}{}_{bc}\delta_{1a} + \underset{(\beta 2)}{K}{}^{a}{}_{bc}\dot{\partial}_{2a}, (\beta = 1,2), \delta_{2a} = \dot{\partial}_{2a}, \end{aligned}$$

$$\begin{aligned} \nabla_{\delta_{1c}}\delta_b &= \underset{(0)}{\overset{1}{M}}{}^{a}{}_{bc}\delta_a + \underset{(1)}{\overset{1}{M}}{}^{a}{}_{bc}\delta_{1a} + \underset{(2)}{\overset{1}{M}}{}^{a}{}_{bc}\dot{\partial}_{2a}, \\ \nabla_{\delta_{1c}}\delta_{\beta b} &= \underset{(\beta 0)}{\overset{1}{F}}{}^{a}{}_{bc}\delta_a + \underset{(\beta 1)}{\overset{1}{F}}{}^{a}{}_{bc}\delta_{1a} + \underset{(\beta 2)}{\overset{1}{F}}{}^{a}{}_{bc}\dot{\partial}_{2a}, (\beta = 1,2), \delta_{2a} = \dot{\partial}_{2a}, \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
\nabla_{\dot\partial_{2c}}\delta_b &= \overset{2}{\underset{(0)}{M}}{}^{a}{}_{bc}\delta_a + \overset{2}{\underset{(1)}{M}}{}^{a}{}_{bc}\delta_{1a} + \overset{2}{\underset{(2)}{M}}{}^{a}{}_{bc}\dot\partial_{2a},\\
\nabla_{\dot\partial_{2c}}\delta_{\beta b} &= \overset{2}{\underset{(\beta 0)}{F}}{}^{a}{}_{bc}\delta_a + \overset{2}{\underset{(\beta 1)}{F}}{}^{a}{}_{bc}\delta_{1a} + \overset{2}{\underset{(\beta 2)}{F}}{}^{a}{}_{bc}\dot\partial_{2a}, (\beta=1,2), \delta_{2a}=\dot\partial_{2a},
\end{aligned}
$$

Writing the system of equations (3.2) in the adapted basis $(\delta_a, \delta_{1a}, \dot\partial_{2a}\ )$ one gets a system of equations which allows us to determine the local components of ∇_X. Inserting these local components in the local form of the equation (3.1) one obtains:

Theorem 3.1 *The local coefficients of the Levi-Civita connection ∇ of the metric structure G on the manifold T^2M are as follows:*

$$
\begin{aligned}
\underset{(0)}{L}{}^{a}{}_{bc} &= \overset{c}{\underset{(00)}{L}}{}^{a}{}_{bc},\ \underset{(\beta)}{L}{}^{a}{}_{bc} = -\tfrac{1}{2}\overset{c}{\underset{(0\beta)}{R}}{}^{a}{}_{bc} - \overset{c}{\underset{(0\beta)}{C}}{}^{f}{}_{bd}\underset{(0)}{g}{}_{fc}\underset{(\beta)}{g}{}^{da}, (\beta=1,2),\\
\underset{(\beta 0)}{K}{}^{a}{}_{bc} &= (\tfrac{1}{2}\overset{c}{\underset{(0\beta)}{R}}{}^{f}{}_{dc}\underset{(\beta)}{g}{}_{fb} + \overset{c}{\underset{(0\beta)}{C}}{}^{f}{}_{db}\underset{(0)}{g}{}_{fc})\underset{(0)}{g}{}^{da},\ \underset{(\beta\beta)}{K}{}^{a}{}_{bc} = \overset{c}{\underset{(\beta 0)}{L}}{}^{a}{}_{bc} + \overset{\beta}{\underset{(1)}{O}}{}^{fa}_{bd}\overset{c}{\underset{(\beta\beta)}{P}}{}^{a}{}_{cf},\\
&\qquad (\beta=1,2)\\
\underset{(12)}{K}{}^{a}{}_{bc} &= \tfrac{1}{2}(\overset{c}{\underset{(12)}{P}}{}^{a}{}_{cb} - \overset{c}{\underset{(21)}{P}}{}^{f}{}_{cd}\underset{(1)}{g}{}_{fb}\underset{(2)}{g}{}^{da}),\ \underset{(21)}{K}{}^{a}{}_{bc}, = (\tfrac{1}{2}\overset{c}{\underset{(21)}{P}}{}^{a}{}_{cb} - \overset{c}{\underset{(12)}{P}}{}^{f}{}_{cd}\underset{(2)}{g}{}_{fb}\underset{(1)}{g}{}^{da}),\\
\overset{\beta}{\underset{(0)}{M}}{}^{a}{}_{bc} &= \underset{(\beta 0)}{K}{}^{a}{}_{cb}, (\beta=1,2),\ \overset{\beta}{\underset{(\beta)}{M}}{}^{a}{}_{bc} = -\overset{\beta}{\underset{(2)}{O}}{}^{fa}_{cd}\overset{c}{\underset{(\beta\beta)}{P}}{}^{d}{}_{bf}, (\beta=1,2),\\
\overset{1}{\underset{(2)}{M}}{}^{a}{}_{bc} &= -\tfrac{1}{2}(\overset{c}{\underset{(12)}{P}}{}^{a}{}_{bc} + \overset{c}{\underset{(21)}{P}}{}^{f}{}_{bd}\underset{(1)}{g}{}_{fc}\underset{(2)}{g}{}^{da}),\ \overset{2}{\underset{(1)}{M}}{}^{a}{}_{bc} = -\tfrac{1}{2}(\overset{c}{\underset{(21)}{P}}{}^{a}{}_{bc} + \overset{c}{\underset{(12)}{P}}{}^{f}{}_{bd}\underset{(2)}{g}{}_{fc}\underset{(1)}{g}{}^{da}),\\
\overset{\beta}{\underset{(\gamma 0)}{F}}{}^{a}{}_{bc} &= \tfrac{1}{2}(\overset{c}{\underset{(\gamma\beta)}{P}}{}^{f}{}_{db}\underset{(\beta)}{g}{}_{fc} + \overset{c}{\underset{(\beta\gamma)}{P}}{}^{f}{}_{dc}\underset{(\gamma)}{g}{}_{fb})g^{da},\ \overset{\beta}{\underset{(\beta\beta)}{F}}{}^{a}{}_{bc} = \overset{c}{\underset{(\beta\beta)}{C}}{}^{a}{}_{bc}, (\beta,\gamma=1,2),\\
\overset{1}{\underset{(12)}{F}}{}^{a}{}_{bc} &= -\tfrac{1}{2}\overset{c}{\underset{(12)}{S}}{}^{a}{}_{bc} - \tfrac{1}{2}(\overset{c}{\underset{(12)}{C}}{}^{f}{}_{cd}\underset{(1)}{g}{}_{bf} + \overset{c}{\underset{(12)}{C}}{}^{f}{}_{bd}\underset{(1)}{g}{}_{cf})\underset{(2)}{g}{}^{da},\\
\overset{2}{\underset{(11)}{F}}{}^{a}{}_{bc} &= \tfrac{1}{2}\overset{c}{\underset{(12)}{S}}{}^{f}{}_{db}\underset{(2)}{g}{}_{fc}\underset{(1)}{g}{}^{da} + \overset{1}{\underset{(2)}{O}}{}^{fa}_{bd}\overset{c}{\underset{(12)}{C}}{}^{d}{}_{fc},\ \overset{2}{\underset{(12)}{F}}{}^{a}{}_{bc} = -\overset{2}{\underset{(2)}{O}}{}^{fa}_{cd}\underset{(22)}{Q}{}^{d}{}_{bf},\\
\overset{1}{\underset{(21)}{F}}{}^{a}{}_{bc} &= \tfrac{1}{2}\overset{c}{\underset{(12)}{S}}{}^{f}{}_{dc}\underset{(2)}{g}{}_{fb}\underset{(1)}{g}{}^{da} + \overset{1}{\underset{(2)}{O}}{}^{fa}_{cd}\overset{c}{\underset{(12)}{C}}{}^{d}{}_{fb},\\
\overset{1}{\underset{(22)}{F}}{}^{a}{}_{bc} &= \overset{c}{\underset{(21)}{C}}{}^{a}{}_{bc} + \overset{2}{\underset{(1)}{O}}{}^{fa}_{bd}\overset{c}{\underset{(22)}{Q}}{}^{d}{}_{cf},\ \overset{2}{\underset{(21)}{F}}{}^{a}{}_{bc} = \tfrac{1}{2}(\overset{c}{\underset{(22)}{Q}}{}^{f}{}_{db}\underset{(2)}{g}{}_{fc} + \overset{c}{\underset{(22)}{Q}}{}^{f}{}_{dc}\underset{(2)}{g}{}_{fb})\underset{(1)}{g}{}^{da}.
\end{aligned}
\tag{3.4}
$$

3.4 Some remarkable metrics on T^2M

Recall that a given metrical structure G on the manifold T^2M determines a nonlinear connection and with respect to it G decomposes into a sum of three d-tensor fields which may be viewed as defining metrical structures in horizontal and verticals distributions, respectively. Conversely, if a nonlinear connection, as well as some metrical structures in horizontal and verticals distributions are given, a metrical structure on T^2M may be obtained.

From now on we fix a nonlinear connection $\mathrm{N}(\underset{1}{N}{}^a_b, \underset{2}{N}{}^a_b)$ in the tangent bundle of second order (T^2M, π^2, M).

Definition 4.1.

1°. An h-metric on T^2M is a d-tensor field $G^H = \underset{(0)}{g}{}_{ab} dx^a \otimes dx^b$, where $\underset{(0)}{g}{}_{ab}\left(x, y^{(1)}, y^{(2)}\right) = \underset{(0)}{g}{}_{ba}\left(x, y^{(1)}, y^{(2)}\right)$, $det(g_{ab}(x, y^{(1)}, y^{(2)})) \neq 0$ and the quadratic form $\underset{(0)}{g}{}_{ab}\xi^a\xi^b$ has constant signature.

2°. A v_1-metric on T^2M is a d-tensor field $G^{V_1} = \underset{(1)}{g}{}_{ab}\delta y^{(1)a} \otimes \delta y^{(1)b}$, where $\underset{(1)}{g}{}_{ab}$ has the same properties as $\underset{(0)}{g}{}_{ab}$.

3°. A v_2-metric on T^2M is a d-tensor field $G^{V_2} = \underset{(2)}{g}{}_{ab}\delta y^{(2)a} \otimes \delta y^{(2)b}$, where $\underset{(2)}{g}{}_{ab}$ has the same properties as $\underset{(0)}{g}{}_{ab}$.

4°. An (h,v_1,v_2)- metric on T^2M is the d-tensor field $G = G^H + G^{V_1} + G^{V_2}$, i.e.

$$G = \underset{(0)}{g}{}_{ab}\left(x, y^{(1)}, y^{(2)}\right) dx^a \otimes dx^b + \sum_{\beta=1}^{2} \underset{(\beta)}{g}{}_{ab}\left(x, y^{(1)}, y^{(2)}\right) \delta y^{(\beta)a} \otimes \delta y^{(\beta)b}. \quad (4.1)$$

Obviously, the metric structure (9.9), Ch.1

$$G = g_{ab} dx^a \otimes dx^b + g_{ab}\delta y^{(1)a} \otimes \delta y^{(1)b} + h_{ab}\delta y^{(2)a} \otimes \delta y^{(2)b}, \quad (4.2)$$

the metric structure (9.8), Ch. 1

$$G = g_{ab} dx^a \otimes dx^b + h_{ab}\delta y^{(1)a} \otimes \delta y^{(2)b} + g_{ab}\delta y^{(2)a} \otimes \delta y^{(2)b}, \quad (4.3)$$

the metric structure (9.7), Ch. 1

$$G = g_{ab} dx^a \otimes dx^b + h_{ab}\delta y^{(1)a} \otimes \delta y^{(2)b} + h_{ab}\delta y^{(2)a} \otimes \delta y^{(2)b}, \quad (4.4)$$

and the metric structure (9.10), Ch. 1

$$G = g_{ab} dx^a \otimes dx^b + g_{ab}\delta y^{(1)a} \otimes \delta y^{(1)b} + g_{ab}\delta y^{(2)a} \otimes \delta y^{(2)b}, \quad (4.5)$$

where $g_{ab}\left(x, y^{(1)}, y^{(2)}\right)$ and $h_{ab}\left(x, y^{(1)}, y^{(2)}\right)$ has the same properties as $\underset{(0)}{g}{}_{ab}$, are the (h,v_1,v_2)-metric structures on T^2M.

By using Theorem 1.3 we can write the metric N-linear connections depending only on G given by (4.2)-(4.5)

For instance, we have

Theorem 4.1. *If the manifold T^2M is endowed with the metric structure G given by (4.4) then the metrical canonical N-linear connection has the coefficients:*

$$
\begin{aligned}
\overset{c}{\underset{(00)}{L}}{}^{a}{}_{bc} &= \tfrac{1}{2}g^{ad}(\delta_c g_{bd} + \delta_b g_{dc} - \delta_d g_{bc}),\\
\overset{c}{\underset{(\beta 0)}{L}}{}^{a}{}_{bc} &= \underset{(\beta\beta)}{B}{}^{a}{}_{cb} + \tfrac{1}{2}h^{ad}(\delta_c h_{bd} - \underset{(\beta\beta)}{B}{}^{f}{}_{cb} h_{fd} - \underset{(\beta\beta)}{B}{}^{f}{}_{cd} h_{bf}), (\beta = 1,2)\,,\\
\overset{c}{\underset{(0\beta)}{C}}{}^{a}{}_{bc} &= \tfrac{1}{2}g^{ad}\delta_{\beta c} g_{bd}, \left(\beta = 1,2, \delta_{2a} = \dot{\partial}_{2a}\right),\\
\overset{c}{\underset{(21)}{C}}{}^{a}{}_{bc} &= \tfrac{1}{2}h^{ad}\delta_{1c}h_{bd}, \overset{c}{\underset{(12)}{C}}{}^{a}{}_{bc} = \tfrac{1}{2}h^{ad}\dot{\partial}_{2c}h_{bd},\\
\overset{c}{\underset{(\beta\beta)}{C}}{}^{a}{}_{bc} &= \tfrac{1}{2}h^{ad}(\delta_{\beta c}h_{bd} + \delta_{\beta b}h_{dc} - \delta_{\beta d}h_{bc}), (\beta = 1,2, \delta_{2a} = \dot{\partial}_{2a}).
\end{aligned}
\tag{4.6}
$$

Theorem 4.2. *If the manifold T^2M is endowed with the metric structure G given by (4.5) then the metrical canonical N-linear connection has the coefficients:*

$$
\begin{aligned}
\overset{c}{\underset{(00)}{L}}{}^{a}{}_{bc} &= \tfrac{1}{2}g^{ad}(\delta_c g_{bd} + \delta_b g_{dc} - \delta_d g_{bc}),\\
\overset{c}{\underset{(\beta 0)}{L}}{}^{a}{}_{bc} &= \underset{(\beta\beta)}{B}{}^{a}{}_{cb} + \tfrac{1}{2}g^{ad}(\delta_c g_{bd} - \underset{(\beta\beta)}{B}{}^{f}{}_{cb} g_{fd} - \underset{(\beta\beta)}{B}{}^{f}{}_{cd} g_{bf}), (\beta = 1,2)\,,\\
\overset{c}{\underset{(01)}{C}}{}^{a}{}_{bc} &= \overset{c}{\underset{(21)}{C}}{}^{a}{}_{bc} = \tfrac{1}{2}g^{ad}\delta_{1c}g_{bd},\\
\overset{c}{\underset{(02)}{C}}{}^{a}{}_{bc} &= \overset{c}{\underset{(12)}{C}}{}^{a}{}_{bc} = \tfrac{1}{2}g^{ad}\dot{\partial}_{2c}g_{bd},\\
\overset{c}{\underset{(\beta\beta)}{C}}{}^{a}{}_{bc} &= \tfrac{1}{2}g^{ad}(\delta_{\beta c}g_{bd} + \delta_{\beta b}g_{dc} - \delta_{\beta d}g_{bc}),\\
&\left(\beta = 1,2, \delta_{2a} = \dot{\partial}_{2a}\right)
\end{aligned}
\tag{4.7}
$$

Definition 4.2.

1°. The (h,v_1,v_2)-metric G given by (4.1) is said to be h-Riemannian if ($\underset{(0)}{g}{}_{ab}$) do not depend on $y^{(1)a}$ and $y^{(2)a}$.

2°. The (h,v_1,v_2)-metric G given by (4.1) is said to be v_1-Riemannian if ($\underset{(1)}{g}{}_{ab}$) do not depend on $y^{(1)a}$ and $y^{(2)a}$.

3°. The (h,v_1,v_2)-metric G given by (4.1) is said to be v_2-Riemannian if ($\underset{(2)}{g}{}_{ab}$) do not depend on $y^{(1)a}$ and $y^{(2)a}$.

It is now clearly what G is (h,v_1,v_2)-Riemannian means.
We have

Proposition 4.1.

a. G is an h-Riemannian metric if and only if $\overset{c}{\underset{(01)}{C}}{}^{a}{}_{bc}$ *and* $\overset{c}{\underset{(02)}{C}}{}^{a}{}_{bc}$ *from (1.15) vanish.*

b. G is a v_1-Riemannian metric if and only if $\overset{c}{\underset{(11)}{C}}{}^{a}{}_{bc}$ *and* $\overset{c}{\underset{(12)}{C}}{}^{a}{}_{bc}$ *from (1.15) vanish.*

c. G is a v_2-Riemannian metric if and only if $\overset{c}{\underset{(21)}{C}}{}^{a}{}_{bc}$ *and* $\overset{c}{\underset{(22)}{C}}{}^{a}{}_{bc}$ *from (1.15) vanish.*

d. G is an (h,v_1,v_2)-Riemannian metric on T^2M if and only if

$$\overset{c}{\underset{(0\beta)}{C}}{}^{a}{}_{bc}=0,\ \overset{c}{\underset{(12)}{C}}{}^{a}{}_{bc}=0,\ \overset{c}{\underset{(21)}{C}}{}^{a}{}_{bc}=0,\ \overset{c}{\underset{(\beta\beta)}{C}}{}^{a}{}_{bc}=0,(\beta=1,2)\,. \tag{4.8}$$

Coming back to the Theorem 1.3, we obtain

Proposition 4.2. *If the (h,v_1,v_2)-metric G given by (4.1) is (h,v_1,v_2)-Riemannian metric then about (4.8) we have also*

i) $\overset{c}{\underset{(00)}{L}}{}^{a}{}_{bc}=\left\{{a \atop bc}\right\},\ \overset{c}{\underset{(\beta 0)}{L}}{}^{a}{}_{bc}=\frac{1}{2}\underset{(\beta)}{g}{}^{ad}\partial_c \underset{(\beta)}{g}{}_{bd},(\beta=1,2)\,,$

ii) $\overset{c}{\underset{(00)}{T}}{}^{a}{}_{bc}=0,\ \overset{c}{\underset{(10)}{P}}{}^{a}{}_{bc}=0,\ \overset{c}{\underset{(20)}{P}}{}^{a}{}_{bc}=0,$

$\overset{c}{\underset{(11)}{S}}{}^{a}{}_{bc}=0,\ \overset{c}{\underset{(21)}{Q}}{}^{a}{}_{bc}=0,\ \overset{c}{\underset{(22)}{Q}}{}^{a}{}_{bc}=\underset{(21)}{B}{}^{a}{}_{bc},\ \overset{c}{\underset{(22)}{S}}{}^{a}{}_{bc}=0,$

iii) $\overset{c}{\underset{(00)}{R}}{}^{a}_{b\,cd}=r^{a}_{b\,cd},\ \overset{c}{\underset{(\beta\alpha)}{P}}{}^{a}_{b\,cd}=0$

$\overset{c}{\underset{(2\alpha)}{Q}}{}^{a}_{b\,cd}=0\quad,\overset{c}{\underset{(\beta\alpha)}{S}}{}^{a}_{b\,cd}=0,\left(\alpha=0,1,2;\beta=1,2,\delta_{2a}=\dot{\partial}_{2a}\right),$

where $\left\{{a \atop bc}\right\}$ are the Christofell symbols and the $r^{a}_{b\,cd}$ the curvature tensor constructed with $(g_{ab}(x))$. The superscript c refer to the metrical canonical N-linear connection $\overset{c}{D\Gamma}(N)$.

As in the case of the tangent bundle $Osc^1M=TM$, cf. with S. Ikeda, [Some Physical Aspects Underlying the Lagrangian Theory of Relativity, in the R. Miron and M. Anastasiei's book: Fibrate Vectoriale. Spatii Lagrange. Aplicatii in teoria relativitatii, Ed. Acad. Romania, 1987], (see [87]), the cases when G is h-, v_1- and v_2- Riemannian "seams to have no essential physical meaning", but these are of theoretic interest.

Definition 4.3.

1°. The (h,v_1,v_2)-metric G given by (4.1) is said to be h- not accelerate metric if $(\underset{(0)}{g}{}_{ab})$ *do not depend on* $y^{(2)a}$.

2°. The (h,v$_1$,v$_2$)-metric G given by (4.1) is said to be v$_1$- not accelerate metric if ($\underset{(1)}{g}_{ab}$) do not depend on $y^{(2)a}$.

3°. The (h,v$_1$,v$_2$)-metric G given by (4.1) is said to be v$_2$- not accelerate metric if ($\underset{(2)}{g}_{ab}$) do not depend on $y^{(2)a}$.

It is evidently what means G is (h,v$_1$,v$_2$) - not accelerate metric.

We have

Proposition 4.3.

a. *G is an h-not accelerate metric if and only if $\overset{c}{\underset{(02)}{C}}{}^{a}{}_{bc}$ from (1.15) vanishes.*

b. *G is a v_1-not accelerate metric if and only if $\overset{c}{\underset{(12)}{C}}{}^{a}{}_{bc}$ from (1.15) vanishes.*

c. *G is a v_2-not accelerate metric if and only if $\overset{c}{\underset{(22)}{C}}{}^{a}{}_{bc}$ from (1.15) vanishes.*

d. *G is an (h,v$_1$,v$_2$)-not accelerate metric if and only if*

$$\overset{c}{\underset{(02)}{C}}{}^{a}{}_{bc} = 0,\ \overset{c}{\underset{(12)}{C}}{}^{a}{}_{bc} = 0,\ \overset{c}{\underset{(22)}{C}}{}^{a}{}_{bc} = 0 \tag{4.9}$$

If is not difficult to write the metrical canonical N-linear connections in the above cases for G in the forms (4.1) - (4.5). For example, we have

Theorem 4.3. *If the manifold T^2M is endowed with the (h,v$_1$,v$_2$)- not accelerate metric structure G:*

$$G = g_{ab}\left(x, y^{(1)}\right) dx^a \otimes dx^b + g_{ab}\left(x, y^{(1)}\right) \delta y^{(1)a} \otimes \delta y^{(1)b} + g_{ab}\left(x, y^{(1)}\right) \delta y^{(2)a} \otimes \delta y^{(2)b},$$

then the metrical canonical N-linear connection has the coefficients given by (4.9) and by following expressions:

$$\overset{c}{\underset{(00)}{L}}{}^{a}{}_{bc} = \tfrac{1}{2} g^{ad}\left(\partial_c g_{bd} + \partial_b g_{dc} - \partial_d g_{bc}\right) - \underset{(0)}{\sigma}{}^{a}{}_{bc}$$

$$\begin{array}{rcl} \overset{c}{\underset{(\beta 0)}{L}}{}^{a}{}_{bc} & = & \underset{(\beta\beta)}{B}{}^{a}{}_{cb} + \frac{1}{2} g^{ad}(\partial_c g_{bd} - \underset{(\beta\beta)}{B}{}^{f}{}_{cb} g_{fd} - \underset{(\beta\beta)}{B}{}^{f}{}_{cd} g_{bf}) - \underset{(\beta)}{\sigma}{}^{a}{}_{bc} \\ \overset{c}{\underset{(01)}{C}}{}^{a}{}_{bc} & = & \overset{c}{\underset{(21)}{C}}{}^{a}{}_{bc} = \frac{1}{2} g^{ad} \dot{\partial}_{1c} g_{bd} \end{array} \tag{4.10}$$

$$\overset{c}{\underset{(11)}{C}}{}^{a}{}_{bc} = \tfrac{1}{2} g^{ad}\left(\dot{\partial}_{1c} g_{bd} + \dot{\partial}_{1b} g_{dc} - \dot{\partial}_{1d} g_{bc}\right), (\beta = 1, 2),$$

where

$$\begin{array}{rcl} \underset{(0)}{\sigma}{}^{a}{}_{bc} & = & \frac{1}{2} g^{ab}(\underset{1}{N}{}^{f}{}_{c} \dot{\partial}_f g_{bd} + \underset{1}{N}{}^{f}{}_{b} \dot{\partial}_f g_{dc} - \underset{1}{N}{}^{f}{}_{d} \dot{\partial}_f g_{bc}), \\ \underset{(1)}{\sigma}{}^{a}{}_{bc} & = & \underset{(2)}{\sigma}{}^{a}{}_{bc} = \frac{1}{2} g^{ad} \underset{1}{N}{}^{f}{}_{c} \dot{\partial}_{1f} g_{bd}. \end{array}$$

The following metric structures can be interesting for physics., [44], [118], [136], [137].

Definition 4.4. *We shall say that the metric G given by (4.1) is v_β-locally Minkowski if for every point $u \in T^2M$ there exists a local chart around it on T^2M such that on its domain ($\underset{(\beta)}{g}{}_{ab}$) depends on $y^{(1)}$ only, ($\beta = 1$ or/and 2).*

Since on the manifold $Osc^1M = TM$, there exists h-Riemannian and v-locally Minkowski metric, [cf with R. Miron, M. Anastasiei: The Geometry of Lagrange Space. Theory and Applications, Kluwer Acad. Publ., FTPH, no. 59, 1994], [88] , by the prolongations to $Osc^2M = T^2M$ of this metric structure (see, [93]), one obtains an h-Riemannian and v_1-, v_2- locally Minkowski metric on T^2M. This prove the existence of v_β-locally Minkowski metric ($\beta = 1$ or/and 2) on T^2M.

Theorem 4.4. *If the (h,v_1,v_2)-metric G given by (4.1) is h-Riemannian and v_1-, v_2- locally Minkowski metric then the metrical canonical N-linear connection has the coefficients:*

$$\begin{aligned}
\underset{(00)}{\overset{c}{L}}{}^{a}{}_{bc} &= \tfrac{1}{2}\underset{(0)}{g}{}^{ad}(\partial_c \underset{(0)}{g}{}_{bd} + \partial_b \underset{(0)}{g}{}_{dc} - \partial_d \underset{(0)}{g}{}_{bc}),\\
\underset{(\beta 0)}{\overset{c}{L}}{}^{a}{}_{bc} &= \underset{(\beta\beta)}{B}{}^{a}{}_{bc} - \tfrac{1}{2}\underset{(\beta)}{g}{}^{ad}(\underset{1}{N}{}^{f}{}_{c}\dot{\partial}_{1f}\underset{(\beta)}{g}{}_{bd} + \underset{(\beta\beta)}{B}{}^{f}{}_{bc}\underset{(\beta)}{g}{}_{fd} + \underset{(\beta\beta)}{B}{}^{f}{}_{cd}\underset{(\beta)}{g}{}_{bc}), (\beta = 1, 2)\,,
\end{aligned}$$

$$\underset{(01)}{\overset{c}{C}}{}^{a}{}_{bc} = 0,\ \underset{(02)}{\overset{c}{C}}{}^{a}{}_{bc} = 0,\ \underset{(12)}{\overset{c}{C}}{}^{a}{}_{bc} = 0,\ \underset{(22)}{\overset{c}{C}}{}^{a}{}_{bc} = 0 \tag{4.11}$$

$$\begin{aligned}
\underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc} &= \tfrac{1}{2}\underset{(1)}{g}{}^{ad}(\dot{\partial}_{1c}\underset{(1)}{g}{}_{bd} + \dot{\partial}_{1b}\underset{(1)}{g}{}_{dc} - \dot{\partial}_{1d}\underset{(1)}{g}{}_{bc})\\
\underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc} &= \tfrac{1}{2}\underset{(2)}{g}{}^{ad}\dot{\partial}_{1c}\underset{(2)}{g}{}_{bd}.
\end{aligned}$$

Proof. Indeed, by (1.15) we get (4.11).

Also, we get

Theorem 4.5. *If the manifold T^2M is endowed with the (h,v_1,v_2)-metric structure G, h-Riemannian and v_1-, v_2-locally Minkowski given by*

$$G = g_{ab}(x)dx^a \otimes dx^b + h_{ab}(y^{(1)})\delta y^{(1)a} \otimes \delta y^{(1)b} + h_{ab}(y^{(1)})\delta y^{(2)a} \otimes \delta y^{(2)b},$$

then we have

i) $\underset{(00)}{\overset{c}{L}}{}^{a}{}_{bc} = \{{}^{\,a}_{bc}\}_x$, $\underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc} = \{{}^{\,a}_{bc}\}_{y^{(1)}}$, $\underset{(21)}{C}{}^{a}{}_{bc} = \frac{1}{2}h^{ad}\dot{\partial}_{1c}h_{bd}$,

$\underset{(\beta 0)}{\overset{c}{L}}{}^{a}{}_{bc} = \underset{(\beta\beta)}{B}{}^{a}{}_{cb} - \frac{1}{2}h^{ad}(\underset{1}{N}{}^{f}{}_{c}\dot{\partial}_{1f}h_{bd} + \underset{(\beta\beta)}{B}{}^{f}{}_{cb}h_{fd} + \underset{(\beta\beta)}{B}{}^{f}{}_{cd}h_{bc}), (\beta = 1, 2)\,,$

$\underset{(01)}{\overset{c}{C}}{}^{a}{}_{bc} = 0$, $\underset{(02)}{\overset{c}{C}}{}^{a}{}_{bc} = 0$, $\underset{(12)}{\overset{c}{C}}{}^{a}{}_{bc} = 0$, $\underset{(22)}{\overset{c}{C}}{}^{a}{}_{bc} = 0$,

ii) $\underset{(00)}{\overset{c}{T}}{}^{a}{}_{bc} = 0$, $\underset{(10)}{\overset{c}{P}}{}^{a}{}_{bc} = 0$, $\underset{(20)}{\overset{c}{P}}{}^{a}{}_{bc} = 0$,

$$\overset{c}{\underset{(11)}{S}}{}^{a}{}_{bc}=0,\ \overset{c}{\underset{(21)}{Q}}{}^{a}{}_{bc}=0,\ \overset{c}{\underset{(22)}{S}}{}^{a}{}_{bc}=0,$$

iii) $\overset{c}{\underset{(00)}{R}}{}_{b}{}^{a}{}_{cd}=\underset{(0)}{r}{}_{b}{}^{a}{}_{cd}\left(x\right),$

$$\overset{c}{\underset{(0\beta)}{R}}{}_{b}{}^{a}{}_{cd}=\delta_d\overset{c}{\underset{(\beta0)}{L}}{}^{a}{}_{bc}-\delta_c\overset{c}{\underset{(\beta0)}{L}}{}^{a}{}_{bd}+\overset{c}{\underset{(\beta0)}{L}}{}^{f}{}_{bc}\overset{c}{\underset{(\beta0)}{L}}{}^{a}{}_{fd}-\overset{c}{\underset{(\beta0)}{L}}{}^{f}{}_{bd}\overset{c}{\underset{(\beta0)}{L}}{}^{a}{}_{fc}+$$
$$+\overset{c}{\underset{(\beta1)}{C}}{}^{a}{}_{bf}\underset{(01)}{R}{}^{f}{}_{cd},\left(\beta=1,2\right),$$

$$\overset{c}{\underset{(\beta0)}{P}}{}_{b}{}^{a}{}_{cd}=0,\left(\beta=1,2\right),$$

$$\overset{c}{\underset{(11)}{P}}{}_{b}{}^{a}{}_{cd}=\delta_{1d}\overset{c}{\underset{(10)}{L}}{}^{a}{}_{bc}-\overset{c}{\underset{(11)}{C}}{}^{a}{}_{bd|1c}+\overset{c}{\underset{(11)}{C}}{}^{a}{}_{bf}\overset{c}{\underset{(11)}{P}}{}^{f}{}_{cd},$$

$$\overset{c}{\underset{(22)}{P}}{}_{b}{}^{a}{}_{cd}=\dot{\partial}_{2d}\overset{c}{\underset{(20)}{L}}{}^{a}{}_{bc}+\overset{c}{\underset{(21)}{C}}{}^{a}{}_{bf}\overset{c}{\underset{(21)}{P}}{}^{f}{}_{cd},$$

$$\overset{c}{\underset{(2\alpha)}{Q}}{}_{b}{}^{a}{}_{cd}=0,\left(\alpha=0,1,2\right),\ \overset{c}{\underset{(\beta0)}{S}}{}_{b}{}^{a}{}_{cd}=0,\ \overset{c}{\underset{(\beta2)}{S}}{}_{b}{}^{a}{}_{cd}=0,(\beta=1,2),$$

$$\overset{c}{\underset{(11)}{S}}{}_{b}{}^{a}{}_{cd}=\underset{(1)}{r}{}_{b}{}^{a}{}_{cd}\left(y^{(1)}\right),$$

where $\{{}^{a}_{bc}\}_x$ (resp. $\{{}^{a}_{bc}\}_{y^{(1)}}$) are the Christofell symbols and $\underset{(0)}{r}{}_{b}{}^{a}{}_{cd}\left(x\right)$ (resp. $\underset{(1)}{r}{}_{b}{}^{a}{}_{cd}\left(y^{(1)}\right)$) the curvature tensor constructed with $g_{ab}\left(x\right)$ and ∂_a (resp. $h_{ab}\left(y^{(1)}\right)$ and $\dot{\partial}_{1a}$). The superscript c refers to the metrical canonical N-linear connection $D\overset{c}{\Gamma}\left(N\right)$.

Proof. Since g_{ab} (resp. h_{ab}) depend only x (resp $y^{(1)}$) it follows $\delta_a=\partial_a$ (resp. $\delta_{1a}=\dot{\partial}_{1a}$) and by (2.3) one gets $\overset{c}{\underset{(00)}{L}}{}^{a}{}_{bc}=\{{}^{a}_{bc}\}_x$ (resp. $\overset{c}{\underset{(11)}{C}}{}^{a}{}_{bc}=\{{}^{a}_{bc}\}_{y^{(1)}}$) and $\overset{c}{\underset{(22)}{C}}{}^{a}{}_{bc}=0$. Then a glance to (4.6), (7.2) Ch. 2 and (7.11) Ch.2, say us the other equations.

q.e.d.

Definition 4.5. *We shall say that the metric G given by (4.1) is v_β- locally accelerate if for every point $u\in T^2M$ there exist a local chart around it on T^2M such that on its domain $\underset{(\beta)}{g}{}_{ab}$ depends on $y^{(2)}$ only, ($\beta=1$ or/and 2).*

Let (T^2M,π^2,M) be the tangent bundle of second order endowed with a nonlinear connection N. Suppose that its vectorial distribution V_2 is endowed with a norm $||\,||:V_2\to R_+$. If $v_2=v_2^a e_a$, where (e_a) is a basis of V_2, we set $||v_2||=f(v_2^1,...,v_2^n)=f(v^a)$ and suppose that f is differentiable at least of class C^3 for $v_2\neq0$. The set

$$\{T/T\in GL(n,R),||Tv_2||=||v_2||,v_2\in V\}$$

is a Lie group. Let H_2 be a subgroup of it.

Definition 4.6. *We say that the tangent bundle of second order* (T^2M, π^2, M) *admits a* H_2*-structure (or it is a* $\{V_2, H_2\}$*-bundle) if there exists a bundle atlas* $\{(U_\alpha, \varphi_\alpha, V_2)\}$ *such that the mappings* $\varphi_{\beta,x} \circ \varphi^{-1}_{\alpha,x}$ *belongs to* H_2 *for every* $x \in U_\alpha \cap U_\beta \neq \Phi$.

The local fibres of a $\{v_2, H_2\}$- bundle are isomorphic and isometric each of others.

Using the bundle atlas $(U_\alpha, \varphi_\alpha, V_2)$ it comes out that the norm f defines a function on T^2M, say F, such that in any atlas on T^2M the matrix

$$h_{ab}\left(x, y^{(1)}, y^{(2)}\right) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^{(2)a}\partial y^{(2)b}} \tag{4.12}$$

is nonsingular and the quadratic form $h_{ab}\eta^a\eta^b$ $(\eta \in R^n)$ is positive definite.

If we consider on T^2M the atlas indeed by the bundle atlas $(U_\alpha, \varphi_\alpha, V_2)$ then F and (h_{ab}) depend on $y^{(2)}$ only.

Now let $g_{ab}(x)$ be the local coefficients of a Riemannian metric on the manifold M. It is clear that G given by (4.4) with this $g_{ab}(x)$ and $h_{ab}\left(y^{(2)}\right)$ from (4.12) is an h-Riemannian, v$_1$- and v$_2$- locally accelerate metric on T^2M.

If is not difficult to prove

Theorem 4.6. *If the (h,*v_1*,*v_2*)-metric G given by (4.1) is h-Riemannian,* v_1*-locally Minkowski and* v_2*-locally accelerate metric then the metrical canonical N-linear connection has the coefficients:*

$$\begin{aligned}
\overset{c}{\underset{(00)}{L}}{}^a{}_{bc} &= \tfrac{1}{2}(\partial_c \underset{(0)}{g}{}_{bd} + \partial_{bc} \underset{(0)}{g}{}_{dc} - \partial_d \underset{(0)}{g}{}_{bc}),\\
\overset{c}{\underset{(\beta 0)}{L}}{}^a{}_{bc} &= \underset{(\beta\beta)}{B}{}^a{}_{bc} - \tfrac{1}{2}\underset{(\beta)}{g}{}^{ad}(\underset{(\beta)}{N}{}^f{}_c\dot{\partial}_{\beta f}\underset{(\beta)}{g}{}_{bd} + \underset{(\beta\beta)}{B}{}^f{}_{cb}\underset{(\beta)}{g}{}_{fd} + \underset{(\beta\beta)}{B}{}^f{}_{cd}\underset{(\beta)}{g}{}_{bf}),\\
\overset{c}{\underset{(01)}{C}}{}^a{}_{bc} &= 0,\ \overset{c}{\underset{(02)}{C}}{}^a{}_{bc} = 0,\ \overset{c}{\underset{(12)}{C}}{}^a{}_{bc} = 0,\\
\overset{c}{\underset{(11)}{C}}{}^a{}_{bc} &= \tfrac{1}{2}\underset{(1)}{g}{}^{ad}(\dot{\partial}_{1c}\underset{(1)}{g}{}_{bd} + \dot{\partial}_{1b}\underset{(1)}{g}{}_{dc} - \dot{\partial}_{1d}\underset{(1)}{g}{}_{bc}),\\
\overset{c}{\underset{(21)}{C}}{}^a{}_{bc} &= -\tfrac{1}{2}\underset{(2)}{g}{}^{ad}\underset{1}{N}{}^f{}_c\dot{\partial}_{2f}\underset{(2)}{g}{}_{bd},\\
\overset{c}{\underset{(22)}{C}}{}^a{}_{bc} &= \tfrac{1}{2}\underset{(2)}{g}{}^{ad}(\dot{\partial}_{2c}\underset{(2)}{g}{}_{bd} + \dot{\partial}_{2b}\underset{(2)}{g}{}_{dc} - \dot{\partial}_{2d}\underset{(2)}{g}{}_{bc}).
\end{aligned} \tag{4.13}$$

Finally, we recall if G given by (4.1) satisfies the equations (1.1) and (1.4) we get

$$G = \underset{(0)}{g}{}_{ab}(x, y^{(1)}, y^{(2)})dx^a \otimes dx^b + \sum_{\beta=1}^{2}\underset{(\beta)}{g}{}_{ab}\left(x, y^{(1)}, y^{(2)}\right)dy^{(\beta)a} \otimes dy^{(\beta)b}. \tag{4.14}$$

We can prove:

Theorem 4.7. *If the manifold T^2M is endowed with the (h,v_1,v_2)-metric structure G, h-Riemannian, v_1-locally Minkowski and v_2-locally accelerate metrics given by*

$$G = g_{ab}(x)\, dx^a \otimes dx^b + h_{ab}(y^{(1)}) dy^{(1)a} \otimes dy^{(1)b} + m_{ab}(y^{(2)}) dy^{(2)a} \otimes dy^{(2)b}, \quad (4.15)$$

then

i) The metrical canonical N-linear connection $D\overset{c}{\Gamma}(N)$ has the coefficients:

$$\begin{aligned}
&\underset{(00)}{\overset{c}{L}}{}^{a}{}_{bc} = \{^{a}_{bc}\},\ \underset{(10)}{\overset{c}{L}}{}^{a}{}_{bc} = 0,\ \underset{(20)}{\overset{c}{L}}{}^{a}{}_{bc} = 0,\\
&\underset{(01)}{\overset{c}{C}}{}^{a}{}_{bc} = 0,\ \underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc} = \tfrac{1}{2} h^{ad}(\dot{\partial}_{1c} h_{bd} + \dot{\partial}_{1b} h_{dc} - \dot{\partial}_{1d} h_{bc}),\ \underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc} = 0,\\
&\underset{(02)}{\overset{c}{C}}{}^{a}{}_{bc} = 0,\ \underset{(12)}{\overset{c}{C}}{}^{a}{}_{bc} = 0,\ \underset{(22)}{\overset{c}{C}}{}^{a}{}_{bc} = \tfrac{1}{2} m^{ad}(\dot{\partial}_{2c} m_{bd} + \dot{\partial}_{2b} m_{dc} - \dot{\partial}_{2d} h_{bc}).
\end{aligned}$$

ii) The metrical N-linear connection $D\overset{c}{\Gamma}(N)$ coincides with the Levi-Civita connection on G, that is, all its tensors of torsion vanish.

iii) The tensors of curvature of $D\overset{c}{\Gamma}(N)$ are as follows:

$$\begin{aligned}
&\underset{(00)}{R}{}_{b}{}^{a}{}_{cd} = r_{b}{}^{a}{}_{cd},\ \underset{(0\beta)}{R}{}_{b}{}^{a}{}_{cd} = 0, (\beta = 1,2),\\
&\underset{(\beta 0)}{P}{}_{b}{}^{a}{}_{cd} = 0,\ \underset{(\beta\beta)}{P}{}_{b}{}^{a}{}_{cd} = -\underset{(\beta\beta)}{\overset{c}{C}}{}^{a}{}_{bd|\beta c}, (\beta = 1,2),\\
&\underset{(21)}{P}{}_{b}{}^{a}{}_{cd} = 0,\ \underset{(12)}{P}{}_{b}{}^{a}{}_{cd} = 0,\ \underset{(2\alpha)}{Q}{}_{b}{}^{a}{}_{cd} = 0, (\alpha = 0,1,2),\\
&\underset{(\beta 0)}{S}{}_{b}{}^{a}{}_{cd} = 0, (\beta = 1,2),\ \underset{(21)}{S}{}_{b}{}^{a}{}_{cd} = 0,\ \underset{(12)}{S}{}_{b}{}^{a}{}_{cd} = 0,\\
&\underset{(\beta\beta)}{S}{}_{b}{}^{a}{}_{cd} = \dot{\partial}_{\beta d} \underset{(\beta\beta)}{\overset{c}{C}}{}^{a}{}_{bc} - \dot{\partial}_{\beta c} \underset{(\beta\beta)}{\overset{c}{C}}{}^{a}{}_{bd} + \underset{(\beta\beta)}{\overset{c}{C}}{}^{f}{}_{bc} \underset{(\beta\beta)}{\overset{c}{C}}{}^{a}{}_{fd} - \underset{(\beta\beta)}{\overset{c}{C}}{}^{f}{}_{bd} \underset{(\beta\beta)}{\overset{c}{C}}{}^{a}{}_{fc},\\
&\qquad (\beta = 1,2).
\end{aligned}$$

Proof. By the Theorem 9.5, Ch.1, we have $\underset{1}{N}{}^{a}{}_{b} = 0, \underset{2}{N}{}^{a}{}_{b} = 0, \delta_a = \partial_a, \delta_{1a} = \dot{\partial}_a$ and then by (4.13) we obtain i). Then an easy computation shows that the tensors of torsion of $D\overset{c}{\Gamma}(N)$ vanish, that is $D\overset{c}{\Gamma}(N)$ coincides with the Levi-Civita connection of the Riemannian metric G on T^2M. By i) and ii) we obtain iii) on account of (7.11), Ch.2.

q.e.d.

Remark. The previous theorem has, as a consequence, the fact that $D\overset{c}{\Gamma}(N)$ is just the Riemannian-Christoffel connection of G on T^2M. Therefore, $D\overset{c}{\Gamma}(N)$ is very convenient for an anisotropic theory of relativity. It seams that this case corresponds to the stand-point of "unified" field obtained by direct-product of the external (x)-field domined by $g_{ab}(x)$ and the internal $(y^{(1)})$- and $(y^{(2)})$- fields dominated by $h_{ab}\left(y^{(1)}\right)$ and $m_{ab}\left(y^{(2)}\right)$, respectively.

Chapter 4

The dual bundle of a $2-$tangent bundle

4.1 The manifold $T^{*2}M$

Let M be a real differentiable manifold of dimension n. A point of M will be denoted by x and its local coordinate system by $(U, \varphi), \varphi(x) = (x^a)$. The indices $a, b, ...$ run over set $\{1, ..., n\}$ and Einstein convention of summarizing is adapted all over this work. Let (TM, π, M) be the tangent bundle of the manifold M and (T^*M, π^*, M) its cotangent bundle, [80], [82], [134].

Definition 1.1 *We call* ***the dual bundle*** *of the $2-$tangent bundle (T^2M, π^2, M), the differentiable bundle $(T^{*2}M, \pi^{*2}, M)$ whose total space is*

$$T^{*2}M = TM \ \times_M \ T^*M. \tag{1.1}$$

Sometime we denote $\left(T^{*2}M, \pi^{*2}, M\right)$ by $T^{*2}M$.A point $u \in T^{*2}M$ will be denoted by $u = (x, y, p)$ having the local coordinates (x^a, y^a, p_a). The projection $\pi^{*2}(u) = \pi^{*2}(x, y, p) = x$. Evidently, we take the projections on the factors of the fibered products (1.1):$\pi_1^{*2} : T^{*2}M \longrightarrow TM$, $\pi : TM \longrightarrow M$ as being $\pi_1^{*2}(x, y, p) = (x, y)$ and $\pi^*(x, y) = x$; also, $\overline{\pi}^* : T^{*2}M \longrightarrow T^*M$ is given by $\overline{\pi}^*(u) = \overline{\pi}^*(x, y, p) = (x, p)$.

The change of local coordinates on the manifold $T^{*2}M$ is:

$$\begin{cases} \widetilde{x}^a = \widetilde{x}^a\left(x^1, ..., x^n\right) \ , \ \det\left(\dfrac{\partial \widetilde{x}^a}{\partial x^b}\right) \neq 0, \\ \widetilde{y}^a = \dfrac{\partial \widetilde{x}^a}{\partial x^b} y^b, \\ \widetilde{p}_a = \dfrac{\partial x^b}{\partial \widetilde{x}^a} p_b. \end{cases} \tag{1.2}$$

The dimension of the manifold $T^{*2}M$ is $3n$.

The null section $0 : M \to T^{*2}M$ of the projection π^{*2} is defined by $0 : (x) \in M \to (x, 0, 0) \in T^{*2}M$ we denote by $\widetilde{T^{*2}M} = T^{*2}M \setminus \{0\}$.

Let us consider the tangent bundle of the differentiable manifold $T^{*2}M$, $\left(TT^{*2}M, \tau^{*2}, T^{*2}M\right)$, where τ^{*2} is the canonical projection and the vertical distribution $V : u \in T^{*2}M \longrightarrow V(u) \subset T_uT^{*2}M$,, generated by the vector fields$\left\{\frac{\partial}{\partial y^a}|_u, \frac{\partial}{\partial p_a}|_u\right\}$, $\forall u \in T^{*2}M$. As usually, the natural basis, let us denote

$$\partial_a = \frac{\partial}{\partial x^a},\ \dot{\partial}_a = \frac{\partial}{\partial y^a},\ \dot{\partial}^a = \frac{\partial}{\partial p_a}. \tag{1.3}$$

By means of (1.2), we can consider the following subdistributions of V :

$$V_1 : u \in T^{*2}M \longrightarrow V_1(u) \subset T_uT^{*2}M, \tag{1.4}$$

and

$$W_2 : u \in T^{*2}M \longrightarrow W_2(u) \subset T_uT^{*2}M, \tag{1.4'}$$

locally generated by the vector fields $\left\{\dot{\partial}_a \;|_u, u \in T^{*2}M\right\}$ and $\left\{\dot{\partial}^a \;|_u, u \in T^{*2}M\right\}$, respectively. Clearly,we have

$$V(u) = V_1(u) \oplus W_2(u), \forall u \in T^{*2}M. \tag{1.5}$$

Some important geometrical objects fields can be introduced:

(i) the Liouville vector field on $T^{*2}M$:

$$\mathbb{C}(u) = y^a \dot{\partial}_a \;|_u,\ \forall u \in T^{*2}M, \tag{1.6}$$

(ii) the Hamilton vector field on $T^{*2}M$:

$$\mathbb{C}^*(u) = p_a \dot{\partial}^a \;|_u, \forall u \in T^{*2}M, \tag{1.7}$$

(iii) the scalar field

$$\varphi = p_a y^a. \tag{1.8}$$

We remark that $\mathbb{C} \in \mathbb{V}_1$ and $C^* \in W_2$.

Also, let us consider the following forms

$$\omega = p_a dx^a \text{ (Liouville 1-form)}, \tag{1.9}$$

$$\theta = d\omega = dp_a \wedge dx^a. \tag{1.10}$$

Theorem 1.1 $1°$.*The differential forms ω and θ are globally defined on the manifold $T^{*2}M$.*

$2°$. *The 2-form θ is closed and rank θ is $2n$.*

$3°$.*θ is a presymplectic structures on $T^{*2}M$.*

The two Poisson bracket $\{\}_0, \{\}_1$, can be defined on the manifold $T^{*2}M$ by

$$\begin{aligned}\{f,g\}_0 &= \frac{\partial f}{\partial x^\alpha}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial x^\alpha},\\ \{f,g\}_1 &= \frac{\partial f}{\partial y^\alpha}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial y^\alpha}.\end{aligned} \tag{1.11}$$

Theorem 1.2 *Every bracket* $\{\}_0$ *and* $\{\}_1$ *defines a canonical Poisson structure on the manifold* $T^{*2}M$.

Now, the following $\mathcal{F}\left(T^{*2}M\right)$ -linear mapping

$$J : \mathcal{X}\left(T^{*2}M\right) \to \mathcal{X}\left(T^{*2}M\right),$$

defined by

$$J\left(\partial_a\right) = \dot{\partial}_a,\ J\left(\dot{\partial}_a\right) = 0,\ J\left(\dot{\partial}^a\right) = 0, \forall u \in \widetilde{T^{*2}M}, \tag{1.12}$$

has geometrical meaning. It is not difficult to prove:

Theorem 1.3 *1°. J is a tensor field of type* $(1,1)$ *on manifold* $T^{*2}M$.
2°. J is a tangent structure on $T^{*2}M$, i.e. $J_0 J = 0$.
3°. J is a integrable structure.
4°. $J_0 J = J^2 = 0$.
5°. $KerJ = V_1 \oplus W_2$, $\mathrm{Im}\, J = V_1$.

With these object fields we can construct the geometry of the manifold $T^{*2}M$.

4.2 Nonlinear connections on $T^{*2}M$

We extend the classical definition [123], of the nonlinear connection on the total space of the dual bundle $\left(T^{*2}M, \pi^{*2}, M\right)$.

Definition 2.1 *A nonlinear connection of the manifold $T^{*2}M$ is a regular distribution N on $T^{*2}M$, supplementary to the vertical distribution V, i.e.*

$$T_u T^{*2}M = N(u) \oplus V(u), \forall u \in T^{*2}M. \tag{2.1}$$

Taking into account (1.5) it follows that the distribution N has the property

$$T_u T^{*2}M = N(u) \oplus V_1(u) \oplus W_2(u), \forall u \in T^{*2}M. \tag{2.2}$$

Therefore, the main geometrical objects on $T^{*2}M$ will be reported to the direct sum (2.2) of vector spaces.

We denote by

$$\left\{\frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^a}, \frac{\partial}{\partial p_a}\right\}, (a = 1, ..., n), \tag{2.3}$$

a local adapted basis to N, V_1, W_2. Clearly, we have

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - N^b{}_a \frac{\partial}{\partial y^b} + N_{ab}\frac{\partial}{\partial p_b}. \tag{2.4}$$

The system of functions $\left(N^{b}_{\ a}(x,y,p), N_{ab}(x,y,p)\right)$ are the coefficients of the nonlinear connection N.

With respect to the coordinate transformations (1.2), we have the rule:

$$\frac{\delta}{\delta x^a} = \frac{\partial \widetilde{x}^b}{\partial x^a}\frac{\delta}{\delta \widetilde{x}^b}, \ \frac{\partial}{\partial y^a} = \frac{\partial \widetilde{x}^b}{\partial x^a}\frac{\partial}{\partial \widetilde{y}^b}, \ \frac{\partial}{\partial p_a} = \frac{\partial x^a}{\partial \widetilde{x}^b}\frac{\partial}{\partial \widetilde{p}_b}. \tag{2.4'}$$

Theorem 2.1 *With respect to (1.2) the coefficients* $(N^a_{\ b}, N_{ab})$ *of a nonlinear connection* N *on* $T^{*2}M$ *obey the rule*

$$\begin{aligned} \widetilde{N}^a_{\ c}\frac{\partial \widetilde{x}^c}{\partial x^b} &= N^c_{\ b}\frac{\partial \widetilde{x}^a}{\partial x^c} - \frac{\partial \widetilde{y}^a}{\partial x^b}, \\ \widetilde{N}_{ab} &= \frac{\partial x^c}{\partial \widetilde{x}^a}\frac{\partial x^d}{\partial \widetilde{x}^b}N_{cd} + p_c\frac{\partial^2 x^c}{\partial \widetilde{x}^a \partial \widetilde{x}^b}. \end{aligned} \tag{2.5}$$

Conversely, if the system of functions $(N^a_{\ b}, N_{ab})$ *are given on the every domain of local chart of the manifold* $T^{*2}M$*, such that the equations* (2.5) *hold, then* $(N^a_{\ b}, N_{ab})$ *are the coefficients of a nonlinear connection on* $T^{*2}M$.

Assuming that the manifold M is paracompact it follows that the manifold $T^{*2}M$ is paracompact, too. Let $\gamma_{ab}(x)$, $x \in M$ be a Riemannian metric on M and $\gamma^a_{bc}(x)$ its Christoffel symbols. Setting

$$f_b = \gamma^a_{bc}(x)\, p_a y^c. \tag{2.6}$$

Then, the system of functions

$$N^a_{\ b} = \dot{\partial}^a f_b, \ N_{ab} = \dot{\partial}_b f_a, \tag{2.7}$$

are geometrical object fields on $T^{*2}M$, having the rules of transformations (2.5), with respect to the changing of local coordinates (1.2). Hence:

Theorem 2.2 *If the base manifold* M *is paracompact, then there exist nonlinear connection on the manifold* $T^{*2}M$.

From now we denote the basis (2.3) by:

$$\left\{\delta_a, \dot{\partial}_a, \dot{\partial}^a\right\}. \tag{2.3'}$$

The dual basis of the adapted basis (2.3) is given by

$$\{dx^a, \delta y^a, \delta p_a\}, \tag{2.8}$$

where

$$\delta y^a = dy^a + N^a_{\ b}dx^b, \ \delta p_a = dp_a - N_{ba}dx^b. \tag{2.8'}$$

With respect to (1.2), the covector fields (2.8) are transformed by the rules:

$$d\widetilde{x}^a = \frac{\partial \widetilde{x}^a}{\partial x^b}dx^b, \ \delta\widetilde{y}^a = \frac{\partial \widetilde{x}^a}{\partial x^b}\delta y^b, \ \delta\widetilde{p}_a = \frac{\partial x^b}{\partial \widetilde{x}^a}\delta p_b, \tag{2.8''}$$

Also, we remark that the differential of a function $f \in \mathcal{F}\left(T^{*2}M\right)$ can be written in the form

$$df = \frac{\delta f}{\delta x^a} dx^a + \frac{\partial f}{\partial y^a} \delta y^a + \frac{\partial f}{\partial p_a} \delta p_a. \tag{2.9}$$

4.3 The distinguished vector and covector fields. The algebra of $d-$tensor fields

Let N be a nonlinear connection on $T^{*2}M$. Let h, v_1, w_2 be the projectors defined by the distributions N, V_1, W_2 of the direct decomposition (2.2). We have

$$\begin{gathered} h + v_1 + w_2 = I,\ h^2 = h,\ v_1^2 = v_1,\ w_2^2 = w_2, \\ h \circ v_1 = v_1 \circ h = 0,\ h \circ w_2 = w_2 \circ h = 0,\ v_1 \circ w_2 = w_2 \circ v_1 = 0. \end{gathered} \tag{3.1}$$

If $X \in \chi\left(\widetilde{T^{*2}M}\right)$ we denote

$$X^H = hX,\ X^{V_1} = v_1 X,\ X^{W_2} = w_2 X. \tag{3.2}$$

Therefore we have the unique decomposition:

$$X = X^H + X^{V_1} + X^{W_2}. \tag{3.3}$$

Each of components X^H, X^{V_1}, X^{W_2} are called **d−vector fields** on $\widetilde{T^{*2}M}$.

In the adapted basis (2.3) we get

$$X^H = X^{(0)a} \delta_a,\ X^{V_1} = X^{(1)a} \dot{\partial}_a,\ X^{W_2} = \underset{(2)}{X}{}_a \dot{\partial}^a. \tag{3.3'}$$

By means of (2.4') we have

$$\widetilde{X}^{(0)a} = \frac{\partial \widetilde{x}^a}{\partial x^b} X^{(0)b},\ \widetilde{X}^{(1)a} = \frac{\partial \widetilde{x}^a}{\partial x^b} X^{(1)b},\ \underset{(2)}{\widetilde{X}}{}_a = \frac{\partial x^b}{\partial \widetilde{x}^a} \underset{(2)}{X}{}_b, \tag{3.4}$$

i.e., the classical rules of the transformations of the local coordinates of vector and covector fields on M. Therefore, $X^{(0)a}, X^{(1)a}$ are called **d−vector fields** and $\underset{(2)}{X}{}_a$ is called a **d−covector field** on the manifold $T^{*2}M$.

For instant, the Liouville vector field $\mathbb{C}$ and the Hamilton vector field $\mathbb{C}^*$ have the properties

$$\begin{aligned} \mathbb{C}^H &= 0,\ \mathbb{C}^{V_1} = y^a \dot{\partial}_a = \mathbb{C},\ \mathbb{C}^{W_2} = 0, \\ \mathbb{C}^{*H} &= 0,\ \mathbb{C}^{*V_1} = 0,\ \mathbb{C}^{*W_2} = p_a \dot{\partial}^a = \mathbb{C}^*. \end{aligned}$$

The following result is important

Proposition 3.1 The distribution N is integrable if and only if for any vector fields $X, Y \in \chi\left(T^{*2}M\right)$ we have:

$$\left[X^H, Y^H\right]^{V_1} = 0,\ \left[X^H, Y^H\right]^{W_2} = 0.$$

Indeed, the Lie bracket of any two horizontal vector fields X^H, Y^H belongs to the horizontal distribution N if and only if the last two equations hold.

We remark that the distributions V_1 and W_2 are both integrable.

A similar theory can be done for distinguished $1-$forms.

With respect to the direct decomposition (2.2) a $1-$form $\omega \in \chi^*\left(T^{*2}M\right)$ can be uniquely written in the form:

$$\omega = \omega^H + \omega^{V_1} + \omega^{W_2}, \tag{3.5}$$

where

$$\omega^H = \omega \circ h,\ \omega^{V_1} = \omega \circ v_1,\ \omega^{W_2} = \omega \circ w_2. \tag{3.5'}$$

În the adapted cobasis (2.8), we have

$$\omega = \underset{(0)}{\omega}_a dx^a + \underset{(1)}{\omega}_a \delta y^a + \omega^{(2)a}\delta p_a. \tag{3.6}$$

The quantities $\omega^H, \omega^{V_1}, \omega^{W_2}$ are called $\mathbf{d}-1-$**forms.**

The coefficients $\underset{(0)}{\omega}_a, \underset{(1)}{\omega}_a, \omega^{(2)a}$ are transformed by (1.2) as follows:

$$\underset{(0)}{\omega}_a = \frac{\partial \widetilde{x}^b}{\partial x^a}\underset{(0)}{\widetilde{\omega}}_a,\ \underset{(1)}{\omega}_a = \frac{\partial \widetilde{x}^b}{\partial x^a}\underset{(1)}{\widetilde{\omega}}_b,\ \widetilde{\omega}^{(2)a} = \frac{\partial \widetilde{x}^a}{\partial x^b}\omega^{(2)b}. \tag{3.7}$$

Hence $\underset{(0)}{\omega}_a$ and $\underset{(1)}{\omega}_a$ are called **d−covector fields** and $\omega^{(2)a}$ is called a **d−vector field**.

If the nonlinear connection N is a priori given, then some remarkable $d-1-$forms can be associated in a natural way. Namely, let us consider:

$$\begin{array}{l} \omega = \omega^H = p_a dx^a \\ \alpha = \alpha^{V_1} = p_a \delta y^a \\ \beta = \beta^{W_2} = y^a \delta p_a \end{array} \tag{3.8}$$

One use these $d-$forms for studying the Hamilton geometry of order 2 on $T^{*2}M$. (see [114], [116], [129]).

Now, let us consider a function f on $T^{*2}M$. Its differential can be written in the form (2.9). Therefore

$$\begin{cases} df = (df)^H + (df)^{V_1} + (df)^{W_2}, \qquad \text{where} \\ (df)^H = (\delta_a f)\, dx^a,\ (df)^{V_1} = \left(\dot{\partial}_a f\right)\delta y^a,\ (df)^{W_2} = \left(\dot{\partial}^a f\right)\delta p_a. \end{cases} \tag{3.9}$$

As an application, let us consider a smooth parametrized curve $\gamma : I \subset R \longrightarrow \widetilde{T^{*2}M}$, such that $\operatorname{Im} \gamma \subset \left(\pi^{*2}\right)^{-1}(U)$. It can be analytical represented by:

$$x^a = x^a(t), \ y^a = y^a(t), \ p_a = p_a(t), \ t \in I. \tag{3.10}$$

The tangent vector $\dfrac{d\gamma}{dt}$, in a point of the curve γ, can be written in the form:

$$\frac{d\gamma}{dt} = \left(\frac{d\gamma}{dt}\right)^H + \left(\frac{d\gamma}{dt}\right)^{V_1} + \left(\frac{d\gamma}{dt}\right)^{W_2} = \frac{dx^a}{dt}\delta_a + \frac{\delta y^a}{dt}\dot{\partial}_a + \frac{\delta p_a}{dt}\dot{\partial}^a, \tag{3.11}$$

where

$$\frac{\delta y^a}{dt} = \frac{dy^a}{dt} + N^a_b \frac{dx^b}{dt}, \frac{\delta p_a}{dt} = \frac{dp_a}{dt} - N_{ba}\frac{dx^b}{dt}. \tag{3.12}$$

The curve (3.10) is called horizontal if $\dfrac{d\gamma}{dt} = \left(\dfrac{d\gamma}{dt}\right)^H$ in every point of the curve γ.

Proposition 3.2 *An horizontal curve on* $\widetilde{T^{*2}M}$ *is characterized by the following system of differentiable equations:*

$$x^a = x^a(t), \ \frac{\delta y^a}{\delta t} = 0, \ \frac{\delta p_a}{\delta t} = 0, \ t \in I. \tag{3.13}$$

Clearly, the system of differential equations (3.13) has local solutions, if the initial points $x^a_0 = x^a(t_0)$, y^a_0, p^0_a on $T^{*2}M$ are given, $t_0 \in I$.

The horizontal curves with the property

$$y^a = \frac{dx^a}{dt} \tag{3.14}$$

are called **autoparallel** curves of the nonlinear connection N. These curves are characterized by (3.13) with supplementary condition (3.14).

Definition 3.1 *A distinguished tensor (briefly,* **d−*tensor field***) *on the manifold* $T^{*2}M$ *is a* $d-$*tensor field* T of type (r,s) *on* $T^{*2}M$, *with the property:*

$$T\left(\overset{1}{\omega},...,\overset{r}{\omega}, \underset{1}{X},...,\underset{s}{X}\right) = T\left(\overset{1}{\omega}{}^H,...,\overset{r}{\omega}{}^{W_2}, \underset{1}{X}{}^H,...,\underset{s}{X}{}^{W_2}\right), \tag{3.15}$$
$$\forall\overset{1}{\omega},...,\overset{r}{\omega} \in \chi^*\left(T^{*2}M\right), \ \forall\underset{1}{X},...,\underset{s}{X} \in \chi\left(T^{*2}M\right).$$

For instance, every components X^H, X^{V_1}, X^{W_2} of a vector field X is a $d-$tensor field of type $(1,0)$, and every components $\omega^H, \omega^{V_1}, \omega^{W_2}$ of a $1-$form ω is a $d-$tensor field of type $(0,1)$.

In the adapted basis $\left(\delta_a, \dot{\partial}_a, \dot{\partial}^a\right)$ and its dual basis $(dx^a, \delta y^a, \delta p_a)$ a $d-$tensor field T of type (r,s) can written in the form:

$$T = T^{\ a_1 \ldots a_r}_{\ b_1 \ldots b_s}\,(x, y, p)\, \delta_{a_1} \otimes \ldots \otimes \dot{\partial}^{b_s} \otimes dx^{b_1} \otimes \ldots \otimes \delta p_{a_r}, \tag{3.16}$$

where

$$T^{\ a_1 \ldots a_r}_{\ b_1 \ldots b_s}\,(x, y, p) = T\left(dx^{b_1}, \ldots, \delta p_{a_r}, \delta_{a_1}, \ldots, \dot{\partial}^{b_s}\right).$$

It follows that the set $\left\{1, \delta_a, \dot{\partial}_a, \dot{\partial}^a\right\}$ generates **the algebra** of the $d-$tensor fields over the ring of functions $\mathcal{F}\left(T^{*2}M\right)$, (see R. Miron [86], [97]).

With respect to the transformation of the coordinates on $T^{*2}M$, the local coefficients $T^{\ a_1 \ldots a_r}_{\ b_1 \ldots b_s}$ of T are transformed by classical rule:

$$\widetilde{T}^{\ c_1 \ldots c_r}_{\ d_1 \ldots d_s} = \frac{\partial \widetilde{x}^{c_1}}{\partial x^{a_1}} \cdots \frac{\partial \widetilde{x}^{c_r}}{\partial x^{a_r}} \frac{\partial x^{b_1}}{\partial \widetilde{x}^{d_1}} \cdots \frac{\partial x^{b_s}}{\partial \widetilde{x}^{d_s}} T^{\ a_1 \ldots a_r}_{\ b_1 \ldots b_s}. \tag{3.17}$$

4.4 Lie brackets. Exterior differentials

In applications, the Lie brackets of the vector fields $\left(\delta_a, \dot{\partial}_a, \dot{\partial}^a\right)$, from the adapted basis to the direct decomposition (2.2), are important.

By a direct calculus, we have:

Proposition 4.1 *The Lie brackets of the vector fields of the adapted basis are given by*

$$\begin{aligned}
&[\delta_b, \delta_c] = \underset{(01)}{R}{}^{a}{}_{bc} \dot{\partial}_a + \underset{(02)}{R}{}_{abc} \dot{\partial}^a, \\
&\left[\delta_b, \dot{\partial}_c\right] = \underset{(11)}{B}{}^{a}{}_{bc} \dot{\partial}_a + \underset{(12)}{B}{}_{abc} \dot{\partial}^a, \\
&\left[\delta_b, \dot{\partial}^c\right] = \underset{(21)}{B}{}^{a}{}_{b}{}^{c} \dot{\partial}_a + \underset{(22)}{B}{}_{ab}{}^{c} \dot{\partial}^a, \\
&\left[\dot{\partial}_b, \dot{\partial}_c\right] = 0, \ \left[\dot{\partial}_b, \dot{\partial}^c\right] = 0, \ \left[\dot{\partial}^b, \dot{\partial}^c\right] = 0,
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
&\underset{(01)}{R}{}^{a}{}_{bc} = \delta_c N^a_b - \delta_b N^a_c, && \underset{(02)}{R}{}_{abc} = \delta_b N_{ca} - \delta_c N_{ba}, \\
&\underset{(11)}{B}{}^{a}{}_{bc} = \dot{\partial}_c N^a_b, && \underset{(12)}{B}{}_{abc} = -\dot{\partial}_c N_{ba}, \\
&\underset{(21)}{B}{}^{a}{}_{b}{}^{c} = \dot{\partial}^c N^a_b, && \underset{(22)}{B}{}_{ab}{}^{c} = -\dot{\partial}^c N_{ba}.
\end{aligned} \tag{4.2}$$

Proposition 4.2 *The exterior differentials of the* $1-$*forms* $\{dx^a, \delta y^a, \delta p_a\}$ *which determine the adapted cobasis (2.8'), are given by*

$$\begin{aligned}
&d\,(dx^a) = 0,\\
&d\,(\delta y^a) = \left\{\frac{1}{2}\underset{(01)}{R}{}^a{}_{bc}dx^c + \underset{(11)}{B}{}^a{}_{bc}\delta y^c + \underset{(21)}{B}{}^a{}_b{}^c\delta p_c\right\}\wedge dx^b,\\
&d\,(\delta p_a) = \left\{\frac{1}{2}\underset{(0)}{R}{}_{abc}dx^c + \underset{(1)}{B}{}_{abc}\delta y^{(c)} + \underset{(2)}{B}{}_{ab}{}^c\delta p_c\right\}\wedge dx^b.
\end{aligned} \tag{4.3}$$

Let us consider the followings coefficients from (4.1):

$$\underset{(11)}{B}{}^a{}_{bc} = \dot{\partial}_c N^a_b,\ -\underset{(22)}{B}{}_{ab}{}^c = \dot{\partial}^c N_{ba}\ \left(= -\underset{(22)}{B}{}^c{}_{ab}\right). \tag{4.4}$$

By means of (2.5) it follows

Proposition 4.3 *The coefficients* $\underset{(11)}{B}{}^a{}_{cb} = \underset{(11)}{U}{}^a{}_{bc}$, $-\underset{(22)}{B}{}^a{}_{bc} = \underset{(22)}{U}{}^a{}_{bc}$ *have the same rule of transformation with respect to the local changing of coordinates (1.2)* on $T^{*2}M$. *This is*

$$\underset{(\beta\beta)}{\widetilde{U}}{}^a{}_{df}\frac{\partial x^d}{\partial x^b}\frac{\partial x^f}{\partial x^c} = \frac{\partial \widetilde{x}^a}{\partial x^d}\underset{(\beta\beta)}{U}{}^d{}_{bc} - \frac{\partial^2\widetilde{x}^a}{\partial x^b\partial x^c},\ (\beta = 1, 2)\,. \tag{4.5}$$

We will be see that these coefficients are the horizontal coefficients of an $N-$linear connections on $T^{*2}M$.

By a direct computation, we obtain

Proposition 4.4 *The coefficients* $\underset{(01)}{R}{}^a{}_{bc}$, $\underset{(02)}{R}{}_{abc}$ and

$$\underset{(21)}{B}{}^a{}_b{}^c = \dot{\partial}^c N^a_b,\ \underset{(12)}{B}{}_{abc} = -\dot{\partial}_c N_{ba}, \tag{4.6}$$

are $d-$*tensor fields* on $T^{*2}M$, *of type* $(1,2), (0,3), (2,1)$ *and, respectively,* $(0,3)$ *i.e.*

$$\underset{(01)}{\widetilde{R}}{}^d{}_{cf} = \frac{\partial \widetilde{x}^d}{\partial x^a}\frac{\partial x^b}{\partial \widetilde{x}^c}\frac{\partial x^c}{\partial \widetilde{x}^f}\underset{(01)}{R}{}^a{}_{bc},\ \text{etc.}$$

We will see that (4.6) can be the vertical coefficients of $N-$linear connection on $T^{*2}M$.

By (4.1) and the Proposition 3.1, we get

Theorem 4.1 *The nonlinear connection* N *is integrable if and only if the following* $d-$*tensor fields vanish:*

$$\underset{(01)}{R}{}^a{}_{bc} = 0,\ \underset{(0)}{R}{}_{abc} = 0. \tag{4.7}$$

4.5 The almost product $\mathbb{P}$. The almost contact structure $\mathbb{F}$.

Assuming that a nonlinear connection N is given, we define a $\mathcal{F}\left(T^{*2}M\right)-$ linear mapping

$$\mathbb{P} : \chi\left(T^{*2}M\right) \longrightarrow \chi\left(T^{*2}M\right),$$

by defined

$$\mathbb{P}\left(X^H\right) = X^H,\ \mathbb{P}\left(X^{V_1}\right) = -X^{V_1},\ \mathbb{P}\left(X^{W_2}\right) = -X^{W_2},\ \forall X \in \chi\left(T^{*2}M\right). \quad (5.1)$$

We have

$$\begin{cases} \mathbb{P} \circ \mathbb{P} = I, \\ \mathbb{P} = \mathbb{I} - 2\left(v_1 + w_2\right) = 2h - I, \\ rang\ \mathbb{P} = 3n. \end{cases} \quad (5.2)$$

Theorem 5.1 *A nonlinear connection N on $T^{*2}M$ is characterized by the existence of an almost product structure $\mathbb{P}$ on $T^{*2}M$ whose eigenspaces corresponding to the eigenvalue-1 coincide with the linear spaces of the vertical distribution V on $T^{*2}M$.*

Much more, taking into account that the Nijenhuis tensor of the structure $\mathbb{P}$ is given by

$$N_{\mathbb{F}}\left(X,Y\right) = \mathbb{P}^2\left[X,Y\right] + \left[\mathbb{P}X, \mathbb{P}Y\right] - \mathbb{P}\left[\mathbb{P}X, Y\right] - \mathbb{P}\left[X, \mathbb{P}Y\right], \quad (5.3)$$

we obtain

$$\begin{aligned} & N_{\mathbb{F}}\left(X^H, Y^H\right) = 4v\left[X^H, Y^H\right], \quad (5.4)\\ & N_{\mathbb{F}}\left(X^H, Y^V\right) = 0, \\ & N_{\mathbb{F}}\left(X^V, Y^V\right) = 0, \end{aligned}$$

and we can formulate

Proposition 5.1 *The almost product structure $\mathbb{P}$ is integrable if and only if the horizontal distribution N is integrable.*

The nonlinear connection N being fixed we have the direct decomposition (2.1), (2.2) and the corresponding adapted basis (2.3).

Let us consider the $\mathcal{F}\left(T^{*2}M\right)-$linear mapping:

$$\mathbb{F} : \chi\left(T^{*2}M\right) \longrightarrow \chi\left(T^{*2}M\right),$$

determined by

$$\mathbb{F}\left(\delta_a\right) = -\dot{\partial}_a,\ \mathbb{F}\left(\dot{\partial}_a\right) = \delta_a,\ \mathbb{F}\left(\dot{\partial}^a\right) = 0. \quad (5.5)$$

Then, we deduce

Theorem 5.2 *The mapping* $\mathbb{F}$ *has the following properties:*

1°. It is globally defined on $\widetilde{T^{*2}M}$.
2°. $\mathbb{F}$ *is a tensor field of type* $(1,1)$.
3°. $Ker\ \mathbb{F} = W_2,\ \mathrm{Im}\,\mathbb{F} = N \oplus V_1$.
4°. $rank\ \mathbb{F} = 2n$.
5°. $\mathbb{F}^3 + \mathbb{F} = 0$.

Proof. 1°. Taking into account (2.4') we have $\dfrac{\partial x^a}{\partial \widetilde{x}^b}\mathbb{F}\left(\dfrac{\delta}{\delta x^a}\right) = -\dfrac{\partial x^a}{\partial \widetilde{x}^b}\dfrac{\partial}{\partial y^a}$, implies $\mathbb{F}\left(\dfrac{\delta}{\delta \widetilde{x}^b}\right) = -\dfrac{\partial}{\partial \widetilde{y}^b}$. Also, $\dfrac{\partial x^a}{\partial \widetilde{x}^b}\mathbb{F}\left(\dfrac{\partial}{\partial y^a}\right) = \dfrac{\partial x^a}{\partial \widetilde{x}^b}\dfrac{\delta}{\delta x^a}$ and $\dfrac{\partial \widetilde{x}^b}{\partial x^a}\mathbb{F}\left(\dfrac{\partial}{\partial p_a}\right) = 0$, lead to $\mathbb{F}\left(\dfrac{\partial}{\partial \widetilde{y}^b}\right) = \dfrac{\delta}{\delta \widetilde{x}^b}$ and $\mathbb{F}\left(\dfrac{\partial}{\partial \widetilde{p}_a}\right) = 0$.

For $2° - 5°$ see [97] pg.259.

We can say that $\mathbb{F}$ is a **natural almost contact structure** determined by the nonlinear connection N.

4.6 The Riemann structures on $\widetilde{T^{*2}M}$.

Let us consider a Riemannian structure $\mathbb{G}$ on the manifold $\widetilde{T^{*2}M}$.

In the natural basis, $\mathbb{G}$ is given locally by

$$\mathbb{G} = \underset{(00)}{g}{}_{ab}dx^a \otimes dx^b + \underset{(01)}{g}{}_{ab}dx^a \otimes dy^b + \underset{(02)}{g}{}_a{}^b dx^a \otimes dp_b + ... + \underset{(22)}{g}{}^{ab}dp_a \otimes dp_b, \quad (6.1)$$

where the matrix $\| \underset{(\alpha\beta)}{g} \|$ is positively defined.

Let $\{\delta_a\}, (a = 1, ..., n)$, be the adapted basis on N :

$$\delta_a = \partial_a - N^b{}_a \dot{\partial}_b + N_{ab}\dot{\partial}^b. \quad (6.2)$$

The following problem is arises: Can the Riemannian structure $\mathbb{G}$ determine a nonlinear connection N on $T^{*2}M$?

The conditions of orthogonality between N and V :

$$\mathbb{G}\left(\delta_a, \dot{\partial}_b\right) = 0,\ \mathbb{G}\left(\delta_a, \dot{\partial}^b\right) = 0,\ (a, b = 1, ..., n), \quad (6.3)$$

give us the following system of equations for determining the coefficients $N^b{}_a$ and N_{ab} :

$$\begin{cases} \underset{(11)}{g}{}_{cb}N^c{}_a - \underset{(12)}{g}{}_b{}^c N_{ac} = \underset{(01)}{g}{}_{ab}, \\ \underset{(21)}{g}{}_c{}^b N^c{}_a - \underset{(22)}{g}{}^{cb} N_{ac} = \underset{(02)}{g}{}_a{}^b, \end{cases} \quad (6.4)$$

where, the matrix

$$\left\| \begin{matrix} \underset{(11)}{g}{}_{cb} & \underset{(12)}{g}{}_{b}{}^{c} \\ \underset{(21)}{g}{}_{c}{}^{b} & \underset{(22)}{g}{}^{cb} \end{matrix} \right\| \tag{6.4'}$$

is nonsingular.

Therefore, the system (6.4) has an unique solution. Whether, take into account the rule of transformation of the coefficients $\underset{(\alpha\beta)}{g}$ from $\mathbb{G}$ we can prove that the solution $(N^a{}_b, N_{ab})$ of (6.4) has the rule of transformation (2.5), by means of the transformations of local coordinates on $T^{*2}M$. Consequently, we have:

Theorem 6.1 *A Riemannian structure* $\mathbb{G}$ on $T^{*2}M$ *determines uniquely a nonlinear connection* N *if the distribution* N *is orthogonal to the distribution* V. *The coefficients* $N^a{}_b$and N_{ab} *of* N *are given by the system of equations* (6.4).

Let $\mathbb{F}$ be the natural contact structure determined by the previous nonlinear connection N.

The following problem arises: When the pair $(\mathbb{G}, \mathbb{F})$ is a Riemannian almost structure?

Evidently, is necessary to have:

$$\mathbb{G}(\mathbb{F}X, Y) = -\mathbb{G}(X, \mathbb{F}Y), \ \forall X, Y \in \chi\left(\widetilde{T^{*2}M}\right).$$

Consequently, we get:

Theorem 6.2 *The pair* $(\mathbb{G}, \mathbb{F})$ *is a Riemannian almost structure if and only if in the adapted basis determined by* N *and* V *the tensor* $\mathbb{G}$ *has the form*

$$\mathbb{G} = g_{ab}dx^a \otimes dx^b + g_{ab}\delta y^a \otimes \delta y^b + h^{ab}\delta p_a \otimes \delta p_b. \tag{6.5}$$

Corollary 6.1 *With respect to the Riemannian structure (6.5) the distributions* N, V_1, W_2 *are orthogonal respectively.*

Chapter 5

Linear connections on the manifold $T^{*2}M$

5.1 $N-$linear connections

A **linear connection** on $T^{*2}M$ is an application

$$D : \chi\left(T^{*2}M\right) \times \chi\left(T^{*2}M\right) \longrightarrow \chi\left(T^{*2}M\right),\ (X,Y) \longmapsto D_X Y,$$

with the properties:

1. $D_{X_1+X_2}Y = D_{X_1}Y + D_{X_2}Y,$
   $D_{fX}Y = fD_XY,\ \forall f \in \mathcal{F}\left(T^{*2}M\right),\ \forall X, X_1, X_2, Y \in \chi\left(T^{*2}M\right).$
2. $D_X\left(Y_1+Y_2\right) = D_XY_1 + D_XY_2,\ \forall X, Y_1, Y_2 \in \chi\left(T^{*2}M\right).$
3. $D_X\left(fY\right) = \left(Xf\right)Y + fD_XY,\ \forall X, Y \in \chi\left(T^{*2}M\right),\ \forall f \in \mathcal{F}\left(T^{*2}M\right).$

We consider $X, Y \in \chi\left(T^{*2}M\right)$. With respect to decompositions of type (2.2), §4.2, we have

$$D_XY = \sum_{\alpha=0}^{2}\left(D_{X^H}Y^{V_\alpha} + D_{X^{V_1}}Y^{V_\alpha} + D_{X^{W_2}}Y^{V_\alpha}\right), \tag{1.1}$$

where $V_0 = H$ and $V_2 = W_2$.

The components $D_{X^H}Y^{V_\alpha},\ D_{X^{V_1}}Y^{V_\alpha},\ D_{X^{W_2}}Y^{V_\alpha},\ \left(V_0 = H,\ V_2 = W_2\right)$, are vector fields, not necessary distinguished.

The linear connection D on $T^{*2}M$ is uniquely determined by its **27 coefficients**, written in the adapted basis in the form

$$\left\{\begin{array}{l}
D_{\delta_c}\delta_b = \underset{(00)}{\overset{0}{H}}{}^a{}_{bc}\delta_a + \underset{(00)}{\overset{1}{H}}{}^a{}_{bc}\dot{\partial}_a + \underset{(00)}{\overset{2}{H}}{}_{abc}\dot{\partial}^a, \\
D_{\delta_c}\dot{\partial}_b = \underset{(10)}{\overset{0}{H}}{}^a{}_{bc}\delta_a + \underset{(10)}{\overset{1}{H}}{}^a{}_{bc}\dot{\partial}_a + \underset{(10)}{\overset{2}{H}}{}_{abc}\dot{\partial}^a, \\
D_{\delta_c}\dot{\partial}^b = \underset{(20)}{\overset{0}{H}}{}^{ab}{}_c\delta_a + \underset{(20)}{\overset{1}{H}}{}^{ab}{}_c\dot{\partial}_a - \underset{(20)}{\overset{2}{H}}{}_a{}^b{}_c\dot{\partial}^a,
\end{array}\right. \tag{$(1.2)_0$}$$

$$\left\{\begin{array}{l}
D_{\dot{\partial}_c}\delta_b = \underset{(01)}{\overset{0}{C}}{}^a{}_{bc}\delta_a + \underset{(01)}{\overset{1}{C}}{}^a{}_{bc}\dot{\partial}_a + \underset{(01)}{\overset{2}{C}}{}_{abc}\dot{\partial}^a, \\
D_{\dot{\partial}_c}\dot{\partial}_b = \underset{(11)}{\overset{0}{C}}{}^a{}_{bc}\delta_a + \underset{(11)}{\overset{1}{C}}{}^a{}_{bc}\dot{\partial}_a + \underset{(11)}{\overset{2}{C}}{}_{abc}\dot{\partial}^a, \\
D_{\dot{\partial}_c}\dot{\partial}^b = \underset{(21)}{\overset{0}{C}}{}^{ab}{}_c\delta_a + \underset{(21)}{\overset{1}{C}}{}^{ab}{}_c\dot{\partial}_a - \underset{(21)}{\overset{2}{C}}{}_a{}^b{}_c\dot{\partial}^a,
\end{array}\right. \tag{$(1.2)_1$}$$

$$\left\{\begin{array}{l}
D_{\dot{\partial}_c}\delta_b = \underset{(02)}{\overset{0}{C}}{}^a{}_b{}^c\delta_a + \underset{(02)}{\overset{1}{C}}{}^a{}_b{}^c\dot{\partial}_a + \underset{(02)}{\overset{2}{C}}{}_{ab}{}^c\dot{\partial}^a, \\
D_{\dot{\partial}_c}\dot{\partial}_b = \underset{(12)}{\overset{0}{C}}{}^a{}_b{}^c\delta_a + \underset{(12)}{\overset{1}{C}}{}^a{}_b{}^c\dot{\partial}_a + \underset{(12)}{\overset{2}{C}}{}_{ab}{}^c\dot{\partial}^a, \\
D_{\dot{\partial}_c}\dot{\partial}^b = \underset{(22)}{\overset{0}{C}}{}^{abc}\delta_a + \underset{(22)}{\overset{1}{C}}{}^{abc}\dot{\partial}_a - \underset{(22)}{\overset{2}{C}}{}_a{}^{bc}\dot{\partial}^a.
\end{array}\right. \tag{$(1.2)_2$}$$

To work with these 27 coefficients is not impossible, but is laborious.

We will use in continuation the $N-$linear connections whose coefficients are much easy to shunt.

Let N be a nonlinear connection on $T^{*2}M$.

Definition 1.1 *A linear connection D on $T^{*2}M$ is called an **N-linear connection** if it preserves by parallelism the horizontal and vertical distributions N, V_1 and W_2* on $T^{*2}M$.

By other words, a linear connection D is $N-$linear connection if and only if, for any $X \in \chi\left(T^{*2}M\right)$, D_X carries the horizontal vector fields to the horizontal vector fields and the vertical vector fields to the vertical vectors. Thus $D_X Y^H \in N$, $D_X Y^{V_1} \in V_1$ and $D_X Y^{W_2} \in W_2$, written in the form

$$\begin{array}{l}
D_X(hY) = hD_XY, \\
D_X(v_1Y) = v_1D_XY, \\
D_X(w_2Y) = w_2D_XY.
\end{array} \tag{1.3}$$

Consequently we have

Theorem 1.1 *A linear connection D is an $N-$linear connection if and only if, for any $X \in \chi\left(T^{*2}M\right)$, we have*

$$D_Xh = 0,\ D_Xv_1 = 0,\ D_Xw_2 = 0. \tag{1.4}$$

Corollary 1.1 *For any $N-$linear connection D we obtain*

$$D_X\mathbb{P} = 0,\ \forall X \in \chi\left(T^{*2}M\right). \tag{1.5}$$

Also, we have

Theorem 1.2 *A linear connection D on $T^{*2}M$ is an $N-$linear connection if and only if*

$$\begin{array}{l}\left(D_X Y^H\right)^{V_\beta}=0,\ \left(D_X Y^{V_\beta}\right)^H=0,\ \left(\beta=1,2;V_2=W_2\right),\\ \left(D_X Y^{V_1}\right)^{W_2}=0,\ \left(D_X Y^{W_2}\right)^{V_1}=0,\ \forall X,Y\in\chi\left(T^{*2}M\right).\end{array}\tag{1.6}$$

From (1.6) result, for an $N-$linear connection D, in decomposition (1.1), the components

$$\underset{\alpha}{D}_{X^H}Y:=D_{X^H}Y^{V_\alpha},\ \underset{\alpha}{D}_{X^{V_\beta}}Y:=D_{X^{V_\beta}}Y^{V_\alpha},\tag{1.7}$$

$(\alpha=0,1,2,\ \beta=1,2;\ V_0=H,\ V_2=W_2)$, are $d-$vector fields and thus

$$\underset{\alpha}{D}{}^H_X:=\underset{\alpha}{D}_{X^H},\ \underset{\alpha}{D}{}^{V_1}_X:=\underset{\alpha}{D}_{X^{V_1}},\ \underset{\alpha}{D}{}^{W_2}_X:=\underset{\alpha}{D}_{X^{W_2}},\left(\alpha=0,1,2\right),\tag{1.8}$$

are derivation operators in the algebra of $d-$tensor fields. We have

$$\begin{cases}\underset{0}{D}{}^H_X Y=D_{X^H}Y^H,\ \underset{0}{D}{}^{V_1}_X Y=D_{X^{V_1}}Y^H,\ \underset{0}{D}{}^{W_2}_X Y=D_{X^{W_2}}Y^H,\\ \underset{\beta}{D}{}^H_X Y=D_{X^H}Y^{V_\beta},\ \underset{\beta}{D}{}^{V_1}_X Y=D_{X^{V_1}}Y^{V_\beta},\ \underset{\beta}{D}{}^{W_2}_X Y=D_{X^{W_2}}Y^{V_\beta},\\ \qquad\qquad\left(\beta=1,2;V_2=W_2\right).\end{cases}\tag{1.9}$$

These operators are not covariant derivations, because for $f\in\mathcal{F}\left(T^{*2}M\right)$, we have $D^H_X f=X^H f\neq Xf$, etc., but they have similar properties with the covariant derivations, respectively:

1. The operators $\underset{\alpha}{D}{}^H_X,\ \underset{\alpha}{D}{}^{V_1}_X,\ \underset{\alpha}{D}{}^{W_2}_X$, verified the equalities (1.6).
2. $\underset{\alpha}{D}{}^H_X f=X^H f,\ \underset{\alpha}{D}{}^{V_1}_X f=X^{V_1}f,\ \underset{\alpha}{D}{}^{W_2}_X f=X^{W_2}f.$
3. $\underset{\alpha}{D}{}^H_X\left(fY\right)=X^H\left(fY\right)+f\underset{\alpha}{D}{}^H_X Y,\ \underset{\alpha}{D}{}^{V_1}_X\left(fY\right)=X^{V_1}\left(fY\right)+f\underset{\alpha}{D}{}^{V_1}_X Y,\ \underset{\alpha}{D}{}^{W_2}_X\left(fY\right)=$ $=X^{W_2}\left(fY\right)+f\underset{\alpha}{D}{}^{W_2}_X Y.$
4. $\left(\underset{\alpha}{D}{}^H_X Y\right)_{|U}=\underset{\alpha}{D}{}^H_{X|U}Y_{|U},\ \left(\underset{\alpha}{D}{}^{V_1}_X Y\right)_{|U}=\underset{\alpha}{D}{}^{V_1}_{X|U}Y_{|U},\ \left(\underset{\alpha}{D}{}^{W_2}_X Y\right)_{|U}=$ $\underset{\alpha}{D}{}^{W_2}_{X|U}Y_{|U}$, for any open set $U\subset T^{*2}M$.
5. $\underset{\alpha}{D}{}^H_{X+Y}=\underset{\alpha}{D}{}^H_X+\underset{\alpha}{D}{}^H_Y,\ \underset{\alpha}{D}{}^{V_1}_{X+Y}=\underset{\alpha}{D}{}^{V_1}_X+\underset{\alpha}{D}{}^{V_2}_Y,\ \underset{\alpha}{D}{}^{W_2}_{X+Y}=\underset{\alpha}{D}{}^{W_2}_X+\underset{\alpha}{D}{}^{W_2}_Y.$
6. $\underset{\alpha}{D}{}^H_{fX}=f\underset{\alpha}{D}{}^H_X,\ \underset{\alpha}{D}{}^{V_1}_{fX}=f\underset{\alpha}{D}{}^H_X,\ \underset{\alpha}{D}{}^{W_2}_{fX}=f\underset{\alpha}{D}{}^{W_2}_X,$

for any $f\in\mathcal{F}\left(T^{*2}M\right)$ and any vector fields $X,Y\in\chi\left(T^{*2}M\right),\left(\alpha=0,1,2\right).$

Definition 1.2 *The operators $\underset{\alpha}{D}{}^H,\underset{\alpha}{D}{}^{V_1},\underset{\alpha}{D}{}^{W_2}$ are called the* $\mathbf{h}_\alpha-$, $\mathbf{v_1}_\alpha-$, *and,* $\mathbf{w_2}_\alpha-$***covariant derivatives,*** $\left(\alpha=0,1,2\right).$

For the $1-$form filed $\omega\in\chi^*\left(T^{*2}M\right)$ we have

$$\begin{aligned}
&\left(\underset{\alpha}{D}{}_X^H\omega\right)(Y)=X^H\omega(Y)-\omega\left(D_X^HY\right),\\
&\left(\underset{\alpha}{D}{}_X^{V_1}\omega\right)(Y)=X^{V_1}\omega(Y)-\omega\left(D_X^{V_1}Y\right),\\
&\left(\underset{\alpha}{D}{}_X^{W_2}\omega\right)(Y)=X^{W_2}\omega(Y)-\omega\left(D_X^{W_2}Y\right).
\end{aligned}\tag{1.10}$$

The action of operators $D_X^H, D_X^{V_1}, D_X^{W_2}$ can be extended to any tensor field, particularly to any $d-$tensor field on $T^{*2}M$.

Let $T\in\mathcal{T}_s^r\left(T^{*2}M\right)$ be a $d-$tensor field .

1. For $X=X^H$ we have

$$\begin{aligned}
&\left(\underset{\alpha}{D}{}_X^HT\right)\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right):=X^HT\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right)-\\
&-T\left(\underset{\alpha}{D}{}_X^H\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right)-...-T\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{\alpha}{D}{}_X^H\underset{s}{X}\right),\ (\alpha=0,1,2).
\end{aligned}\tag{1.11}$$

2. For $X=X^{V_1}$ we have

$$\begin{aligned}
&\left(\underset{\alpha}{D}{}_X^{V_1}T\right)\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right):=X^{V_1}T\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right)-\\
&-T\left(\underset{\alpha}{D}{}_X^{V_1}\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right)-...-T\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{\alpha}{D}{}_X^{V_1}\underset{s}{X}\right),\ (\alpha=0,1,2).
\end{aligned}\tag{1.11'}$$

3. For $X=X^{W_2}$ we have

$$\begin{aligned}
&\left(\underset{\alpha}{D}{}_X^{W_2}T\right)\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right):=X^{W_2}T\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right)-\\
&-T\left(\underset{\alpha}{D}{}_X^{W_2}\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{s}{X}\right)-...-T\left(\overset{1}{\omega},...,\overset{r}{\omega},\underset{1}{X},...,\underset{\alpha}{D}{}_X^{W_2}\underset{s}{X}\right),\ (\alpha=0,1,2).
\end{aligned}\tag{1.11''}$$

5.2 The coefficients of an $N-$linear connection

Let D be an $N-$linear connection on $T^{*2}M$. In the adapted basis

$$\left\{\delta_a=\frac{\delta}{\delta x^a},\ \dot{\partial}_a=\frac{\partial}{\partial y^a},\ \dot{\partial}^a=\frac{\partial}{\partial p_a}\right\},$$

have places the relations $(1.2)_\alpha,\ (\alpha=0,1,2)$, and taking into account that, for example, $D_{\delta_c}\delta_b$ belongs to the horizontal distribution. Hence, $\underset{(00)}{\overset{1}{H}}{}^a{}_{bc}=0,\ \underset{(00)}{\overset{2}{H}}{}_{abc}=0$, this means that for an $N-$linear connection D on $T^{*2}M$ we have

$$D_{\delta_c}\delta_b=\underset{(00)}{\overset{0}{H}}{}^a{}_{bc}\delta_a=:\underset{(00)}{H}{}^a{}_{bc}\delta_a.$$

We proceed analogous with other relations $(1.2)_\alpha,\ (\alpha=0,1,2)$, and we obtain

Theorem 2.1 1°. *An $N-$linear connection can be uniquely written in the adapted basis* $\left(\delta_a, \dot{\partial}_a, \dot{\partial}^a\right)$ *in the form*

$$
\begin{cases}
D_{\delta_c}\delta_b = \underset{(00)}{H}{}^{a}{}_{bc}\delta_a,\ D_{\delta_c}\dot{\partial}_b = \underset{(10)}{H}{}^{a}{}_{bc}\dot{\partial}_a,\ D_{\delta_c}\dot{\partial}^b = -\underset{(00)}{H}{}_{a}{}^{b}{}_{c}\dot{\partial}^a,\\
D_{\dot{\partial}_c}\delta_b = \underset{(01)}{C}{}^{a}{}_{bc}\delta_a,\ D_{\dot{\partial}_c}\dot{\partial}_b = \underset{(11)}{C}{}^{a}{}_{bc}\dot{\partial}_a,\ D_{\dot{\partial}_c}\dot{\partial}^b = -\underset{(21)}{C}{}_{a}{}^{b}{}_{c}\dot{\partial}^a,\\
D_{\dot{\partial}^c}\delta_b = \underset{(02)}{C}{}^{a}{}_{b}{}^{c}\delta_a,\ D_{\dot{\partial}^c}\dot{\partial}_b = \underset{(12)}{C}{}^{a}{}_{b}{}^{c}\dot{\partial}_a,\ D_{\dot{\partial}^c}\dot{\partial}^b = -\underset{(22)}{C}{}_{a}{}^{bc}\dot{\partial}^a.
\end{cases}
\tag{2.1}
$$

2°. *With respect to the coordinate transformation (1.2),*§ 4.1, *the coefficients* $\underset{(\alpha0)}{H}{}^{a}{}_{bc}$, $\left(\alpha = 0,1,2;\ \underset{(20)}{H}{}^{a}{}_{bc} := \underset{(20)}{H}{}_{b}{}^{a}{}_{c}\right)$ *obey the rule of transformation:*

$$
\underset{(\alpha0)}{\widetilde{H}}{}^{a}{}_{de}\frac{\partial\widetilde{x}^d}{\partial x^b}\frac{\partial\widetilde{x}^e}{\partial x^c} = \frac{\partial\widetilde{x}^a}{\partial x^e}\underset{(\alpha0)}{H}{}^{e}{}_{bc} - \frac{\partial^2\widetilde{x}^a}{\partial x^b\partial x^c}. \tag{2.2}
$$

3°.*The system of functions* $\underset{(\alpha1)}{C}{}^{a}{}_{bc}$, $\underset{(\alpha2)}{C}{}^{a}{}_{b}{}^{c}$, $(\alpha = 0,1,2;\ \underset{(21)}{C}{}^{a}{}_{bc} := \underset{(21)}{C}{}_{b}{}^{a}{}_{c};$ $\underset{(22)}{C}{}^{a}{}_{b}{}^{c} := \underset{(22)}{C}{}_{b}{}^{ac}$) *are $d-$tensor fields of type* $(1,2)$ *and* $(2,1)$, *respectively.*

The assertions 2° and 3° can be prove by a direct calculus, taking into account the rule of transformations (2.4'), § 4.1, for $\left(\delta a, \dot{\partial}_a, \dot{\partial}^a\right)$.

The system of functions

$$
D\Gamma(N) := \left(\underset{(00)}{H}{}^{a}{}_{bc}, \underset{(10)}{H}{}^{a}{}_{bc}, \underset{(20)}{H}{}^{a}{}_{bc}, \underset{(01)}{C}{}^{a}{}_{bc}, \underset{(11)}{C}{}^{a}{}_{bc}, \underset{(21)}{C}{}^{a}{}_{bc}, \underset{(02)}{C}{}_{a}{}^{bc}, \underset{(12)}{C}{}_{a}{}^{bc}, \underset{(22)}{C}{}_{a}{}^{bc}\right), \tag{2.3}
$$

are called the **coefficients** of the $N-$linear connection D on $T^{*2}M$.

The inverse statement of Theorem 2.1 holds also: if on each domain of local chart on $T^{*2}M$ having the system of functions of type (2.3), which of local chart (1.2),§ 4.1, on $T^{*2}M$ have been transformed by the rule of transformations 2° and 3° of Theorem 2.1, then, on $T^{*2}M$, there exists an unique $N-$linear connection D whose admit these functions as coefficients

Taking into account (1.10), (2.1) and the condition of duality between vectors of adapted basis and $1-$forms of cobasis we assuming the rule of covariant derivatives for cobasis fields $(dx^a, \delta y^a, \delta p_a)$ as following:

$$
\begin{aligned}
&D_{\delta_c}dx^a = -\underset{(00)}{H}{}^{a}{}_{bc}dx^b,\ D_{\delta_c}\delta y^a = -\underset{(10)}{H}{}^{a}{}_{bc}\delta y^b,\ D_{\delta_c}\delta p_b = \underset{(20)}{H}{}^{a}{}_{bc}\delta p_a,\\
&D_{\dot{\partial}_c}dx^a = -\underset{(01)}{C}{}^{a}{}_{bc}dx^b,\ D_{\dot{\partial}_c}\delta y^a = -\underset{(11)}{C}{}^{a}{}_{bc}\delta y^b,\ D_{\dot{\partial}_c}\delta p_b = \underset{(21)}{C}{}^{a}{}_{bc}\delta p_a,\\
&D_{\dot{\partial}^c}dx^a = -\underset{(02)}{C}{}^{a}{}_{b}{}^{c}dx^b,\ D_{\dot{\partial}^c}\delta y^a = -\underset{(12)}{C}{}^{a}{}_{b}{}^{c}\delta y^b,\ D_{\dot{\partial}^c}\delta p_b = \underset{(22)}{C}{}^{a}{}_{b}{}^{c}\delta p_a.
\end{aligned}
\tag{2.4}
$$

We have the following theorem **of existence** of an $N-$linear connection on $T^{*2}M$.

Theorem 2.2 *If the manifold M is paracompact and N is a nonlinear connection on $T^{*2}M$ with coefficients N^a_b, N_{ab}, then there exists an $N-$linear connection on $T^{*2}M$.*

Proof. Because M is paracompact, there exists an a linear connection on M of local coefficients, say $\Gamma^a_{bc}(x)$. Let $N^a_b(x,y,p)$ and $N_{ab}(x,y,p)$ be the local coefficients of the nonlinear connection N. We set $\underset{(00)}{H}{}^a{}_{bc} = \Gamma^a_{bc}(x)$, $\underset{(10)}{H}{}^a{}_{bc} = \dot{\partial}_b N^a_c$, $\underset{(20)}{H}{}^a{}_{bc} = \dot{\partial}^a N_{bc}$. Thus, taking into account Proposition 4.3, § 4.4, we obtain three set of function which transform, with respect to (1.2),§ 4.1, by (2.1) $(\alpha = 0,1,2)$. It result that $D\Gamma(N)$ given by

$$D\Gamma(N) = \left(\Gamma^a_{bc}(x), \underset{(11)}{B}{}^a{}_{cb}, -\underset{(22)}{B}{}^a{}_{bc}, 0,0,0,0,0,0\right), \tag{2.5}$$

defines an $N-$linear connection on $T^{*2}M$

q.e.d.

In applications, we will use the $N-$linear connections of the form

$$B\Gamma(N) = \left(\underset{(00)}{L}{}^a{}_{bc}, \underset{(11)}{B}{}^a{}_{cb}, -\underset{(22)}{B}{}^a{}_{bc}, 0, \underset{(11)}{C}{}^a{}_{bc}, 0,0,0, \underset{(22)}{C}{}^{ac}{}_b\right) \tag{2.6}$$

called $N-$linear connection of **Berwald type** on $T^{*2}M$.

5.3 The local expressions of $h_\alpha-, v_{1\alpha}-$ and $w_{2\alpha}-$ covariant derivatives, $(\alpha = 0,1,2)$

Let us consider a $d-$tensor fields T of type (r,s). In the adapted basis (2.3), and (2.8), § 4.2, T can be written in the form

$$T = T^{a_1...a_r}_{\ b_1...b_s}(x,y,p)\,\delta a_1 \otimes ... \otimes \dot{\partial}^{b_s} \otimes dx^{b_1} \otimes ... \otimes \delta p_{a_r}. \tag{3.0}$$

Let $X = X^H = X^{(0)a}\delta_a$ be a $d-$vector field. Taking into account the properties of the operators $\underset{\alpha}{D}{}^H_X T = X^{(0)d}\underset{\alpha}{D}{}^{W_2}_{\delta_d} T$, $(\alpha = 0,1,2)$ and the formulae (2.1), (2.4) we obtain

$$\underset{\alpha}{D}{}^H_X T = X^{(0)d} T^{a_1...a_r}_{\ b_1...b_s|\alpha d}\delta_{a_1} \otimes ... \otimes \dot{\partial}^{b_s} \otimes dx^{b_1} \otimes ... \otimes \delta p_{a_r}, \tag{3.1}$$

where

$$\begin{aligned} T^{a_1...a_r}_{\ b_1...b_s|\alpha d} = \delta_d T^{a_1...a_r}_{\ b_1...b_s} + \underset{(\alpha 0)}{H}{}^{a_1}{}_{cd} T^{ca_2...a_r}_{\ b_1...b_s} + ... + \underset{(\alpha 0)}{H}{}^{a_r}{}_{cd} T^{a_1...c}_{\ b_1...b_s} - \\ - \underset{(\alpha 0)}{H}{}^c{}_{b_1 d} T^{a_1...a_r}_{\ cb_2...b_s} - ... - \underset{(\alpha 0)}{H}{}^c{}_{b_s d} T^{a_1...a_r}_{\ b_1...c}. \end{aligned} \tag{3.2}$$

The operators "$_{|\alpha}$" are called the $\mathbf{h}_\alpha$**−covariant derivatives** with respect to $D\Gamma(N), (\alpha = 0,1,2)$.

Let us consider the operators $\underset{\alpha}{D}{}_X^{V_1}$, for the $d-$vector field $X^{V_1} = X^{(1)d}\dot{\partial}_d$, applied of T given by (3.0). We obtain

$$\underset{\alpha}{D}{}_X^{V_1} T = X^{(1)d} T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} |_{\alpha d}\ \delta_{a_1} \otimes \dots \otimes \dot{\partial}^{b_s} \otimes dx^{a_1} \otimes \dots \otimes \delta p_{a_r}, \tag{3.1'}$$

where

$$T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} |_{\alpha d} = \dot{\partial}_d T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} + \underset{(\alpha 1)}{C}{}^{a_1}{}_{cd} T^{\ c a_2 \dots a_r}_{\ b_1 \dots b_s} + \dots + \underset{(\alpha 1)}{C}{}^{a_r}{}_{cd} T^{\ a_1 \dots c}_{\ b_1 \dots b_s} -$$
$$- \underset{(\alpha 1)}{C}{}^{c}{}_{b_1 d} T^{\ a_1 \dots a_r}_{\ c b_2 \dots b_s} - \dots - \underset{(\alpha 1)}{C}{}^{c}{}_{b_s d} T^{\ a_1 \dots a_r}_{\ b_1 \dots c}. \tag{3.2'}$$

The operators "$|_\alpha$" are called $\mathbf{v}_{1\alpha}$**−covariant derivatives** with respect to $D\Gamma(N)$, $(\alpha = 0,1,2)$.

Analogous, for $X = X^{W_2} = \underset{(2)}{X}{}_a \dot{\partial}^a$, we have

$$\underset{\alpha}{D}{}_X^{W_2} T = \underset{(2)}{X}{}_d T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} |^{\alpha d}\ \delta_{a_1} \otimes \dots \otimes \dot{\partial}^{b_s} \otimes dx^{a_1} \otimes \dots \otimes \delta p_{a_r}, \tag{3.1''}$$

where

$$T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} |^{\alpha d} = \dot{\partial}^d T^{\ a_1 \dots a_r}_{\ b_1 \dots b_r} + \underset{(\alpha 2)}{C}{}_c{}^{a_1 d} T^{\ c a_2 \dots a_r}_{\ b_1 \dots b_s} + \dots + \underset{(\alpha 2)}{C}{}_c{}^{a_r d} T^{\ a_1 \dots c}_{\ b_1 \dots b_s} -$$
$$- \underset{(\alpha 2)}{C}{}_{b_1}{}^{cd} T^{\ a_1 \dots a_r}_{\ c b_2 \dots b_s} - \dots - \underset{(\alpha 2)}{C}{}_{b_s}{}^{cd} T^{\ a_1 \dots a_r}_{\ b_1 \dots c}. \tag{3.2''}$$

The operators "$|^\alpha$" are called $\mathbf{w}_{2\alpha}$**−covariant derivatives** with respect to $D\Gamma(N), (\alpha = 0,1,2)$.

By a direct calculus, we obtain

Proposition 3.1 *The quantities*

$$T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s | \alpha d},\ T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} |_{\alpha d},\ T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} |^{\alpha d},\ (\alpha = 0,1,2),$$

are $d-$tensor fields. The first six are of type $(r, s+1)$, the last three are of type $(r+1, s)$.

Proposition 3.2 *The operators $_{|\alpha}, |_\alpha, |^\alpha, (\alpha = 0,1,2)$ have the properties*

1°. $f_{|\alpha d} = \delta_d f,\ f\,|_{\alpha d} = \dot{\partial}_d f,\ f\,|^{\alpha d} = \dot{\partial}^d f,\ \forall f \in \mathcal{F}\left(T^{*2}M\right)$.

2°. *These operators are distributive with respect to the adition of the $d-$tensor field of the same type.*

3°. They commute with the operation of contaction.

4°. They verify the Leibnitz rule with respect to the tensor product.

Now, we shall give two applications of this paragraph.

Application 3.1 Let us study the **"$(\underset{\alpha}{\mathbf{y}})-$ deflection tensor fields"** of the $N-$linear connection $D\Gamma(N)$, where y^a is the Liouville $d-$vector field (**the speed**). They are defined by

$$\underset{\alpha}{D}{}^a{}_b = y^a{}_{|\alpha b},\ \underset{\alpha}{d}{}^a{}_b = y^a\,|_{\alpha b},\ \underset{\alpha}{d}{}^{ab} = y^a\,|^{\alpha b},\ (\alpha = 0,1,2)\,. \tag{3.3}$$

Proposition 3.3 *The* $(\underset{\alpha}{y})-$*deflection tensor fields,* $(\alpha = 0,1,2)$, *have the following expressions*

$$\underset{\alpha}{D}{}^a{}_b = -N^a_b + y^c \underset{(\alpha 0)}{H}{}^a{}_{cb},\ \underset{\alpha}{d}{}^a{}_b = \delta^a_b + y^c \underset{(\alpha 1)}{C}{}^a{}_{cb},\ \underset{\alpha}{d}{}^{ab} = y^c \underset{(\alpha 2)}{C}{}_c{}^{ab}. \tag{3.3'}$$

These equalities are easy to prove, if we notice that

$$y^a{}_{|\alpha b} = \delta_b y^a + y^c \underset{(\alpha 0)}{H}{}^a{}_{cb},\ y^a\,|_{\alpha b} = \dot{\partial}_d y^a + y^c \underset{(\alpha 1)}{C}{}^a{}_{cb},\ y^a\,|^{\alpha b} = \dot{\partial}^d y^a + y^c \underset{(\alpha 2)}{C}{}^a{}_{cb},$$
$$(\alpha = 0,1,2)\,.$$

Analogous, we introduced **"$(\underset{\alpha}{\mathbf{p}})-$ deflection tensor fields"**, where p_a is Hamilton $d-$covector field (**the momentum**), by

$$\underset{\alpha}{\triangle}{}_{ab} = p_{a|\alpha b},\ \underset{\alpha}{\vartheta}{}_{ab} = p_a\,|_{\alpha b},\ \underset{\alpha}{\vartheta}{}_a{}^b = p_a\,|^{\alpha b}, (\alpha = 0,1,2)\,. \tag{3.4}$$

Proposition 3.4 $(\underset{\alpha}{p})-$*deflection tensor fields,* $(\alpha = 0,1,2)$, *have the following expressions*

$$\underset{\alpha}{\triangle}{}_{ab} = N_{ba} - p_c \underset{(\alpha 0)}{H}{}^c{}_{ab},\ \vartheta_{ab} = -p_c \underset{(\alpha 1)}{C}{}^c{}_{ab},\ \vartheta_a{}^b = \delta^b_a - p_c \underset{(\alpha 2)}{C}{}^c{}_{ab},$$
$$(\alpha = 0,1,2)\,. \tag{3.4'}$$

Proof. Using (3.2), (3.2') and (3.2") applied of p_a we obtain the expressions (3.4').

q.e.d.

The deflection tensors will used in determination of some important identities as particular case of Ricci identities, applied of $d-$tensor fields y^a and $p_a, (a = 1,...,n)$.

Application 3.2 The distinguished tensor field $g_{ab}(x,y,p)$ used in § 4.6, formula (6.5), has $h_\alpha-, v_{1\alpha}-$ and $w_{2\alpha}-$ covariant derivatives with respect to the $N-$linear connection $D\Gamma(N)$, (2.3), given by:

$$\begin{cases} g_{ab|\alpha c} = \delta_c g_{ab} - \underset{(\alpha 0)}{H}{}^d{}_{ac} g_{db} - \underset{(\alpha 0)}{H}{}^d{}_{bc} g_{ad}, \\ g_{ab}\,|_{\alpha c} = \dot{\partial}_c g_{ab} - \underset{(\alpha 1)}{C}{}^d{}_{ac} g_{db} - \underset{(\alpha 1)}{C}{}^d{}_{bc} g_{ad}, \\ g_{ab}\,|^{\alpha c} = \dot{\partial}^c g_{ab} - \underset{(\alpha 2)}{C}{}_a{}^{dc} g_{db} - \underset{(\alpha 2)}{C}{}_b{}^{dc} g_{ad}. \end{cases} \tag{3.5}$$

5.4 $N-$linear connections of Miron type on $T^{*2}M$

An important particular case of $N-$linear connection is given by

Definition 4.1 *An $N-$linear connection D on $T^{*2}M$ is called an* **MN$-$linear connection** *($N-$linear connection of* **Miron type** *) if:*

1°. The $1-$tangent structure J is absolute parallel with respect to D.

2°. The presymplectic structure θ is absolute parallel with respect to D.

Because

$$\begin{array}{l} (D_X J)\left(Y^H\right) = D_X\left(JY^H\right) - J\left(D_X Y^H\right), \\ (D_X J)\left(Y^{V_1}\right) = D_X\left(JY^{V_1}\right) - J\left(D_X Y^{V_1}\right), \\ (D_X J)\left(Y^{W_2}\right) = D_X\left(JY^{W_2}\right) - J\left(D_X Y^{W_2}\right), \end{array}$$

we can formulate

Theorem 4.1 *An $N-$linear connection D is an $MN-$linear connection on the manifold $T^{*2}M$ if and only if:*

$$\begin{array}{rl} 1^{\circ}.\ D_X\left(JY^H\right) = J\left(D_X Y^H\right), & \\ D_X\left(JY^{V_1}\right) = J\left(D_X Y^{V_1}\right), & \\ D_X\left(JY^{W_2}\right) = J\left(D_X Y^{W_2}\right), & \forall X, Y \in \chi\left(T^{*2}M\right). \end{array} \tag{4.1}$$

$$2^{\circ}.\ D\theta = 0. \tag{4.2}$$

We remark that the latest two equalities from (4.1) are coorsely, because a $N-$linear connection D preserves by parallelism the distributions V_1 and W_2 and $J\left(Y^{V_1}\right) = 0,\ J\left(Y^{W_2}\right) = 0.$

Theorem 4.2 *An $MN-$linear connection on $T^{*2}M$ is characterized by the coefficients $MD\Gamma(N)$ given by (2.3) where*

$$\begin{array}{l} \underset{(00)}{H}{}^{a}{}_{bc} = \underset{(10)}{H}{}^{a}{}_{bc} = \underset{(20)}{H}{}^{a}{}_{bc} =: H^{a}{}_{bc}, \\ \underset{(01)}{C}{}^{a}{}_{bc} = \underset{(11)}{C}{}^{a}{}_{bc} = \underset{(21)}{C}{}^{a}{}_{bc} =: C^{a}{}_{bc}, \\ \underset{(02)}{C}{}^{a}{}_{b}{}^{c} = \underset{(12)}{C}{}^{a}{}_{b}{}^{c} = \underset{(22)}{C}{}^{a}{}_{b}{}^{c} =: C^{a}{}_{b}{}^{c}, \end{array} \tag{4.3}$$

Proof. By the first equalities (4.1) we can write

$D_{\delta_c}(J\delta_b) = J(D_{\delta_c}\delta_b)$, that is $D_{\delta_c}\dot{\partial}_b = \underset{(00)}{H}{}^{a}{}_{bc}J(\delta_a)$, where $\underset{(10)}{H}{}^{a}{}_{bc} = \underset{(00)}{H}{}^{a}{}_{bc}$.

$D_{\dot{\partial}_c}(J\delta_b) = J\left(D_{\dot{\partial}_c}\delta_b\right)$, that is $D_{\dot{\partial}_c}\dot{\partial}_b = \underset{(01)}{C}{}^{a}{}_{bc}J(\delta_a)$, and therefore $\underset{(11)}{C}{}^{a}{}_{bc} = \underset{(01)}{C}{}^{a}{}_{bc}$.

$D_{\dot{\partial}^c}(J\delta_b) = J\left(D_{\dot{\partial}^c}\delta_b\right)$, that is $D_{\dot{\partial}^c}\dot{\partial}_b = \underset{(02)}{C}{}^{a}{}_{b}{}^{c}J(\delta_a)$, where $\underset{(12)}{C}{}^{a}{}_{b}{}^{c} = \underset{(02)}{C}{}^{a}{}_{b}{}^{c}$.

The equalities $\underset{(20)}{H}{}^{a}{}_{bc} = \underset{(00)}{H}{}^{a}{}_{bc}$, $\underset{(21)}{C}{}^{a}{}_{bc} = \underset{(01)}{C}{}^{a}{}_{bc}$and $\underset{(22)}{C}{}^{a}{}_{b}{}^{c} = \underset{(02)}{C}{}^{a}{}_{b}{}^{c}$ are obtaining from $D_{\delta_c}\theta = 0$.

q.e.d.

Also, we obtain

Proposition 4.1 *For any MN-linear connection $MD\Gamma(N) = \left(H^a{}_{bc}, C^a{}_{bc}, C_a{}^{bc}\right)$ we have*

$$D_X\mathbb{F} = 0,\ \forall X \in \chi\left(T^{*2}M\right). \tag{4.4}$$

Indeed, by $(D_X\mathbb{F})(Y) = D_X\mathbb{F}(Y) - \mathbb{F}(D_XY)$, we obtain

$(D_{\delta_c}\mathbb{F})(\delta_b) = D_{\delta_c}\mathbb{F}(\delta_b) - \mathbb{F}(D_{\delta_c}\delta_b) = D_{\delta_c}\left(-\dot{\partial}_b\right) - H^a{}_{bc}\mathbb{F}(\delta_b) =$

$= -\left(H^a{}_{bc} - H^a{}_{cb}\right)\dot{\partial}_a = 0$, etc.

Remark 4.1 1°.We have $\{MD\Gamma(N)\} \subset \{D\Gamma(N)\}$.

2°.For any $MN-$linear connection, the $h_\alpha-, v_{1\alpha}-$ and $w_{2\alpha}-$covariant derivatives, $(\alpha = 0, 1, 2)$, one reduce to $h-, v_1-$ and w_2- covariant derivatives. Also " $_{\shortmid\alpha c}$ ", $(\alpha = 0, 1, 2)$ one reduce to " $_{\shortmid c}$ " only, " $|_{\alpha c}$ ", $(\alpha = 0, 1, 2)$ one reduce to " $|_c$ " and " $|^{\alpha c}$ " one reduce to " $|^c$ ", $(c = 1, ..., n)$, respectively.

3°. For any $MN-$linear connection with the coefficients

$$MD\Gamma(N) = \left(H^a{}_{bc}, C^a{}_{bc}, C_a{}^{bc}\right), \tag{4.5}$$

the deflection $d-$tensor fields have the expressions

$$\begin{aligned} D^a{}_b &= y^a{}_{\shortmid b},\ d^a{}_b = y^a\ |_b,\ d^{ab} = y^a\ |^b, \\ \triangle_{ab} &= p_{a\shortmid b},\ \delta_{ab} = p_a\ |_b,\ \delta_a{}^b = p_a\ |^b, \end{aligned} \tag{4.6}$$

etc.

Whole these correspond of R. Miron theory on the Hamilton spaces of higher order recently achieved and published in prestigious Kluwer Acad. Press, in two volume of speciality [86], [97].

From this paragraph we clearly see how results of this paper generalize the works remarked before. To work with nine coefficients for a linear connection on $T^{*2}M$ (replaced three) is an advantage in the physical applications in electrodynamics [103], [104], elasticity [105],quantum field theories [109], [119], in the deviations of geodesics [29], [30], [133], etc., because, after who shall see, the torsion, the curvature, remarkable identities, etc., are much more substantial.

5.5 The torsion of an $N-$linear connection

Let D be an $N-$linear connection. **The torsion** of D is given by

$$\mathbb{T}(X, Y) = D_XY - D_YX - [X, Y],\ \forall X, Y \in \chi\left(T^{*2}M\right). \tag{5.1}$$

It can be evaluated for the pairs of $d-$vector fields $\left(X^H, Y^H\right)$, $\left(X^H, Y^{V_1}\right)$, $\left(X^H, Y^{W_2}\right)$, $\left(X^{V_1}, Y^{V_1}\right)$, $\left(X^{V_1}, Y^{W_2}\right)$ and $\left(X^{W_2}, Y^{W_2}\right)$. We obtain the vector fields,

$\mathbb{T}\left(X^H,Y^H\right),\ \mathbb{T}\left(X^H,Y^{V_\beta}\right),\ \mathbb{T}\left(X^{V_\beta},Y^{V_\gamma}\right),\ (\beta,\gamma=1,2,\ \beta\leq\gamma,V_2=W_2).$

Since, D preserves by parallelism the distributions N,V_1,W_2 and the distributions V_1,W_2 are integrable it follows

Proposition 5.1 *The following properties of the torsion* $\mathbb{T}$ holds:

$$h\mathbb{T}\left(X^{V_1},Y^{V_1}\right)=0,\ h\mathbb{T}\left(X^{V_1},Y^{W_2}\right)=0,\ h\mathbb{T}\left(X^{W_2},Y^{W_2}\right)=0,$$
$$w_2\mathbb{T}\left(X^{V_1},Y^{V_1}\right)=0,\ v_1\mathbb{T}\left(X^{W_2},Y^{W_2}\right)=0. \quad (5.2)$$

From this assertion, we deduce

Proposition 5.2 *The tensor field of torsion* $\mathbb{T}$ *of an* $N-$*linear connection* D *is uniquely determined by the following components*

$$\begin{cases}\mathbb{T}\left(X^H,Y^H\right)=h\mathbb{T}\left(X^H,Y^H\right)+v_1\mathbb{T}\left(X^H,Y^H\right)+w_2\mathbb{T}\left(X^H,Y^H\right),\\ \mathbb{T}\left(X^H,Y^{V_1}\right)=h\mathbb{T}\left(X^H,Y^{V_1}\right)+v_1\mathbb{T}\left(X^H,Y^{V_1}\right)+w_2\mathbb{T}\left(X^H,Y^{V_1}\right),\\ \mathbb{T}\left(X^H,Y^{W_2}\right)=h\mathbb{T}\left(X^H,Y^{W_2}\right)+v_1\mathbb{T}\left(X^H,Y^{W_2}\right)+w_2\mathbb{T}\left(X^H,Y^{W_2}\right),\end{cases} \quad ((5.3)_1)$$

$$\begin{cases}\mathbb{T}\left(X^{V_1},Y^{V_1}\right)= & v_1\mathbb{T}\left(X^{V_1},Y^{V_1}\right)\\ \mathbb{T}\left(X^{V_1},Y^{W_2}\right)= & v_1\mathbb{T}\left(X^{V_1},Y^{W_2}\right)+w_2\mathbb{T}\left(X^{V_1},Y^{W_2}\right),\end{cases} \quad ((5.3)_2)$$

$$\mathbb{T}\left(X^{W_2},Y^{W_2}\right)= \quad w_2\mathbb{T}\left(X^{W_2},Y^{W_2}\right), \quad ((5.3)_3)$$

where in the right part of each equalities we have $d-$*tensor fields of type* $(1,2)$.

These terms, will call **d−tensors of torsion** of the $N-$linear connection D. More exactly, $h\mathbb{T}\left(X^H,Y^H\right)$ is called $h\left(hh\right)-$tensor of torsion of D, $v_1\mathbb{T}\left(X^H,Y^H\right)$ is called $v_1\left(hh\right)-$tensor of torsion of D and so on.

By direct calculus we prove

Theorem 5.1 The $d-$*tensors of torsion of an* $N-$*linear connection* D *on* $T^{*2}M$, *with the coefficients*

$$D\Gamma\left(N\right)=\left(\underset{(00)}{H}{}^a{}_{bc},\underset{(10)}{H}{}^a{}_{bc},\underset{(20)}{H}{}^a{}_{bc},\underset{(01)}{C}{}^a{}_{bc},\underset{(11)}{C}{}^a{}_{bc},\underset{(21)}{C}{}^a{}_{bc},\underset{(02)}{C}{}^a{}_b{}^c,\underset{(12)}{C}{}^a{}_b{}^c,\underset{(22)}{C}{}^a{}_b{}^c\right),$$

in the adapted basis (2.3), § 4.2, have the expressions:

$$\begin{cases}h\mathbb{T}\left(\delta_c,\delta_b\right)=\underset{(00)}{\overset{0}{T}}{}^a{}_{bc}\delta_a,\\ \qquad v_1\mathbb{T}\left(\delta_c,\delta_b\right)=\underset{(00)}{\overset{1}{T}}{}^a{}_{bc}\dot{\partial}_a,\ w_2\mathbb{T}\left(\delta_c,\delta_b\right)=\underset{(00)}{\overset{2}{T}}{}_{abc}\dot{\partial}^a,\\ h\mathbb{T}\left(\dot{\partial}_c,\delta_b\right)=\underset{(01)}{\overset{0}{P}}{}^a{}_{bc}\delta_a,\\ \qquad v_1\mathbb{T}\left(\dot{\partial}_c,\delta_b\right)=\underset{(01)}{\overset{1}{P}}{}^a{}_{bc}\dot{\partial}_a,\ w_2\mathbb{T}\left(\dot{\partial}_c,\delta_b\right)=\underset{(01)}{\overset{2}{P}}{}_{abc}\dot{\partial}^a,\\ h\mathbb{T}\left(\dot{\partial}^c,\delta_b\right)=\underset{(02)}{\overset{0}{P}}{}^a{}_b{}^c\delta_a,\\ \qquad v_1\mathbb{T}\left(\dot{\partial}^c,\delta_b\right)=\underset{(02)}{\overset{1}{P}}{}^a{}_b{}^c\dot{\partial}_a,\ w_2\mathbb{T}\left(\dot{\partial}^c,\delta_b\right)=\underset{(02)}{\overset{2}{P}}{}_{ab}{}^c\dot{\partial}^a,\end{cases} \quad ((5.4)_1)$$

$$\begin{cases} v_1\mathbb{T}\left(\dot{\partial}_c, \dot{\partial}_b\right) = \underset{(11)}{\overset{1}{S}}{}^a{}_{bc}\dot{\partial}_a, \\ v_1\mathbb{T}\left(\dot{\partial}^c, \dot{\partial}_b\right) = \underset{(12)}{\overset{1}{Q}}{}^a{}_b{}^c\dot{\partial}_a, w_2\mathbb{T}\left(\dot{\partial}^c, \dot{\partial}_b\right) = \underset{(12)}{\overset{2}{Q}}{}_{ab}{}^c\dot{\partial}^a, \end{cases} \tag{(5.4)$_2$}$$

$$w_2\mathbb{T}\left(\dot{\partial}^c, \dot{\partial}^b\right) = \underset{(22)}{\overset{2}{S}}{}_a{}^{bc}\dot{\partial}^a, \tag{(5.4)$_3$}$$

where

$$\begin{cases} \underset{(00)}{\overset{0}{T}}{}^a{}_{bc} = \underset{(00)}{H}{}^a{}_{bc} - \underset{(00)}{H}{}^a{}_{cb}, \ \underset{(00)}{\overset{1}{T}}{}^a{}_{bc} = \underset{(01)}{R}{}^a{}_{bc}, \ \underset{(00)}{\overset{2}{T}}{}_{abc} = \underset{(02)}{R}{}_{abc}, \\ \underset{(01)}{\overset{0}{P}}{}^a{}_{bc} = \underset{(01)}{C}{}^a{}_{bc}, \ \underset{(01)}{\overset{1}{P}}{}^a{}_{bc} = \underset{(11)}{B}{}^a{}_{bc} - \underset{(10)}{H}{}^a{}_{cb}, \ \underset{(01)}{\overset{2}{P}}{}_{abc} = \underset{(12)}{B}{}_{abc}, \\ \underset{(02)}{\overset{0}{P}}{}^a{}_b{}^c = \underset{(02)}{C}{}^a{}_b{}^c, \ \underset{(02)}{\overset{1}{P}}{}^a{}_b{}^c = \underset{(21)}{B}{}^a{}_b{}^c, \ \underset{(02)}{\overset{2}{P}}{}_{ab}{}^c = \underset{(22)}{B}{}_{ab}{}^c + \underset{(20)}{H}{}^c{}_{ab}, \end{cases} \tag{(5.5)$_1$}$$

$$\begin{cases} \underset{(11)}{\overset{1}{S}}{}^a{}_{bc} = \underset{(11)}{C}{}^a{}_{bc} - \underset{(11)}{C}{}^a{}_{cb}, \\ \underset{(12)}{\overset{1}{Q}}{}^a{}_b{}^c = \underset{(12)}{C}{}^a{}_b{}^c =: \underset{(12)}{C}{}^{ac}{}_b, \ \underset{(12)}{\overset{2}{Q}}{}_{ab}{}^c = \underset{(21)}{C}{}_a{}^c{}_b =: \underset{(21)}{C}{}^c{}_{ab}, \end{cases} \tag{(5.5)$_2$}$$

$$\underset{(22)}{\overset{2}{S}}{}_a{}^{bc} = -\left(\underset{(22)}{C}{}_a{}^{bc} - \underset{(22)}{C}{}_a{}^{cb}\right). \tag{(5.5)$_3$}$$

Proof. We take into account the Lie brackets, Proposition 4.1, the formulae (4.1), (4.2), § 4.4,and of the write of an $N-$linear connection in the adapted basis, the formulae (2.1), § 5.2. We obtain, successively

$$h\mathbb{T}(\delta_c, \delta_b) = hD_{\delta_c}\delta_b - hD_{\delta_b}\delta_c - h[\delta_c, \delta_b] = \underset{(00)}{H}{}^a{}_{bc} - \underset{(00)}{H}{}^a{}_{cb},$$

and the first equality $(5.5)_1$ is true. Now

$$v_1\mathbb{T}\left(\dot{\partial}_c, \delta_b\right) = v_1D_{\dot{\partial}_c}\delta_b - v_1D_{\delta_b}\dot{\partial}_c - v_1\left[\dot{\partial}_c, \delta_b\right] = -\underset{(10)}{H}{}^a{}_{cb} + \underset{(11)}{B}{}^a{}_{bc},$$

and the 5^{th}equality $(5.5)_1$ is correct. Then, for exemple

$$w_2\mathbb{T}\left(\dot{\partial}^c, \delta_b\right) = w_2D_{\dot{\partial}^c}\delta_b - w_2D_{\delta_b}\dot{\partial}^c - w_2\left[\dot{\partial}^c, \delta_b\right] = \underset{(20)}{H}{}_a{}^c{}_b + B_{ab}{}^c,$$

and the 9^{th}equality $(5.5)_1$ is true. In same manner, we obtain the other equalities **q.e.d.**

Remark 5.1 The tensor field of torsion $\mathbb{T}$ of an $N-$linear connection D on $T^{*2}M$ is formed by **13** $d-$tensor fields of torsion, components of $\mathbb{T}$. Especially, we have

$$\underset{(00)}{\overset{0}{T}}{}^{a}{}_{bc} = \underset{(00)}{H}{}^{a}{}_{bc} - \underset{(00)}{H}{}^{a}{}_{cb},\ \underset{(11)}{\overset{1}{S}}{}^{a}{}_{bc} = \underset{(11)}{C}{}^{a}{}_{bc} - \underset{(11)}{C}{}^{a}{}_{cb},\ \underset{(22)}{\overset{2}{S}}{}_{a}{}^{bc} = -\left(\underset{(22)}{C}{}_{a}{}^{bc} - \underset{(22)}{C}{}_{a}{}^{cb}\right). \tag{5.6}$$

Therefore,

Proposition 5.3 *The following statements are equivalent*

1°. $\underset{(00)}{\overset{0}{T}}{}^{a}{}_{bc} = 0,\ \underset{(11)}{\overset{1}{S}}{}^{a}{}_{bc} = 0,\ \underset{(22)}{\overset{2}{S}}{}_{a}{}^{bc} = 0.$

2°. $\underset{(00)}{H}{}^{a}{}_{bc} = \underset{(00)}{H}{}^{a}{}_{cb},\ \underset{(11)}{C}{}^{a}{}_{bc} = \underset{(11)}{C}{}^{a}{}_{cb},\ \underset{(22)}{C}{}_{a}{}^{bc} = \underset{(22)}{C}{}_{a}{}^{cb}.$

We pay attention to the $N-$linear connection given in following definition.

Definition 5.1 *An $N-$linear connection on $T^{*2}M$ is called* ***semisymmetric*** *if*

$$\underset{(00)}{\overset{0}{T}}{}^{a}{}_{bc} = \frac{1}{2}\left(\delta^a_b\sigma_c - \delta^a_c\sigma_b\right),$$
$$\underset{(11)}{\overset{1}{S}}{}^{a}{}_{bc} = \frac{1}{2}\left(\delta^a_b\tau_c - \delta^a_c\tau_b\right),\ \underset{(22)}{\overset{2}{S}}{}_{a}{}^{bc} = -\frac{1}{2}\left(\delta^b_a\upsilon^c - \delta^c_a\upsilon^b\right) \tag{5.7}$$

where $\sigma,\tau \in \chi^*\left(T^{*2}M\right)$ *and* $\upsilon \in \chi\left(T^{*2}M\right)$.

In the next calculus we have need of the following $d-$tensor fields:

$$\underset{(\alpha 0)}{\overset{0}{T}}{}^{a}{}_{bc} = \underset{(\alpha 0)}{H}{}^{a}{}_{bc} - \underset{(\alpha 0)}{H}{}^{a}{}_{cb},\ \underset{(\alpha 1)}{P}{}^{a}{}_{bc} = \underset{(11)}{B}{}^{a}{}_{bc} - \underset{(\alpha 0)}{H}{}^{a}{}_{cb},\ \underset{(\alpha 2)}{P}{}_{ab}{}^{c} = \underset{(22)}{B}{}_{ab}{}^{c} + \underset{(\alpha 0)}{H}{}^{c}{}_{ab},$$
$$\underset{(\alpha 1)}{\overset{\alpha}{S}}{}^{a}{}_{bc} = \underset{(\alpha 1)}{C}{}^{a}{}_{bc} - \underset{(\alpha 1)}{C}{}^{a}{}_{cb},\ \underset{(\alpha 2)}{\overset{\alpha}{S}}{}_{a}{}^{bc} = -\left(\underset{(\alpha 2)}{C}{}_{a}{}^{bc} - \underset{(\alpha 2)}{C}{}_{a}{}^{cb}\right),$$
$$(\alpha = 0,1,2). \tag{5.8}$$

We remark that we have

$$\underset{(11)}{P}{}^{a}{}_{bc} = \underset{(01)}{\overset{1}{P}}{}^{a}{}_{bc},\ \ \underset{(22)}{P}{}_{ab}{}^{c} = \underset{(02)}{\overset{2}{P}}{}_{ab}{}^{c}. \tag{5.9}$$

Particularly, we have

Proposition 5.4 *For any $N-$linear connection of Miron type,* $MD\Gamma(N) = \left(H^a{}_{bc}, C^a{}_{bc}, C_a{}^{bc}\right)$ *we obtain*

$$\begin{aligned}
&\underset{(00)}{\overset{0}{T}}{}^{a}{}_{bc} = \underset{(10)}{\overset{0}{T}}{}^{a}{}_{bc} = \underset{(20)}{\overset{0}{T}}{}^{a}{}_{bc} = H^a{}_{bc} - H^a{}_{cb} =: T^a{}_{bc}\\
&\underset{(01)}{P}{}^{a}{}_{bc} = \underset{(11)}{P}{}^{a}{}_{bc} = \underset{(21)}{P}{}^{a}{}_{bc} = \underset{(11)}{B}{}^{a}{}_{bc} - H^a{}_{cb} =: \underset{(1)}{P}{}^{a}{}_{bc}\\
&\underset{(02)}{P}{}_{ab}{}^{c} = \underset{(12)}{P}{}^{a}{}_{bc} = \underset{(22)}{P}{}^{a}{}_{bc} = \underset{(22)}{B}{}_{ab}{}^{c} + H^c{}_{ab} =: \underset{(2)}{P}{}_{ab}{}^{c}
\end{aligned} \tag{5.10}$$

Now, it is easy to write the $d-$tensors of torsion of an $N-$linear connection of Miron type. By the formulae (4.3), the Theorem 4.2, we have

Proposition 5.5 *The $d-$tensors of torsion, components of torsion tensor field $\mathbb{T}$ of an $MN-$linear connection with coefficients $MD\Gamma(N) = \left(H^a{}_{bc}, C^a{}_{bc}, C_a{}^{bc}\right)$ are*

$$\begin{array}{l} H^a{}_{bc} - H^a{}_{cb} =: T^a{}_{bc},\ \underset{(01)}{R}{}^a{}_{bc},\ \underset{(01)}{R}{}_{abc}, \\ C^a{}_{bc},\ \underset{(11)}{B}{}^a{}_{bc} - H^a{}_{cb} =: \underset{(1)}{P}{}^a{}_{bc},\ \underset{(12)}{B}{}_{abc}, \\ C^a{}_b{}^c =: C_b{}^{ac},\ \underset{(21)}{B}{}^a{}_b{}^c =: \underset{(21)}{B}{}_b{}^{ac},\ \underset{(22)}{B}{}_{ab}{}^c + H^c{}_{ab} =: \underset{(2)}{P}{}^c{}_{ab}, \\ C^a{}_{bc} - C^a{}_{cb} =: S^a{}_{bc}, -\left(C_a{}^{bc} - C_a{}^{cb}\right) =: S_a{}^{bc}. \end{array} \tag{5.11}$$

Remark 5.2 The tensor field $\mathbb{T}$ of an $N-$linear connection of Miron type on $T^{*2}M$ is given by **11** $d-$torsion tensor fields, components of $\mathbb{T}$.

5.6 The curvature of an $N-$linear connection

Let D be an $N-$linear connection on $T^{*2}M$. **The curvature** of D is given by

$$\mathbb{R}(X,Y)Z = (D_X D_Y - D_Y D_X)Z - D_{[X,Y]}Z,\ \forall X,Y,Z \in \chi\left(T^{*2}M\right). \tag{6.1}$$

We will express $\mathbb{R}$ by his components, taking into account the decomposition (3.3), §4.3, for the vector fields on $T^{*2}M$.

We prove

Theorem 6.1 *The curvature tensor field $\mathbb{R}$ of the $N-$linear connection on $T^{*2}M$ have the properties*

$$\begin{array}{l} v_1\mathbb{R}(X,Y)Z^H = 0,\ w_2\mathbb{R}(X,Y)Z^H = 0, \\ h\mathbb{R}(X,Y)Z^{V_1} = 0,\ h\mathbb{R}(X,Y)Z^{W_2} = 0, \\ \mathbb{R}(X,Y)Z = h\mathbb{R}(X,Y)Z^H + v_1\mathbb{R}(X,Y)Z^{V_1} + w_2\mathbb{R}(X,Y)Z^{W_2}. \end{array} \tag{6.2}$$

Proof. Because D preserves by parallelism the horizontal and verticals distributions, by (6.1), the operator $\mathbb{R}(X,Y)$ carries horizontal vector fields to horizontal and verticals vector fields to verticals. Thus the first four equations form (6.2) hold. The next one is an easy consequence of the first four.

q.e.d.

By Theorem 6.1 and the equation $\mathbb{R}(X,Y) = -\mathbb{R}(Y,X), \forall X,Y \in T^{*2}M$, we get

Theorem 6.2 *The curvature tensor $\mathbb{R}$ of an $N-$linear connection D on the total space $T^{*2}M$ of a $2-$cotangent bundle $\left(T^{*2}M, \pi^{*2}, M\right)$ is completely determined by the following $d-$tensor fields:*

$$
\begin{aligned}
&\mathbb{R}\left(X^H, Y^H\right) Z^H = \underset{0}{D}{}^H_X \underset{0}{D}{}^H_Y Z^H - \underset{0}{D}{}^H_Y \underset{0}{D}{}^H_X Z^H - \sum_{\alpha=0}^{2} \underset{0}{D}{}^{V_\alpha}_{[X^H, Y^H]} Z^H, \\
&\mathbb{R}\left(X^H, Y^H\right) Z^{V_\gamma} = \underset{\gamma}{D}{}^H_X \underset{\gamma}{D}{}^H_Y Z^{V_\gamma} - \underset{\gamma}{D}{}^H_Y \underset{\gamma}{D}{}^H_X Z^{V_\gamma} - \sum_{\alpha=0}^{2} \underset{\gamma}{D}{}^{V_\alpha}_{[X^H, Y^H]} Z^{V_\gamma}, (\gamma = 1, 2), \\
&\mathbb{R}\left(X^{V_\beta}, Y^H\right) Z^H = \underset{0}{D}{}^{V_\beta}_X \underset{0}{D}{}^H_Y Z^H - \underset{0}{D}{}^H_Y \underset{0}{D}{}^{V_\beta}_X Z^H - \sum_{\alpha=0}^{2} \underset{0}{D}{}^{V_\alpha}_{[X^H, Y^H]} Z^H, (\beta = 1, 2), \\
&\mathbb{R}\left(X^{V_\beta}, Y^H\right) Z^{V_\gamma} = \underset{\gamma}{D}{}^{V_\beta}_X \underset{\gamma}{D}{}^H_Y Z^{V_\gamma} - \underset{\gamma}{D}{}^H_Y \underset{\gamma}{D}{}^{V_\beta}_X Z^{V_\gamma} - \sum_{\alpha=0}^{2} \underset{\gamma}{D}{}^{V_\alpha}_{\left[X^{V_\beta}, Y^H\right]} Z^{V_\gamma}, \\
&\hfill (\beta, \gamma = 1, 2), \\
&\mathbb{R}\left(X^{V_\beta}, Y^{V_\gamma}\right) Z^H = \underset{0}{D}{}^{V_\beta}_X \underset{0}{D}{}^{V_\gamma}_Y Z^H - \underset{0}{D}{}^{V_\gamma}_Y \underset{0}{D}{}^{V_\beta}_X Z^H - \sum_{\varepsilon=1}^{2} \underset{0}{D}{}^{V_\varepsilon}_{\left[X^{V_\beta}, Y^{V_\gamma}\right]} Z^H, \\
&\hfill (\beta, \gamma = 1, 2, \beta \leq \gamma), \\
&\mathbb{R}\left(X^{V_\beta}, Y^{V_\gamma}\right) Z^{V_\delta} = \underset{\delta}{D}{}^{V_\beta}_X \underset{\delta}{D}{}^{V_\gamma}_Y Z^{V_\delta} - \underset{\delta}{D}{}^{V_\gamma}_Y \underset{\delta}{D}{}^{V_\beta}_X Z^{V_\delta} - \sum_{\varepsilon=1}^{2} \underset{\delta}{D}{}^{V_\varepsilon}_{\left[X^{V_\beta}, Y^{V_\gamma}\right]} Z^{V_\delta}, \\
&\hfill (\beta, \gamma, \delta = 1, 2, \beta \leq \gamma),
\end{aligned}
\tag{6.3}
$$

where $V_0 = H$ *and* $V_2 = W_2$.

The $d-$tensors fields from the right part of the following equalities are called **d−tensors of curvature** of the $N-$linear connection D.

Local expressions of $d-$tensors of curvature of $N-$linear connection D with the coefficients

$$
D\Gamma(N) = \left(\underset{(00)}{H}{}^a{}_{bc}, \underset{(10)}{H}{}^a{}_{bc}, \underset{(20)}{H}{}^a{}_{bc}, \underset{(01)}{C}{}^a{}_{bc}, \underset{(11)}{C}{}^a{}_{bc}, \underset{(21)}{C}{}^a{}_{bc}, \underset{(02)}{C}{}_a{}^{bc}, \underset{(12)}{C}{}_a{}^{bc}, \underset{(22)}{C}{}_a{}^{bc} \right)
$$

can be obtained by (6.3). With notations

$$
\begin{aligned}
&R(\delta_d, \delta_c)\delta_b = \underset{(000)}{R}{}_b{}^a{}_{cd}\delta_a, \; R(\delta_d, \delta_c)\dot{\partial}_b = \underset{(100)}{R}{}_b{}^a{}_{cd}\dot{\partial}_a, \\
&\qquad R(\delta_d, \delta_c)\dot{\partial}^a = -\underset{(200)}{R}{}_b{}^a{}_{cd}\dot{\partial}^b, \\
&R\left(\dot{\partial}_d, \delta_c\right)\delta_b = \underset{(001)}{R}{}_b{}^a{}_{cd}\delta_a, \; R\left(\dot{\partial}_d, \delta_c\right)\dot{\partial}_b = \underset{(101)}{R}{}_b{}^a{}_{cd}\dot{\partial}_a, \\
&\qquad R\left(\dot{\partial}_d, \delta_c\right)\dot{\partial}^a = -\underset{(201)}{R}{}_b{}^a{}_{cd}\dot{\partial}^b, \\
&R\left(\dot{\partial}^d, \delta_c\right)\delta_b = \underset{(002)}{R}{}_b{}^a{}_c{}^d\delta_a, \; R\left(\dot{\partial}^d, \delta_c\right)\dot{\partial}_b = \underset{(102)}{R}{}_b{}^a{}_c{}^d\dot{\partial}_a, \\
&\qquad R\left(\dot{\partial}^d, \delta_c\right)\dot{\partial}^a = -\underset{(202)}{R}{}_b{}^a{}_c{}^d\dot{\partial}^b, \\
&R\left(\dot{\partial}_d, \dot{\partial}_c\right)\delta_b = \underset{(011)}{R}{}_b{}^a{}_{cd}\delta_a, \; R\left(\dot{\partial}_d, \dot{\partial}_c\right)\dot{\partial}_b = \underset{(111)}{R}{}_b{}^a{}_{cd}\dot{\partial}_a, \\
&\qquad R\left(\dot{\partial}_d, \dot{\partial}_c\right)\dot{\partial}^a = -\underset{(211)}{R}{}_b{}^a{}_{cd}\dot{\partial}^b, \\
&R\left(\dot{\partial}^d, \dot{\partial}_c\right)\delta_b = \underset{(012)}{R}{}_b{}^a{}_c{}^d\delta_a, \; R\left(\dot{\partial}^d, \dot{\partial}_c\right)\dot{\partial}_b = \underset{(112)}{R}{}_b{}^a{}_c{}^d\dot{\partial}_a,
\end{aligned}
\tag{6.4}
$$

$$R\left(\dot{\partial}^d, \dot{\partial}_c\right)\dot{\partial}^a = -\underset{(212)}{R}{}_b{}^a{}_c{}^d\dot{\partial}^b,$$
$$R\left(\dot{\partial}^d, \dot{\partial}^c\right)\delta_b = \underset{(022)}{R}{}_b{}^{acd}\delta_a,\ R\left(\dot{\partial}^d, \dot{\partial}^c\right)\dot{\partial}_b = \underset{(122)}{R}{}_b{}^{acd}\dot{\partial}_a,$$
$$R\left(\dot{\partial}^d, \dot{\partial}^c\right)\dot{\partial}^a = -\underset{(222)}{R}{}_b{}^{acd}\dot{\partial}^b,$$

we get

Theorem 6.3 *An $N-$linear connection D with the coefficients*

$$D\Gamma(N) = \left(\underset{(00)}{H}{}^a{}_{bc}, \underset{(10)}{H}{}^a{}_{bc}, \underset{(20)}{H}{}^a{}_{bc}, \underset{(01)}{C}{}^a{}_{bc}, \underset{(11)}{C}{}^a{}_{bc}, \underset{(21)}{C}{}^a{}_{bc}, \underset{(02)}{C}{}_a{}^{bc}, \underset{(12)}{C}{}_a{}^{bc}, \underset{(22)}{C}{}_a{}^{bc}\right)$$

has the $d-$tensors of curvature (6.4) expressed by the following formulae

$$\begin{cases} \underset{(\alpha00)}{R}{}_b{}^a{}_{cd} = \delta_d \underset{(\alpha0)}{H}{}^a{}_{bc} - \delta_c \underset{(\alpha0)}{H}{}^a{}_{bd} + \underset{(\alpha0)}{H}{}^f{}_{bc}\underset{(\alpha0)}{H}{}^a{}_{fd} - \underset{(\alpha0)}{H}{}^f{}_{bd}\underset{(\alpha0)}{H}{}^a{}_{fc} + \\ \qquad + \underset{(\alpha1)}{C}{}^a{}_{bf}\underset{(01)}{R}{}^f{}_{cd} + \underset{(\alpha2)}{C}{}_b{}^{af}\underset{(02)}{R}{}_{fcd}, \\ \underset{(\alpha01)}{R}{}_b{}^a{}_{cd} = \dot{\partial}_d \underset{(\alpha0)}{H}{}^a{}_{bc} - \underset{(\alpha1)}{C}{}^a{}_{bd|\alpha c} + \underset{(\alpha1)}{C}{}^a{}_{bf}\underset{(\alpha1)}{P}{}^f{}_{bc} + \underset{(\alpha2)}{C}{}_b{}^{af}\underset{(12)}{B}{}_{fcd}, \\ \underset{(\alpha00)}{R}{}_b{}^a{}_c{}^d = \dot{\partial}^d \underset{(\alpha0)}{H}{}^a{}_{bc} - \underset{(\alpha2)}{C}{}_b{}^{ad}{}_{|\alpha c} + \underset{(\alpha1)}{C}{}^a{}_{bf}\underset{(21)}{B}{}^f{}_c{}^d + \underset{(\alpha2)}{C}{}_b{}^{af}\underset{(\alpha2)}{P}{}_{fc}{}^d, \\ \qquad (\alpha = 0,1,2), \end{cases} \tag{6.5}$$

$$\begin{cases} \underset{(\alpha11)}{R}{}_b{}^a{}_{cd} = \dot{\partial}_d \underset{(\alpha1)}{C}{}^a{}_{bc} - \dot{\partial}_c \underset{(\alpha1)}{C}{}^a{}_{bd} + \underset{(\alpha1)}{C}{}^f{}_{bc}\underset{(\alpha1)}{C}{}^a{}_{fd} - \underset{(\alpha1)}{C}{}^f{}_{bd}\underset{(\alpha1)}{C}{}^a{}_{fc}, \\ \underset{(\alpha12)}{R}{}_b{}^a{}_c{}^d = \dot{\partial}^d \underset{(\alpha1)}{C}{}^a{}_{bc} - \dot{\partial}_c \underset{(\alpha2)}{C}{}_b{}^{ad} + \underset{(\alpha1)}{C}{}^f{}_{bc}\underset{(\alpha2)}{C}{}_f{}^{ad} - \underset{(\alpha2)}{C}{}_b{}^{fd}\underset{(\alpha1)}{C}{}^a{}_{fc}, \\ \underset{(\alpha22)}{R}{}_b{}^{acd} = \dot{\partial}^d \underset{(\alpha2)}{C}{}_b{}^{ac} - \dot{\partial}^c \underset{(\alpha2)}{C}{}_b{}^{ad} + \underset{(\alpha2)}{C}{}_b{}^{fc}\underset{(\alpha2)}{C}{}_f{}^{ad} - \underset{(\alpha2)}{C}{}_b{}^{fd}\underset{(\alpha2)}{C}{}_f{}^{ac}, \\ \qquad (\alpha = 0,1,2). \end{cases} \tag{6.6}$$

Proof. Taking into account the formulae (4.1),(4.2), §4.4, (2.1), §5.2, and the notations (5.8) and (6.4). We obtain, for example

$$R\left(\dot{\partial}_d, \delta_c\right)\dot{\partial}^a = -\underset{(201)}{R}{}_b{}^a{}_{cd}\dot{\partial}^d = \left(D_{\dot{\partial}_d}D_{\delta_c} - D_{\delta_c}D_{\dot{\partial}_d}\right)\dot{\partial}^a - D_{\dot{\partial}_d,\delta_c}\dot{\partial}^a =$$
$$= -D_{\dot{\partial}_d}\left(\underset{(20)}{H}{}^a{}_{fc}\dot{\partial}^f\right) + D_{\delta_c}\left(\underset{(21)}{C}{}^a{}_{fd}\dot{\partial}^f\right) + D_{\underset{(11)}{B}{}^f{}_{cd}\dot{\partial}_f + \underset{(12)}{B}{}_{fcd}\dot{\partial}^f}\dot{\partial}^a =$$
$$= -\left(\dot{\partial}_d \underset{(20)}{H}{}^a{}_{bc}\right)\dot{\partial}^b + \underset{(20)}{H}{}^a{}_{fc}\underset{(21)}{C}{}^f{}_{bd}\dot{\partial}^b + \left(\delta_c \underset{(21)}{C}{}^a{}_{bd}\right)\dot{\partial}^b - \underset{(21)}{C}{}^a{}_{fd}\underset{(20)}{H}{}^f{}_{bc}\dot{\partial}^b -$$
$$- \underset{(11)}{B}{}^f{}_{cd}\underset{(21)}{C}{}^a{}_{bf}\dot{\partial}^b - \underset{(12)}{B}{}_{fcd}\underset{(22)}{C}{}_b{}^{af}\dot{\partial}^b.$$

We have, therefore

$$\underset{(201)}{R}{}_b{}^a{}_{cd} = \dot{\partial}_d \underset{(20)}{H}{}^a{}_{bc} - \underline{\underset{(20)}{H}{}^a{}_{fc} \underset{(21)}{C}{}^f{}_{bd} - \delta_c \underset{(21)}{C}{}^a{}_{bd} + \underset{(20)}{H}{}^f{}_{bc} \underset{(21)}{C}{}^a{}_{fd}} +$$

$$+ \underset{(21)}{C}{}^a{}_{bf} \underset{(11)}{B}{}^f{}_{cd} + \underset{(22)}{C}{}_b{}^{af} \underset{(12)}{B}{}_{fcd}.$$

But, by formula (3.2) we get

$$\underset{(21)}{C}{}^a{}_{bd|2c} = \underline{\delta_c \underset{(21)}{C}{}^a{}_{bd} + \underset{(20)}{H}{}^a{}_{fc} \underset{(21)}{C}{}^f{}_{bd} - \underset{(20)}{H}{}^f{}_{bc} \underset{(21)}{C}{}^a{}_{fd}} - \underset{(20)}{H}{}^f{}_{dc} \underset{(21)}{C}{}^a{}_{bf}.$$

Interchanging the underline terms, in the last equations, results

$$\underset{(201)}{R}{}_b{}^a{}_{cd} = \dot{\partial}_d \underset{(20)}{H}{}^a{}_{bc} - \underset{(21)}{C}{}^a{}_{bd|2c} + \underset{(21)}{C}{}^a{}_{bf} \underset{(21)}{P}{}^f{}_{bc} + \underset{(22)}{C}{}_b{}^{af} \underset{(12)}{B}{}_{fcd},$$

namely the 6^{th} relation of the lot (6.5) $(\alpha = 2)$. The other equalities are given in same manner.

q.e.d.

Remark 6.1 The tensor field of curvature $\mathbb{R}$ of an $N-$linear connection D on $T^{*2}M$ is given by **18** $d-$tensor fields of curvature, components of $\mathbb{R}$.

Particularly, we have

Proposition 6.1 *For any $N-$linear connection of Miron type , $MD\Gamma(N) = \left(H^a{}_{bc}, C^a{}_{bc}, C_a{}^{bc}\right)$ we obtain*

$$\begin{cases} \underset{(000)}{R}{}_b{}^a{}_{cd} = \underset{(100)}{R}{}_b{}^a{}_{cd} = \underset{(200)}{R}{}_b{}^a{}_{cd} =: R_b{}^a{}_{cd}, \\ \underset{(001)}{R}{}_b{}^a{}_{cd} = \underset{(100)}{R}{}_b{}^a{}_{cd} = \underset{(200)}{R}{}_b{}^a{}_{cd} =: P_b{}^a{}_{cd}, \\ \underset{(002)}{R}{}_b{}^a{}_c{}^d = \underset{(102)}{R}{}_b{}^a{}_c{}^d = \underset{(202)}{R}{}_b{}^a{}_c{}^d =: P_b{}^a{}_c{}^d, \end{cases} \tag{6.7}$$

$$\begin{cases} \underset{(011)}{R}{}_b{}^a{}_{cd} = \underset{(111)}{R}{}_b{}^a{}_{cd} = \underset{(211)}{R}{}_b{}^a{}_{cd} =: S_b{}^a{}_{cd}, \\ \underset{(012)}{R}{}_b{}^a{}_c{}^d = \underset{(112)}{R}{}_b{}^a{}_c{}^d = \underset{(212)}{R}{}_b{}^a{}_c{}^d =: S_b{}^a{}_c{}^d, \\ \underset{(022)}{R}{}_b{}^{acd} = \underset{(122)}{R}{}_b{}^{acd} = \underset{(222)}{R}{}_b{}^{acd} =: S_b{}^{acd}. \end{cases} \tag{6.8}$$

The writing of $d-$tensors of curvature from an $N-$linear connection of Miron type, is immediately by Theorem 6.3 and formulae (4.3), § 5.4.

Proposition 6.2 There exist only 6 the essentially *$d-$tensors of curvature, components of the tensor field of curvature $\mathbb{R}$ of an $MN-$linear connection with coefficients $MD\Gamma(N) = \left(H^a{}_{bc}, C^a{}_{bc}, C_a{}^{bc}\right)$ namely*

$$
\begin{cases}
R_b{}^a{}_{cd} = \delta_d H^a{}_{bc} - \delta_c H^a{}_{bd} + H^f{}_{bc} H^a{}_{fd} - H^f{}_{bd} H^a{}_{fc} + \\
\qquad\qquad + C^a{}_{bf} \underset{(01)}{R}{}^f{}_{cd} + C_b{}^{af} \underset{(02)}{R}{}_{fcd}, \\
P_b{}^a{}_{cd} = \dot{\partial}_d H^a{}_{bc} - C^a{}_{bd|c} + C^a{}_{bf} \underset{(1)}{P}{}^f{}_{bc} + C_b{}^{af} \underset{(12)}{B}{}_{fcd}, \\
P_b{}^a{}_c{}^d = \dot{\partial}^d H^a{}_{bc} - C_b{}^{ad}{}_{|c} + C^a{}_{bf} \underset{(21)}{B}{}^f{}_c{}^d + C_b{}^{af} \underset{(2)}{P}{}_{fc}{}^d,
\end{cases}
\tag{6.9}
$$

$$
\begin{cases}
S_b{}^a{}_{cd} = \dot{\partial}_d C^a{}_{bc} - \dot{\partial}_c C^a{}_{bd} + C^f{}_{bc} C^a{}_{fd} - C^f{}_{bd} C^a{}_{fc}, \\
S_b{}^a{}_c{}^d = \dot{\partial}^d C^a{}_{bc} - \dot{\partial}_c C_b{}^{ad} + C^f{}_{bc} C_f{}^{ad} - C_b{}^{fd} C^a{}_{fc}, \\
S_b{}^{acd} = \dot{\partial}^d C_b{}^{ac} - \dot{\partial}^c C_b{}^{ad} + C_b{}^{fc} C_f{}^{ad} - C_b{}^{fd} C_f{}^{ac}.
\end{cases}
\tag{6.10}
$$

5.7 The Ricci identities

In the application it is suitable to consider the equalities (6.3) as **Ricci identities.** A simple aranjament of (6.3), gives us

Theorem 7.1 *For any $N-$linear connection D on $T^{*2}M$ the following Ricci identities hold:*

$$
\begin{aligned}
&\underset{0}{D}{}^{H}_{X} \underset{0}{D}{}^{H}_{Y} Z^H - \underset{0}{D}{}^{H}_{Y} \underset{0}{D}{}^{H}_{X} Z^H = \mathbb{R}\left(X^H, Y^H\right) Z^H + \underset{0}{D}{}^{H}_{[X^H, Y^H]} Z^H + \\
&\qquad + \sum_{\varepsilon=1}^{2} \underset{0}{D}{}^{V_\varepsilon}_{[X^H, Y^H]} Z^H, \\
&\underset{0}{D}{}^{V_\beta}_{X} \underset{0}{D}{}^{H}_{Y} Z^H - \underset{0}{D}{}^{H}_{Y} \underset{0}{D}{}^{V_\beta}_{X} Z^H = \mathbb{R}\left(X^{V_\beta}, Y^H\right) Z^H + \underset{0}{D}{}^{H}_{[X^{V_\beta}, Y^H]} Z^H + \\
&\qquad + \sum_{\varepsilon=1}^{2} \underset{0}{D}{}^{V_\varepsilon}_{[X^{V_\beta}, Y^H]} Z^H, (\beta = 1, 2), \\
&\underset{0}{D}{}^{V_\beta}_{X} \underset{0}{D}{}^{V_\gamma}_{Y} Z^H - \underset{0}{D}{}^{V_\gamma}_{Y} \underset{0}{D}{}^{V_\beta}_{X} Z^H = \mathbb{R}\left(X^{V_\beta}, Y^{V_\gamma}\right) Z^H + \sum_{\varepsilon=1}^{2} \underset{0}{D}{}^{V_\varepsilon}_{[X^{V_\beta}, Y^{V_\gamma}]} Z^H, \\
&\qquad (\beta, \gamma = 1, 2, \beta \leq \gamma),
\end{aligned}
\tag{(7.1)$_1$}
$$

$$
\begin{aligned}
&\underset{\gamma}{D}{}^{H}_{X} \underset{\gamma}{D}{}^{H}_{Y} Z^{V_\gamma} - \underset{\gamma}{D}{}^{H}_{Y} \underset{\gamma}{D}{}^{H}_{X} Z^{V_\gamma} = \mathbb{R}\left(X^H, Y^H\right) Z^{V_\gamma} + \underset{\gamma}{D}{}^{H}_{[X^H, Y^H]} Z^{V_\gamma} + \\
&\qquad + \sum_{\varepsilon=1}^{2} \underset{\gamma}{D}{}^{V_\varepsilon}_{[X^H, Y^H]} Z^{V_\gamma}, (\gamma = 1, 2), \\
&\underset{\gamma}{D}{}^{V_\beta}_{X} \underset{\gamma}{D}{}^{H}_{Y} Z^{V_\gamma} - \underset{\gamma}{D}{}^{H}_{Y} \underset{\gamma}{D}{}^{V_\beta}_{X} Z^{V_\gamma} = \mathbb{R}\left(X^{V_\beta}, Y^H\right) Z^{V_\gamma} + \underset{\gamma}{D}{}^{H}_{[X^{V_\beta}, Y^H]} Z^{V_\gamma} + \\
&\qquad + \sum_{\varepsilon=1}^{2} \underset{\gamma}{D}{}^{V_\varepsilon}_{[X^{V_\beta}, Y^H]} Z^{V_\gamma}, (\beta, \gamma = 1, 2), \\
&\underset{\delta}{D}{}^{V_\beta}_{X} \underset{\delta}{D}{}^{V_\gamma}_{Y} Z^{V_\delta} - \underset{\delta}{D}{}^{V_\gamma}_{Y} \underset{\delta}{D}{}^{V_\beta}_{X} Z^{V_\delta} = \mathbb{R}\left(X^{V_\beta}, Y^{V_\gamma}\right) Z^{V_\delta} + \sum_{\varepsilon=1}^{2} \underset{\delta}{D}{}^{V_\varepsilon}_{[X^{V_\beta}, Y^{V_\gamma}]} Z^{V_\delta}, \\
&\qquad (\beta, \gamma, \delta = 1, 2, \beta \leq \gamma),
\end{aligned}
\tag{(7.1)$_2$}
$$

where $V_0 = H$ *and* $V_2 = W_2$.

We can establish these identities for a vector field, although they could be write for any tensor field.

Theorem 7.2 *For any vector field* $X \in \chi\left(T^{*2}M\right)$ *we have following Ricci identities*

$$\begin{aligned}
&X^a{}_{\shortmid\alpha b\shortmid\alpha c} - X^a{}_{\shortmid\alpha c\shortmid\alpha b} = X^f \underset{(\alpha 00)}{R}{}_f{}^a{}_{bc} - \overset{0}{\underset{(\alpha 0)}{T}}{}^f{}_{bc} X^a{}_{\shortmid\alpha f} - \\
&\qquad\qquad - \underset{(01)}{R}{}^f{}_{bc} X^a|_{\alpha f} - \underset{(02)}{R}{}_{fbc} X^a|^{\alpha f}, \\
&X^a{}_{\shortmid\alpha b}|_{\alpha c} - X^a|_{\alpha c\shortmid\alpha b} = X^f \underset{(\alpha 01)}{R}{}_f{}^a{}_{bc} - \underset{(\alpha 1)}{C}{}^f{}_{bc} X^a{}_{\shortmid\alpha f} - \\
&\qquad\qquad - \underset{(\alpha 1)}{P}{}^f{}_{bc} X^a|_{\alpha f} - \underset{(12)}{B}{}_{fbc} X^a|^{\alpha f}, \\
&X^a{}_{\shortmid\alpha b}|^{\alpha c} - X^a|^{\alpha c}{}_{\shortmid\alpha b} = X^f \underset{(\alpha 02)}{R}{}_f{}^a{}_b{}^c - \underset{(\alpha 2)}{C}{}_b{}^{fc} X^a{}_{\shortmid\alpha f} - \\
&\qquad\qquad - \underset{(21)}{B}{}^f{}_b{}^c X^a|_{\alpha f} - \underset{(\alpha 2)}{P}{}_{fb}{}^c X^a|^{\alpha f}, \\
&X^a|_{\alpha b}|_{\alpha c} - X^a|_{\alpha c}|_{\alpha b} = X^f \underset{(\alpha 11)}{R}{}_f{}^a{}_{bc} - \underset{(\alpha 1)}{S}{}^f{}_{bc} X^a|_{\alpha f}, \\
&X^a|_{\alpha b}|^{\alpha c} - X^a|^{\alpha c}|_{\alpha b} = X^f \underset{(\alpha 12)}{R}{}_f{}^a{}_b{}^c - \underset{(\alpha 2)}{C}{}_b{}^{fc} X^a|_{\alpha f} - \underset{(\alpha 1)}{C}{}^c{}_{fb} X^a|^{\alpha f}, \\
&X^a|^{\alpha b}|^{\alpha c} - X^a|^{\alpha c}|^{\alpha b} = X^f \underset{(\alpha 22)}{R}{}_f{}^{abc} - \underset{(\alpha 2)}{S}{}_f{}^{bc} X^a|^{\alpha f}, \\
&\qquad\qquad\qquad\qquad (\alpha = 0, 1, 2).
\end{aligned} \tag{7.2}$$

Remark 7.1 Using the previous considerations we can express the Ricci identities for any $MN-$linear connections with coefficients $MD\Gamma(N)$ $= \left(H^a{}_{bc}, C^a{}_{bc}, C_a{}^{bc}\right)$, [86], [97].

Application 7.1 As a first application let us consider a Riemannian metric $\mathbb{G}$ on $T^{*2}M$ in the form

$$\mathbb{G} = \underset{(0)}{g}{}_{ab} dx^a \otimes dx^b + \underset{(1)}{g}{}_{ab} \delta y^a \otimes \delta y^b + \underset{(2)}{g}{}^{ab} \delta p_a \otimes \delta p_b, \tag{7.3}$$

having the properties

$$\underset{(\alpha)}{g}{}_{ab\shortmid\alpha c} = 0, \ \underset{(\alpha)}{g}{}_{ab}|_{\alpha c} = 0, \ \underset{(\alpha)}{g}{}_{ab}|^{\alpha c} = 0, (\alpha = 0, 1, 2), \tag{7.4}$$

where $\| \underset{(2)}{g}{}_{ab} \| = \| \underset{(2)}{g}{}^{ab} \|^{-1}$.

Then we have

Theorem 7.3 *If the Riemannian metric* G, *(7.3), verifies the conditions*

(7.4), then the following d−tensor fields

$$\begin{array}{ll}
\underset{(\alpha 00)}{R}{}_{abcd} = \underset{(\alpha)}{g}{}_{fb}\underset{(\alpha 00)}{R}{}_{a}{}^{f}{}_{cd}, & \underset{(\alpha 01)}{R}{}_{abcd} = \underset{(\alpha)}{g}{}_{fb}\underset{(\alpha 01)}{R}{}_{a}{}^{f}{}_{cd}, \\
 & \underset{(\alpha 02)}{R}{}_{abc}{}^{d} = \underset{(\alpha)}{g}{}_{fb}\underset{(\alpha 02)}{R}{}_{a}{}^{f}{}_{c}{}^{d}, \\
\underset{(\alpha 00)}{R}{}_{abcd} = \underset{(\alpha)}{g}{}_{fb}\underset{(\alpha 11)}{R}{}_{a}{}^{f}{}_{cd}, & \underset{(\alpha 12)}{R}{}_{abc}{}^{d} = \underset{(\alpha)}{g}{}_{fb}\underset{(\alpha 12)}{R}{}_{a}{}^{f}{}_{c}{}^{d}, \\
 & \underset{(\alpha 22)}{R}{}_{ab}{}^{cd} = \underset{(\alpha)}{g}{}_{fb}\underset{(\alpha 22)}{R}{}_{a}{}^{fcd}, \\
 & (\alpha = 0, 1, 2).
\end{array} \tag{7.5}$$

are skew-symmetrics in the first two indices (ab).

Indeed, writing the Ricci identities for $d-$tensor fields $\underset{(\alpha)}{g}{}_{ab}$ and taking into account the equations (7.4). We deduce

$$\underset{(\alpha)}{g}{}_{af}\underset{(\alpha 00)}{R}{}_{b}{}^{f}{}_{cd} + \underset{(\alpha)}{g}{}_{bf}\underset{(\alpha 00)}{R}{}_{a}{}^{f}{}_{cd} = 0, ...$$

Using (7.5) and we get $\underset{(\alpha 00)}{R}{}_{bacd} + \underset{(\alpha 00)}{R}{}_{abcd} = 0, (\alpha = 0, 1, 2)$, etc.

q.e.d.

Application 7.2 The Ricci identities applied to the Liouville $d-$vector field y_a and the Hamilton $d-$covector field p_a lead to the some fundamental identities for electromagnetic theory on $T^{*2}M$.

Theorem 7.4 *For any* $N-$*linear connection* D *with the coefficients*

$$D\Gamma(N) = \left(\underset{(00)}{H}{}^{a}{}_{bc}, \underset{(10)}{H}{}^{a}{}_{bc}, \underset{(20)}{H}{}^{a}{}_{bc}, \underset{(01)}{C}{}^{a}{}_{bc}, \underset{(11)}{C}{}^{a}{}_{bc}, \underset{(21)}{C}{}^{a}{}_{bc}, \underset{(02)}{C}{}_{a}{}^{bc}, \underset{(12)}{C}{}_{a}{}^{bc}, \underset{(22)}{C}{}_{a}{}^{bc}\right)$$

the $d-$*deflection tensor fields (3.3), and (3.4), § 5.3, satisfies the following identities*

$$\begin{array}{l}
\underset{\alpha}{D}{}^{a}{}_{b\shortmid\alpha c} - \underset{\alpha}{D}{}^{a}{}_{c\shortmid\alpha b} = y^{f}\underset{(\alpha 00)}{R}{}_{f}{}^{a}{}_{bc} - \underset{\alpha}{D}{}^{a}{}_{f}\underset{(\alpha 0)}{\overset{0}{T}}{}^{f}{}_{bc} - \\
\qquad - \underset{\alpha}{d}{}^{a}{}_{f}\underset{(01)}{R}{}^{f}{}_{bc} - \underset{\alpha}{d}{}^{af}\underset{(01)}{R}{}_{fbc}, \\
\underset{\alpha}{D}{}^{a}{}_{b}\mid_{\alpha c} - \underset{\alpha}{d}{}^{a}{}_{c\shortmid\alpha b} = y^{f}\underset{(\alpha 01)}{R}{}_{f}{}^{a}{}_{bc} - \underset{\alpha}{D}{}^{a}{}_{f}\underset{(\alpha 1)}{C}{}^{f}{}_{bc} - \\
\qquad - \underset{\alpha}{d}{}^{a}{}_{f}\underset{(\alpha 1)}{P}{}^{f}{}_{bc} - \underset{\alpha}{d}{}^{af}\underset{(12)}{B}{}_{fbc}, \\
\underset{\alpha}{D}{}^{a}{}_{b}\mid^{\alpha c} - \underset{\alpha}{d}{}^{ac}{}_{\shortmid\alpha b} = y^{f}\underset{(\alpha 02)}{R}{}_{f}{}^{a}{}_{b}{}^{c} - \underset{\alpha}{D}{}^{a}{}_{f}\underset{(\alpha 2)}{C}{}^{f}{}_{bc} - \\
\qquad - \underset{\alpha}{d}{}^{a}{}_{f}\underset{(21)}{B}{}^{f}{}_{b}{}^{c} - \underset{\alpha}{d}{}^{af}\underset{(\alpha 2)}{P}{}_{fb}{}^{c}, \\
\underset{\alpha}{d}{}^{a}{}_{b}\mid_{\alpha c} - \underset{\alpha}{d}{}^{a}{}_{c}\mid_{\alpha b} = y^{f}\underset{(\alpha 11)}{R}{}_{f}{}^{a}{}_{bc} - \underset{\alpha}{d}{}^{a}{}_{f}\underset{(\alpha 1)}{S}{}^{f}{}_{bc}, \\
\underset{\alpha}{d}{}^{a}{}_{b}\mid^{\alpha c} - \underset{\alpha}{d}{}^{ac}\mid_{\alpha b} = y^{f}\underset{(\alpha 12)}{R}{}_{f}{}^{a}{}_{b}{}^{c} - \underset{\alpha}{d}{}^{a}{}_{f}\underset{(\alpha 2)}{C}{}_{b}{}^{fc} - \underset{\alpha}{d}{}^{af}\underset{(\alpha 1)}{C}{}^{c}{}_{fb}, \\
\underset{\alpha}{d}{}^{ab}\mid^{\alpha c} - \underset{\alpha}{d}{}^{ac}\mid^{\alpha b} = y^{f}\underset{(\alpha 22)}{R}{}_{f}{}^{abc} - \underset{\alpha}{d}{}^{af}\underset{(\alpha 2)}{S}{}_{f}{}^{bc}, \\
\qquad (\alpha = 0, 1, 2),
\end{array} \tag{7.6}$$

respectively

$$
\begin{aligned}
&\underset{\alpha}{\triangle}_{ab|\alpha c} - \underset{\alpha}{\triangle}_{ac|\alpha b} = -p_f \underset{(\alpha 00)}{R}{}_a{}^f{}_{bc} - \underset{\alpha}{\triangle}_{af} \overset{0}{\underset{(\alpha 0)}{T}}{}^f{}_{bc} - \\
&\qquad\qquad - \underset{\alpha}{\vartheta}_{af} \underset{(01)}{R}{}^f{}_{bc} - \underset{\alpha}{\vartheta}_a{}^f \underset{(02)}{R}{}_{fbc}, \\
&\underset{\alpha}{\triangle}_{ab}\,|_{\alpha c} - \underset{\alpha}{\vartheta}_{ac|\alpha b} = -p_f \underset{(\alpha 01)}{R}{}_a{}^f{}_{bc} - \underset{\alpha}{\triangle}_{af} \underset{(\alpha 1)}{C}{}^f{}_{bc} - \\
&\qquad\qquad - \underset{\alpha}{\vartheta}_{af} \underset{(\alpha 1)}{P}{}^f{}_{bc} - \underset{\alpha}{\vartheta}_a{}^f \underset{(12)}{B}{}_{fbc}, \\
&\underset{\alpha}{\triangle}_{ab}\,|^{\alpha c} - \underset{\alpha}{\vartheta}_a{}^c{}_{|\alpha b} = -p_f \underset{(\alpha 02)}{R}{}_a{}^f{}_b{}^c - \underset{\alpha}{\triangle}_{af} \underset{(\alpha 2)}{C}{}_b{}^{fc} - \\
&\qquad\qquad - \underset{\alpha}{\vartheta}_{af} \underset{(21)}{B}{}^f{}_b{}^c - \underset{\alpha}{\vartheta}_a{}^f \underset{(02)}{P}{}_{fb}{}^c, \\
&\underset{\alpha}{\vartheta}_{ab}\,|_{\alpha c} - \underset{\alpha}{\vartheta}_{ac}\,|_{\alpha b} = -p_f \underset{(\alpha 11)}{R}{}_a{}^f{}_{bc} - \underset{\alpha}{\vartheta}_{af} \underset{(\alpha 1)}{S}{}^f{}_{bc}, \\
&\underset{\alpha}{\vartheta}_{ab}\,|^{\alpha c} - \underset{\alpha}{\vartheta}_a{}^c\,|_{\alpha b} = -p_f \underset{(\alpha 12)}{R}{}_a{}^f{}_b{}^c - \underset{\alpha}{\vartheta}_{af} \underset{(\alpha 2)}{C}{}_b{}^{fc} - \underset{\alpha}{\vartheta}_a{}^f \underset{(\alpha 1)}{C}{}^c_{fb}, \\
&\underset{\alpha}{\vartheta}_a{}^b\,|^{\alpha c} - \underset{\alpha}{\vartheta}_a{}^c\,|^{\alpha b} = -p_f \underset{(\alpha 22)}{R}{}_a{}^{fbc} - \underset{\alpha}{\vartheta}_a{}^f \underset{(\alpha 2)}{S}{}_f{}^{bc}, \\
&\qquad\qquad\qquad\qquad (\alpha = 0, 1, 2)\,.
\end{aligned}
\tag{7.7}
$$

We pay attention to an important particular case.

If the (y^a)- deflection tensor and the (p_a) $-$deflection tensor have the following particular forms

$$
\begin{cases}
\underset{\alpha}{D}{}^a{}_b = 0,\ \underset{\alpha}{d}{}^a{}_b = \delta^a_b,\ \underset{\alpha}{d}{}^{ab} = 0, \\
\underset{\alpha}{\triangle}_{ab} = 0,\ \underset{\alpha}{\vartheta}_{ab} = 0,\ \underset{\alpha}{\vartheta}_a{}^b = \delta^a_b.
\end{cases}
\tag{7.8}
$$

then, the fundamental identities from Theorem 7.4 are very important, especially for applications.

Proposition 7.1 *If the $d-$deflection tensors are given by (7.8) then, the following identities hold:*

$$
\begin{cases}
y^f \underset{(\alpha 00)}{R}{}_f{}^a{}_{bc} = \underset{(01)}{R}{}^a{}_{bc},\ y^f \underset{(\alpha 01)}{R}{}_f{}^a{}_{bc} = \underset{(\alpha 1)}{P}{}^a{}_{bc},\ y^f \underset{(\alpha 02)}{R}{}_f{}^a{}_b{}^c = \underset{(21)}{B}{}^a{}_b{}^c, \\
y^f \underset{(\alpha 11)}{R}{}_f{}^a{}_{bc} = \underset{(\alpha 1)}{S}{}^a{}_{bc},\ y^f \underset{(\alpha 12)}{R}{}_f{}^a{}_b{}^c = \underset{(\alpha 2)}{C}{}_b{}^{ac},\ y^f \underset{(\alpha 22)}{R}{}_f{}^{abc} = 0, \\
\qquad\qquad\qquad\qquad (\alpha = 0, 1, 2)\,,
\end{cases}
\tag{7.9}
$$

and respectively

$$
\begin{cases}
p_f \underset{(\alpha 00)}{R}{}_a{}^f{}_{bc} = -\underset{(02)}{R}{}_{abc},\ p_f \underset{(\alpha 01)}{R}{}_a{}^f{}_{bc} = -\underset{(12)}{B}{}_{abc},\ p_f \underset{(\alpha 02)}{R}{}_a{}^f{}_b{}^c = -\underset{(\alpha 2)}{P}{}_{ab}{}^c, \\
p_f \underset{(\alpha 11)}{R}{}_a{}^f{}_{bc} = 0,\ p_f \underset{(\alpha 12)}{R}{}_a{}^f{}_b{}^c = -\underset{(\alpha 1)}{C}{}^c{}_{ab},\ p_f \underset{(\alpha 22)}{R}{}_a{}^{fbc} = -\underset{(\alpha 2)}{S}{}_a{}^{bc}, \\
\qquad\qquad\qquad\qquad (\alpha = 0, 1, 2)\,.
\end{cases}
\tag{7.10}
$$

5.8 Parallelism of vector fields on the manifold $T^{*2}M$

Let D be an $N-$linear connection on cotangent bundle of second order, with the coefficients $D\Gamma\left(N\right)$ given by (2.3), § 5.2.

Let us consider a smooth parametrized curve $\gamma : I \longrightarrow T^{*2}M$ having the image in a domain of a chart of $T^{*2}M$. Thus, γ has an analytical expression of the form

$$x^a = x^a\left(t\right),\ y^a = y^a\left(t\right),\ p_a = p_a\left(t\right),\ t \in I. \tag{8.1}$$

The tangent vector field $\dot{\gamma} = \dfrac{d\gamma}{dt}$, by means of form (3.11), § 3.4, can be written as follows

$$\dot{\gamma} = \frac{dx^a}{dt}\delta_a + \frac{\delta y^a}{dt}\dot{\partial}_a + \frac{\delta p_a}{dt}\dot{\partial}^a, \tag{8.2}$$

where

$$\frac{\delta y^a}{dt} = \frac{dy^a}{dt} + N^a{}_b\frac{dx^a}{dt},\ \frac{\delta p_a}{dt} = \frac{dp_a}{dt} - N_{ab}\frac{dx^a}{dt}. \tag{8.3}$$

Let us denote

$$D_{\dot{\gamma}}X = \frac{DX}{dt},\ D_X = \frac{DX}{dt}dt,\ \forall X \in \chi\left(T^{*2}M\right). \tag{8.4}$$

The quantity DX is **the covariant differential** of the vector X and $\dfrac{DX}{dt}$ is the **covariant differential** along the curve **along the curve** γ.

If X is written in the form

$$X = X^H + X^{V_1} + X^{W_2} = X^{(0)a}\delta_a + X^{(1)a}\dot{\partial}_a + \underset{(2)}{X}{}_a\dot{\partial}^a$$

and we put

$$D_{\dot{\gamma}} = D_{\dot{\gamma}H} + D_{\dot{\gamma}V_1} + D_{\dot{\gamma}W_2} = D^H_{\dot{\gamma}} + D^{V_1}_{\dot{\gamma}} + D^{W_2}_{\dot{\gamma}} = \\ = \frac{dx^a}{dt}D_{\delta_a} + \frac{\delta y^a}{dt}D_{\dot{\partial}_a} + \frac{\delta p_a}{dt}D_{\dot{\partial}^a},$$

then, after a straightforward calculus, we have

$$DX = \left(dX^{(0)a} + X^{(0)f}\underset{(0)}{\omega}{}^a{}_f\right)\delta_a + \\ + \left(dX^{(1)a} + X^{(1)f}\underset{(1)}{\omega}{}^a{}_f\right)\dot{\partial}_a + \\ + \left(d\underset{(2)}{X}{}_a - \underset{(2)}{X}{}_f\underset{(2)}{\omega}{}^f{}_a\right)\dot{\partial}^a, \tag{8.5}$$

where

$$\underset{(\alpha)}{\omega}{}^a{}_b = \underset{(\alpha 0)}{H}{}^a{}_{bc} dx^c + \underset{(\alpha 1)}{C}{}^a{}_{bc} \delta y^c + \underset{(\alpha 2)}{C}{}_b{}^{ac} \delta p_c, \ (\alpha = 0, 1, 2). \tag{8.6}$$

The 1-forms $\underset{(0)}{\omega}{}^a{}_b, \underset{(1)}{\omega}{}^a{}_b, \underset{(2)}{\omega}{}^a{}_b$ from (8.6) are called **1−forms of connections** of D.

Putting

$$\frac{\underset{(\alpha)}{\omega}{}^a{}_b}{dt} = \underset{(\alpha 0)}{H}{}^a{}_{bc} \frac{dx}{dt} + \underset{(\alpha 1)}{C}{}^a{}_{bc} \frac{\delta y^c}{dt} + \underset{(\alpha 2)}{C}{}_b{}^{ac} \frac{\delta p_c}{dt}, \tag{8.6'}$$

then, the covariant differential along the curve γ is given by

$$\frac{DX}{dt} = \left(\frac{DX^{(0)a}}{dt} + X^{(0)f} \frac{\underset{(\alpha)}{\omega}{}^a{}_f}{dt} \right) + \\ + \left(\frac{dX^{(1)a}}{dt} + X^{(1)f} \frac{\underset{(1)}{\omega}{}^a{}_f}{dt} \right) + \left(\frac{d\underset{(2)}{X}{}_a}{dt} - \underset{(2)}{X}{}_f \frac{\underset{(2)}{\omega}{}^f{}_a}{dt} \right). \tag{8.7}$$

From (8.7) results that the parallelism of the vector fields along the curve γ, can be used. We obtain, directly

Theorem 8.1 *The vector field*

$$X = X^{(0)a} \delta_a + X^{(1)a} \dot{\partial}_a + \underset{(2)}{X}{}_a \dot{\partial}^a$$

is parallel along the parametrized curve γ, with respect to D, if and only if its coordinates $X^{(0)a}, X^{(1)a}, \underset{(2)}{X}{}_a$ are solutions of the differential equations

$$\frac{dX^{(\beta)a}}{dt} + X^{(\beta)f} \frac{\underset{(\beta)}{\omega}{}^a{}_f}{dt} = 0, \ \frac{\underset{(2)}{X}{}_a}{dt} - \underset{(2)}{X}{}_f \frac{\underset{(2)}{\omega}{}^f{}_a}{dt} = 0, \ (\beta = 0, 1). \tag{8.8}$$

A theorem of existence and uniqueness for the parallel vector fields along a given parametrized curve on $T^{*2}M$ can be formulated in the classical manner.

The vector field $X \in \chi\left(T^{*2}M\right)$ is called **absolute parallel** with respect to the $N-$linear connection $D\Gamma(N)$, if $DX = 0$ for any curve γ. It is equivalent to the fact that the following system of Pfaff equations is integrable:

$$dX^{(\beta)a} + X^{(\beta)f} \underset{(\beta)}{\omega}{}^a{}_f = 0, \ d\underset{(2)}{X}{}_a - \underset{(2)}{X}{}_f \underset{(2)}{\omega}{}^f{}_a = 0, \ (\beta = 0, 1). \tag{8.9}$$

The system (8.9) is equivalent to the system

$$\begin{cases} X^{(\beta)a}{}_{\shortmid \alpha b} = X^{(\beta)a} \,|_{\alpha b} = X^{(\beta)a} \,|^{\alpha b} = 0, \\ \underset{(2)}{X}{}_{a \shortmid \alpha b} = \underset{(2)}{X}{}_a \,|_{\alpha b} = \underset{(2)}{X}{}_a \,|^{\alpha b} = 0, \\ \qquad (\beta = 0, 1; \alpha = 0, 1, 2). \end{cases} \tag{8.9'}$$

Using Ricci identities, the system (8.9') is integrable if and only if the coordinates $X^{(\beta)a}$, $\underset{(2)}{X}_a$, $(\beta = 0,1)$ of the vector field X satisfy the following equations

$$\begin{cases} X^{(\beta)f} \underset{(\beta 00)}{R}{}_f{}^a{}_{bc} = 0, \ X^{(\beta)f} \underset{(\beta 01)}{R}{}_f{}^a{}_{bc} = 0, \ X^{(\beta)f} \underset{(\beta 02)}{R}{}_f{}^a{}_b{}^c = 0, \\ X^{(\beta)f} \underset{(\beta 11)}{R}{}_f{}^a{}_{bc} = 0, \ X^{(\beta)f} \underset{(\beta 12)}{R}{}_f{}^a{}_b{}^c = 0, \ X^{(\beta)f} \underset{(\beta 22)}{R}{}_f{}^{abc} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (\beta = 0,1)\,, \end{cases} \tag{8.10}$$

and

$$\begin{cases} \underset{(2)}{X}_f \underset{(200)}{R}{}_a{}^f{}_{bc} = 0, \ \underset{(2)}{X}_f \underset{(201)}{R}{}_a{}^f{}_{bc} = 0, \ \underset{(2)}{X}_f \underset{(202)}{R}{}_a{}^f{}_{bc} = 0, \\ \underset{(2)}{X}_f \underset{(211)}{R}{}_a{}^f{}_{bc} = 0, \ \underset{(2)}{X}_f \underset{(212)}{R}{}_a{}^f{}_b{}^c = 0, \ \underset{(2)}{X}_f \underset{(222)}{R}{}_a{}^{fbc} = 0. \\ \qquad\qquad\qquad\qquad\qquad\qquad (\beta = 0,1)\,. \end{cases} \tag{8.11}$$

The manifold $T^{*2}M$ **is** called with **absolute parallelism** of vectors with respect to D, if any vector field on $T^{*2}M$ is absolute parallel. In this case the system of equations (8.10) and (8.11) are verified for any vector field X. It follows:

Theorem 8.2 *The manifold $T^{*2}M$ is with absolute parallelism of vectors, with respect to the $N-$linear connection D, if and only if , all $d-$tensors of curvature of D, vanish.*

The curvature γ is **autoparallel** with respect to D if $D_{\dot\gamma}\dot\gamma = 0$.

By means of (8.2) and (8.7) we deduce

$$\frac{D\dot\gamma}{Dt} = \left(\frac{d^2x^a}{dt^2} + \frac{dx^f}{dt}\frac{\underset{(0)}{\omega}{}^a{}_f}{dt}\right)\delta_a + $$
$$+\left(\frac{d}{dt}\frac{\delta y^a}{dt} + \frac{\delta y^f}{dt}\frac{\underset{(1)}{\omega}{}^a{}_f}{dt}\right)\dot\partial_a + \left(\frac{d}{dt}\frac{\delta p_a}{dt} - \frac{\delta p_a}{dt}\frac{\underset{(2)}{\omega}{}^f{}_a}{dt}\right), \tag{8.12}$$

which we permit to formulate

Theorem 8.3 *A smooth parametrized curve (8.1) is an autoparallel curve with respect to the $N-$linear connection D if and only if the functions $x^a(t)\,,\ y^a(t)\,,$*
$p_a(t)\,,\ t \in I$, verify the following system of differential equations

$$\frac{d^2x^a}{dt^2} + \frac{dx^f}{dt}\frac{\underset{(0)}{\omega}{}^a{}_f}{dt} = 0,$$
$$\frac{d}{dt}\frac{\delta y^a}{dt} + \frac{\delta y^f}{dt}\frac{\underset{(1)}{\omega}{}^a{}_f}{dt} = 0,$$
$$\frac{d}{dt}\frac{\delta p_a}{dt} - \frac{\delta p_f}{dt}\frac{\underset{(2)}{\omega}{}^f{}_a}{dt} = 0. \tag{8.13}$$

Evidently, the theorem of existence and uniqueness for the autoparallel curve can be easily formulated.

We recall that γ is an **horizontal curve** if $\dot{\gamma} = \dot{\gamma}^H$. The horizontal curve are characterized by

$$x^a = x^a(t), \ \frac{\delta y^a}{dt} = 0, \ \frac{\delta p_a}{dt} = 0. \tag{8.14}$$

We pay attention to the special horizontal curves:

Definition 8.1 *The **horizontal path** of an $N-$linear connection D, is an horizontal autoparallel curve with respect to D.*

We have

Theorem 8.4 *The horizontal paths of an $N-$linear connection D on $T^{*2}M$ are characterized by the system of differential equations:*

$$\frac{d^2x^a}{dt^2} + \underset{(\alpha 0)}{H}{}^a{}_{bc}(x,y,p)\frac{dx^b}{dt}\frac{dx^c}{dt} = 0, \ \frac{\delta y^a}{dt} = 0, \ \frac{\delta p_a}{dt} = 0, \quad (\alpha = 0,1,2). \tag{8.15}$$

Proof. The equations (8.14), (8.6') and (8.13) imply (8.15)

q.e.d.

Now, the following notions are easily explained.

A parametrized curve γ is $\mathbf{v}_1-$**vertical curve** in the point $x_0 \in M$ if its tangent vector field $\dot{\gamma}$ is belongs to the distributions V_1.

A parametrized curve γ is $\mathbf{w}_2-$**vertical curve** in the point $x_0 \in M$ if its tangent vector field $\dot{\gamma}$ is belongs to the distributions W_2.

Of course, a v_1-vertical curve γ in the point $x_0 \in M$ is analytically represented by the equations of the form

$$x^a = x^a_0, \ y^a = y^a(t), \ p_a = 0, \ t \in I, \tag{8.16}$$

and a w_2-vertical curve γ in the point $x_0 \in M$ is analytically represented by the equations of the form

$$x^a = x^a_0, \ y^a = 0, \ p_a = p_a(t), \ t \in I. \tag{8.17}$$

e define a $\mathbf{v}_1-$**path** in the point $x_0 \in M$ with respect to D, to be a v_1-vertical curve in the point $x_0 \in M$, which is an autoparallel curve with respect to D. It is clear, what is mean $\mathbf{w}_2-$**path** in the point $x_0 \in M$ with respect to D.

By means of (8.16), (8.17) and (8.11) we can immediately prove

Theorem 8.5 1°.The *v_1-vertical paths in the point $x_0 \in M$ are characterized by the system of differentiable equations*

$$x^a = x^a_0, \ \frac{d^2y^a}{dt^2} + \underset{(\alpha 1)}{C}{}^a{}_{bc}(x_0,y,0)\frac{dy^b}{dt}\frac{dy^c}{dt} = 0, \ p_a = 0, \quad (\alpha = 0,1,2) \tag{8.18}$$

2°.The w_2-*vertical paths in the point* $x_0 \in M$ *are characterized by the system of differentiable equations*

$$x^a = x_0^a,\ y^a = 0,\ \frac{d^2 p_a}{dt^2} - \underset{(\alpha 2)}{C}{}_a{}^{bc}(x_0, 0, p_a)\frac{dp_b}{dt}\frac{dp_c}{dt} = 0,\quad (\alpha = 0,1,2) \tag{8.19}$$

Remark 8.1 In Theorem 8.5, we assume that there exists the coefficients $\underset{(\alpha 1)}{C}(x_0, y, 0)$ and $\underset{(\alpha 2)}{C}(x_0, 0, p)$, $(\alpha = 0,1,2)$.

Remark 8.2 By Theorem 4.2, and formulae (4.3), § 5.4, we can obtain the results of this section, with respect to the $MN-$linear connection $MD\Gamma(N) = \left(H^a{}_{bc}, C^a{}_{bc}, C_a{}^{bc}\right)$ on $T^{*2}M$. These coincide with the results of R. Miron and his collaborators [86], [97].

Remark 8.3 In the case of Berwald connection (2.5), § 5.2, the characterizations for v_1-paths and w_{2-}paths appear in a very simple form, because $\underset{(\alpha 1)}{C}{}^a{}_{bc} = 0$ and $\underset{(\alpha 2)}{C}{}_a{}^{bc} = 0$.

5.9 Structure equations of an $N-$linear connection

For an $N-$linear connection D, with the coefficients $D\Gamma(N)$ given by the formulaes (2.3), in the adapted basis $\left(\delta_a, \dot{\partial}_a . \dot{\partial}^a\right)$ on $T^{*2}M$ we can prove

Lemma 9.1 1°. *Each of the geometrical object fields*

$$d(dx^a) - dx^b \wedge \underset{(\alpha)}{\omega}{}^a{}_b,\ d(\delta y^a) - \delta y^b \wedge \underset{(\alpha)}{\omega}{}^a{}_b,$$

$(\alpha = 0,1,2)$, *is a* $d-$*vector field, and each of geometrical object fields*

$$d(dp_a) + \delta p_b \wedge \underset{(\alpha)}{\omega}{}^b{}_a,$$

$(\alpha = 0,1,2)$, *is a* $d-$*covector field.*

2°. *The geometrical object fields*

$$d\underset{(\alpha)}{\omega}{}^a{}_b - \underset{(\alpha)}{\omega}{}^c{}_b \wedge \underset{(\alpha)}{\omega}{}^a{}_c,\ (\alpha = 0,1,2),$$

are $d-$*tensor fields,with respect to indices* a *and* b.

Using the previous Lemma we can prove, by a straightforward calculus, a fundamental result in the geometry of 2−cotangent bundle.

Theorem 9.1 *For any* $N-$linear connection $D\Gamma\left(N\right)$ *the following structure equations hold good:*

$$\begin{array}{rcl} d\left(dx^{a}\right)-dx^{b}\wedge \underset{(\alpha)}{\omega}{}^{a}{}_{b} &=& -\overset{0}{\underset{(\alpha)}{\Omega}}{}^{a}, \\ d\left(\delta y^{a}\right)-\delta y^{b}\wedge \underset{(\alpha)}{\omega}{}^{a}{}_{b} &=& -\overset{1}{\underset{(\alpha)}{\Omega}}{}^{a}, \\ d\left(\delta p_{a}\right)+\delta p_{b}\wedge \underset{(\alpha)}{\omega}{}^{b}{}_{a} &=& -\underset{(\alpha)}{\Omega}{}_{a}, \end{array} \qquad (9.1)$$

and

$$d\,\underset{(\alpha)}{\omega}{}^{a}{}_{b}-\underset{(\alpha)}{\omega}{}^{a}{}_{c}\wedge \underset{(\alpha)}{\omega}{}^{c}{}_{b}=-\underset{(\alpha)}{\Omega}{}^{a}{}_{b}, \qquad (9.2)$$

where $\overset{0}{\underset{(\alpha)}{\Omega}}{}^{a}, \overset{1}{\underset{(\alpha)}{\Omega}}{}^{a},$ *and* $\underset{(\alpha)}{\Omega},\left(\alpha=0,1,2\right)$ *are the* **2−forms of torsion**

$$\begin{array}{rcl} \overset{0}{\underset{(\alpha)}{\Omega}}{}^{a} &=& \dfrac{1}{2}\underset{(\alpha 0)}{T}{}^{a}{}_{bc}dx^{b}\wedge dx^{c}+ \\ && +\underset{(\alpha 1)}{C}{}^{a}{}_{bc}dx^{b}\wedge \delta y^{c}+\underset{(\alpha 2)}{C}{}_{b}{}^{ac}dx^{b}\wedge \delta p_{c}, \\ \overset{1}{\underset{(\alpha)}{\Omega}}{}^{a} &=& \dfrac{1}{2}\underset{(01)}{R}{}^{a}{}_{bc}dx^{b}\wedge dx^{c}+ \\ && +\underset{(\alpha 1)}{P}{}^{a}{}_{bc}dx^{b}\wedge \delta y^{c}+\underset{(21)}{B}{}^{a}{}_{b}{}^{c}dx^{b}\wedge \delta p_{c}+ \\ && +\dfrac{1}{2}\underset{(\alpha 1)}{S}{}^{a}{}_{bc}\delta y^{b}\wedge \delta y^{c}+\underset{(\alpha 2)}{C}{}_{b}{}^{ac}\delta y^{b}\wedge \delta p_{c}, \\ \underset{(\alpha)}{\Omega}{}_{a} &=& \dfrac{1}{2}\underset{(02)}{R}{}_{abc}dx^{b}\wedge dx^{c}+ \\ && +\underset{(12)}{B}{}_{abc}dx^{b}\wedge \delta y^{c}+\underset{(21)}{P}{}_{ab}{}^{c}dx^{b}\wedge \delta p_{c}+ \\ && +\underset{(\alpha 1)}{C}{}^{c}{}_{ab}\delta y^{b}\wedge \delta p_{c}+\dfrac{1}{2}\underset{(\alpha 2)}{S}{}_{a}{}^{bc}\delta p_{b}\wedge \delta p_{c}, \end{array} \qquad (9.3)$$

and where $\underset{(\alpha)}{\Omega}{}^{a}{}_{b},\left(\alpha=0,1,2\right)$ *are* **2−forms of curvature**

$$\begin{array}{l} \underset{(\alpha)}{\Omega}{}^{a}{}_{b}=\dfrac{1}{2}\underset{(\alpha 00)}{R}{}_{b}{}^{a}{}_{cd}dx^{c}\wedge dx^{d}+\underset{(\alpha 01)}{R}{}_{b}{}^{a}{}_{cd}dx^{c}\wedge \delta y^{d}+\underset{(\alpha 02)}{R}{}_{b}{}^{a}{}_{c}{}^{d}dx^{c}\wedge \delta p_{a}+ \\ \qquad +\dfrac{1}{2}\underset{(\alpha 11)}{R}{}_{b}{}^{a}{}_{cd}\delta y^{a}\wedge \delta y^{d}+\underset{(\alpha 12)}{R}{}_{b}{}^{a}{}_{c}{}^{d}\delta y^{b}\wedge \delta p_{b}+\dfrac{1}{2}\underset{(\alpha 22)}{R}{}_{b}{}^{acd}\delta p_{c}\wedge \delta p_{a}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left(\alpha=0,1,2\right) \end{array} \qquad (9.4)$$

Remark 9.1 The theorem 9.1 is extremely important in a theory of submanifold embedding in the total space $T^{*2}M$ of the $2-$cotangent bundle $\left(T^{*2}M,\pi^{*2},M\right)$.

Remark 9.2 For any $MN-$linear connection with coefficients

$M\Gamma\left(N\right)=\left(H^{a}{}_{bc},C^{a}{}_{bc},C_{a}{}^{bc},\right)$ we have

$$\begin{array}{lll} \overset{0}{\underset{(0)}{\Omega}}{}^{a} & = & \overset{0}{\underset{(1)}{\Omega}}{}^{a}=\overset{0}{\underset{(2)}{\Omega}}{}^{a}=:\ \overset{0}{\Omega}{}^{a}, \\ \overset{1}{\underset{(0)}{\Omega}}{}^{a} & = & \overset{1}{\underset{(1)}{\Omega}}{}^{a}=\overset{1}{\underset{(2)}{\Omega}}{}^{a}=:\ \overset{1}{\Omega}{}^{a}, \\ \underset{(0)}{\Omega}{}_{a} & = & \underset{(1)}{\Omega}{}_{a}=\underset{(2)}{\Omega}{}_{a}=:\ \Omega_{a}, \\ \underset{(0)}{\Omega}{}^{a}{}_{b} & = & \underset{(1)}{\Omega}{}^{a}{}_{b}=\underset{(2)}{\Omega}{}^{a}{}_{b}=:\ \Omega^{a}{}_{b}. \end{array} \tag{9.5}$$

and then, by Theorem 4.2, formulae (4.3), § 5.4, we obtain the structure equations of an $N-$linear connection of Miron type ([97], pg. 282, formulae (8.6) and (8.7)).

Chapter 6

Metric structures on the manifold $T^{*2}M$

6.1 Metric $N-$linear connections on $T^{*2}M$

Definition 1.1 *A* **metric structure** *on the manifold $T^{*2}M$ is a symmetric covariant tensor field $\mathbb{G}$ of the type* (0.2) , *which is non degenerate at each point $u = (x, y, p) \in T^{*2}M$ and of constant signature* on $T^{*2}M$. *If $\mathbb{G}$ is positive definite we say it $\mathbb{G}$ defines a Riemannian structure on $T^{*2}M$.*

As in the Section 4.6, where was used a Riemannian structure on $T^{*2}M$, we can prove that there is an uniquely nonlinear connection such that the distribution N will be orthogonal to distribution $V = V_1 \oplus W_2$, namely orthogonal on both V_1 and W_2 :

$$\mathbb{G}\left(X^H, Y^{V_1}\right) = 0,\ \mathbb{G}\left(X^H, Y^{W_2}\right) = 0. \tag{1.1}$$

By using adapted basis $\left(\delta_a, \dot{\partial}_a, \dot{\partial}^a\right)$, we have

$$\mathbb{G}\left(\delta_a, \dot{\partial}_a\right) = 0,\ \mathbb{G}\left(\delta_a, \dot{\partial}^a\right) = 0. \tag{1.1'}$$

The system of equations (1.1') is equivalently with the following system of equations for the determination of coefficients ${N^a}_b$ and N_{ab}

$$\begin{cases} \underset{(11)}{g}{}_{cb}{N^c}_a - \underset{(12)}{g}{}_b{}^c N_{ac} = \underset{(01)}{g}{}_{ab} \\ \underset{(21)}{g}{}_c{}^b {N^c}_a - \underset{(22)}{g}{}^{cb} N_{ac} = \underset{(02)}{g}{}_a{}^b, \end{cases} \tag{1.2}$$

where, matrix

$$\begin{Vmatrix} \underset{(11)}{g}{}_{cb} & \underset{(12)}{g}{}_b{}^c \\ \underset{(21)}{g}{}_c{}^b & \underset{(22)}{g}{}^{cb} \end{Vmatrix} \tag{1.3}$$

is nonsingular.

We denoted

$$\begin{aligned} &\underset{(00)}{g}{}_{ab} = \mathbb{G}\left(\partial_a, \partial_b\right), \ \underset{(01)}{g}{}_{ab} = \mathbb{G}\left(\partial_a, \dot{\partial}_a\right), \ \underset{(02)}{g}{}_{a}{}^{b} = \mathbb{G}\left(\partial_a, \dot{\partial}^b\right), \\ &\underset{(12)}{g}{}_{a}{}^{b} = \mathbb{G}\left(\dot{\partial}_a, \dot{\partial}^b\right), \text{etc.} \end{aligned} \tag{1.4}$$

Also, we suppose that in V the distributions V_1 and W_2 are orthogonal with respect to $\mathbb{G}$, namely

$$\mathbb{G}\left(X^{V_1}, Y^{W_2}\right) = 0 \tag{1.5}$$

We have

$$\mathbb{G}\left(\dot{\partial}_a, \dot{\partial}^b\right) = 0 \Longleftrightarrow \underset{(12)}{g}{}_{a}{}^{b} = 0 \tag{1.6}$$

relation which together with (1.2) we permit to formulate

Theorem 1.1 *A metric structure $\mathbb{G}$ on $T^{*2}M$ determine an unique nonlinear connection N, if the distributions horizontal N and verticals V_1 and W_2 are orthogonal in pairs. The coefficients $N^a{}_b$ and N_{ab} of N are given by*

$$N^a{}_b = \underset{(11)}{g}{}^{ac} \underset{(01)}{g}{}_{bc}, \ N_{ab} = \underset{(22)}{g}{}_{bc} \underset{(02)}{g}{}_{a}{}^{c} \tag{1.7}$$

where

$$rank \parallel \underset{(11)}{g}{}_{ab} \parallel = rank \parallel \underset{(22)}{g}{}^{ab} \parallel = n, \ \parallel \underset{(11)}{g}{}^{ab} \parallel = \parallel \underset{(11)}{g}{}_{ab} \parallel^{-1}, \ \parallel \underset{(22)}{g}{}_{ab} \parallel = \parallel \underset{(22)}{g}{}^{ab} \parallel^{-1}.$$

In this chapter we shall use only this nonlinear connection.

Let us consider a metric structure $\mathbb{G}$ on $T^{*2}M$ and the distributions N, V_1, W_2 are orthogonal in pairs with respect to the metric structure $\mathbb{G}$. By (1.1) and (1.5) we have the following decomposition of $\mathbb{G}$:

$$\begin{aligned} &\mathbb{G}(X, Y) = \mathbb{G}\left(X^H, Y^H\right) + \mathbb{G}\left(X^{V_1}, Y^{V_1}\right) + \mathbb{G}\left(X^{W_2}, Y^{W_2}\right), \\ &\quad \forall X, Y \in \chi\left(T^{*2}M\right) \end{aligned} \tag{1.8}$$

With the other words, $\mathbb{G}$ decomposes in a sum of three $d-$tensor fields:

(0) $\mathbb{G}^H$ of type $(0,2)$ defined by $\mathbb{G}^H(X,Y) = \mathbb{G}\left(X^H, Y^H\right)$,
(1) $\mathbb{G}^{V_1}$of type $(0,2)$ defined by $\mathbb{G}^{V_1}(X,Y) = \mathbb{G}\left(X^{V_1}, Y^{V_1}\right)$,
(2) $\mathbb{G}^{W_2}$of type $(0,2)$ defined by $\mathbb{G}^{W_2}(X,Y) = \mathbb{G}\left(X^{W_2}, Y^{W_2}\right)$.

Locally, these $d-$tensor fields can be written as

$$\begin{aligned} \mathbb{G}^H &= \underset{(0)}{g}{}_{ab} dx^a \otimes dx^b, \\ \mathbb{G}^{V_1} &= \underset{(1)}{g}{}_{ab} \delta y^a \otimes \delta y^b, \\ \mathbb{G}^{W_2} &= \underset{(2)}{g}{}^{ab} \delta p_a \otimes \delta p_b, \end{aligned} \tag{1.9}$$

where

$$\underset{(0)}{g}_{ab} = \mathbb{G}\left(\delta_a, \delta_b\right), \ \underset{(1)}{g}_{ab} = \mathbb{G}\left(\dot{\partial}_a, \dot{\partial}_b\right), \ \underset{(2)}{g}^{ab} = \mathbb{G}\left(\dot{\partial}^a, \dot{\partial}^b\right), \tag{1.9'}$$

$$rank \parallel \underset{(\alpha)}{g}_{ab} \parallel = n, (\alpha = 0, 1, 2), \ \parallel \underset{(2)}{g}_{ab} \parallel = \parallel \underset{(2)}{g}^{ab} \parallel^{-1}. \tag{1.10}$$

Thus, the decomposition (1.8) looks locally as following:

$$\mathbb{G} = \underset{(0)}{g}_{ab} dx^a \otimes dx^b + \underset{(1)}{g}_{ab} \delta y^a \otimes \delta y^b + \underset{(2)}{g}^{ab} \delta p_a \otimes \delta p_b. \tag{1.11}$$

Definition 1.2 *An $N-$linear connection D on $T^{*2}M$ endowed with a metric structure G is said to be a metric $N-$linear connection if $D_X G = 0$ for every $X \in T^{*2}M$.*

Proposition 1.1 *If a linear connection D on $T^{*2}M$ has the proprieties:*

(a) D preserves by parallelism the vertical distributions V_1 and W_2,

(b) $D_X G = 0$, $\forall X \in T^{*2}M$, then it is a metric $N-$linear connection.

Proof. It is enough to prove that D preserves by parallelism the horizontal distribution $u \longrightarrow N(u)$. Using (a), (1.1) and (1.5) in the equalities

$$0 = (D_X\mathbb{G})\left(Y^H, Z^{V_1}\right) = X\mathbb{G}\left(Y^H, Z^{V_1}\right) - \mathbb{G}\left(D_X Y^H, Z^{V_1}\right) - \mathbb{G}\left(Y^H, D_X Z^{V_1}\right),$$
$$0 = (D_X\mathbb{G})\left(Y^H, Z^{W_2}\right) = X\mathbb{G}\left(Y^H, Z^{W_2}\right) - \mathbb{G}\left(D_X Y^H, Z^{W_2}\right) - \mathbb{G}\left(Y^H, D_X Z^{W_2}\right),$$

one gets

$$\mathbb{G}\left(D_X Y^H, Z^{V_1}\right) = 0, \ \mathbb{G}\left(D_X Y^H, Z^{W_2}\right) = 0, \ \forall Z \in \chi\left(T^{*2}M\right).$$

Thus, by (1.1) and (1.5) we have that $D_X Y^H$ is an horizontal vector field

q.e.d.

Proposition 1.2 *An $N-$linear connection D on $T^{*2}M$ endowed with a metric structure $\mathbb{G}$ is a metric $N-$linear connection if and only if*

$$\begin{array}{l}
\underset{0}{D}{}^H_X \mathbb{G}^H = 0, \ \underset{0}{D}{}^{V_1}_X \mathbb{G}^H = 0, \ \underset{0}{D}{}^{W_2}_X \mathbb{G}^H = 0, \\
\underset{1}{D}{}^H_X \mathbb{G}^{V_1} = 0, \ \underset{1}{D}{}^{V_1}_X \mathbb{G}^{V_1} = 0, \ \underset{1}{D}{}^{W_2}_X \mathbb{G}^{V_1} = 0, \\
\underset{2}{D}{}^H_X \mathbb{G}^{W_2} = 0, \ \underset{2}{D}{}^{V_1}_X \mathbb{G}^{W_2} = 0, \ \underset{2}{D}{}^{W_2}_X \mathbb{G}^{W_2} = 0.
\end{array} \tag{1.12}$$

Proof. The equation $D_X\mathbb{G} = 0$ implies

$$D^H_X \mathbb{G} = 0, \ D^{V_1}_X \mathbb{G} = 0, \ D^{W_2}_X \mathbb{G} = 0.$$

By (3.3), Ch. 4 and (1.8) we have

$$\begin{array}{r}
0 = \left(D^H_X \mathbb{G}\right)(Y, Z) = \left(\underset{0}{D}{}^H_X \mathbb{G}^H\right)(Y, Z) + \left(\underset{1}{D}{}^H_X \mathbb{G}^{V_1}\right)(Y, Z) \\
+ \left(\underset{2}{D}{}^H_X \mathbb{G}^{W_2}\right)(Y, Z),
\end{array} \tag{(*)}$$

$$0=\left(D_X^{V_1}\mathbb{G}\right)(Y,Z)=\left(\underset{0}{D}{}_X^{V_1}\mathbb{G}^H\right)(Y,Z)+\left(\underset{1}{D}{}_X^{V_1}\mathbb{G}^{V_1}\right)(Y,Z)+ \\ +\left(\underset{2}{D}{}_X^{V_1}\mathbb{G}^{W_2}\right)(Y,Z), \tag{(**)}$$

$$0=\left(D_X^{W_2}\mathbb{G}\right)(Y,Z)=\left(\underset{0}{D}{}_X^{W_2}\mathbb{G}^H\right)(Y,Z)+\left(\underset{1}{D}{}_X^{W_2}\mathbb{G}^{V_1}\right)(Y,Z)+ \\ +\left(\underset{2}{D}{}_X^{W_2}\mathbb{G}^{W_2}\right)(Y,Z). \tag{(***)}$$

Taking in (*), $Y=Y^H, Z=Z^H$, one gets $\underset{0}{D}{}_X^H\mathbb{G}^H=0$, taking $Y=Y^{V_1}$, $Z=Z^{V_1}$, one gets $\underset{1}{D}{}_X^H\mathbb{G}^{V_1}=0$ and taking $Y=Y^{W_2},\ Z=Z^{W_2}$, we obtain $\underset{2}{D}{}_X^H\mathbb{G}^{W_2}=0$.Now, putting in (**), $Y=Y^H,\ Z=Z^H$ one obtains $\underset{0}{D}{}_X^{V_1}\mathbb{G}^H=0$, putting $Y=Y^{V_1},\ Z=Z^{V_1}$ one gets $\underset{1}{D}{}_X^{V_1}\mathbb{G}^{V_1}=0$ and if $Y=Y^{W_2},\ Z=Z^{W_2}$, we obtain $\underset{2}{D}{}_X^{V_1}\mathbb{G}^{W_2}=0$. Analogous, putting in (***), $Y=Y^H,\ Z=Z^H$ and results $\underset{2}{D}{}_X^{W_2}\mathbb{G}^{W_2}=0$, putting $Y=Y^{V_1},\ Z=Z^{V_1}$ one gets $\underset{1}{D}{}_X^{W_2}\mathbb{G}^{V_1}=0$ and then $Y=Y^{W_2},\ Z=Z^{W_2}$, we obtain $\underset{2}{D}{}_X^{W_2}\mathbb{G}^{W_2}=0$. Conversely using (1.12) in (*) one results $D_X^H\mathbb{G}=0$, using (1.12) in (**) one results $D_X^{V_1}\mathbb{G}=0$ and then by (***) one deduce $D_X^{W_2}\mathbb{G}=0$.From these lasts three equations it follows $D_XG=0$.

q.e.d.

We shall now discuss the existence of metric $N-$linear connections on $T^{*2}M$. First, we prove

Theorem 1.2 *If $\overset{*}{D}$ is a fixed $N-$linear connection on $T^{*2}M$, then the $N-$linear connection given by the following formulae is metric with respect to $\mathbb{G}$:*

$$2\mathbb{G}^H\left(\underset{0}{D}{}_X^HY,Z\right)=X^H\left(\mathbb{G}^H\right)(Y,Z)+Y^H\left(\mathbb{G}^H\right)(Z,X)-Z^H\left(\mathbb{G}^H\right)(X,Y)- \\ -\mathbb{G}^H\left(X,\left[Y^H,Z^H\right]\right)+\mathbb{G}^H\left(Y,\left[Z^H,X^H\right]\right)+\mathbb{G}^H\left(Z,\left[X^H,Y^H\right]\right),$$

$$\underset{1}{D}{}_X^HY=\underset{1}{\overset{*}{D}}{}_X^HY+\underset{(10)}{A}\left(Y^{V_1},X^H\right),\ \textit{such that}$$
$$2\left(\mathbb{G}^{V_1}\right)\left(\underset{(10)}{A}\left(Y^{V_1},X^H\right),Z\right)=\left(\underset{1}{\overset{*}{D}}{}_X^H\mathbb{G}^{V_1}\right)\left(Y^{V_1},Z^{V_1}\right),$$

$$\underset{2}{D}{}_X^HY=\underset{2}{\overset{*}{D}}{}_X^HY+\underset{(20)}{A}\left(Y^{W_2},X^H\right),\ \textit{such that}$$
$$2\left(\mathbb{G}^{W_2}\right)\left(\underset{(20)}{A}\left(Y^{W_2},X^H\right),Z\right)=\left(\underset{2}{\overset{*}{D}}{}_X^H\mathbb{G}^{W_2}\right)\left(Y^{W_2},Z^{W_2}\right),$$

$$\underset{0}{D}{}_X^{V_1}Y=\underset{0}{\overset{*}{D}}{}_X^{V_1}Y+\underset{(01)}{A}\left(Y^H,X^{V_1}\right),\ \textit{such that}$$
$$2\left(\mathbb{G}^H\right)\left(\underset{(01)}{A}\left(Y^H,X^{V_1}\right),Z\right)=\left(\underset{0}{\overset{*}{D}}{}_X^{V_1}\mathbb{G}^H\right)\left(Y^H,Z^H\right),$$

$$2\mathbb{G}^{V_1}\left(\underset{1}{D}{}_X^{V_1}Y,Z\right)=X^{V_1}\left(\mathbb{G}^{V_1}\right)(Y,Z)+Y^{V_1}\left(\mathbb{G}^{V_1}\right)(Z,X)-Z^{V_1}\left(\mathbb{G}^{V_1}\right)(X,Y)- \\ -\mathbb{G}^{V_1}\left(X,\left[Y^{V_1},Z^{V_1}\right]\right)+\mathbb{G}^{V_1}\left(Y,\left[Z^{V_1},X^{V_1}\right]\right)+\mathbb{G}^{V_1}\left(Z,\left[X^{V_1},Y^{V_1}\right]\right),$$

$$\begin{aligned}
&\underset{2}{D}{}^{V_1}_X Y = \overset{*}{\underset{2}{D}}{}^{V_1}_X Y + \underset{(21)}{A}\left(Y^{W_2}, X^{V_1}\right), \textit{ such that}\\
&\qquad 2\left(\mathbb{G}^{W_2}\right)\left(\underset{(21)}{A}\left(Y^{W_2},X^{V_1}\right),Z\right) = \left(\overset{*}{\underset{2}{D}}{}^{V_1}_X \mathbb{G}^{W_2}\right)\left(Y^{W_2},Z^{W_2}\right),\\
&\underset{0}{D}{}^{W_2}_X Y = \overset{*}{\underset{0}{D}}{}^{W_2}_X Y + \underset{(02)}{A}\left(Y^{H}, X^{W_2}\right), \textit{ such that}\\
&\qquad 2\left(\mathbb{G}^{H}\right)\left(\underset{(02)}{A}\left(Y^{H},X^{W_2}\right),Z\right) = \left(\overset{*}{\underset{0}{D}}{}^{W_2}_X \mathbb{G}^{H}\right)\left(Y^{H},Z^{H}\right),\\
&\underset{0}{D}{}^{W_2}_X Y = \overset{*}{\underset{1}{D}}{}^{W_2}_X Y + \underset{(12)}{A}\left(Y^{V_1}, X^{W_2}\right), \textit{ such that}\\
&\qquad 2\left(\mathbb{G}^{V_1}\right)\left(\underset{(12)}{A}\left(Y^{V_1},X^{W_2}\right),Z\right) = \left(\overset{*}{\underset{1}{D}}{}^{W_2}_X \mathbb{G}^{V_1}\right)\left(Y^{V_1},Z^{V_1}\right),\\
&2\mathbb{G}^{w_2}\left(\underset{2}{D}{}^{W_2}_X Y, Z\right) = X^{w_2}\left(\mathbb{G}^{w_2}\right)(Y,Z) + Y^{w_2}\left(\mathbb{G}^{w_2}\right)(Z,X) - Z^{w_2}\left(\mathbb{G}^{w_2}\right)(X,Y) -\\
&\qquad -\mathbb{G}^{w_2}\left(X,\left[Y^{w_2},Z^{w_2}\right]\right) + \mathbb{G}^{w_2}\left(Y,\left[Z^{w_2},X^{w_2}\right]\right) + \mathbb{G}^{w_2}\left(Z,\left[X^{w_2},Y^{w_2}\right]\right).
\end{aligned} \tag{1.13}$$

Proof. It is obvious that the formulae (1.13) uniquely determine on D^H_X, $D^{V_1}_X$ and $D^{W_2}_X$, hence they uniquely determine an $N-$linear connection on $T^{*2}M$. By a direct computation one checks D^H_X, $D^{V_1}_X$ and $D^{W_2}_X$ verify (1.12). Thus D is a metric $N-$linear connection

q.e.d.

We note that $h\,(hh)-$, $v_1\,(v_1v_1)-$ and $w_2\,(w_2w_2)-$ tensors of torsion of D vanish.

Next we have:

Theorem 1.3 *Let $\mathbb{G}$ be a metric structure on $T^{*2}M$. There exist metric $N-$linear connections on $T^{*2}M$ depending only on $\mathbb{G}$. One of them is given by (1.13) in which*

$$\begin{aligned}
&\overset{*}{\underset{1}{D}}{}^{H}_X Y = \left[X^H, Y^{V_1}\right]^{V_1}, \ \overset{*}{\underset{2}{D}}{}^{H}_X Y = \left[X^H, Y^{W_2}\right]^{W_2},\\
&\overset{*}{\underset{0}{D}}{}^{V_1}_X Y = \left[X^{V_1}, Y^{H}\right]^{H}, \ \overset{*}{\underset{2}{D}}{}^{V_1}_X Y = \left[X^{V_1}, Y^{W_2}\right]^{H},\\
&\overset{*}{\underset{0}{D}}{}^{W_2}_X Y = \left[X^{W_2}, Y^{H}\right]^{H}, \ \overset{*}{\underset{1}{D}}{}^{W_2}_X Y = \left[X^{W_2}, Y^{V_1}\right]^{V_1},
\end{aligned} \tag{1.14}$$

Proof. Ii is evident that $\underset{0}{D}{}^{H}_X = D_{X^H}Y^H$, $\underset{1}{D}{}^{V_1}_X = D_{X^{V_1}}Y^{V_1}$ and $\underset{2}{D}{}^{W_2}_X = D_{X^{W_2}}Y^{W_2}$ given by the first, the fifth and the ninth equations from (1.13) depend on $\mathbb{G}$ only. If we chose the $N-$linear connection $\overset{*}{D}_X$ such that

$v_1\overset{*}{T}\left(X^H, Y^{V_1}\right) = 0$, $w_2\overset{*}{T}\left(X^H, Y^{W_2}\right) = 0$, $v_1\overset{*}{T}\left(Y^{V_1}, X^{V_1}\right) = \left[X^{V_1}, Y^{V_1}\right]^{V_1}$,
$w_2\overset{*}{T}\left(Y^{W_2}, X^{V_1}\right) = \left[X^{V_1}, Y^{W_2}\right]^{W_2}$, $h\overset{*}{T}\left(X^H, Y^{V_1}\right) = 0$, $h\overset{*}{T}\left(X^H, Y^{W_2}\right) = 0$,
then the equations (1.14) hold by the second, the sixth, the seventh and the eight equations from (1.13), $\underset{1}{D}{}^{H}_X Y$, $\underset{2}{D}{}^{H}_X Y$, $\underset{0}{D}{}^{V_1}_X Y$, $\underset{2}{D}{}^{V_1}_X Y$ and $\underset{0}{D}{}^{W_2}_X Y$, $\underset{1}{D}{}^{W}_X Y$, respectively, depend by $\mathbb{G}$ only. **q.e.d.**

Now, we shall express a metric $N-$linear connection and related results in terms of local coordinate system.

As we have seen, a metric structure $\mathbb{G}$ uniquely determines a nonlinear connection N and if this metric satisfy (1.7), then $\mathbb{G}$ takes the local form (1.11), where the dual basis $(dx^a, \delta y^a, \delta p_a)$ was used.

Translating the Proposition 1.2 in local coordinate, we obtain

Proposition 1.3 *An $N-$linear connection on $T^{*2}M$ is a metric $N-$linear connection if and only if*

$$\underset{(\beta)}{g}_{ab\shortmid\alpha c} = 0,\quad \underset{(\beta)}{g}_{ab}|_{\alpha c}= 0,\quad \underset{(2)}{g}^{ab}|^{\alpha c}= 0, \tag{1.15}$$

$$(\beta = 0, 1; \alpha = 0, 1, 2)\,.$$

Remark 1.1 The conditions (1.15) are, respectively, equivalent with the conditions

$$\underset{(\beta)}{g}^{ab}{}_{\shortmid\alpha c} = 0,\quad \underset{(\beta)}{g}^{ab}|_{\alpha c}= 0,\quad \underset{(2)}{g}_{ab}|^{\alpha c}= 0, \tag{1.15'}$$

where $\| \underset{(\alpha)}{g}^{ab} \|=\| \underset{(\alpha)}{g}_{ab} \|^{-1},\ (\alpha = 0, 1, 2)\,.$

If we proceed similarly with Theorem 1.2 we deduce

Theorem 1.4 *If the manifold $T^{*2}M$ is endowed with the metric structure $\mathbb{G}$ given by (1.11), then there exists on $T^{*2}M$ a metric $N-$linear connection , depending only on $\mathbb{G}$, whose $h\,(hh)-$, $v_1\,(v_1v_1)-$ and $w_2\,(w_2w_2)-$ tensors of torsion vanish. Its local coefficients*

$$D\Gamma\,(N) = \left(\underset{(00)}{H}{}^a{}_{bc}, \underset{(10)}{H}{}^a{}_{bc}, \underset{(20)}{H}{}^a{}_{bc}, \underset{(01)}{C}{}^a{}_{bc}, \underset{(11)}{C}{}^a{}_{bc}, \underset{(21)}{C}{}^a{}_{bc}, \underset{(02)}{C}{}_a{}^{bc}, \underset{(12)}{C}{}_a{}^{bc}, \underset{(22)}{C}{}_a{}^{bc}\right),$$

are as follows:

$$\begin{aligned}
&\overset{c}{\underset{(00)}{H}}{}^a{}_{bc} = \frac{1}{2}g^{ad}\left(\delta_c \underset{(0)}{g}_{bd} + \delta_b \underset{(0)}{g}_{dc} - \delta_d \underset{(0)}{g}_{bc}\right),\\
&\overset{c}{\underset{(10)}{H}}{}^a{}_{bc} = \underset{(11)}{B}{}^a{}_{cb} + \frac{1}{2}\underset{(1)}{g}{}^{ad}\left(\delta_c \underset{(1)}{g}_{bd} - \underset{(11)}{B}{}^f{}_{cb}\underset{(1)}{g}_{fd} - \underset{(11)}{B}{}^f{}_{cd}\underset{(1)}{g}_{bf}\right),\\
&\overset{c}{\underset{(20)}{H}}{}^a{}_{bc} = -\underset{(22)}{B}{}^a{}_{bc} + \frac{1}{2}\underset{(2)}{g}{}^{ad}\left(\delta_c \underset{(2)}{g}_{bd} + \underset{(22)}{B}{}^f{}_{bc}\underset{(2)}{g}_{fd} + \underset{(22)}{B}{}^f{}_{dc}\underset{(2)}{g}_{bf}\right),\\
&\overset{c}{\underset{(01)}{C}}{}^a{}_{bc} = \frac{1}{2}\underset{(0)}{g}{}^{ad}\dot{\partial}_c \underset{(0)}{g}_{bd},\\
&\overset{c}{\underset{(11)}{C}}{}^a{}_{bc} = \frac{1}{2}\underset{(1)}{g}{}^{ad}\left(\dot{\partial}_c \underset{(1)}{g}_{bd} + \dot{\partial}_b \underset{(1)}{g}_{dc} - \dot{\partial}_d \underset{(1)}{g}_{bc}\right),\\
&\overset{c}{\underset{(21)}{C}}{}^a{}_{bc} = \frac{1}{2}\underset{(2)}{g}{}^{ad}\dot{\partial}_c \underset{(2)}{g}_{bd},\\
&\overset{c}{\underset{(02)}{C}}{}_a{}^{bc} = -\frac{1}{2}\underset{(0)}{g}_{ad}\dot{\partial}_c \underset{(0)}{g}{}^{bd},
\end{aligned} \tag{1.16}$$

$$\overset{c}{\underset{(12)}{C}}{}_a{}^{bc} = -\frac{1}{2}\underset{(1)}{g}{}_{ad}\dot{\partial}^c \underset{(1)}{g}{}^{bd},$$

$$\overset{c}{\underset{(22)}{C}}{}_a{}^{bc} = -\frac{1}{2}\underset{(2)}{g}{}_{ad}\left(\dot{\partial}^c \underset{(2)}{g}{}^{bd} + \dot{\partial}^b \underset{(2)}{g}{}^{dc} - \dot{\partial}^d \underset{(2)}{g}{}^{bc}\right).$$

Definition 1.3 *The metric $N-$linear connection given by (1.16) will be called the* ***canonical*** *$N-$linear connection associated with $\mathbb{G}$.*

Let

$$D\overset{*}{\Gamma}(N) = \left(\overset{*}{\underset{(00)}{H}}{}^a{}_{bc}, \overset{*}{\underset{(10)}{H}}{}^a{}_{bc}, \overset{*}{\underset{(20)}{H}}{}^a{}_{bc}, \overset{*}{\underset{(01)}{C}}{}^a{}_{bc}, \overset{*}{\underset{(11)}{C}}{}^a{}_{bc}, \overset{*}{\underset{(21)}{C}}{}^a{}_{bc}, \overset{*}{\underset{(02)}{C}}{}_a{}^{bc}, \overset{*}{\underset{(12)}{C}}{}_a{}^{bc}, \overset{*}{\underset{(22)}{C}}{}_a{}^{bc}\right)$$

be an N-linear connection on $T^{*2}M$ which is endowed with a metric structure $\mathbb{G}$.

If we denote by $\overset{*}{|}_{\alpha c}, \overset{*}{|}_{\alpha c}, \overset{*}{|}{}^{\alpha c}$ the $h_\alpha-$,$v_{1\alpha}-$ and $w_{2\alpha}-$ covariant derivations with respect to $D\overset{*}{\Gamma}(N)$, then by a direct calculation one checks that the $N-$linear connection whose local coefficients are given by

$$\begin{aligned}
\underset{(\alpha 0)}{H}{}^a{}_{bc} &= \overset{*}{\underset{(\alpha 0)}{H}}{}^a{}_{bc} + \frac{1}{2}\underset{(\alpha)}{g}{}^{ad}\underset{(\alpha)}{g}{}_{db\overset{*}{|}\alpha c},\\
\underset{(\alpha 1)}{C}{}^a{}_{bc} &= \overset{*}{\underset{(\alpha 1)}{C}}{}^a{}_{bc} + \frac{1}{2}\underset{(\alpha)}{g}{}^{ad}\underset{(\alpha)}{g}{}_{db}\overset{*}{|}_{\alpha c},\\
\underset{(\alpha 2)}{C}{}_a{}^{bc} &= \overset{*}{\underset{(\alpha 2)}{C}}{}_a{}^{bc} - \frac{1}{2}\underset{(\alpha)}{g}{}_{ad}\underset{(\alpha)}{g}{}^{db}\overset{*}{|}{}^{\alpha c}.
\end{aligned} \tag{1.17}$$

is a metrical $N-$linear connection .

This method of metrisation of an $N-$linear connection is called the Kawaguchi metrisation process, [7], [86].

Let us associate to $\mathbb{G}$ the following operators of Obata type:

$$\overset{\alpha}{\underset{1}{O}}{}^{cd}_{ab} = \frac{1}{2}\left(\delta^c_a\delta^d_b - \underset{(\alpha)}{g}{}_{ab}\underset{(\alpha)}{g}{}^{cd}\right), \overset{\alpha}{\underset{2}{O}}{}^{cd}_{ab} = \frac{1}{2}\left(\delta^c_a\delta^d_b + \underset{(\alpha)}{g}{}_{ab}\underset{(\alpha)}{g}{}^{cd}\right), \tag{1.18}$$

$(\alpha = 0, 1, 2)$.

Theorem 1.5 *The set of all metric $N-$linear connections with respect to $\mathbb{G}$ on the manifold $T^{*2}M$ is given by* $(\alpha = 0, 1, 2)$:

$$\begin{aligned}
\underset{(\alpha 0)}{H}{}^a{}_{bc} &= \overset{c}{\underset{(\alpha 0)}{H}}{}^a{}_{bc} + \overset{\alpha}{\underset{1}{O}}{}^{fa}_{bd}\overset{\alpha}{X}{}^d{}_{fc},\\
\underset{(\alpha 1)}{C}{}^a{}_{bc} &= \overset{c}{\underset{(\alpha 1)}{C}}{}^a{}_{bc} + \overset{\alpha}{\underset{1}{O}}{}^{fa}_{bd}\overset{\alpha}{Y}{}^d{}_{fc},\\
\underset{(\alpha 2)}{C}{}_a{}^{bc} &= \overset{c}{\underset{(\alpha 2)}{C}}{}_a{}^{bc} + \overset{\alpha}{\underset{1}{O}}{}^{bd}_{fa}\overset{\alpha}{Z}{}_d{}^{fc},
\end{aligned} \tag{1.19}$$

where $\left(\overset{c}{\underset{(\alpha 0)}{H}}{}^a{}_{bc}, \overset{c}{\underset{(\alpha 1)}{C}}{}^a{}_{bc}, \overset{c}{\underset{(\alpha 2)}{C}}{}_a{}^{bc}\right)$ *is the canonical $N-$linear connection (1.16), $\overset{\alpha}{X}{}^a{}_{bc}, \overset{\alpha}{Y}{}^a{}_{bc}$ are arbitrary $d-$tensor fields of type* $(1,2)$ *and $\overset{\alpha}{Z}{}_a{}^{bc}$ are arbitrary $d-$tensor fields of type* $(2,1)$.

For demonstration we can see V. Cruceanu, R. Miron [44], V. Oproiu [110], [111].

6.2 Metric $N-$linear connections with the torsion prescribed

In the previous paragraph we have proved the existence of metric $N-$linear connection whose $h\,(hh)-$, $v_1\,(v_1v_1)-$ and $w_2\,(w_2w_2)-$ tensors of torsion vanish. But there are certain problems, especially related to the theory of relativity, in which metrical $N-$linear connections with $h\,(hh)-$, $v_1\,(v_1v_1)-$ and $w_2\,(w_2w_2)-$ tensors of torsion prescribed are needed.

In the following we show that such metric $N-$linear connections do exists.

Definition 2.1 *An $N-$linear connection D on $T^{*2}M$ is called an* $\mathbf{h_0v_{11}w_{22}}-$**metric** *with respect to a metric structure $\mathbb{G}$ if*

$$\underset{0}{D}{}^H_X\mathbb{G}^H=0,\ \underset{1}{D}{}^{V_1}_X\mathbb{G}^{V_1}=0,\ \underset{2}{D}{}^{W_2}_X\mathbb{G}^{W_2}=0,\ \forall X\in\chi\left(T^{*2}M\right). \tag{2.1}$$

An easy computation in local coordinates leads to

Proposition 2.1 *An $N-$linear connection $D\Gamma\,(N)=\left(\underset{(\alpha 0)}{H}{}^a{}_{bc},\underset{(\alpha 1)}{C}{}^a{}_{bc},\underset{(\alpha 2)}{C}{}_a{}^{bc}\right)$, $(\alpha=0,1,2)$ is $h_0v_{11},w_{22}-$ metric with respect to $\mathbb{G}=\underset{(0)}{g}{}_{ab}dx^a\otimes dx^b+\underset{(1)}{g}{}_{ab}\delta y^a\otimes\delta y^b+\underset{(2)}{g}{}^{ab}\delta p_a\otimes\delta p_a$ if and only if*

$$\underset{(0)}{g}{}_{ab|0c}=0,\ \underset{(1)}{g}{}_{ab}\,|_{1c}=0,\ \underset{(2)}{g}{}^{ab}\,|^{2c}=0. \tag{2.2}$$

Let us consider an $N-$linear connection of the Berwald type (formula (2.6), §5.2)

$$B\overset{c}{\Gamma}\,(N)=\left(\overset{c}{\underset{(\alpha 0)}{H}}{}^a{}_{bc},\underset{(11)}{B}{}^a{}_{cb},-\underset{(22)}{B}{}^a{}_{cb},0,\overset{c}{\underset{(11)}{C}}{}^a{}_{bc},0,0,0,\overset{c}{\underset{(22)}{C}}{}_a{}^{bc}\right), \tag{2.3}$$

where

$$\begin{aligned}
\overset{c}{\underset{(00)}{H}}{}^a{}_{bc}&=\frac{1}{2}\underset{(0)}{g}{}^{ad}\left(\delta_b\underset{(0)}{g}{}_{dc}+\delta_c\underset{(0)}{g}{}_{bd}-\delta_d\underset{(0)}{g}{}_{bc}\right),\\
\overset{c}{\underset{(11)}{C}}{}^a{}_{bc}&=\frac{1}{2}\underset{(1)}{g}{}^{ad}\left(\dot\partial_b\underset{(1)}{g}{}_{dc}+\dot\partial_c\underset{(1)}{g}{}_{bd}-\dot\partial_d\underset{(1)}{g}{}_{bc}\right),\\
\overset{c}{\underset{(22)}{C}}{}_a{}^{bc}&=-\frac{1}{2}\underset{(2)}{g}{}_{ad}\left(\dot\partial^b\underset{(2)}{g}{}^{dc}+\dot\partial^c\underset{(1)}{g}{}^{bd}-\dot\partial^d\underset{(2)}{g}{}^{bc}\right).
\end{aligned} \tag{2.4}$$

Taking into account (1.15), (1.16), §6.1, the formulae (5.5), (6.5) and (6.6), §5.5 and §5.6, we have

Proposition 2.2 1°.*The $N-$linear connection of the Berwald type (2.3) is $h_0v_{11}w_{22}-$metric. It depends on the metric $\mathbb{G}$, only.*

2°.*The $d-$tensors of torsions of $B\overset{c}{\Gamma}(N)$ are given by*

$$\begin{array}{l}
\overset{0}{\underset{(00)}{T}}{}^{a}{}_{bc}=0,\ \underset{(01)}{R}{}^{a}{}_{bc},\ \underset{(02)}{R}{}_{abc},\\
\overset{0}{\underset{(01)}{P}}{}^{a}{}_{bc}=0,\ \overset{1}{\underset{(01)}{P}}{}^{a}{}_{bc}=0,\ \underset{(12)}{B}{}_{abc},\\
\overset{0}{\underset{(02)}{P}}{}^{a}{}_{b}{}^{c}=0,\ \underset{(21)}{B}{}^{a}{}_{b}{}^{c},\ \overset{2}{\underset{(02)}{P}}{}_{ab}{}^{c}=0,\\
\overset{1}{\underset{(11)}{S}}{}^{a}{}_{bc}=0,\ \overset{1}{\underset{(12)}{Q}}{}^{a}{}_{b}{}^{c}=0,\ \overset{2}{\underset{(12)}{Q}}{}_{ab}{}^{c}=0,\ \overset{2}{\underset{(22)}{S}}{}_{a}{}^{bc}=0,
\end{array}\tag{2.5}$$

3°. The *$d-$tensors of curvature of $B\overset{c}{\Gamma}(N)$ has the following expressions:*

$$\left\{\begin{array}{l}
\underset{(000)}{R}{}_{b}{}^{a}{}_{cd}=\delta_d\overset{c}{\underset{(00)}{H}}{}^{a}{}_{bc}-\delta_c\overset{c}{\underset{(00)}{H}}{}^{a}{}_{bd}+\overset{c}{\underset{(00)}{H}}{}^{f}{}_{bc}\overset{c}{\underset{(00)}{H}}{}^{a}{}_{fd}-\overset{c}{\underset{(00)}{H}}{}^{f}{}_{bd}\underset{(00)}{H}{}^{a}{}_{fc},\\
\underset{(100)}{R}{}_{b}{}^{a}{}_{cd}=\delta_d\underset{(11)}{B}{}^{a}{}_{cb}-\delta_c\underset{(11)}{B}{}^{a}{}_{db}+\underset{(11)}{B}{}^{f}{}_{cb}\underset{(11)}{B}{}^{a}{}_{df}-\underset{(11)}{B}{}^{f}{}_{db}\underset{(11)}{B}{}^{a}{}_{cf}+\\
\qquad\qquad+\overset{c}{\underset{(11)}{C}}{}^{a}{}_{bf}\underset{(01)}{R}{}^{f}{}_{cd},\\
\underset{(200)}{R}{}_{b}{}^{a}{}_{cd}=-\delta_d\underset{(22)}{B}{}^{a}{}_{bc}+\delta_c\underset{(22)}{B}{}^{a}{}_{bd}+\underset{(22)}{B}{}^{f}{}_{bc}\underset{(22)}{B}{}^{a}{}_{fd}-\underset{(22)}{B}{}^{f}{}_{bd}\underset{(22)}{B}{}^{a}{}_{fc}+\\
\qquad\qquad+\overset{c}{\underset{(22)}{C}}{}_{b}{}^{af}\underset{(02)}{R}{}_{fcd},
\end{array}\right.\tag{(2.6)$_1$}$$

$$\left\{\begin{array}{l}
\underset{(001)}{R}{}_{b}{}^{a}{}_{cd}=\dot{\partial}_d\overset{c}{\underset{(00)}{H}}{}^{a}{}_{bc},\ \underset{(201)}{R}{}_{b}{}^{a}{}_{cd}=\dot{\partial}_d\overset{c}{\underset{(20)}{H}}{}^{a}{}_{bc}+\overset{c}{\underset{(22)}{C}}{}_{b}{}^{af}\underset{(12)}{B}{}_{fcd},\\
\underset{(101)}{R}{}_{b}{}^{a}{}_{cd}=\dot{\partial}_d\overset{c}{\underset{(10)}{H}}{}^{a}{}_{bc}-\overset{c}{\underset{(11)}{C}}{}^{a}{}_{bd\shortmid 1c}+\overset{c}{\underset{(11)}{C}}{}^{a}{}_{bf}\left(\underset{(11)}{B}{}^{f}{}_{cd}-\overset{c}{\underset{(10)}{C}}{}^{f}{}_{dc}\right)
\end{array}\right.\tag{(2.6)$_2$}$$

$$\left\{\begin{array}{l}
\underset{(002)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=\dot{\partial}^d\overset{c}{\underset{(00)}{H}}{}^{a}{}_{bc},\ \underset{(102)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=\dot{\partial}^d\overset{c}{\underset{(10)}{H}}{}^{a}{}_{bc}+\overset{c}{\underset{(11)}{C}}{}^{a}{}_{bf}\underset{(21)}{B}{}^{f}{}_{c}{}^{d},\\
\underset{(202)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=\dot{\partial}^d\overset{c}{\underset{(20)}{H}}{}^{a}{}_{bc}-\overset{c}{\underset{(22)}{C}}{}_{b}{}^{ad}{}_{\shortmid 2c}+\overset{c}{\underset{(22)}{C}}{}_{b}{}^{af}\left(\underset{(22)}{B}{}^{d}{}_{fc}+\overset{c}{\underset{(20)}{H}}{}^{d}{}_{fc}\right),
\end{array}\right.\tag{(2.6)$_3$}$$

$$\left\{\begin{array}{l}
\underset{(011)}{R}{}_{b}{}^{a}{}_{cd}=0,\ \underset{(211)}{R}{}_{b}{}^{a}{}_{cd}=0,\\
\underset{(111)}{R}{}_{b}{}^{a}{}_{cd}=\dot{\partial}_d\underset{(11)}{C}{}^{a}{}_{bc}-\dot{\partial}_c\underset{(11)}{C}{}^{a}{}_{bd}+\underset{(11)}{C}{}^{f}{}_{bc}\underset{(11)}{C}{}^{a}{}_{fd}-\underset{(11)}{C}{}^{f}{}_{bd}\underset{(11)}{C}{}^{a}{}_{fc}
\end{array}\right.\tag{(2.6)$_4$}$$

$$\underset{(012)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=0,\ \underset{(112)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=\dot{\partial}^d\overset{c}{\underset{(11)}{C}}{}^{a}{}_{bc},\ \underset{(212)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=-\dot{\partial}_c\overset{c}{\underset{(22)}{C}}{}_{b}{}^{ad},\tag{(2.6)$_5$}$$

$$\left\{\begin{array}{l}
\underset{(022)}{R}{}_{b}{}^{acd}=0,\ \underset{(122)}{R}{}_{b}{}^{acd}=0,\\
\underset{(222)}{R}{}_{b}{}^{acd}=\dot{\partial}^d\overset{c}{\underset{(22)}{C}}{}_{b}{}^{ac}-\dot{\partial}^c\overset{c}{\underset{(22)}{C}}{}_{b}{}^{ad}+\overset{c}{\underset{(22)}{C}}{}_{b}{}^{fc}\underset{(22)}{C}{}_{f}{}^{ad}-\underset{(22)}{C}{}_{b}{}^{fd}\underset{(22)}{C}{}_{f}{}^{ac}.
\end{array}\right.\tag{(2.6)$_6$}$$

The $N-$linear connection (2.3) will be called **the canonical Berwald type connection** on $T^{*2}M$

Now we shall prove:

Theorem 2.1 *There exists an unique $h_0v_{11}w_{22}-$metric $N-$linear connection of the Berwald type with $h\,(hh)-,\ v_1\,(v_1v_1)-$ and $w_2\,(w_2w_2)-$tensors of torsion prescribed.*

Proof. Let us fix the Berwald connection $B\overset{c}{\Gamma}(N)$ introduced in the above. Then, by the general theory of connections, every other $N-$linear connection of the Berwald type is of the form

$$\left(\underset{(00)}{H}{}^{a}{}_{bc}+\underset{(0)}{\tau}{}^{a}{}_{bc},\ \underset{(11)}{B}{}^{a}{}_{cb},\ -\underset{(22)}{B}{}^{a}{}_{bc},0,\ \underset{(11)}{C}{}^{a}{}_{bc}+\underset{(1)}{\tau}{}^{a}{}_{bc},0,0,0,\ \underset{(22)}{C}{}_{a}{}^{bc}+\underset{(2)}{\tau}{}_{a}{}^{bc}\right),$$

where $\underset{(0)}{\tau}{}^{a}{}_{bc},\ \underset{(1)}{\tau}{}^{a}{}_{bc}$ and $\underset{(2)}{\tau}{}_{a}{}^{bc}$ are arbitrary $d-$tensor fields. Let $\underset{(00)}{T}{}^{a}{}_{bc},\ \underset{(11)}{S}{}^{a}{}_{bc}$ and $\underset{(22)}{S}{}_{a}{}^{bc}$ be three $d-$tensor fields which are skew-symmetric,first and second with respect to the covariant indices the third with respect to the contravariant indices. We shall determine the $d-$tensor fields $\underset{(0)}{\tau}{}^{a}{}_{bc},\ \underset{(1)}{\tau}{}^{a}{}_{bc}$ and $\underset{(2)}{\tau}{}_{a}{}^{bc}$ such that the Berwald type connection of general form given in the above to be $h_0v_{11}w_{22}-$metric and to have $\underset{(00)}{T}{}^{a}{}_{bc},\ \underset{(11)}{S}{}^{a}{}_{bc}$ and $\underset{(22)}{S}{}_{a}{}^{bc}$ as $h\,(hh)-,\ v_1\,(v_1v_1)-$ and $w_2\,(w_2w_2)-$tensors of torsion.

These conditions show us that $\underset{(0)}{\tau}{}^{a}{}_{bc},\ \underset{(1)}{\tau}{}^{a}{}_{bc}$ and $\underset{(2)}{\tau}{}_{a}{}^{bc}$ must satisfies the following systems of equations:

$$\underset{(0)}{\tau}{}^{a}{}_{bc}-\underset{(0)}{\tau}{}^{a}{}_{cb}=\underset{(00)}{T}{}^{a}{}_{bc},\ \ \underset{(0)}{\tau}{}^{d}{}_{bc}\,\underset{(0)}{g}{}_{ad}+\underset{(0)}{\tau}{}^{d}{}_{ac}\,\underset{(0)}{g}{}_{db}=0, \tag{2.7}$$

$$\underset{(1)}{\tau}{}^{a}{}_{bc}-\underset{(1)}{\tau}{}^{a}{}_{cb}=\underset{(11)}{S}{}^{a}{}_{bc},\ \ \underset{(1)}{\tau}{}^{d}{}_{bc}\,\underset{(1)}{g}{}_{ad}+\underset{(1)}{\tau}{}^{d}{}_{ac}\,\underset{(1)}{g}{}_{db}=0, \tag{2.8}$$

$$\underset{(2)}{\tau}{}_{a}{}^{bc}-\underset{(2)}{\tau}{}_{a}{}^{cb}=-\underset{(2)}{S}{}_{a}{}^{bc},\ \ \underset{(2)}{\tau}{}_{d}{}^{bc}\,\underset{(2)}{g}{}^{ad}+\underset{(2)}{\tau}{}_{d}{}^{ac}\,\underset{(2)}{g}{}^{db}=0. \tag{2.9}$$

If in the second equation (2.7) (resp., the second equation (2.8)) we cyclicly permute the indices a,b,c, then we add by the Christoffel sum method (we sum the first equation with the third equation and we subtract the second equation) we obtain

$$\underset{(0)}{\tau}{}^{a}{}_{bc}=\frac{1}{2}\underset{(0)}{g}{}^{ad}\left(\underset{(0)}{g}{}_{df}\,\underset{(00)}{T}{}^{f}{}_{bc}-\underset{(0)}{g}{}_{bf}\,\underset{(00)}{T}{}^{f}{}_{dc}+\underset{(0)}{g}{}_{fc}\,\underset{(00)}{T}{}^{f}{}_{bd}\right), \tag{2.7'}$$

$$\underset{(1)}{\tau}{}^{a}{}_{bc}=\frac{1}{2}\underset{(1)}{g}{}^{ad}\left(\underset{(1)}{g}{}_{df}\,\underset{(11)}{S}{}^{f}{}_{bc}-\underset{(1)}{g}{}_{bf}\,\underset{(11)}{S}{}^{f}{}_{dc}+\underset{(1)}{g}{}_{fc}\,\underset{(11)}{S}{}^{f}{}_{bd}\right), \tag{2.8'}$$

If we similarly proceed with the equation (2.9) we deduce

$$\underset{(2)}{\tau}{}_a{}^{bc} = -\frac{1}{2}\underset{(2)}{g}{}_{ad}\left(\underset{(2)}{g}{}^{df}\underset{(22)}{S}{}_f{}^{bc} - \underset{(2)}{g}{}^{bf}\underset{(22)}{S}{}_f{}^{dc} + \underset{(2)}{g}{}^{fc}\underset{(22)}{S}{}_f{}^{bd}\right). \tag{2.9'}$$

Consequently $\underset{(0)}{\tau}{}^a{}_{bc}$, $\underset{(1)}{\tau}{}^a{}_{bc}$ and $\underset{(2)}{\tau}{}_a{}^{bc}$ are uniquely determined.

q.e.d.

Now, from (1.17) we see directly that the Kawaguchi metrisation process leaves unchanged the $h\,(hh)-$, $v_1\,(v_1v_1)-$ and $w_2\,(w_2w_2)-$tensors of torsion. Thus we have:

Theorem 2.2 *Let $T^{*2}M$ be endowed with a metric structure $\mathbb{G}$. There exists on $T^{*2}M$ a metric $N-$linear connection completely determined by $\mathbb{G}$ whose $h\,(hh)-$, $v_1\,(v_1v_1)-$ and $w_2\,(w_2w_2)-$tensors of torsion are prescribed. It is obtained from the $h_0v_{11}w_{22}$ $-$metric Berwald type connection given by Theorem 2.1 via the Kawaguchi metrisation process and has the following local coefficients:*

$$\begin{cases} \underset{(00)}{H}{}^a{}_{bc} = \dfrac{1}{2}\underset{(0)}{g}{}^{ad}\left(\delta_c \underset{(0)}{g}{}_{bd} + \delta_b \underset{(0)}{g}{}_{dc} - \delta_d \underset{(0)}{g}{}_{bc}\right) + \underset{(0)}{\tau}{}^a{}_{bc}, \\ \underset{(10)}{H}{}^a{}_{bc} = \underset{(10)}{\overset{c}{H}}{}^a{}_{bc}, \ \underset{(20)}{H}{}^a{}_{bc} = \underset{(20)}{\overset{c}{H}}{}^a{}_{bc}, \end{cases} \tag{$(2.10)_0$}$$

$$\begin{cases} \underset{(01)}{C}{}^a{}_{bc} = \underset{(01)}{\overset{c}{C}}{}^a{}_{bc}, \ \underset{(21)}{C}{}^a{}_{bc} = \underset{(21)}{\overset{c}{C}}{}^a{}_{bc}, \\ \underset{(11)}{C}{}^a{}_{bc} = \dfrac{1}{2}\underset{(1)}{g}{}^{ad}\left(\dot{\partial}_c \underset{(1)}{g}{}_{bd} + \dot{\partial}_b \underset{(1)}{g}{}_{dc} - \dot{\partial}_d \underset{(1)}{g}{}_{bc}\right) + \underset{(1)}{\tau}{}^a{}_{bc}, \end{cases} \tag{$(2.10)_1$}$$

$$\begin{cases} \underset{(02)}{C}{}_a{}^{bc} = \underset{(02)}{\overset{c}{C}}{}_a{}^{bc}, \ \underset{(12)}{C}{}_a{}^{bc} = \underset{(21)}{\overset{c}{C}}{}_a{}^{bc}, \\ \underset{(22)}{C}{}_a{}^{bc} = -\dfrac{1}{2}\underset{(2)}{g}{}_{ad}\left(\dot{\partial}^c \underset{(2)}{g}{}^{bd} + \dot{\partial}^b \underset{(2)}{g}{}^{dc} - \dot{\partial}^d \underset{(2)}{g}{}^{bc}\right) + \underset{(2)}{\tau}{}_a{}^{bc}, \end{cases} \tag{$(2.10)_2$}$$

where $\underset{(0)}{\tau}{}^a{}_{bc}$, $\underset{(1)}{\tau}{}^a{}_{bc}$ and $\underset{(2)}{\tau}{}_a{}^{bc}$ are given by (2.7'), (2.8') and (2.9'), respectively.

6.3 The Levi-Civita connection on $T^{*2}M$

It is well known there exists an unique linear connection on $T^{*2}M$ metric with respect to $\mathbb{G}$ and symmetric, it has no tensor of torsion, (torsion is vanish). This is called the Levi-Civita connection of $\mathbb{G}$. Note it is not an $N-$linear connection of $\mathbb{G}$ on $T^{*2}M$.

We shall give the local coefficients of the Levi-Civita connection $\mathbb{G}$ in the adapted basis $\left(\delta_a, \dot{\partial}_a, \dot{\partial}^a\right)$.

These coefficients will be expressed by using the local coefficients of the canonical metrical $N-$linear connection $D\overset{c}{\Gamma}(N)$ given by form (1.16).

If we denote by $\bigtriangledown$ the Levi-Civita connection of $\mathbb{G}$, then by a well known fact about the difference of two linear connections, we can write

$$\bigtriangledown_X = \overset{c}{D}_X + \tau_X,\ \forall X \in \chi\left(T^{*2}M\right), \tag{3.1}$$

where τ_X is a tensor fields of type $(1,1)$ on $T^{*2}M$. Taking into account that the linear connection $\bigtriangledown$ and $\overset{c}{D}$ are metric with respect to $\mathbb{G}$ and $\bigtriangledown$ is without torsion the following system of equations for the determining of τ_X is obtained:

$$\begin{aligned} G\left(\tau\left(Y,X\right),Z\right)+G\left(Y,\tau\left(Z,X\right)\right) &= 0,\\ \tau\left(X,Y\right)-\tau\left(Y,X\right) &= \overset{c}{T}\left(X,Y\right),\\ \forall X,Y &\in \chi\left(T^{*2}M\right)\end{aligned} \tag{3.2}$$

where we have set $\tau_X\left(Y\right)=\tau\left(Y,X\right)$ and $\overset{c}{T}$ is the torsion of $\overset{c}{D}$.

In the adapted basis $\left(\delta_a,\dot{\partial}_a,\dot{\partial}^a\right)$ the Levi-Civita connection looks as follows

$$\begin{cases} \bigtriangledown_{\delta_c}\delta_b = \underset{(000)}{H}{}^a{}_{bc}\delta_a + \underset{(001)}{H}{}^a{}_{bc}\dot{\partial}_a + \underset{(002)}{H}{}_{abc}\dot{\partial}^a,\\ \bigtriangledown_{\delta_c}\dot{\partial}_b = \underset{(100)}{H}{}^a{}_{bc}\delta_a + \underset{(101)}{H}{}^a{}_{bc}\dot{\partial}_a + \underset{(102)}{H}{}_{abc}\dot{\partial}^a,\\ \bigtriangledown_{\delta_c}\dot{\partial}^b = \underset{(200)}{H}{}^{ab}{}_c\delta_a + \underset{(201)}{H}{}^{ab}{}_c\dot{\partial}_a + \underset{(202)}{H}{}_a{}^b{}_c\dot{\partial}^a, \end{cases} \tag{(3.3)$_1$}$$

$$\begin{cases} \bigtriangledown_{\dot{\partial}_c}\delta_b = \underset{(010)}{M}{}^a{}_{bc}\delta_a + \underset{(011)}{M}{}^a{}_{bc}\dot{\partial}_a + \underset{(012)}{M}{}_{abc}\dot{\partial}^a,\\ \bigtriangledown_{\dot{\partial}_c}\dot{\partial}_b = \underset{(110)}{M}{}^a{}_{bc}\delta_a + \underset{(111)}{M}{}^a{}_{bc}\dot{\partial}_a + \underset{(112)}{M}{}_{abc}\dot{\partial}^a,\\ \bigtriangledown_{\dot{\partial}_c}\dot{\partial}^b = \underset{(210)}{M}{}^{ab}{}_c\delta_a + \underset{(211)}{M}{}^{ab}{}_c\dot{\partial}_a + \underset{(212)}{M}{}_{ac}{}^b\dot{\partial}^a, \end{cases} \tag{(3.3)$_2$}$$

$$\begin{cases} \bigtriangledown_{\dot{\partial}^c}\delta_b = \underset{(020)}{K}{}^a{}_b{}^c\delta_a + \underset{(021)}{K}{}^a{}_b{}^c\dot{\partial}_a + \underset{(022)}{K}{}_{ab}{}^c\dot{\partial}^a,\\ \bigtriangledown_{\dot{\partial}^c}\dot{\partial}_b = \underset{(120)}{K}{}^a{}_b{}^c\delta_a + \underset{(121)}{K}{}^a{}_b{}^c\dot{\partial}_a + \underset{(122)}{K}{}_{ab}{}^c\dot{\partial}^a,\\ \bigtriangledown_{\dot{\partial}^c}\dot{\partial}^b = \underset{(220)}{K}{}^{abc}\delta_a + \underset{(221)}{K}{}^{abc}\dot{\partial}_a + \underset{(222)}{K}{}_a{}^{bc}\dot{\partial}^a. \end{cases} \tag{(3.3)$_3$}$$

Writing the system of equations (3.2) in the adapted basis $\left(\delta_a,\dot{\partial}_a,\dot{\partial}^a\right)$ we obtain a system of equations which allows us to determine the local components of $\bigtriangledown_X$. Inserting these local components in the local form of the equation (3.1) one obtains:

Theorem 3.1 *The local coefficients of the Levi-Civita connection $\bigtriangledown$ of the metric structure $\mathbb{G}$ on the manifold $T^{*2}M$ one given as follows formulae:*

$$\begin{cases} \underset{(000)}{H}{}^a{}_{bc} = \underset{(00)}{\overset{c}{H}}{}^a{}_{bc},\ \underset{(001)}{H}{}^a{}_{bc} = -\dfrac{1}{2}\underset{(01)}{\overset{c}{R}}{}^a{}_{bc} - \underset{(01)}{\overset{c}{C}}{}^f{}_{bd}\underset{(0)}{g}{}_{fc}\underset{(1)}{g}{}^{da},\\ \qquad\qquad \underset{(002)}{H}{}_{abc} = -\dfrac{1}{2}\underset{(02)}{\overset{c}{R}}{}_{abc} - \underset{(02)}{C}{}_b{}^{fd}\underset{(0)}{g}{}_{fc}\underset{(2)}{g}{}_{da} \end{cases}$$

$$\begin{cases} \underset{(100)}{H}{}^{a}{}_{bc} = \left(\frac{1}{2} \underset{(01)}{\overset{c}{R}}{}^{f}{}_{dc} \underset{(1)}{g}{}_{fb} + \underset{(01)}{\overset{c}{C}}{}^{f}{}_{db} \underset{(0)}{g}{}_{fc} \right) \underset{(0)}{g}{}^{da} \\ \underset{(101)}{H}{}^{a}{}_{bc} = \underset{(10)}{\overset{c}{H}}{}^{a}{}_{bc} + \underset{(1)}{\overset{1}{O}}{}^{fa}_{bd} \underset{(11)}{\overset{c}{P}}{}^{d}{}_{cf}, \ \underset{(102)}{H}{}_{abc} = \frac{1}{2} \left(\underset{(12)}{B}{}_{acb} - \underset{(21)}{B}{}^{f}{}_{c}{}^{d} \underset{(1)}{g}{}_{fb} \underset{(2)}{g}{}_{da} \right), \end{cases}$$

$$\begin{cases} \underset{(200)}{H}{}^{ab}{}_{c} = \left(\frac{1}{2} \underset{(02)}{\overset{c}{R}}{}_{fcd} \underset{(2)}{g}{}^{fb} + \underset{(02)}{\overset{c}{C}}{}^{f}{}_{d}{}^{b} \underset{(0)}{g}{}_{fc} \right) \underset{(0)}{g}{}^{da} \\ \underset{(201)}{H}{}^{ab}{}_{c} = \left(\frac{1}{2} \underset{(21)}{B}{}^{a}{}_{c}{}^{b} - \underset{(12)}{B}{}_{fcd} \underset{(2)}{g}{}^{fb} \underset{(1)}{g}{}^{da} \right), \ \underset{(202)}{H}{}_{b}{}^{a}{}_{c} = \underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc} + \underset{1}{O}{}^{fa}_{bd} \underset{(22)}{\overset{c}{P}}{}^{d}{}_{cf}, \end{cases}$$

$$\begin{cases} \underset{(010)}{M}{}^{a}{}_{bc} = \underset{(100)}{H}{}^{a}{}_{cb}, \ \underset{(011)}{M}{}^{a}{}_{bc} = -\underset{(2)}{\overset{1}{O}}{}^{fa}_{cd} \underset{(11)}{\overset{c}{P}}{}^{d}{}_{bf}, \\ \underset{(012)}{M}{}_{abc} = -\frac{1}{2} \left(\underset{(12)}{B}{}_{abc} + \underset{(21)}{B}{}^{f}{}_{b}{}^{d} \underset{(1)}{g}{}_{fc} \underset{(2)}{g}{}_{da} \right), \end{cases}$$

$$\begin{cases} \underset{(110)}{M}{}^{a}{}_{bc} = \frac{1}{2} \left(\underset{(11)}{\overset{c}{P}}{}^{f}{}_{db} \underset{(1)}{g}{}_{fc} + \underset{(11)}{\overset{c}{P}}{}^{f}{}_{dc} \underset{(1)}{g}{}_{fb} \right) \underset{(1)}{g}{}^{da}, \ \underset{(111)}{M}{}^{a}{}_{bc} = \underset{(11)}{\overset{c}{C}}, \\ \underset{(112)}{M}{}_{abc} = -\frac{1}{2} \underset{(12)}{B}{}_{abc} - \frac{1}{2} \left(\underset{(12)}{\overset{c}{C}}{}_{c}{}^{fd} \underset{(1)}{g}{}_{bf} + \underset{(12)}{\overset{c}{C}}{}_{b}{}^{fd} \underset{(1)}{g}{}_{cf} \right) \underset{(2)}{g}{}_{da}, \end{cases}$$

$$\begin{cases} \underset{(210)}{M}{}^{ab}{}_{c} = \frac{1}{2} \left(\underset{(21)}{B}{}^{f}{}_{d}{}^{b} \underset{(1)}{g}{}_{fc} + \underset{(12)}{B}{}_{fdc} \underset{(2)}{g}{}^{fb} \right) \underset{(1)}{g}{}^{da}, \\ \underset{(211)}{M}{}^{ab}{}_{c} = \frac{1}{2} \underset{(12)}{B}{}_{fdc} \underset{(2)}{g}{}^{fb} \underset{(1)}{g}{}^{da} + \underset{(2)}{\overset{1}{O}}{}^{fa}_{cd} \underset{(12)}{\overset{c}{C}}{}_{f}{}^{bd}, \\ \underset{(212)}{M}{}_{bc}{}^{a} = \underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc} + \underset{(1)}{\overset{2}{O}}{}^{fa}_{bd} \underset{(21)}{\overset{c}{C}}{}^{d}{}_{cf} \end{cases}$$

$$\begin{cases} \underset{(020)}{K}{}^{a}{}_{c}{}^{b} = \underset{(200)}{H}{}^{ab}{}_{c}, \ \underset{(021)}{K}{}^{a}{}_{c}{}^{b} = -\frac{1}{2} \left(\underset{(21)}{B}{}^{a}{}_{c}{}^{b} + \underset{(12)}{B}{}_{fcd} \underset{(2)}{g}{}^{fb} \underset{(1)}{g}{}^{da} \right), \\ \underset{(022)}{K}{}_{bc}{}^{a} = -\underset{(2)}{\overset{2}{O}}{}^{fa}_{cd} \underset{(22)}{P}{}^{d}{}_{bf}, \end{cases}$$

$$\begin{cases} \underset{(120)}{K}{}^{a}{}_{c}{}^{b} = \frac{1}{2} \left(\underset{(12)}{B}{}_{fdc} \underset{(2)}{g}{}^{fb} + \underset{(21)}{B}{}^{f}{}_{d}{}^{b} \underset{(1)}{g}{}_{fc} \right) \underset{(1)}{g}{}^{da}, \\ \underset{(121)}{K}{}^{a}{}_{c}{}^{b} = \frac{1}{2} \underset{(12)}{\overset{c}{C}}{}_{d}{}^{fb} \underset{(2)}{g}{}_{fc} \underset{(1)}{g}{}^{da} + \underset{(2)}{\overset{1}{O}}{}^{fa}_{cd} \underset{(12)}{\overset{c}{C}}{}_{f}{}^{da}, \ \underset{(122)}{K}{}_{bc}{}^{a} = -\underset{(2)}{\overset{2}{O}}{}^{fa}_{cd} \underset{(21)}{\overset{c}{C}}{}^{d}{}_{bf} \end{cases}$$

$$\begin{cases} \underset{(220)}{K}{}^{abc} = \frac{1}{2} \left(\underset{(22)}{\overset{c}{P}}{}^{d}{}_{fd} \underset{(2)}{g}{}^{fc} + \underset{(22)}{\overset{c}{P}}{}^{c}{}_{fd} \underset{(2)}{g}{}^{fb} \right) \underset{(0)}{g}{}^{da} \\ \underset{(221)}{K}{}^{abc} = \frac{1}{2} \left(\underset{(21)}{\overset{c}{C}}{}^{b}{}_{fd} \underset{(2)}{g}{}^{fc} + \underset{(21)}{\overset{c}{C}}{}^{c}{}_{fd} \underset{(2)}{g}{}^{fb} \right) \underset{(1)}{g}{}^{da}, \ \underset{(222)}{K}{}_{a}{}^{bc} = \underset{(22)}{\overset{c}{C}}{}_{a}{}^{bc}. \end{cases}$$

6.4 Some remarkable metrics on $T^{*2}M$

Recall that a given metrical structure $\mathbb{G}$ on the manifold $T^{*2}M$ determines a nonlinear connection and with respect to it $\mathbb{G}$ decomposes into a sum of three $d-$tensor fields which may be viewed as defining metrical structures in horizontal and verticals distributions, respectively. Conversely if a nonlinear connection, as well as some metrical structures in horizontal and verticals distributions are given, a metrical structure on $T^{*2}M$ may be obtained.

From now on we fix a nonlinear connection $N\left(N_{b}^{c}, N_{ab}\right)$ in the cotangent bundle of second order $T^{*2}M$.

Definition 4.1 1°. *An* **h−metric** *on* $T^{*2}M$ *is a* $d-$*tensor field* $\mathbb{G}^{H}=\underset{(0)}{g}_{ab}dx^{a}\otimes dx^{b}$, *where* $\underset{(0)}{g}_{ab}\left(x,y,p\right)=\underset{(0)}{g}_{ba}\left(x,y,p\right),\det\parallel\underset{(0)}{g}_{ab}\left(x,y,p\right)\parallel\neq 0$ *and the quadratic form* $\underset{(0)}{g}_{ab}\xi^{a}\xi^{b}$ *has constant signature.*

2°. *A* $\mathbf{v}_1-$**metric** *on* $T^{*2}M$ *is a* $d-$*tensor field* $\mathbb{G}^{V_1}=\underset{(1)}{g}_{ab}\delta y^{a}\otimes\delta y^{b}$, *where* $\underset{(1)}{g}_{ab}$ *has the same properties as* $\underset{(0)}{g}_{ab}$.

3°. *A* $\mathbf{w}_2-$**metric** *on* $T^{*2}M$ *is a* $d-$*tensor field* $\mathbb{G}^{W_2}=\underset{(2)}{g}^{ab}\delta p_{a}\otimes\delta p_{b}$, *where* $\underset{(2)}{g}_{ab}$,with$\parallel\underset{(2)}{g}_{ab}\parallel=\parallel\underset{(2)}{g}^{ab}\parallel^{-1}$, *has the same properties as* $\underset{(0)}{g}_{ab}$.

4°. *An* $\left(\mathbf{h},\mathbf{v_1},\mathbf{w_2}\right)-$**metric** *on* $T^{*2}M$ *is the* $d-$ *tensor field* $\mathbb{G}=\mathbb{G}^{H}+\mathbb{G}^{V_1}+\mathbb{G}^{W_2}$, *i.e.*

$$\mathbb{G}=\underset{(0)}{g}_{ab}\left(x,y,p\right)dx^{a}\otimes dx^{b}+\underset{(1)}{g}_{ab}\left(x,y,p\right)\delta y^{a}\otimes\delta y^{b}+\underset{(2)}{g}^{ab}\left(x,y,p\right)\delta p_{a}\otimes\delta p_{b}, \quad (4.1)$$

Obvious, the metric structure (6.5),§4.6

$$\mathbb{G}=g_{ab}dx^{a}\otimes dx^{b}+g_{ab}\delta y^{a}\otimes\delta y^{b}+h^{ab}\delta p_{a}\otimes\delta p_{b}, \quad (4.2)$$

and the metric structure

$$\mathbb{G}=g_{ab}dx^{a}\otimes dx^{b}+g_{ab}\delta y^{a}\otimes\delta y^{b}+g^{ab}\delta p_{a}\otimes\delta p_{b}, \quad (4.3)$$

where $g_{ab}\left(x,y,p\right)$ and $h_{ab}\left(x,y,p\right)$,with$\parallel h_{ab}\parallel=\parallel h^{ab}\parallel^{-1}$, has the same properties as $\underset{(0)}{g}_{ab}$, are the $\left(h,v_1,w_2\right)-$metric structures on $T^{*2}M$.

By using Theorem 1.4, we can written the metric $N-$linear connections depending only on $\mathbb{G}$ given by (4.2), respectively (4.3).

For instance, we have

Theorem 4.1 *If the manifold* $T^{*2}M$ *is endowed with the metric structure* $\mathbb{G}$ *given by (4.2) then the metrical canonical* $N-$*linear connection has the coefficients:*

$$\begin{cases} \underset{(00)}{\overset{c}{H}}{}^{a}{}_{bc} = \dfrac{1}{2}g^{ad}\left(\delta_c g_{bd} + \delta_b g_{dc} - \delta_d g_{bc}\right), \\ \underset{(10)}{\overset{c}{H}}{}^{a}{}_{bc} = \underset{(11)}{B}{}^{a}{}_{cb} + \dfrac{1}{2}g^{ad}\left(\delta_c g_{bd} - \underset{(11)}{B}{}^{f}{}_{cb} g_{fd} - \underset{(11)}{B}{}^{f}{}_{cd} g_{bf}\right), \\ \underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc} = -\underset{(22)}{B}{}^{a}{}_{bc} + \dfrac{1}{2}h^{ad}\left(\delta_c h_{bd} + \underset{(22)}{B}{}^{f}{}_{bc} h_{fd} + \underset{(22)}{B}{}^{f}{}_{dc} h_{bf}\right), \end{cases}$$

$$\begin{cases} \underset{(01)}{\overset{c}{C}}{}^{a}{}_{bc} = \dfrac{1}{2}g^{ad}\dot{\partial}_c g_{bd}, \quad \underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc} = \dfrac{1}{2}h^{ad}\dot{\partial}_c h_{bd}, \\ \underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc} = \dfrac{1}{2}g^{ad}\left(\dot{\partial}_c g_{bd} + \dot{\partial}_c g_{dc} - \dot{\partial}_d g_{bc}\right), \end{cases} \tag{4.4}$$

$$\begin{cases} \underset{(02)}{\overset{c}{C}}{}_{a}{}^{bc} = \underset{(12)}{\overset{c}{C}}{}_{a}{}^{bc} = -\dfrac{1}{2}g_{ad}\dot{\partial}^c g^{bd}, \\ \underset{(22)}{\overset{c}{C}}{}_{a}{}^{bc} = -\dfrac{1}{2}h_{ad}\left(\dot{\partial}^c h^{bd} + \dot{\partial}^b h^{dc} - \dot{\partial}^d h^{bc}\right). \end{cases}$$

Theorem 4.2 *If the manifold $T^{*2}M$ is endowed with the metric structure $\mathbb{G}$ given by (4.3) then the metrical canonical $N-$linear connection has the coefficients:*

$$\begin{cases} \underset{(00)}{\overset{c}{H}}{}^{a}{}_{bc} = \dfrac{1}{2}g^{ad}\left(\delta_c g_{bd} + \delta_b g_{dc} - \delta_d g_{bc}\right), \\ \underset{(10)}{\overset{c}{H}}{}^{a}{}_{bc} = \underset{(11)}{B}{}^{a}{}_{cb} + \dfrac{1}{2}g^{ad}\left(\delta_c g_{bd} - \underset{(11)}{B}{}^{f}{}_{cb} g_{fd} - \underset{(11)}{B}{}^{f}{}_{cd} g_{bf}\right), \\ \underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc} = -\underset{(22)}{B}{}^{a}{}_{bc} + \dfrac{1}{2}g^{ad}\left(\delta_c g_{bd} + \underset{(22)}{B}{}^{f}{}_{bc} g_{fd} + \underset{(22)}{B}{}^{f}{}_{dc} g_{bf}\right), \end{cases}$$

$$\begin{cases} \underset{(01)}{\overset{c}{C}}{}^{a}{}_{bc} = \underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc} = \dfrac{1}{2}g^{ad}\dot{\partial}_c g_{bd}, \\ \underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc} = \dfrac{1}{2}g^{ad}\left(\dot{\partial}_c g_{bd} + \dot{\partial}_c g_{dc} - \dot{\partial}_d g_{bc}\right), \end{cases} \tag{4.5}$$

$$\begin{cases} \underset{(02)}{\overset{c}{C}}{}_{a}{}^{bc} = \underset{(12)}{\overset{c}{C}}{}_{a}{}^{bc} = -\dfrac{1}{2}g_{ad}\dot{\partial}^c g^{bd}, \\ \underset{(22)}{\overset{c}{C}}{}_{a}{}^{bc} = -\dfrac{1}{2}g_{ad}\left(\dot{\partial}^c g^{bd} + \dot{\partial}^b g^{dc} - \dot{\partial}^c g^{bc}\right). \end{cases}$$

In next, we study others (h, v_1, w_2) −metric structures.

Definition 4.2 1°.*The (h, v_1, w_2) −metric $\mathbb{G}$ given by (4.1) is said to be* **h−Riemannian** *if $\underset{(0)}{g}{}_{ab}$ do not depend on y^a and p_a.*

2°.*The (h, v_1, w_2) −metric $\mathbb{G}$ given by (4.1) is said to be* **$\mathbf{v_1}$−Riemannian** *if $\underset{(1)}{g}{}_{ab}$ do not depend on y^a and p_a.*

3°. *The* (h, v_1, w_2) $-$*metric* $\mathbb{G}$ *given by (4.1) is said to be* $\mathbf{w_2}-$**Riemannian** *if* $\underset{(0)}{g}{}^{ab}$ *do not depend on* y^a *and* p_a.

It is now clearly what means $\mathbb{G}$ is $(\mathbf{h}, \mathbf{v_1}, \mathbf{w_2})$ $-$**Riemannian**. We have

Proposition 4.1 *a).* $\mathbb{G}$ *is an* $h-$*Riemannian metric if and only if* $\overset{c}{\underset{(01)}{C}}{}^a{}_{bc}$ *and* $\overset{c}{\underset{(02)}{C}}{}_a{}^{bc}$ *from (1.16) vanish.*

b). $\mathbb{G}$ *is an* $h-$*Riemannian metric if and only if* $\overset{c}{\underset{(11)}{C}}{}^a{}_{bc}$ *and* $\overset{c}{\underset{(12)}{C}}{}_a{}^{bc}$ *from (1.16) vanish.*

c). $\mathbb{G}$ *is an* $h-$*Riemannian metric if and only if* $\overset{c}{\underset{(21)}{C}}{}^a{}_{bc}$ *and* $\overset{c}{\underset{(22)}{C}}{}_a{}^{bc}$ *from (1.16) vanish.*

d). $\mathbb{G}$ *is an* (h, v_1, w_2) $-$*Riemannian metric if and only if*

$$\overset{c}{\underset{(\alpha 1)}{C}}{}^a{}_{bc} = 0,\ \overset{c}{\underset{(\alpha 2)}{C}}{}_a{}^{bc} = 0,\ (\alpha = 0, 1, 2)\,. \tag{4.6}$$

Coming back to the Theorem 1.4, we obtain

Proposition 4.2 *If* (h, v_1, w_2) $-$*metric* $\mathbb{G}$ *given by (4.1) is* (h, v_1, w_2) $-$*Riemannian metric then about (4.6) we have also*

$$\begin{aligned}
&i)\ \overset{c}{\underset{(00)}{H}}{}^a{}_{bc} = \left\{ {a \atop bc} \right\},\ \overset{c}{\underset{(\beta 0)}{H}}{}^a{}_{bc} = \frac{1}{2} \underset{(\beta)}{g}{}^{ad} \partial_c \underset{(\beta)}{g}{}_{bd}, (\beta = 1, 2) \\
&ii)\ \overset{0}{\underset{(00)}{T}}{}^a{}_{bc} = 0,\ \overset{0}{\underset{(01)}{P}}{}^a{}_{bc} = 0,\ \overset{0}{\underset{(02)}{P}}{}^a{}_{bc} = 0 \\
&\quad \underset{(11)}{S}{}^a{}_{bc} = 0,\ \overset{1}{\underset{(12)}{Q}}{}^a{}_b{}^c = 0,\ \overset{2}{\underset{(12)}{Q}}{}_{ab}{}^c = 0,\ \underset{(22)}{S}{}_a{}^{bc} = 0, \\
&iii)\ \underset{(000)}{R}{}_b{}^a{}_{cd} = r_b{}^a{}_{cd},\ \underset{(\alpha 01)}{R}{}_b{}^a{}_{cd} = 0,\ \underset{(\alpha 02)}{R}{}_b{}^a{}_{cd} = 0, \\
&\quad \underset{(\alpha 11)}{R}{}_b{}^a{}_{cd} = 0,\ \underset{(\alpha 12)}{R}{}_b{}^a{}_c{}^d = 0,\ \underset{(\alpha 22)}{R}{}_b{}^{acd} = 0, \\
&\qquad\qquad\qquad\qquad (\alpha = 0, 1, 2)\,,
\end{aligned}$$

where $\left\{ {a \atop bc} \right\}$ *are the Christoffel symbols constructed with* $\underset{(0)}{g}{}_{ab}(x)$ *and the* $r_b{}^a{}_{cd}$ *is the curvature tensors constructed with* $\left\{ {a \atop bc} \right\}$.

As in the case of the tangent bundle of the first order (TM, π, M) (cf. with Satoshi Ikeda from University of Tokyo), the case when $\mathbb{G}$ is $h-, v_1$ and w_2-Riemannian "seams to have no essential physical meaning", but these are theoretically interesting.

Definition 4.3 1°. *The* (h, v_1, w_2) $-$*metric* $\mathbb{G}$ *given by (4.1) is said to be an* $\mathbf{h}-$**elliptical metric of moment**, *if* $\underset{(0)}{g}{}_{ab}$ *do not depend by* p_a.

2°. *(h,v_1,w_2) −metrics $\mathbb{G}$ given by (4.1) is said to be an* **$\mathbf{v_1}$−elliptical metric of moment**, *if $\underset{(1)}{g}_{ab}$ do not depend by p_a.*

3°. *(h,v_1,w_2) −metrics $\mathbb{G}$ given by (4.1) is said to be an* **$\mathbf{w_2}$−elliptical metric of moment**, *if $\underset{(2)}{g}^{ab}$ do not depend by p_a.*

It is evidently what means $\mathbb{G}$ is (h,v_1,w_2) −elliptical metric of moment. Such metrics exist on $T^{*2}M$. They, can be obtained, for example, by the prolongation at $T^{*2}M$ of an Lagrange metric $g_{ab}(x,y)=\dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^a \partial y^b}$ on the tangent bundle of the first order (TM,π,M) at the cotangent bundle of second order $\left(T^{*2}M,\pi^{*2},M\right)$:

$$\mathbb{G}=g_{ab}(x,y)dx^a\otimes dx^b+g_{ab}(x,y)\delta y^a\otimes\delta y^b+g^{ab}(x,y)\delta p_a\otimes\delta p_b, \tag{4.7}$$

We have

Proposition 4.3 *a). $\mathbb{G}$ given by (4.1) is a $h-$elliptical metric of moment if and only if $\underset{(02)}{\overset{c}{C}}{}_a{}^{bc}$ from (1.16) vanish*

b). $\mathbb{G}$ given by (4.1) is a v_1-elliptical metric of moment if and only if $\underset{(12)}{\overset{c}{C}}{}_a{}^{bc}$ from (1.16) vanish

c). $\mathbb{G}$ given by (4.1) is a w_2-elliptical metric of moment if and only if $\underset{(22)}{\overset{c}{C}}{}_a{}^{bc}$ from (1.16) vanish

d). $\mathbb{G}$ given by (4.1) is o (h,v_1,w_2) −elliptical metric of moment if and only if in a form (1.16)

$$\underset{(\alpha 2)}{\overset{c}{C}}{}_a{}^{bc}=0,\ (\alpha=0,1,2). \tag{4.8}$$

Theorem 4.3 *If the manifold $T^{*2}M$ is endowed with the (h,v_1,w_2) −elliptical metric structure of moment $\mathbb{G}$ given by (4.7) then the metrical canonical $N-$linear connection has the coefficients given by (4.8) and the following expressions*

$$\begin{cases}\underset{(00)}{\overset{c}{H}}{}^a{}_{bc}=\left\{{}^{\,a}_{bc}\right\}-\underset{(0)}{\sigma}{}^a{}_{bc}\\ \underset{(10)}{\overset{c}{H}}{}^a{}_{bc}=\underset{(11)}{B}{}^a{}_{cb}+\dfrac{1}{2}g^{ad}\left(\partial_c g_{bd}-\underset{(11)}{B}{}^f{}_{cb}g_{fd}-\underset{(11)}{B}{}^f{}_{cb}g_{bf}\right)-\underset{(1)}{\sigma}{}^a{}_{bc},\\ \underset{(20)}{\overset{c}{H}}{}^a{}_{bc}=-\underset{(22)}{B}{}^a{}_{bc}+\dfrac{1}{2}g^{ad}\left(\partial_c g_{bd}+\underset{(22)}{B}{}^f{}_{bc}g_{fd}+\underset{(22)}{B}{}^f{}_{dc}g_{bf}\right)-\underset{(2)}{\sigma}{}^a{}_{bc},\end{cases} \tag{4.9}$$
$$\begin{cases}\underset{(01)}{\overset{c}{C}}{}^a{}_{bc}=\underset{(21)}{\overset{c}{C}}{}^a{}_{bc}=\dfrac{1}{2}g^{ad}\dot{\partial}_c g_{bd}\\ \underset{(11)}{\overset{c}{C}}{}^a{}_{bc}=\dfrac{1}{2}g^{ad}\left(\dot{\partial}_c g_{bd}+\dot{\partial}_b g_{dc}-\dot{\partial}_d g_{bc}\right)\end{cases}$$

where

$$\begin{aligned} \underset{(0)}{\sigma}{}^{a}{}_{bc} &= \frac{1}{2} g^{ad} \left(N^{f}_{c} \dot{\partial}_f g_{bd} + N^{f}_{b} \dot{\partial}_f g_{dc} - N^{f}_{d} \dot{\partial}_f g_{bc} \right), \\ \underset{(1)}{\sigma}{}^{a}{}_{bc} &= \underset{(2)}{\sigma}{}^{a}{}_{bc} = \frac{1}{2} g^{ad} N^{f}_{c} \dot{\partial}_f g_{bd}. \end{aligned} \tag{4.10}$$

The following metric structures can be interesting for physics.

Definition 4.4 *We shall say that the metric $\mathbb{G}$ given by (4.1) is* $\mathbf{v_1}$**−locally Minkowski** *(resp.,* $\mathbf{w_2}$**−locally Minkowski***) if for every point $u = (x, y, p) \in T^{*2}M$ there exists a local chart around it on $T^{*2}M$ such that on its domain, $\underset{(1)}{g}{}_{ab}$ (resp., $\underset{(2)}{g}{}^{ab}$) depends on y* only.

Theorem 4.4 *If $(h, v_1, w_2)-$metric $\mathbb{G}$ given by (4.1) is $h-$Riemannian and v_1-, w_2-locally Minkowski metric then the metrical cannonical $N-$linear connection has the coefficients:*

$$\begin{cases} \underset{(00)}{\overset{c}{H}}{}^{a}{}_{bc} = \left\{ {}^{a}_{bc} \right\} \\ \underset{(10)}{\overset{c}{H}}{}^{a}{}_{bc} = \underset{(11)}{B}{}^{a}{}_{bc} - \frac{1}{2} \underset{(1)}{g}{}^{ad} \left(N^{f}_{c} \dot{\partial}_f \underset{(1)}{g}{}_{bd} + \underset{(11)}{B}{}^{f}{}_{cb} \underset{(1)}{g}{}_{fd} + \underset{(11)}{B}{}^{f}{}_{cd} \underset{(1)}{g}{}_{bf} \right), \\ \underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc} = - \underset{(22)}{B}{}^{a}{}_{bc} - \frac{1}{2} \underset{(2)}{g}{}^{ad} \left(N^{f}_{c} \dot{\partial}_f \underset{(2)}{g}{}_{bd} - \underset{(22)}{B}{}^{f}{}_{bc} \underset{(2)}{g}{}_{fd} - \underset{(22)}{B}{}^{f}{}_{dc} \underset{(2)}{g}{}_{bf} \right), \end{cases}$$

$$\begin{cases} \underset{(01)}{\overset{c}{C}}{}^{a}{}_{bc} = 0, \ \underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc} = \frac{1}{2} \underset{(2)}{g}{}^{ad} \dot{\partial}_c \underset{(2)}{g}{}_{bd}, \\ \underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc} = \frac{1}{2} \underset{(1)}{g}{}^{ad} \left(\dot{\partial}_c \underset{(1)}{g}{}_{bd} + \dot{\partial}_b \underset{(1)}{g}{}_{dc} - \dot{\partial}_d \underset{(1)}{g}{}_{bc} \right), \end{cases} \tag{4.11}$$

$$\underset{(02)}{\overset{c}{C}}{}_{a}{}^{bc} = 0, \ \underset{(12)}{C}{}_{a}{}^{bc} = 0, \ \underset{(22)}{C}{}_{a}{}^{bc} = 0.$$

Proof. Indeed, because $\underset{(0)}{g}{}_{ab}$ depend of x, and $\underset{(1)}{g}{}_{ab}$, $\underset{(2)}{g}{}^{ab}$ each depend on y only, by (1.16) we get (4.11)

q.e.d.

Also, we get

Theorem 4.5 *If the manifold $T^{*2}M$ is endowed with the $(h, v_1, w_2)-$metric structure $\mathbb{G}$, $h-$Riemannian, v_1-Riemannian and w_2-locally Minkowski given by*

$$\mathbb{G} = g_{ab}(x) dx^a \otimes dx^b + g_{ab}(x) \delta y^a \otimes \delta y^b + h^{ab}(y) \delta p_a \otimes \delta p_b, \tag{4.12}$$

then we have

$$
i)\ \begin{cases}
\underset{(00)}{\overset{c}{H}}{}^{a}{}_{bc} = \left\{ {}^{a}_{bc} \right\}_{x} \\
\underset{(10)}{\overset{c}{H}}{}^{a}{}_{bc} = \underset{(11)}{B}{}^{a}{}_{cb} + \dfrac{1}{2} g^{ad} \left(\partial_c g_{bd} - \underset{(11)}{B}{}^{f}{}_{cb} g_{fd} - \underset{(11)}{B}{}^{f}{}_{cd} \underset{(1)}{g}{}_{bf} \right), \\
\underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc} = - \underset{(22)}{B}{}^{a}{}_{bc} + \dfrac{1}{2} g^{cd} \left(\partial_c g_{bd} + \underset{(22)}{B}{}^{f}{}_{bc} g_{fd} + \underset{(22)}{B}{}^{f}{}_{dc} g_{bf} \right),
\end{cases}
$$

$$
\underset{(01)}{\overset{c}{C}}{}^{a}{}_{bc} = 0,\ \underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc} = 0,\ \underset{(21)}{C}{}^{a}{}_{bc} = \frac{1}{2} h^{ad} \dot{\partial}_c h_{bd}
$$

$$
\underset{(02)}{\overset{c}{C}}{}_{a}{}^{bc} = 0,\ \underset{(12)}{\overset{c}{C}}{}_{a}{}^{bc} = 0,\ \underset{(22)}{\overset{c}{C}}{}_{a}{}^{bc} = 0.
$$

$$
ii)\ \begin{cases}
\underset{(00)}{\overset{0}{T}}{}^{a}{}_{bc} = 0,\ \underset{(01)}{\overset{0}{P}}{}^{a}{}_{bc} = 0,\ \underset{(02)}{\overset{0}{P}}{}^{a}{}_{b}{}^{c} = 0, \\
\underset{(11)}{S}{}^{a}{}_{bc} = 0,\ \underset{(12)}{\overset{1}{Q}}{}^{a}{}_{b}{}^{c} = 0,\ \underset{(12)}{\overset{2}{Q}}{}_{ab}{}^{c} = 0,\ \underset{(22)}{\overset{c}{S}}{}_{a}{}^{bc} = 0.
\end{cases}
$$

$$
iii)\ \begin{cases}
\underset{(000)}{\overset{c}{R}}{}_{b}{}^{a}{}_{cd} = r_{b}{}^{a}{}_{cd}(x), \\
\underset{(100)}{\overset{c}{R}}{}_{b}{}^{a}{}_{cd} = \delta_d \underset{(10)}{\overset{c}{H}}{}^{a}{}_{bc} - \delta_c \underset{(10)}{\overset{c}{H}}{}^{a}{}_{bd} + \underset{(10)}{\overset{c}{H}}{}^{f}{}_{bc} \underset{(10)}{\overset{c}{H}}{}^{a}{}_{fd} - \underset{(10)}{\overset{c}{H}}{}^{f}{}_{bd} \underset{(10)}{\overset{c}{H}}{}^{a}{}_{fc} \\
\underset{(200)}{\overset{c}{R}}{}_{b}{}^{a}{}_{cd} = \delta_d \underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc} - \delta_c \underset{(20)}{\overset{c}{H}}{}^{a}{}_{bd} + \underset{(20)}{\overset{c}{H}}{}^{f}{}_{bc} \underset{(20)}{\overset{c}{H}}{}^{a}{}_{fd} - \underset{(20)}{\overset{c}{H}}{}^{f}{}_{bd} \underset{(20)}{\overset{c}{H}}{}^{a}{}_{fc} + \\
\qquad + \underset{(21)}{C}{}^{a}{}_{bf} \underset{(01)}{R}{}^{f}{}_{cd},
\end{cases}
$$

$$
\begin{cases}
\underset{(001)}{R}{}_{b}{}^{a}{}_{cd} = \dot{\partial}_d \underset{(00)}{H}{}^{a}{}_{bc},\ \underset{(101)}{R}{}_{b}{}^{a}{}_{cd} = \dot{\partial}_d \underset{(10)}{H}{}^{a}{}_{bc}, \\
\underset{(201)}{R}{}_{b}{}^{a}{}_{cd} = \dot{\partial}_d \underset{(20)}{H}{}^{a}{}_{bc} - \underset{(21)}{C}{}^{a}{}_{bd|2c} + \underset{(21)}{C}{}^{a}{}_{bf} \underset{(21)}{P}{}^{f}{}_{cd}
\end{cases}
$$

$$
\begin{cases}
\underset{(002)}{R}{}_{b}{}^{a}{}_{c}{}^{d} = \dot{\partial}^d \underset{(00)}{H}{}^{a}{}_{bc},\ \underset{(102)}{R}{}_{b}{}^{a}{}_{c}{}^{d} = \dot{\partial}^d \underset{(10)}{H}{}^{a}{}_{bc}, \\
\underset{(202)}{R}{}_{b}{}^{a}{}_{c}{}^{d} = \dot{\partial}^d \underset{(20)}{H}{}^{a}{}_{bc} + \underset{(21)}{C}{}^{a}{}_{bf} \underset{(21)}{B}{}^{f}{}_{c}{}^{d},
\end{cases}
$$

$$
\begin{cases}
\underset{(011)}{R}{}_{b}{}^{a}{}_{cd} = 0,\ \underset{(111)}{R}{}_{b}{}^{a}{}_{cd} = 0, \\
\underset{(211)}{R}{}_{b}{}^{a}{}_{cd} = \dot{\partial}_d \underset{(21)}{C}{}^{a}{}_{bc} - \dot{\partial}_c \underset{(21)}{C}{}^{a}{}_{bd} + \underset{(21)}{C}{}^{f}{}_{bc} \underset{(21)}{C}{}^{a}{}_{fd} - \underset{(21)}{C}{}^{f}{}_{bd} \underset{(21)}{C}{}^{a}{}_{fc},
\end{cases}
$$

$$
\underset{(012)}{R}{}_{b}{}^{a}{}_{c}{}^{d} = 0,\ \underset{(112)}{R}{}_{b}{}^{a}{}_{c}{}^{d} = 0,\ \underset{(212)}{R}{}_{b}{}^{a}{}_{c}{}^{d} = \dot{\partial}^d \underset{(21)}{C}{}^{a}{}_{bc},
$$

$$
\underset{(022)}{R}{}_{b}{}^{acd} = 0,\ \underset{(122)}{R}{}_{b}{}^{acd} = 0,\ \underset{(222)}{R}{}_{b}{}^{acd} = 0.
$$

Definition 4.5 *We shall say that the metric* $\mathbb{G}$ *given by (4.1) is* $\mathbf{w_2}$**−locally dependent of moment** *if for every point* $u = (x, y, p) \in T^{*2}M$ *there exist a local chart arround it on* $T^{*2}M$ *such that on its domain* $\underset{(2)}{g}{}^{ab}$ *depends on* p_a, only

It is not difficult to prove

Theorem 4.5 *If the manifold $T^{*2}M$ is endowed with the $(h, v_1, w_2)-$metric structure $\mathbb{G}$, $h-$Riemannian, v_1-locally Minkowski and w_2-locally depending of moment, given by*

$$\mathbb{G} = g_{ab}(x)dx^a \otimes dx^b + m_{ab}(y)\delta y^a \otimes \delta y^b + h^{ab}(p)\delta p_a \otimes \delta p_b, \tag{4.13}$$

then the metrical canonical $N-$linear connection $D\overset{c}{\Gamma}(N)$ has the coefficients

$$\begin{cases} \underset{(00)}{\overset{c}{H}}{}^a{}_{bc} = \left\{{}^{\,a}_{bc}\right\}, \\ \underset{(10)}{\overset{c}{H}}{}^a{}_{bc} = \underset{(11)}{B}{}^a{}_{cb} - \dfrac{1}{2}m^{ad}\left(N^f_c\dot{\partial}_f m_{bd} + \underset{(11)}{B}{}^f{}_{cb}m_{fd} + \underset{(11)}{B}{}^f{}_{cd}\underset{(1)}{m}_{bf}\right), \\ \underset{(20)}{\overset{c}{H}}{}^a{}_{bc} = -\underset{(22)}{B}{}^a{}_{bc} + \dfrac{1}{2}h^{ad}\left(N_{cf}\dot{\partial}^f h_{bd} + \underset{(22)}{B}{}^f{}_{bc}h_{fd} + \underset{(22)}{B}{}^f{}_{dc}h_{bf}\right), \end{cases}$$

$$\begin{array}{l} \underset{(01)}{\overset{c}{C}}{}^a{}_{bc} = 0,\ \underset{(11)}{\overset{c}{C}}{}^a{}_{bc} = \dfrac{1}{2}m^{ad}\left(\dot{\partial}_c m_{bd} + \dot{\partial}_b m_{dc} - \dot{\partial}_d m_{bc}\right),\ \underset{(21)}{\overset{c}{C}}{}^a{}_{bc} = 0, \\ \underset{(02)}{\overset{c}{C}}{}_a{}^{bc} = 0,\ \underset{(12)}{\overset{c}{C}}{}_a{}^{bc} = 0,\ \underset{(22)}{\overset{c}{C}}{}_a{}^{bc} = -\dfrac{1}{2}h_{ad}\left(\dot{\partial}^c h^{bd} + \dot{\partial}^b h^{dc} - \dot{\partial}^d h^{bc}\right). \end{array}$$

One observe that

$$\begin{array}{l} \underset{(00)}{\overset{0}{T}}{}^a{}_{bc} = 0,\ \underset{(01)}{\overset{0}{P}}{}^a{}_{bc} = 0,\ \underset{(02)}{\overset{0}{P}}{}^a{}_b{}^c = 0 \\ \underset{(11)}{S}{}^a{}_{bc} = 0,\ \underset{(12)}{\overset{1}{Q}}{}^a{}_b{}^c = 0,\ \underset{(12)}{\overset{2}{Q}}{}_{ab}{}^c = 0,\ \underset{(22)}{\overset{2}{Q}}{}_{ab}{}^c = 0,\ etc. \end{array}$$

To the end of this section we prove

Theorem 4.6 *If the manifold $T^{*2}M$ is endowed with the $(h, v_1, w_2)-$metric structure $\mathbb{G}$, $h-, v_1-$Riemannian and w_2-locally depending of moment, given by*

$$\mathbb{G} = g_{ab}(x)dx^a \otimes dx^b + g_{ab}(x)\delta y^a \otimes \delta y^b + h^{ab}(p)\delta p_a \otimes \delta p_b, \tag{4.14}$$

then

i) The metrical canonical $N-$ linear connection $D\overset{c}{\Gamma}(N)$ *has the coefficients*

$$\begin{cases} \underset{(00)}{\overset{c}{H}}{}^{a}{}_{bc}=\left\{{}^{a}_{bc}\right\} \\ \underset{(10)}{\overset{c}{H}}{}^{a}{}_{bc}=\dfrac{1}{2}\left[\underset{(11)}{B}{}^{a}{}_{cb}+g^{ad}\left(\partial_c g_{bd}-\underset{(11)}{B}{}^{f}{}_{cb}g_{bf}\right)\right], \\ \underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc}=\dfrac{1}{2}\left[-\underset{(22)}{B}{}^{a}{}_{bc}+h^{ad}\left(N_{cf}\dot{\partial}^{f}h_{bd}+\underset{(22)}{B}{}^{f}{}_{dc}h_{bf}\right)\right], \end{cases}$$

$$\begin{cases} \underset{(01)}{\overset{c}{C}}{}^{a}{}_{bc}=0,\ \underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc}=0,\ \underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc}=0 \\ \underset{(02)}{\overset{c}{C}}{}_{a}{}^{bc}=0,\ \underset{(12)}{C}{}_{a}{}^{bc}=0,\ \underset{(22)}{C}{}_{a}{}^{bc}=-\dfrac{1}{2}h_{ad}\left(\dot{\partial}^{c}h^{bd}+\dot{\partial}^{b}h^{dc}-\dot{\partial}^{d}h^{bc}\right). \end{cases}$$

ii)

$$\begin{cases} \overset{0}{T}{}^{a}{}_{bc}=0,\ \underset{(01)}{\overset{0}{P}}{}^{a}{}_{bc}=0,\ \underset{(02)}{\overset{0}{P}}{}^{a}{}_{b}{}^{c}=0, \\ \underset{(11)}{S}{}^{a}{}_{bc}=0,\ \underset{(12)}{\overset{1}{Q}}{}^{a}{}_{b}{}^{c}=0,\ \underset{(12)}{\overset{2}{Q}}{}_{ab}{}^{c}=0,\ \underset{(22)}{S}{}_{a}{}^{bc}=0 \end{cases}$$

iii) The $d-$ tensors of curvature are as follows

$$\begin{cases} \underset{(000)}{R}{}_{b}{}^{a}{}_{cd}=r_{b}{}^{a}{}_{cd}, \\ \underset{(100)}{R}{}_{b}{}^{a}{}_{cd}=\delta_d\underset{(10)}{\overset{c}{H}}{}^{a}{}_{bc}-\delta_c\underset{(10)}{\overset{c}{H}}{}^{a}{}_{bd}+\underset{(10)}{\overset{c}{H}}{}^{f}{}_{bc}\underset{(10)}{\overset{c}{H}}{}^{a}{}_{fd}-\underset{(10)}{\overset{c}{H}}{}^{f}{}_{bd}\underset{(10)}{\overset{c}{H}}{}^{a}{}_{fc}, \\ \underset{(200)}{R}{}_{b}{}^{a}{}_{cd}=\delta_d\underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc}-\delta_c\underset{(20)}{\overset{c}{H}}{}^{a}{}_{bd}+\underset{(20)}{\overset{c}{H}}{}^{f}{}_{bc}\underset{(20)}{\overset{c}{H}}{}^{a}{}_{fd}-\underset{(20)}{\overset{c}{H}}{}^{f}{}_{bd}\underset{(20)}{\overset{c}{H}}{}^{a}{}_{fc}+ \\ \qquad\qquad\qquad +\underset{(22)}{\overset{c}{C}}{}_{b}{}^{af}\underset{(02)}{R}{}_{fcd}, \end{cases}$$

$$\begin{cases} \underset{(001)}{R}{}_{b}{}^{a}{}_{cd}=0,\ \underset{(101)}{R}{}_{b}{}^{a}{}_{cd}=\dot{\partial}_d\underset{(10)}{\overset{c}{H}}{}^{a}{}_{bc}, \\ \qquad\underset{(201)}{R}{}_{b}{}^{a}{}_{cd}=\dot{\partial}_d\underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc}+\underset{(22)}{\overset{c}{C}}{}_{b}{}^{af}\underset{(12)}{R}{}_{fcd}, \end{cases}$$

$$\begin{cases} \underset{(002)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=0,\ \underset{(102)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=\dot{\partial}^{d}\underset{(10)}{H}{}^{a}{}_{bc}, \\ \underset{(202)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=\dot{\partial}^{d}\underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc}-\underset{(22)}{\overset{c}{C}}{}_{b}{}^{ad}{}_{|c}+\underset{(22)}{\overset{c}{C}}{}_{b}{}^{af}\underset{(22)}{P}{}_{fc}{}^{d}, \end{cases}$$

$$\underset{(011)}{R}{}_{b}{}^{a}{}_{cd}=0,\ \underset{(111)}{R}{}_{b}{}^{a}{}_{cd}=0,\ \underset{(211)}{R}{}_{b}{}^{a}{}_{cd}=0,$$

$$\underset{(012)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=0,\ \underset{(112)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=0,\ \underset{(212)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=0,$$

$$\underset{(022)}{R}{}_{b}{}^{acd}=0,\ \underset{(122)}{R}{}_{b}{}^{acd}=0,$$

$$\underset{(222)}{R}{}_{b}{}^{acd}=\dot{\partial}^{d}\underset{(22)}{\overset{c}{C}}{}_{b}{}^{ac}-\dot{\partial}^{c}\underset{(22)}{\overset{c}{C}}{}_{b}{}^{ad}+\underset{(22)}{\overset{c}{C}}{}_{b}{}^{fc}\underset{(22)}{\overset{c}{C}}{}_{f}{}^{ad}-\underset{(22)}{\overset{c}{C}}{}_{b}{}^{fd}\underset{(22)}{\overset{c}{C}}{}_{f}{}^{ac}$$

Proof. Because g_{ab} depends on x only (resp. h^{ab} depends on p only) follows $\delta_a T^{\cdots}_{\cdots}(x) = \partial_a T^{\cdots}_{\cdots}(x)$, $\dot{\partial}^a T^{\cdots}_{\cdots}(x) = 0$, (resp. $\delta_a T^{\cdots}_{\cdots}(p) = N_{af}\dot{\partial}^f T^{\cdots}_{\cdots}(p)$, $\dot{\partial}_a T^{\cdots}_{\cdots}(p) = 0$) and for (2.4) we have $\underset{(00)}{\overset{c}{H}}{}^a{}_{bc} = \left\{{a \atop bc}\right\}$, $\underset{(11)}{\overset{c}{C}}{}^a{}_{bc} = 0$ (resp. $\underset{(22)}{\overset{c}{C}}{}^a{}_{bc}$ given by the indicated expression in i) Theorem 4.6). Then a look about the formulae (4.4) by Theorem 4.1 and (6.6) by Theorem 6.3, §5.6, determine the other relations.

q.e.d.

Remark 4.1 If we consider the almost contact structure $\mathbb{F}$ introduced by (5.5), §4.5,

$$\mathbb{F}(\delta_a) = -\dot{\partial}_a,\ \mathbb{F}\left(\dot{\partial}_a\right) = \delta_a,\ \mathbb{F}\left(\dot{\partial}^a\right) = 0, \tag{4.15}$$

and take into account the Theorem 6.2, §4.6 result that the pairs $(\mathbb{G}, \mathbb{F})$ with $\mathbb{G}$ given by (4.12), respectively (4.14), are Riemannian almost contact structure on $T^{*2}M$:

$$\mathbb{G}(\mathbb{F}X, Y) = -\mathbb{G}(X, \mathbb{F}Y) \tag{4.16}$$

which constitute an **model** of the cotangent bundle of second order $\left(T^{*2}M, \pi^{*2}, M\right)$, easy to used.

Remark 4.2 If we consider the almost contact structure $\overset{*}{\mathbb{F}}$ introduced by R. Miron, [86], [97]:

$$\overset{*}{\mathbb{F}}(\delta_a) = -g_{ab}\dot{\partial}^b,\ \overset{*}{\mathbb{F}}\left(\dot{\partial}_a\right) = 0,\ \overset{*}{\mathbb{F}}\left(\dot{\partial}^a\right) = g^{ab}\delta_b \tag{4.17}$$

result that the pair $\left(\mathbb{G}, \overset{*}{\mathbb{F}}\right)$ with $\mathbb{G}$ given by (4.3), respectively (4.7), are Riemannian almost contact structure on $T^{*2}M$:

$$\mathbb{G}\left(\overset{*}{\mathbb{F}}X, Y\right) = -\mathbb{G}\left(X, \overset{*}{\mathbb{F}}Y\right) \tag{4.18}$$

which constitute each an **model** of 2−cotangent bundle $\left(T^{*2}M, \pi^{*2}, M\right)$. The first $\left(\mathbb{G}, \overset{*}{\mathbb{F}}\right)$ with $\mathbb{G}$ given by (4.3) is an model in geometry of Hamilton space of second order $H^{(2)n}$, [97] and even more in the generalized Hamilton geometry $GH^{(2)n}$. The second model $\left(\mathbb{G}, \overset{*}{\mathbb{F}}\right)$ with $\mathbb{G}$ given by (4.7) is possible to offer new informations about the generalized Lagrange spaces of first order $GL^{(1)n}$ using $T^{*2}M$.

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The Theory of Linear Connections in the Differential Geometry of Accelerations

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