A MATHEMATICAL DESCRIPTION OF THE FERMIONIC STATE

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The fermionic state is the foundation for the whole of physics. Physics is entirely concerned with fermions and their interactions, and nothing else. It is possible to derive a mathematical expression for the fermionic state, which is an operator only, not an equation, or wavefunction. This operator appears to contain within it all the information needed to construct fermion interactions and particle states. Extensions to particle representations using Finsler geometry could find this formalism a particularly accessible link.

1 The nilpotent version of quantum mechanics

The nilpotent version of quantum mechanics is the most streamlined version available.[1] It is fully relativistic. It uses only (differential) operators. It doesn't need mysterious objects like wavefunctions and spinors. All terms have the same format, based on the operators $E, \mathbf{p}, m$. These terms are compartmentalised using the quaternions $\boldsymbol{k},\ \boldsymbol{i},\ \boldsymbol{j}$, in a similar way to real and imaginary parts. The operators are full quantum field operators – there is no need to apply second quantization. They are also intrinsically supersymmetric. Interactions and particle states are immediately explained, while calculations are relatively easy, easier than nonrelativistic ones. Renormalization of free particles is not needed, while significant divergences simply disappear.

Does this representation relate to Finsler geometry? This is a question to be decided, but, if it does, then the streamlined versions of quantum mechanics and particle physics which become possible through this representation will find it the most accessible link. We only need one operator, and this can find connections with many other formalisms. The fermionic representation is, of course, quadratic, as is conventionally the case, and as derives from special relativity. If a quartic generalisation of special relativity is possible, and if it is found useful, then the same transition can be made with the fermionic representation.

The nilpotent formalism can be derived from the concept of zero using a universal rewrite system, which seems to have much more general applications. This derivation automatically includes quantization and special relativity as part of the abstract formal structure – it doesn't need to assume them. The algebraic structure can be shown to be derived from the algebras of the four fundamental parameters, space, time, mass and charge, and their mathematically symmetric relationships.

While the derivation of the nilpotent formalism from a universal rewrite system is the only truly fundamental one,[2] there are less profound ones available from more conventional theories, e.g. by converting the gamma matrices of the conventional Dirac equation into algebraic operators (based on multivariate 4-vector quaternions or complex double quaternions).[1] The most direct is from the classical relativistic equation:

$$E^2 = p^2 + m^2$$

but it should be remembered that, fundamentally, it is the classical equation that is derived, not the quantum mechanics.

So, let us take the equation in the form:

$$E^2 - p^2 - m^2 = 0$$

and factorize using noncommuting algebraic operators (multivariate 4-vector quaternions or complex double quaternions):

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = 0.$$

To make this a quantum equation, we simply make E and $\mathbf{p}$ into canonical quantum operators, say $i\partial/\partial t$ and $-i\nabla$, with $\hbar = 1$, rather than numerical variables. So, we can, for example, make the first bracket an operator, operating on a phase term of some kind, say $e^{-i(Et-\mathbf{p}.\mathbf{r})}$ for a free particle plane wave, and the second bracket the amplitude which results from this operation.

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = (\pm \boldsymbol{k}\partial/\partial t \pm i\boldsymbol{i}\nabla + \boldsymbol{j}m)(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)e^{-i(Et-\mathbf{p}.\mathbf{r})} = 0.$$

This now becomes equivalent to the Dirac equation for a free particle.

The algebra we need is a tensor product of quaternions and multivariate vectors (i. e. complexified quaternions), or a complexified tensor product between two quaternion algebras. The two algebras are entirely commutative towards each other. Each acts as though the other did not exist. This is intriguingly close to twistor algebra (a complex 4-D space-time), now used in QCD.

Quaternions	**Multivariate vector**
$\boldsymbol{i}\,\boldsymbol{j}\,\boldsymbol{k}$ quaternion units	$\mathbf{i}\,\mathbf{j}\,\mathbf{k}$ vector units
1 scalar	i pseudoscalar

The multiplication rules of the units are as follows:

$$\begin{array}{cc}
\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = 1 & (i\boldsymbol{i})^2 = (i\boldsymbol{j})^2 = (i\boldsymbol{k})^2 = i(i\boldsymbol{i})(i\boldsymbol{j})(i\boldsymbol{k}) = 1 \\
\boldsymbol{ij} = \boldsymbol{ji} = \boldsymbol{k} & (i\boldsymbol{i})(i\boldsymbol{j}) = (i\boldsymbol{j})(i\boldsymbol{i}) = i(i\boldsymbol{k}) \\
\boldsymbol{jk} = \boldsymbol{kj} = \boldsymbol{i} & (i\boldsymbol{j})(i\boldsymbol{k}) = (i\boldsymbol{k})(i\boldsymbol{j}) = i(i\boldsymbol{i}) \\
\boldsymbol{ki} = \boldsymbol{ik} = \boldsymbol{j} & (i\boldsymbol{k})(i\boldsymbol{i}) = (i\boldsymbol{i})(i\boldsymbol{k}) = i(i\boldsymbol{j})
\end{array}$$

and there are 64 possible products of the units:

$(\pm 1, \pm i)$	4 units
$(\pm 1, \pm i) \times (\mathbf{i}, \mathbf{j}, \mathbf{k})$	12 units
$(\pm 1, \pm i) \times (\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$	12 units
$(\pm 1, \pm i) \times (\mathbf{i}, \mathbf{j}, \mathbf{k}) \times (\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$	36 units

Now, if we take

$$(\pm \boldsymbol{k}\partial/\partial t \pm i\boldsymbol{i}\nabla + \boldsymbol{j}m)(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)e^{-i(Et-\mathbf{p}.\mathbf{r})} = 0$$

as the Dirac equation for a free particle, we can interpret the four possible sign variations of $\partial/\partial t$ and ∇ in the first bracket as making up the four components of a row vector; and the four possible sign variations of E and $\mathbf{p}$ in the second bracket as making up the four components of a column vector.

In principle, we have made both the operator and the amplitude into essentially identical 4-component spinors, with the operator applied to a single phase term.

And we can easily identify the meaning of the sign variations:

$$\begin{array}{rl} \text{fermion / antifermion} & \pm E \\ \text{spin up / down} & \pm \mathbf{p} \end{array}$$

These options apply to E and $\mathbf{p}$ as either operators or amplitude eigenvalues.

Relativistic quantum mechanics is now hugely streamlined, because it now depends only on a single term:

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

taken either as operator or as amplitude. Here, E and $\mathbf{p}$ are generic terms, identified by their quaternion labels, which can be covariant derivatives or include field terms or potentials of any kind. The expression

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

which is really a row or column vector, containing the four components

$$\begin{gathered} (\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) \\ (\boldsymbol{k}E - i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) \\ (-\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) \\ (-\boldsymbol{k}E - i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m) \end{gathered}$$

now contains all that can be known about any fermion state.

If we take E and $\mathbf{p}$ as generic operators, then the only way they can operate is by finding a phase factor, such that the resulting amplitude is nilpotent, or squares to zero, i.e.:

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = 0.$$

So, specifying the operator means that we also specify the phase and the amplitude, and the 'wavefunction' becomes redundant.

But the spinor structure is also redundant. We do not need to specify

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

as a row or column vector. Once we specify the first of the four terms, the others follow by automatic sign variation. So, we only need

$$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

for complete specification of the state.

To take a simple example, specifying a state as $(\pm\, \boldsymbol{k}\partial/\partial t \pm i\boldsymbol{i}\nabla + \boldsymbol{j}m)$ (which is a free particle) means that we have automatically created the four linked equations:

$$\begin{aligned} (\pm\, \boldsymbol{k}\partial/\partial t \pm i\boldsymbol{i}\nabla + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)e^{-i(Et-\mathbf{p}.\mathbf{r})} &= 0 \\ (\boldsymbol{k}\partial/\partial t - i\boldsymbol{i}\nabla + \boldsymbol{j}m)(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m))e^{-i(Et-\mathbf{p}.\mathbf{r})} &= 0 \\ (-\boldsymbol{k}\partial/\partial t + i\boldsymbol{i}\nabla + \boldsymbol{j}m)(-i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)e^{-i(Et-\mathbf{p}.\mathbf{r})} &= 0 \\ (-\boldsymbol{k}\partial/\partial t - i\boldsymbol{i}\nabla + \boldsymbol{j}m)(-i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)e^{-i(Et-\mathbf{p}.\mathbf{r})} &= 0 \end{aligned}$$

By comparison with the conventional Dirac equation, we have reduced the number of separate terms required by 98%, because we have reduced a 4×4 matrix operator multiplied by a 4-component spinor wavefunction to an operator with only a single independent term. We have effectively shown that the matrix, the wavefunction, the spinor and the equation itself, are redundant. We need only a single operator.

2 The nilpotent operator and the fundamental physical state

We may ask: what is the physical meaning of defining the fermion as an operator? What is it operating on? The indications are that it is *vacuum,* meaning the rest of the universe. The nilpotency codified in

$$(\pm\, i\boldsymbol{k}E \pm\, \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\pm\, i\boldsymbol{k}E \pm\, \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = 0$$

has built-in Pauli exclusion. No two fermions can have the same state vector. But it also signifies that nilpotency is an expression of the totality zero that is fundamental to the universal rewrite system. Fermion and the rest of the universe (0 – fermion) together make a zero totality, with a zero totality state vector:

$$(\pm\, i\boldsymbol{k}E \pm\, \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(-(\pm\, i\boldsymbol{k}E \pm\, \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)) = 0.$$

The fact that Pauli exclusion is not unique to free fermions then brings us to the most revolutionary step within the nilpotent theory. We assume that *all fermionic amplitudes in all states* are nilpotent. We postulate that the most general form for a state vector is nilpotent, and that we should seek specifically nilpotent solutions for all problems.

Now, for a ‘free’ fermion, the phase factor $(exp(-i(Et - \mathbf{p.r}))$ provides the complete range of space and time translations and rotations, but if the E and $\mathbf{p}$ terms represent covariant derivatives or incorporate field terms, then the phase term is determined by whatever expression is needed to make the amplitude nilpotent.

The operator which defines each fermion is thus a creation operator acting on vacuum (= the rest of the universe). In fact there are four creation operators:

$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion spin up
$(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion spin down
$(-i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	antifermion spin down
$(-i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	antifermion spin up

The nature of the state is determined by which of these is the lead term. The others can be regarded as vacuum states representing ones into which it could transform. So, for example, a real antifermion spin down would be symbolized by a row vector with the following components:

$(-i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	antifermion spin down
$(-i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	antifermion spin up
$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion spin up
$(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion spin down

Because of the way they are defined, nilpotent operators are specified with respect to the entire quantum field. Formal second quantization is unnecessary. We can consider the nilpotency as defining the interaction between the localized fermionic state and the unlocalized vacuum, with which it is uniquely self-dual. The phase is the mechanism through which this is accomplished. So, defining a fermion implies simultaneous definition of vacuum as ‘the rest of the universe’ with which it interacts. The nilpotent structure then provides energy-momentum conservation without requiring the system to be closed. The nilpotent structure is thus naturally thermodynamic, and provides a mathematical route to defining nonequilibrium thermodynamics.

We can now see that the expression

$$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \to 0$$

has at least five independent meanings.

classical	special relativity
operator × operator	Klein-Gordon equation
operator × wavefunction	Dirac equation
wavefunction × wavefunction	Pauli exclusion
fermion × vacuum	thermodynamics

We thus have an operator $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ that potentially incorporates all the physical information available to the fundamental physical state. It is easy to show that we can use our operator to do conventional quantum mechanics, e.g. by defining a probability density by multiplying by its complex quaternion conjugate $(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} - \boldsymbol{j}m)$. We may also note that nilpotent wavefunctions or amplitudes are automatically antisymmetric:

$$(\pm \boldsymbol{ik}E_1 \pm \boldsymbol{i}\mathbf{p}_1 + \boldsymbol{j}m_1)(\pm \boldsymbol{ik}E_2 \pm \boldsymbol{i}\mathbf{p}_2 + \boldsymbol{j}m_2) - (\pm \boldsymbol{ik}E_2 \pm \boldsymbol{i}\mathbf{p}_2 + \boldsymbol{j}m_2)(\pm \boldsymbol{ik}E_1 \pm \boldsymbol{i}\mathbf{p}_1 + \boldsymbol{j}m_1) = \\ = 4\mathbf{p}_1\mathbf{p}_2 - 4\mathbf{p}_2\mathbf{p}_1 = 8i\mathbf{p}_1 \times \mathbf{p}_2$$

This is a particularly striking result, as it implies that all fermionic states have a *spin phase* or 'direction' of $\mathbf{p}$ which is unique.

3 Vacuum and CPT symmetry

The three quaternion operators $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are not just passive mathematical objects in the nilpotent formalism. They have multiple meanings, acting almost as a kind of hypertext.

(1) The primary meaning is as charge generators.
(2) Premultiplying the nilpotent gives vacuum, e.g. $\boldsymbol{k}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ acts as a weak vacuum.
(3) Pre- and postmultiplying the nilpotent transforms via P, C or T operations: e.g. $\boldsymbol{k}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k}$ becomes a T transformation.

If we take $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ and postmultiply it by $\boldsymbol{k}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, the result is $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, multiplied by a scalar, which can be normalized away. This can be done an indefinite number of times. $\boldsymbol{k}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ is an idempotent, which behaves as a vacuum operator. So do $\boldsymbol{i}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ and $\boldsymbol{j}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$.

Previously, we said that the vacuum state vector had the same structure, apart from a scalar factor, as that of the fermion: $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$. How then do these three vacua relate? We can see the three vacuum coefficients $\boldsymbol{k}, \boldsymbol{i}, \boldsymbol{j}$ as originating in (or being responsible for) the concept of discrete (point-like) charge. The operators act as a discrete partitioning of the continuous vacuum responsible for zero-point energy, i.e. $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$. In this sense, they are related to weak, strong and electric localized charges, though they are delocalized.

The 3 vacua also help to explain the meaning of the 4 terms in the Dirac 4-spinor. There is 1 real state (the lead term) and 3 potential (vacuum) states into which the lead term can be transformed by one of the 3 interactions. All possible states are always present, either as real states or vacuum ones, e. g.:

$$\begin{array}{ll} & (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \to (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \\ \text{strong} & \boldsymbol{i}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \to (i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \\ \text{weak} & \boldsymbol{k}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \to (-i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \\ \text{electric} & \boldsymbol{j}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \to (-i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \end{array}$$

We can suggest specific identifications of the interactions on the basis of the pseudoscalar, vector and scalar characteristics of the associated terms.

$$\begin{array}{lll} \boldsymbol{k}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) & \text{weak vacuum} & \text{fermion creation} \\ \boldsymbol{i}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) & \text{strong vacuum} & \text{gluon plasma} \\ \boldsymbol{j}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) & \text{electric vacuum} & \text{SU(2)} \end{array}$$

The 3 additional terms in the Dirac spinor then become strong, weak and electric vacuum 'reflections' of the state defined by the lead term.

CPT symmetry uses the same operators. This is, of course, not a coincidence.

$$\begin{array}{ll} \text{P} & \boldsymbol{i}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{i} = (i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \\ \text{T} & \boldsymbol{k}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k} = (-i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \\ \text{C} & -\boldsymbol{j}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{j} = (-i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \\ \text{CPT} & -\boldsymbol{j}(\boldsymbol{i}(\boldsymbol{k}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k})\boldsymbol{i})\boldsymbol{j} = (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \end{array}$$

It is significant that CPT is defined to connect relativity with causality, but this can only true in a nilpotent theory, which gives the rest mass or proper time term the same algebraic status as the others.

4 Particle states and interactions: the scalar component

The nilpotent theory has many applications, for example, in QED, QFD (weak interaction theory), QCD, and the quantum theory of inertia (QID).[1] It can be shown in only a few lines of calculation that renormalization is not needed for a free particle and that there is therefore no hierarchy problem. In fact, the intrinsic supersymmetry of the theory (each fermion acting in vacuum as its own boson, etc.) suggests that it should be possible, in principle, to eliminate the renormalization process altogether. Also, propagators written in this formalism immediately eliminate the infrared divergence, and allow distinctions to be made between different bosonic propagators. We may additionally anticipate extensions of the theory to condensed matter physics and chemistry. However, at an even more fundamental level, we have the opportunity to resolve some of the problems involved with the symmetry breaking between the different interactions and the particle states with which they are involved.

Here, the major questions is: if the fermionic nilpotent is the most fundamental structure in physics – in effect, its fundamental unit, can it reproduce the fundamental particle states and their interactions? These two questions are not independent of each other. The first thing is to see if the structure of the nilpotent operator can give us any insight into the nature of fermionic interactions. In fact, this is precisely what it can do. But, first, assuming that the constraint of spherical symmetry exists for a point particle, then we can express the momentum term of the operator in polar coordinates, using the Dirac prescription, with an explicit spin term:

$$\boldsymbol{\sigma}.\boldsymbol{\nabla} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \pm i\frac{j + 1/2}{r}.$$

We need the spin term because the multivariate nature of the **p** term cannot be expressed in polar coordinates.

The nilpotent Dirac operator now becomes:

$$\left(\boldsymbol{k}E + \boldsymbol{i}\left(\frac{\partial}{\partial \boldsymbol{r}} + \frac{1}{r} \pm i\frac{j+1/2}{r}\right) + \boldsymbol{ij}m\right).$$

Now, whatever phase factor we apply this to, we will find that we will not get a nilpotent solution unless the $1/r$ term with coefficient $\boldsymbol{i}$ is matched by a similar $1/r$ term with coefficient $\boldsymbol{k}$. So, simply requiring *spherical symmetry* for a point particle, requires a term of the form A/r to be added to E.

If all point particles are spherically symmetric sources, then the minimum nilpotent operator is of the form

$$\left(\boldsymbol{k}\left(E - \frac{A}{r}\right) + \boldsymbol{i}\left(\frac{\partial}{\partial \boldsymbol{r}} + \frac{1}{r} \pm i\frac{j+1/2}{r}\right) + \boldsymbol{ij}m\right).$$

To establish that this is a nilpotent, we must now find the phase to which this must apply to create a nilpotent amplitude. This will quite quickly produce the characteristic solution for the pure Coulomb force (the so-called hydrogen-like atom solution). The solution is straightforward. We try the phase term

$$F = e^{-ar} r^{\gamma} \sum_{\nu=0} a_{\nu} r^{\nu},$$

to find the amplitude, and equate the squared amplitude to zero. Here, we obtain:

$$4\left(E - \frac{A}{r}\right)^2 = -2\left(-a + \frac{\gamma}{r} + \frac{\nu}{r} + ...\frac{1}{r} + i\frac{j+1/2}{r}\right)^2 - 2\left(-a + \frac{\gamma}{r} + \frac{\nu}{r} + ...\frac{1}{r} - i\frac{j+1/2}{r}\right)^2 + 4m^2.$$

Equating constant terms leads to

$$E^2 = -a^2 + m^2,$$

$$a = \sqrt{m^2 - E^2}.$$

Equating terms in $1/r^2$, with $\nu = 0$, we obtain:

$$\left(\frac{A}{r}\right)^2 = -\left(\frac{\gamma+1}{r}\right)^2 + \left(\frac{j+1/2}{r}\right)^2,$$

from which, excluding the negative root (as usual),

$$\gamma = -1 + \sqrt{(j+1/2)^2 - A^2}.$$

Assuming the power series terminates at n', and equating coefficients of $1/r$ for $\nu = n'$,

$$2EA = -2\sqrt{m^2 - E^2}\,(\gamma + 1 + n'),$$

the terms in $(j + \frac{1}{2})$ cancelling over the summation of the four multiplications, with two positive and two negative. From this we may derive

$$\frac{E}{m} = \frac{1}{\sqrt{1 + \frac{A^2}{(\gamma+1+n')^2}}} = \frac{1}{\sqrt{1 + \frac{A^2}{\sqrt{(j+1/2)^2 - A^2} + n'}^{\,2}}}.$$

With $A = Ze^2$, we obtain the hyperfine or fine structure formula for a one-electron nuclear atom or ion:

$$\frac{E}{m} = \frac{1}{\sqrt{1 + \frac{(Ze^2)^2}{(\gamma+1+n')^2}}} = \frac{1}{\sqrt{1 + \frac{(Ze^2)^2}{\sqrt{(j+1/2)^2 - (Ze^2)^2} + n'}^{\,2}}}.$$

We have, of course, without mentioning anything about potentials or interactions, or anything physical at all, and only using the structure of the nilpotent operator, needed to maintain the spherical symmetry of a point-particle source, created the solution for the Coulomb or inverse linear potential. And we have shown that it is absolutely necessary to any fermionic state described as a point source, regardless of what other potentials may be present. We can now proceed to show that another fundamental potential can be derived from the structure of the nilpotent operator alone.

The vector and pseudoscalar components

The vector part of the nilpotent has three components. So a significant question might be: can we have a 3-component state vector?

$$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = 0$$

Clearly, nilpotency makes three identical states impossible. But the following would be possible:

$$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{j}m) \to (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$
$$(i\boldsymbol{k}E + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{j}m) \to (i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$
$$(i\boldsymbol{k}E + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \to (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$$

So we could have a nonzero state vector if we use the vector properties of $\mathbf{p}$ and the arbitrary nature of its sign (+ or −). A state vector of the form

$$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}\mathbf{p}_x + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{j}\mathbf{p}_y + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{k}\mathbf{p}_z + \boldsymbol{j}m),$$

privileging the $\mathbf{p}$ components, has six independent allowed phases, i. e. when

$$\mathbf{p} = \pm\boldsymbol{i}p_x, \quad \mathbf{p} = \pm\boldsymbol{j}p_y, \quad \mathbf{p} = \pm\boldsymbol{k}p_z,$$

but these must be *gauge invariant,* i.e. indistinguishable, or all present at once. Also, we must interpret the $E, \mathbf{p}, m$ symbols as belonging to a totally entangled state, rather than the subcomponents.

In principle, we would be using the concept of spatial (rather than temporal) separation to represent the arbitrary nature of the direction of fermionic spin. One method of picturing the arbitrary nature of the phases (gauge invariance) is to imagine an automatic mechanism of transfer between them.

$$\begin{array}{ll}
(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\boldsymbol{k}p_z + \boldsymbol{j}m) & +RGB \\
(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m)(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{j}m)(i\boldsymbol{k}E - \boldsymbol{i}\boldsymbol{k}p_z + \boldsymbol{j}m) & -RBG \\
(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\boldsymbol{k}p_z + \boldsymbol{j}m) & +BRG \\
(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m)(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{j}m)(i\boldsymbol{k}E - \boldsymbol{i}\boldsymbol{k}p_z + \boldsymbol{j}m) & -GRB \\
(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\boldsymbol{k}p_z + \boldsymbol{j}m) & +GBR \\
(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{j}m)(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{j}m)(i\boldsymbol{k}E - \boldsymbol{i}\boldsymbol{k}p_z + \boldsymbol{j}m) & -BGR
\end{array}$$

This has exactly the same group structure as the standard 'coloured' baryon wavefunction made of R, G and B 'quarks',

$$\psi \sim (RGB - RBG + BRG - GRB + GBR - BGR)$$

That is, it has an SU(3) structure, with 8 generators. And, since the E and $\mathbf{p}$ terms in the state vector really represent time and space derivatives, we can replace these with the covariant derivatives needed for invariance under a local SU(3) gauge transformation. This SU(3) symmetry or strong interaction is entirely nonlocal. That is, the exchange of momentum $\mathbf{p}$ involved is entirely independent of any spatial position of the 3 components of the baryon. We can suppose that the rate of change of momentum (or 'force') is constant with respect to spatial positioning or separation. A constant force is equivalent to a potential which is linear with distance, exactly as is required for the conventional strong interaction.

We can now identify our structures as those that would be required of a baryon in the nilpotent formalism. If we now construct a nilpotent operator, in which spherical symmetry still applies, we will find that the requirement for a term of the form A/r, added to E, remains unchanged; but that we now require another term of the form Br. If we now try to solve for phase and amplitude, we will find that our solution has the characteristics of infrared slavery and asymptotic freedom that we apply to quarks and the strong interaction. We note that the full symmetry between the 3 momentum components can only apply if the momentum operators can be equally positive or negative. With all phases of the interaction present at the same time (perfect gauge invariance), this is equivalent to saying that left-handedness and right-handedness must be present simultaneously in the baryon state (and can be transformed by the parity operator in the term $\boldsymbol{i}\mathbf{p}$). In other words, the baryonic state must have non-zero mass via the Higgs mechanism.

The other significant component of the nilpotent is the pseudoscalar term ($i\boldsymbol{k}E$). The particular significances of this term are:

(1) its necessity to nilpotency
(2) the necessity of removing it by a 'squaring' operation, or multiplication by a complex conjugate
(3) the dipolarity it creates between fermion and vacuum, etc. Ultimately, this leads to the necessity for a term of the form Cr^n, where n may be, say -3, to be added to the E term.

All the fundamental interactions are aspects of the nilpotent structure, in fact, of the $\mathbf{p}$ term. The electric interaction (Coulomb) is spherical symmetry of $\mathbf{p}$ (in polar coordinates). The strong is due to the vector aspect of $\mathbf{p}$. The weak (harmonic oscillator) is due to sign, more particularly, its sign relative to iE.

$$\begin{aligned}
&(\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\\
&(\boldsymbol{k}E - i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\\
&(-\boldsymbol{k}E + i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)\\
&(-\boldsymbol{k}E - i\boldsymbol{i}\mathbf{p} + i\boldsymbol{j}m)
\end{aligned}$$

We have to switch between the 4 components, changing relative signs of E and $\mathbf{p}$, as we do with the classical harmonic oscillator, with force $F = -kx$ and energy proportional to x^2. In fact, any r^n dependence of the potential other than $n = 1$ or $n = -1$ (in addition to the Coulomb term), or any combination of such terms, solves in the nilpotent to a (quantum) harmonic oscillator. For the weak interaction, with its intrinsically dipolar nature, the simplest potential would appear to require a dependence of the form r^{-3}.

There are thus only three point-particle (spherically symmetric) fermionic $\boldsymbol{ik}E$ operators which give us the desired nilpotent solution, and these have the characteristics of the three interactions:

A/r	Coulomb	electric interaction
$A/r + Br$	confinement	strong interaction
$A/r + Cr^n$	harmonic oscillator	weak interaction

We can identify the A/r term with the scalar part of the three operators $\boldsymbol{j}m, \boldsymbol{i}\mathbf{p}, \boldsymbol{ik}E$ (the coupling constant; Br with the vector part of $\mathbf{p}$; and Cr^n with the pseudoscalar part of iE.

5 Interaction vertices

The pseudoscalar part brings us to the consideration of interaction vertices. Because the state vector always represents four terms with the complete variation of signs in E and $\mathbf{p}$, an interaction vertex between any fermion / antifermion and any other

$$(\boldsymbol{ik}E_1 + \boldsymbol{i}\mathbf{p}_1 + \boldsymbol{j}m_1)(\boldsymbol{ik}E_2 + \boldsymbol{i}\mathbf{p}_2 + \boldsymbol{j}m_2)$$

will remove the quaternionic operators, leaving only scalars and vectors. When the $E, \mathbf{p}$ and m values become numerically equal, the vertex can be defined as a new *combined* bosonic state, with a single phase. Where there is an interaction vertex between two fermionic / antifermionic states, the signs of E and $\mathbf{p}$ of the second term, with respect to the first, will also determine the nature of the bosonic or combined state which may be created. Because there are three operators involved – $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ – there are also three possible bosonic states. Any transformation of a fermionic state can be represented as a bosonic state in which the old state is annihilated and the new one created.

spin 1 boson: $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(-\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ T

spin 0 boson: $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(-\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ C

spin 0 Bose-Einstein condensate / Berry phase, etc.:

$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ P

The fermion-fermion state $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ has many physical manifestations: Aharonov-Bohm effect; Jahn-Teller effect; quantum Hall effect; Cooper pairs; even-even nuclei. Even spin 1 He^3 can be accommodated because it has two physically separated components moving independently with opposite directions of momentum (in a harmonic oscillator configuration) and so the two $\boldsymbol{\sigma}.\boldsymbol{p}$ terms can have the same signs, while the two $\mathbf{p}$ terms have opposite.

It now becomes evident that an interaction in which a fermionic state with one set of signs of E and $\mathbf{p}$ is annihilated and replaced by a state with a different sign configuration of E and $\mathbf{p}$ requires an interaction vertex which is equivalent to an intermediate bosonic state, and that such an interaction is required to be a fundamental component of a nilpotent operator which is structured as a 4-component spinor with inherent *zitterbewegung*. In principle, this means that a *weak interaction* of this kind is a consequence of the fundamental structure of the fermionic state – this time of the 4-spinor aspect – in the same way as the pure Coulomb (electric) and strong interactions are respective consequences of the nilpotent magnitude (squaring to zero) and vector aspects. That is, simply by defining an operator which is a nilpotent 4-component spinor with vector properties, we necessarily imply that it is subject to electric, weak and strong interactions.

Significantly, the spin 0 bosonic state cannot be massless, because, if it is nilpotent it automatically becomes zero.

$$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p})(-\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p}) = 0$$

This becomes a significant factor in the Higgs mechanism. It also implies that massless fermions cannot have the same handedness as massless antifermions. The conventional derivation of spin assigns left-handedness to fermions.

The mediators of the strong force will be made up of six bosons of the form:

$$(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_x)(-\boldsymbol{ik}E - \boldsymbol{i}\mathbf{j}p_y)$$

and two combinations of the three bosons of the form:

$$(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_x)(-\boldsymbol{ik}E - \boldsymbol{i}\mathbf{i}p_y)$$

These structures are, of course, identical to an equivalent set in which both brackets undergo a complete sign reversal. The important thing here is that applying any of these mediators will produce a sign change in the $\mathbf{p}$ component that leads to mass.

We can also see how the 3 bosonic states are related to vacua produced by the 3 charge operators:

$$\begin{gathered}
\text{weak} \qquad\qquad\qquad\qquad \text{spin 1} \\
(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m))\boldsymbol{k}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ldots \\
(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(-\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(-\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ldots \\
\text{electric} \qquad\qquad\qquad\qquad \text{spin 0} \\
(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{j}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{j}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{j}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ldots \\
(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(-\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(-\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ldots \\
\text{strong} \qquad\qquad\qquad\qquad \text{B-E condensate} \\
(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{i}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{i}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{i}(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ldots \\
(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ldots
\end{gathered}$$

As stated earlier, all these discrete vacuum states produce virtual boson states which have no effect on the fermion $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$. So, each fermion becomes its *own* supersymmetric bosonic partner, and vice versa. This removes the need for renormalization in the case of free particles, while 'renormalization' of interacting particles is reduced to rescaling – charge values being determined by their interactions with all the others in the universe, while the divergent terms are eliminated by cancellation of boson and fermion loops of opposite sign.

6 Discrete calculus and Finsler geometry

It is in the double nilpotent representation of bosonic states that we may see a possible link with approaches based on Finsler geometry.

It is particularly convenient here to use a version of Kauffman's discrete differential calculus.[3] Here, we define an operator

$$\mathfrak{D} = -\boldsymbol{k}\frac{\partial}{\partial t} + \boldsymbol{i}\boldsymbol{i}\mathbf{i}\frac{\partial}{\partial X_1} + \boldsymbol{i}\boldsymbol{i}\mathbf{j}\frac{\partial}{\partial X_2} + \boldsymbol{i}\boldsymbol{i}\mathbf{k}\frac{\partial}{\partial X_3},$$

where

$$\frac{\partial \psi}{\partial t} = [\psi, H] = [\psi, E] \qquad \text{and} \quad \frac{\partial \psi}{\partial X_i} = [\psi, P_i]\,,$$

and an amplitude

$$\psi = \boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}P_1 + \boldsymbol{i}\mathbf{j}P_2 + \boldsymbol{i}\mathbf{k}P_3 + \boldsymbol{j}m.$$

The actual signs of the differential terms are, of course, arbitrary, but are chosen here to conform with those required using conventional calclulus in the previous sections. Also, since we are not using a velocity operator, we can use $\partial/\partial X_i$, for d/dX_i. When is ψ nilpotent, then

$$\mathfrak{D}\psi = \left(-\boldsymbol{k}\frac{\partial}{\partial t} + i\boldsymbol{i}\nabla\right)\psi = 0,$$

or, allowing for the four sign variations (and reverting to the signs used in the versions based on conventional calculus):

$$\mathfrak{D}\psi = \left(-\boldsymbol{k}\frac{\partial}{\partial t} + i\boldsymbol{i}\nabla\right)\psi = 0. \tag{1}$$

This is then equivalent to the nilpotent Dirac equation in this calculus. Immediately noticeable is the fact that there is no mass term in the differential operator, only in the amplitude. This means that an operator is the exact negative of its charge-conjugated state. Annihilation and creation of a state, defined by an operator, cancel each other in exact mathematical terms. Also, we are able to define quantum differentials without the explicit use of an $i\hbar$ term, as defining $\mathfrak{D}$ in terms of the more conventional quantum operators

$$i\frac{\partial \psi}{\partial t} = [\psi, H] = [\psi, E] \qquad \text{and} \quad -i\frac{\partial \psi}{\partial X_i} = [\psi, P_i]$$

with $\hbar = 1$, will produce equation (4) in the same way. Thus, in this formalism, there is no arbitrary break between quantum and classical domains. The complexity comes solely from the nilpotent amplitude.

We can also extend the definition of $\mathfrak{D}$, following Kauffman, to include covariant terms, such as A_i, so that $\mathfrak{D}$ becomes $\mathfrak{D} - A_i$, and the covariant terms A_i can be seen as representing either a field source or an expression of the distortion of the Euclidean space-time structure, such as that produced by the presence of mass in general relativity, or, in more general terms, the quartic generalisation of the Riemannian structures used for an anisotropic metric in Finsler geometry. So, if we *choose* to use structures of this kind to replace the direct use of mass, then a massless covariant $\mathfrak{D}$ operator provides us with a convenient route to achieving this.

Now, if the Berwald-Moor metric of anisotropic Finsler geometry, $ds = (dx_1dx_2dx_3dx_4)^{1/4}$, is substituted for the isotropic Minkowski metric of Riemannian geometry, $ds = (dt^2 - dr^2)^{1/2}$, the zero interval manifold ($ds = 0$) becomes transformed from the familiar Minkowski light cone to a combination of two pyramids joined at the apex. By introducing exponents into the expression for the metric function, Bogoslovsky has found a geometric phase transition, which could be interpreted as a mass-creating spontaneous-symmetry breaking in a fermion-antifermion consendate.[1)] According to this process, the generalised Lorentz transformations responsible for the process lead directly to the Berwald-Moor metric. The connection with a double nilpotent representation of bosonic states using the massless covariant $\mathfrak{D}$ operator is immediately apparent. In the discrete version of the double nilpotent representation of the bosonic state (or 'fermion-antifernion

[1)] I am grateful to Sergey Siparov for his discussions on Bogoslovsky's ideas, which helped to clarify them for me.

condensate'), no mass term appears in the operator, but the differentials may be replaced by covariant derivatives, and so the opportunity arises to represent the appearance of mass directly in terms of an anisotropic space-time structure.

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