FUNDAMENTAL EQUATIONS FOR A SECOND ORDER GENERALIZED LAGRANGE SPACE ENDOWED WITH A BERWALD-MOOR TYPE METRIC IN INVARIANT FRAMES

Marius Păun

Faculty of Mathematics and Informatics, Transilvania University of Braşov, România
m.paun@unitbv.ro

The purpose of this paper is to study Vranceanu identities and Maxwell equations of a generalized Lagrange space of order 2 endowed with a Berwald Moor type metric in invariant frames end to emphasize their equivalence.

0. Introduction

We introduce distinct non-holonomic frames on the three components of the Whitney's decomposition. This will determine a non-holonomic coordinates system on the total space E and thus its geometry can be studied with methods analogous to the mobile frame. We obtain, in this manner, the representation of geometrical objects (vectors, 1-forms, tensors) and prove that there are invariant at local changes of coordinates and we introduce the invariant covariant derivative. We obtain the expressions of the coefficients of the canonical metrical N-linear connection, we define the d-tensors of torsion and curvature in these non-holonomic frames and prove that there are the invariant components of the d-tensors of torsion and curvature. Using the the non-holonomy coefficients of Vranceanu we write the Vranceanu identities. Defining the invariant electromagnetic tensors in the case of normal frames w.r.t. the canonical metrical connection we can obtain Maxwell equations. The frames introduced here present a theoretical importance for the geometry of a vector bundle because through them we can outline the geometrical properties of the total space of the considerate bundle, invariant to the transformations of the pseudoorthogonal group. We mention here that in the case of this Berwald Moor type metric the Vranceanu identities coincide with the Maxwell equations.

1. General invariant frames

Let us consider M a 4-dimensional differential manifold of C^∞-class, the bundle $E = Osc^2M$, a nonlinear connection N with the coefficients $\left(\underset{(1)}{N}{}^i{}_j, \underset{(2)}{N}{}^i{}_j\right)$ and the duals $\left(\underset{(1)}{M}{}^i{}_j, \underset{(2)}{M}{}^i{}_j\right)$. For a point $u \in E$ let $(x^i, y^{(1)i}, y^{(2)j})$ be its coordinates in a local chart. The nonlinear connection N determines the direct decomposition:

$$T_u(Osc^2M) = N_0(u) \oplus N_1(u) \oplus V_2(u) \quad \forall u \in Osc^2M = E, \tag{1.1}$$

The adapted basis of the direct decomposition (1.1) is:

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}} \right\} \qquad (i = 1, ..., n) \tag{1.2}$$

where:

$$\begin{cases} \dfrac{\delta}{\delta x^i} &= \dfrac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j{}_i \dfrac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N}{}^j{}_i \dfrac{\partial}{\partial y^{(2)j}} \\ \dfrac{\delta}{\delta y^{(1)i}} &= \dfrac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^j{}_i \dfrac{\partial}{\partial y^{(2)j}} \\ \dfrac{\delta}{\delta y^{(2)i}} &= \dfrac{\partial}{\partial y^{(2)i}} \end{cases} \tag{1.3}$$

The dual basis will be:

$$\left\{\delta x^i,\ \delta y^{(1)i},\ \delta y^{(2)i}\right\} \qquad (i = 1, ..., n) \tag{1.4}$$

where:

$$\begin{cases} \delta x^i &= dx^i \\ \delta y^{(1)i} &= dy^{(1)i} + \underset{(1)}{M}{}^i{}_j dx^j \\ \delta y^{(2)i} &= dy^{(2)i} + \underset{1}{M}{}^i{}_j dy^{(1)j} + \underset{2}{M}{}^i{}_j dx^j \end{cases} \tag{1.5}$$

We shall consider a metric

$$G = g_{ij} dx^i \otimes dx^i + g_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j} + g_{ij} \delta y^{(2)i} \otimes \delta y^{(2)j}$$

where $g_{ij} = g_{ij}(x, y^{(1)}, y^{(2)})$ such that the distributions $N_0,\ N_1,\ V_2$ generated by the nonlinear connection be orthogonal in pairs with respect to G.

Let also

$$F = \sqrt[4]{y^{(1)1} y^{(1)2} y^{(1)3} y^{(1)4}}$$

be the Berwald-Moor function and the generalized Lagrange metric on M given by

$$h_{ij} = \frac{1}{12F^4} \frac{\partial^2 F^4}{\partial y^i \partial y^j}$$

In this paper we shall use a particular kind of metric on $E \setminus \{0\}$

$$g_{ij}(x, y^{(1)}, y^{(2)}) = h_{ij}(y^{(1)}).$$

The invariant frames adapted to the direct decomposition (1.1)

$$\mathcal{R} = (e_\alpha^{(0)i},\ e_\alpha^{(1)i},\ e_\alpha^{(2)i}),$$

where i is a component index and α counter index, are defined in this shape:

$$\begin{aligned} e_\alpha^{(0)} &:\ u \in E \to e_\alpha^{(0)}(u) \subset N_0(u) \\ e_\alpha^{(1)} &:\ u \in E \to e_\alpha^{(1)}(u) \subset N_1(u) \\ e_\alpha^{(2)} &:\ u \in E \to e_\alpha^{(2)}(u) \subset V_2(u) \\ & \qquad (\alpha = 1, ..., n) \end{aligned} \tag{1.6}$$

For this frames we have $e_\alpha^{(A)}(u) = e_\alpha^{(A)i} \dfrac{\delta}{\delta y^{(A)i}}\ |_u$ where $y^{(0)i} = x^i$. Denote by $\mathcal{R}^* = (f_i^{(0)\alpha},\ f_i^{(1)\alpha},\ f_i^{(2)\alpha})$ the dual frames of $\mathcal{R}$. For $\mathcal{R}^*,\ i$ is the counter index and α is a component index.

The duality conditions are:

$$< e^{(A)i}_{\ \alpha}, f^{(B)\alpha}_{\ \ j} > = \delta^i_j \delta^A_B \quad (A, B = 0, 1, 2) \tag{1.7}$$

The frames $\mathcal{R}$ şi $\mathcal{R}^*$ are non-holonomic and thru them we can introduce a non-holonomic coordinate system $(s^{(0)\alpha}, s^{(1)\alpha}, s^{(2)\alpha})$ in Vrânceanu sense. In this frames the adapted basis and the cobasis have the representations:

$$\frac{\delta}{\delta x^i} = f^{(0)\alpha}_{\ \ i} \frac{\delta}{\delta s^{(0)\alpha}}; \quad \frac{\delta}{\delta y^{(A)i}} = f^{(A)\alpha}_{\ \ i} \frac{\delta}{\delta s^{(A)\alpha}}, \quad (A = 1, 2) \tag{1.8}$$

$$\delta x^i = e^{(0)i}_{\ \ \alpha} \delta s^{(0)\alpha}; \quad \delta y^{(A)i} = e^{(A)i}_{\ \ \alpha} \delta s^{(A)\alpha}, \quad (A = 1, 2);$$

The following relations hold:

$$\left\langle \frac{\delta}{\delta s^{(A)\alpha}}, \delta s^{(B)\beta} \right\rangle = \delta^\beta_\alpha \delta^B_A, \qquad (A, B = 0, 1, 2) \tag{1.9}$$

These representations lead us to a transformation group of invariant frames $\mathcal{R} \rightarrow \bar{\mathcal{R}}$. Analytical expression of this group are:

$$\overline{e}^{(A)i}_{\ \ \alpha} = \overset{A}{C}{}^\beta_\alpha \,(x, y^{(1)}, y^{(2)}) e^{(A)i}_{\ \ \beta}; \quad f^{(B)\alpha}_{\ \ j} = \overline{\overset{B}{C}}{}^\alpha_\beta \, \overline{f}^{(B)\beta}_{\ \ j}, \quad (A, B = 0, 1, 2) \tag{1.10}$$

Proposition 1.1 *The set of frame transformations (II.2.6) togheter with the product of transformations is a group isomorphic with the multiplicative group of nonsingular matrices:*

$$\begin{pmatrix} \overset{0}{C}{}^\alpha_\beta & 0 & 0 \\ 0 & \overset{1}{C}{}^\alpha_\beta & 0 \\ 0 & 0 & \overset{2}{C}{}^\alpha_\beta \end{pmatrix}$$

So we have:

$$\frac{\delta}{\delta s^{(A)\alpha}} = e^{(A)i}_{\ \ \alpha} \frac{\delta}{\delta y^{(A)i}} \quad (A = 0, 1, 2) \quad (y^{(0)i} = x^i) \tag{1.11}$$

and

$$\frac{\delta}{\delta \bar{s}^{(A)\alpha}} = \overset{A}{C}{}^\beta_\alpha \frac{\delta}{\delta s^{(A)\beta}}, \quad (A = 0, 1, 2) \tag{1.12}$$

together with:

$$\delta \bar{s}^{(A)\alpha} = \overline{\overset{A}{C}}{}^\alpha_\beta \delta s^{(A)\beta} \quad (A = 0, 1, 2) \tag{1.13}$$

The context in which we are working and a straightforward computing of Lie brackets lead us to the introduction of the non-holonomy coefficients of Vranceanu. We obtain:

$$\left[\frac{\delta}{\delta s^{(A)\alpha}}, \frac{\delta}{\delta s^{(B)\beta}} \right] = \underset{(AB)}{\overset{C}{W}}{}^{\gamma}_{\alpha\beta} \frac{\delta}{\delta s^{(C)\gamma}} \tag{1.14}$$

$(A, B, C = 0, 1, 2; \quad A \leq B; \quad$ *sum on* $C)$ where

$$\underset{(00)}{\overset{0}{W}}{}^\gamma_{\beta\alpha} = f^{(0)\gamma}_{\ \ l} \left(\frac{\delta e^{(0)l}_{\ \ \beta}}{\delta s^{(0)\alpha}} - \frac{\delta e^{(0)l}_{\ \ \alpha}}{\delta s^{(0)\beta}} \right); \quad \underset{(00)}{\overset{1}{W}}{}^\gamma_{\beta\alpha} = \underset{(00)}{\overset{2}{W}}{}^\gamma_{\beta\alpha} = 0. \tag{1.15}$$

$$\overset{0}{\underset{(01)}{W}}{}^{\gamma}_{\beta\alpha} = f^{(0)\gamma}_{\ \ l} \frac{\delta e^{(0)l}_{\ \ \alpha}}{\delta s^{(1)\beta}} \ ; \quad \overset{1}{\underset{(01)}{W}}{}^{\gamma}_{\beta\alpha} = -f^{(1)\gamma}_{\ \ l} \frac{\delta e^{(1)l}_{\ \ \beta}}{\delta s^{(0)\alpha}} \ ; \quad \overset{2}{\underset{(01)}{W}}{}^{\gamma}_{\alpha\beta} = 0. \tag{1.16}$$

$$\overset{0}{\underset{(02)}{W}}{}^{\gamma}_{\beta\alpha} = f^{(0)\gamma}_{\ \ l} \frac{\delta e^{(0)l}_{\ \ \alpha}}{\delta s^{(2)\beta}} \ ; \quad \overset{1}{\underset{(02)}{W}}{}^{\gamma}_{\alpha\beta} = 0 \ ; \quad \overset{2}{\underset{(02)}{W}}{}^{\gamma}_{\beta\alpha} = -f^{(2)\gamma}_{\ \ l} \frac{\delta e^{(2)l}_{\ \ \beta}}{\delta s^{(0)\alpha}}. \tag{1.17}$$

$$\overset{1}{\underset{(11)}{W}}{}^{\gamma}_{\beta\alpha} = f^{(1)\gamma}_{\ \ l} \left(\frac{\delta e^{(1)l}_{\ \ \beta}}{\delta s^{(1)\alpha}} - \frac{\delta e^{(1)l}_{\ \ \alpha}}{\delta s^{(1)\beta}} \right) \ ; \quad \overset{2}{\underset{(11)}{W}}{}^{\gamma}_{\beta\alpha} = 0 \tag{1.18}$$

$$\overset{1}{\underset{(12)}{W}}{}^{\gamma}_{\alpha\beta} = f^{(1)\gamma}_{\ \ l} \frac{\delta e^{(1)l}_{\ \ \alpha}}{\delta s^{(2)\beta}} \ ; \quad \overset{2}{\underset{(12)}{W}}{}^{\gamma}_{\beta\alpha} = -f^{(2)\gamma}_{\ \ l} \frac{\delta e^{(2)l}_{\ \ \beta}}{\delta s^{(1)\alpha}}; \tag{1.19}$$

$$\overset{2}{\underset{(22)}{W}}{}^{\gamma}_{\beta\alpha} = f^{(2)\gamma}_{\ \ l} \left(\frac{\delta e^{(2)l}_{\ \ \beta}}{\delta s^{(2)\alpha}} - \frac{\delta e^{(2)l}_{\ \ \alpha}}{\delta s^{(2)\beta}} \right); \tag{1.20}$$

All other coefficients are 0. We observe that:

$$\overset{A}{\underset{(BB)}{W}}{}^{\ \gamma}_{\alpha\beta} - \overset{A}{\underset{(BB)}{W}}{}^{\ \gamma}_{\beta\alpha} = 0 \quad (A, B = 0, 1, 2) \tag{1.21}$$

$$\overset{2}{\underset{(01)}{W}}{}^{\gamma}_{\alpha\beta} - \overset{2}{\underset{(01)}{W}}{}^{\gamma}_{\beta\alpha} = 0;$$

$$\overset{1}{\underset{(02)}{W}}{}^{\gamma}_{\alpha\beta} - \overset{1}{\underset{(02)}{W}}{}^{\gamma}_{\beta\alpha} = 0;$$

Proposition 1.2 *Transformation laws of the Vranceanu coefficients at frames transformations are:*

$$\overset{A}{\underset{(AA)}{\bar{W}}}{}^{\ \lambda}_{\beta\alpha} = \left(\overset{A}{C}{}^{\gamma}_{\alpha} \overset{A}{C}{}^{\varphi}_{\gamma} \overset{A}{\underset{(AA)}{W}}{}^{\ \eta}_{\varphi\gamma} + \overset{A}{C}{}^{\gamma}_{\alpha} \frac{\delta \overset{A}{C}{}^{\eta}_{\beta}}{\delta s^{(A)\gamma}} - \overset{A}{C}{}^{\varphi}_{\beta} \frac{\delta \overset{A}{C}{}^{\eta}_{\alpha}}{\delta s^{(A)\varphi}} \right) \overset{A}{\bar{C}}{}^{\lambda}_{\eta}, \quad (A = 0, 1, 2) \tag{1.22}$$

$$\overset{A}{\underset{(AB)}{\bar{W}}}{}^{\ \lambda}_{\beta\alpha} = \left(\overset{A}{C}{}^{\gamma}_{\alpha} \overset{B}{C}{}^{\varphi}_{\gamma} \overset{A}{\underset{(AB)}{W}}{}^{\ \eta}_{\varphi\gamma} - \overset{A}{C}{}^{\varphi}_{\beta} \frac{\delta \overset{A}{C}{}^{\eta}_{\alpha}}{\delta s^{(A)\varphi}} \right) \overset{A}{\bar{C}}{}^{\lambda}_{\eta}, \quad (A, B = 0, 1, 2 \quad A < B) \tag{1.23}$$

$$\overset{A}{\underset{(BA)}{\bar{W}}}{}^{\ \lambda}_{\beta\alpha} = \left(\overset{B}{C}{}^{\gamma}_{\alpha} \overset{A}{C}{}^{\varphi}_{\gamma} \overset{A}{\underset{(BA)}{W}}{}^{\ \eta}_{\varphi\gamma} + \overset{B}{C}{}^{\gamma}_{\alpha} \frac{\delta \overset{A}{C}{}^{\eta}_{\beta}}{\delta s^{(B)\gamma}} \right) \overset{A}{\bar{C}}{}^{\lambda}_{\eta}, \quad (A, B = 0, 1, 2 \quad A > B) \tag{1.24}$$

$$\overset{1}{\underset{(00)}{\bar{W}}}{}^{\lambda}_{\beta\alpha} = \overset{0}{C}{}^{\gamma}_{\alpha} \overset{0}{C}{}^{\varphi}_{\beta} \overset{1}{\underset{(00)}{W}}{}^{\eta}_{\varphi\gamma} \overset{1}{\bar{C}}{}^{\lambda}_{\eta}; \quad \overset{2}{\underset{(00)}{\bar{W}}}{}^{\lambda}_{\beta\alpha} = \overset{0}{C}{}^{\gamma}_{\alpha} \overset{0}{C}{}^{\varphi}_{\beta} \overset{2}{\underset{(00)}{W}}{}^{\eta}_{\varphi\gamma} \overset{2}{\bar{C}}{}^{\lambda}_{\eta}; \quad \overset{2}{\underset{(01)}{\bar{W}}}{}^{\lambda}_{\beta\alpha} = \overset{0}{C}{}^{\gamma}_{\alpha} \overset{1}{C}{}^{\varphi}_{\beta} \overset{2}{\underset{(01)}{W}}{}^{\eta}_{\varphi\gamma} \overset{2}{\bar{C}}{}^{\lambda}_{\eta}; \tag{1.25}$$

$$\overset{1}{\underset{(02)}{\bar{W}}}{}^{\lambda}_{\beta\alpha} = \overset{0}{C}{}^{\gamma}_{\alpha} \overset{2}{C}{}^{\varphi}_{\beta} \overset{1}{\underset{(02)}{W}}{}^{\eta}_{\varphi\gamma} \overset{1}{\bar{C}}{}^{\lambda}_{\eta}; \quad \overset{2}{\underset{(11)}{\bar{W}}}{}^{\lambda}_{\beta\alpha} = \overset{1}{C}{}^{\gamma}_{\alpha} \overset{2}{C}{}^{\varphi}_{\beta} \overset{2}{\underset{(11)}{W}}{}^{\eta}_{\varphi\gamma} \overset{2}{\bar{C}}{}^{\lambda}_{\eta}$$

Theorem 1.1 *In the case* $e^{(0)i}_{\ \ \alpha} = e^{(1)i}_{\ \ \alpha} = e^{(2)i}_{\ \ \alpha} = e^{i}_{\alpha}$ *and* $f^{(0)\alpha}_{\ \ i} = f^{(1)\alpha}_{\ \ i} = f^{(2)\alpha}_{\ \ i} = f^{\alpha}_{i}$ *the frame* $\mathcal{R}$ *is holonomic if and only if*

$$\overset{0}{\underset{(00)}{W}}{}^{\alpha}_{\beta\gamma} = \overset{0}{\underset{(01)}{W}}{}^{\alpha}_{\beta\gamma} = \overset{0}{\underset{(02)}{W}}{}^{\alpha}_{\beta\gamma} = 0 \tag{1.26}$$

Observation 1.1 *$\mathcal{R}$ are holonomic if and only if the Lie brackets are vertical and the conditions imposed by the previous theorem lead us to*

$$\underset{(11)}{\overset{1}{W}}{}^{\gamma}_{\beta\alpha} = \underset{(22)}{\overset{2}{W}}{}^{\gamma}_{\beta\alpha} = 0$$

i.e. N_1 and V_2 are integrabile. The horizontal distribution is integrable if and only if $\underset{(01)}{R}{}^{i}_{jk} = \underset{(02)}{R}{}^{i}_{jk} = 0$ condition wich is independent of the frame $\mathcal{R}$.

Generally we have:

Theorem 1.2 *The frames $\mathcal{R}$ are holonomic if and only if :*

$$\underset{(00)}{\overset{0}{W}}{}^{\gamma}_{\beta\alpha} = \underset{(01)}{\overset{0}{W}}{}^{\gamma}_{\beta\alpha} = \underset{(02)}{\overset{0}{W}}{}^{\gamma}_{\beta\alpha} = \underset{(11)}{\overset{1}{W}}{}^{\gamma}_{\beta\alpha} = \underset{(12)}{\overset{1}{W}}{}^{\gamma}_{\beta\alpha} = \underset{(22)}{\overset{2}{W}}{}^{\gamma}_{\beta\alpha} = 0$$

2. The representation of geometric objects in invariant frames $\mathcal{R}$

Let $X \in \chi(E)$ a vector field. Then for X we have the local representations

$$X = X^{(0)i}\frac{\delta}{\delta x^i} + X^{(1)i}\frac{\delta}{\delta y^{(1)i}} + X^{(2)i}\frac{\delta}{\delta y^{(2)i}} \tag{2.1}$$

in adapted basis and

$$X = X^{(0)\alpha}\frac{\delta}{\delta s^{\alpha}} + X^{(1)\alpha}\frac{\delta}{\delta s^{(1)\alpha}} + X^{(2)\alpha}\frac{\delta}{\delta s^{(2)\alpha}}. \tag{2.2}$$

in invariant frames.
We have

$$X^{(A)i} = e^{(A)i}_{\alpha} X^{(A)\alpha} \quad \text{or} \quad X^{(A)\alpha} = f^{(A)\alpha}_{i} X^{(A)i} \quad (A = 0, 1, 2). \tag{2.3}$$

At local coordinate changes we have:

$$\bar{X}^{(A)\alpha} = \bar{f}^{(A)\alpha}{}_{i}\bar{X}^{(A)i} = f^{(A)\alpha}{}_{k}\frac{\partial \bar{x}^k}{\partial x^i} X^{(A)l}\frac{\partial x^i}{\partial \bar{x}^l} = f^{(A)\alpha}{}_{k}\delta^k_i X^{(A)i} = X^{(A)\alpha} \tag{2.4}$$

So:

Proposition 2.1 *The nonholonomic components of the h-, v_1-, v_2- projections of the vector field X are invariant at local changes of coordinates.*

Let $\omega \in \chi^*$ be a field of 1-forms. The representation of ω in the invariant frames is:

$$\omega = \omega^{(0)}_{i}\delta x^i + \omega^{(1)}_{i}\delta y^{(1)i} + \omega^{(2)}_{i}\delta y^{(2)i} = \omega^{(0)}_{\alpha}\delta s^{(0)\alpha} + \omega^{(1)}_{\alpha}\delta s^{(1)\alpha} + \omega^{(2)}_{\alpha}\delta s^{(2)\alpha} \tag{2.5}$$

Proposition 2.2 *The non-holonomic components $\omega^{(A)}_{\alpha}$ of the projections of $\omega^{(A)}$ on the three distributions $N_0,\ N_1,\ V_2$ are invariant at local changes of coordinates.*

Dueing the fact that the geometric object studied above have invariant components at local changes of coordinates in the frames $\mathcal{R}$ we call this frames invariant frames.

3. N-linear connections in invariant frames

In the adapted basis the coefficients of a N-linear connection are given by:

$$D_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta x^j} = L^m_{\ ji}\frac{\delta}{\delta x^m}, \quad D_{\frac{\delta}{\delta x^i}}\frac{\delta}{\delta y^{(A)j}} = L^m_{\ ji}\frac{\delta}{\delta y^{(A)m}}, \quad (A=1,2) \tag{3.1}$$

$$D_{\frac{\delta}{\delta y^{(A)i}}}\frac{\delta}{\delta y^{(B)j}} = \underset{(A)}{C}{}^m_{\ ji}\frac{\delta}{\delta y^{(B)m}}, \quad (A,B=1,2) \tag{3.2}$$

$$D_{\frac{\delta}{\delta y^{(A)i}}}\frac{\delta}{\delta x^j} = \underset{(A)}{C}{}^m_{\ ji}\frac{\delta}{\delta x^m}, \quad (A=1,2) \tag{3.3}$$

and transform after the rules

$$\tilde{L}^i_{\ rs}\frac{\partial \tilde{x}^r}{\partial x^j}\frac{\partial \tilde{x}^s}{\partial x^k} = \frac{\partial \tilde{x}^i}{\partial x^r}L^r_{\ jk} - \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} \tag{3.4}$$

$$\underset{(A)}{\tilde{C}}{}^i_{\ rs}\frac{\partial \tilde{x}^r}{\partial x^j}\frac{\partial \tilde{x}^s}{\partial x^k} = \frac{\partial \tilde{x}^i}{\partial x^r}\underset{(A)}{C}{}^r_{\ jk} \quad (A=1,2) \tag{3.5}$$

Proposition 3.1 *In the invariant frames $\mathcal{R}$ şi $\mathcal{R}^*$ the essential components of the N-linear connection are nine and are given by:*

$$\overset{0A}{L}{}^{\gamma}_{\beta\alpha} = f^{(A)\gamma}_{\ \ m}\left(\frac{\delta e^{(A)m}_{\ \ \beta}}{\delta s^{(0)\alpha}} + e^{(0)i}_{\ \ \alpha}e^{(A)j}_{\ \ \beta}L^m_{ij}\right) \quad (A=0,1,2) \tag{3.6}$$

$$\overset{BA}{C}{}^{\gamma}_{\beta\alpha} = f^{(A)\gamma}_{\ \ m}\left(\frac{\delta e^{(A)m}_{\ \ \beta}}{\delta s^{(B)\alpha}} + e^{(B)i}_{\ \ \alpha}e^{(A)j}_{\ \ \beta}\underset{(B)}{C}{}^m_{ij}\right) \quad (A=0,1,2\,;\ B=1,2\,)$$

Considering now the case of the canonical metrical N-linear connection $C\Gamma(N)$ with the coefficients

$$C\Gamma(N) = \left(0, \underset{(1)}{C}{}^m_{\ ji},\ 0\right)$$

we obtain

Corollary 3.1 *The coefficients of the canonical metrical N-linear connection in invariant frames are:*

$$\overset{00}{L}{}^{\gamma}_{\beta\alpha} = \frac{1}{2}\underset{(00)}{\overset{0}{W}}{}^{\gamma}_{\beta\alpha}, \quad \overset{01}{L}{}^{\gamma}_{\beta\alpha} = -\underset{(01)}{\overset{1}{W}}{}^{\gamma}_{\beta\alpha}, \quad \overset{02}{L}{}^{\gamma}_{\beta\alpha} = -\underset{(02)}{\overset{2}{W}}{}^{\gamma}_{\beta\alpha} \tag{3.7}$$

$$\overset{1A}{C}{}^{\gamma}_{\beta\alpha} = f^{(A)\gamma}_{\ \ m}\left(\frac{\delta e^{(A)m}_{\ \ \beta}}{\delta s^{(1)\alpha}} + e^{(1)i}_{\ \ \alpha}e^{(A)j}_{\ \ \beta}\underset{(1)}{C}{}^m_{ij}\right) \quad (A=0,1,2\,;)$$

$$\overset{20}{C}{}^{\gamma}_{\beta\alpha} = \underset{(02)}{\overset{0}{W}}{}^{\gamma}_{\beta\alpha}, \quad \overset{21}{C}{}^{\gamma}_{\beta\alpha} = \underset{(12)}{\overset{1}{W}}{}^{\gamma}_{\beta\alpha}, \quad \overset{22}{C}{}^{\gamma}_{\beta\alpha} = \frac{1}{2}\underset{(22)}{\overset{2}{W}}{}^{\gamma}_{\beta\alpha} \tag{3.8}$$

Denoting by $_|$ and $\overset{(A)}{|}$ (A=1,2) the covariant derivatives w.r.t. $C\Gamma(N)$ we obtain:

Proposition 3.2 *The movement equations of the frames $\mathcal{R}$ and $\mathcal{R}^*$ are:*

$$e^{(A)i}_{\ \ \alpha|m} = \overset{0A}{L}{}^{\gamma}_{\beta\alpha}\, e^{(A)i}_{\ \ \gamma}f^{(0)\beta}_{\ \ m}, \quad e^{(A)i}_{\ \ \alpha}\overset{(B)}{|}m = \overset{AB}{C}{}^{\gamma}_{\beta\alpha}\, e^{(A)i}_{\ \ \gamma}f^{(B)\beta}_{\ \ m} \quad (A=0,1,2;\ B=1,2) \tag{3.9}$$

$$f^{(A)\gamma}_{\ \ i|m} = -\overset{0A}{L}{}^{\gamma}_{\beta\alpha}\, f^{(A)\alpha}_{\ \ i}f^{(0)\beta}_{\ \ m}, \quad f^{(A)\alpha}_{\ \ i}\overset{(B)}{|} = -\overset{AB}{C}{}^{\gamma}_{\beta\alpha}\, f^{(A)\alpha}_{\ \ i}f^{(B)\beta}_{\ \ m} \quad (A=0,1,2;\ B=1,2) \tag{3.10}$$

4 Covariant invariant derivatives

Let $X \in \chi(E)$ and $X^{(0)\alpha},\ X^{(1)\alpha},\ X^{(2)\alpha}$ its invariant components. We denote " $)$", and " $\overset{(A)}{)}$ ", $(A=1,2)$ the operators of h-, v_1- şi v_2- covariant invariant derivative w.r.t. $C\Gamma(N)$ acting in this manner:

Definition 4.1 *The h-, v_1- şi v_2- invariant covariant derivatives of the components $X^{(0)},\ X^{(1)},\ X^{(2)}$ are (A=0,1,2; B=1,2):*

$$X^{(A)\alpha}{}_{)\beta} = \frac{\delta X^{(A)\alpha}}{\delta s^{(0)\beta}} + \overset{(0A)}{L}{}^{\alpha}{}_{\varphi\beta} X^{(A)\varphi} \tag{4.1}$$

$$X^{(A)\alpha}\overset{(B)}{)}{}_{\beta} = \frac{\delta X^{(A)\alpha}}{\delta s^{(B)\beta}} + \overset{(BA)}{C}{}^{\alpha}{}_{\varphi\beta} X^{(A)\varphi} \tag{4.2}$$

If ω is 1-form field and $\omega^{(A)}{}_{\alpha},\ (A=0,1,2)$ its invariant components then:

Definition 4.2 *The h-, v_1- şi v_2- covariant invariant derivatives of the components $\omega^{(0)}{}_{\alpha},\ \omega^{(1)}{}_{\alpha},\ \omega^{(2)}{}_{\alpha}$ are (A=0,1,2; B=1,2):*

$$\omega^{(A)}{}_{\alpha)\beta} = \frac{\delta \omega^{(A)}{}_{\alpha}}{\delta s^{(0)\beta}} - \overset{(0A)}{L}{}^{\varphi}{}_{\alpha\beta}\,\omega^{(A)}{}_{\varphi} \tag{4.3}$$

$$\omega^{(A)}{}_{\alpha}\overset{(B)}{)}{}_{\beta} = \frac{\delta \omega^{(A)}{}_{\alpha}}{\delta s^{(B)\beta}} - \overset{(BA)}{C}{}^{\varphi}{}_{\alpha\beta}\,\omega^{(A)}{}_{\varphi} \tag{4.4}$$

Theorem 4.1 *The h-, v_1- şi v_2- covariant invariant derivatives of $X^{(A)\alpha}$ and $\omega^{(A)}_{\alpha}$ are the invariant components of the h-, v_1- şi v_2- covariant invariant derivatives of the components in the adapted basis of X and ω i. e.*

$$X^{(A)\alpha}{}_{)\beta} = X^{(A)i}{}_{|m} f^{(A)\alpha}_{\ \ i} e^{(0)m}_{\ \ \beta} \tag{4.5}$$

$$X^{(A)\alpha}\overset{(B)}{)}{}_{\beta} = X^{(A)i}\overset{(B)}{|}{}_{m} f^{(A)\alpha}_{\ \ i} e^{(B)m}_{\ \ \beta} \tag{4.6}$$

$$\omega^{(A)}{}_{\alpha)\beta} = \omega^{(A)}{}_{i|m} e^{(A)i}_{\ \ \alpha} e^{(0)m}_{\ \ \beta} \tag{4.7}$$

$$\omega^{(A)}{}_{\alpha}\overset{(B)}{)}{}_{\beta} = \omega^{(A)}{}_{i}\overset{(B)}{|}{}_{m} e^{(A)i}_{\ \ \alpha} e^{(B)m}_{\ \ \beta} \tag{4.8}$$

or equivalent:

$$X^{(A)i}{}_{|m} = X^{(A)\alpha}{}_{)\beta} e^{(A)i}_{\ \ \alpha} f^{(0)\beta}_{\ \ m} \tag{4.9}$$

$$X^{(A)i}\overset{(B)}{|}{}_{m} = X^{(A)\alpha}\overset{(B)}{)}{}_{\beta} e^{(A)i}_{\ \ \alpha} f^{(B)\beta}_{\ \ m}\omega^{(A)}{}_{i|m} = \omega^{(A)}{}_{\alpha)\beta} f^{(A)\alpha}_{\ \ i} f^{(0)\beta}_{\ \ m} \tag{4.10}$$

$$\omega^{(A)}{}_{i}\overset{(B)}{|}{}_{m} = \omega^{(A)}{}_{\alpha}\overset{(B)}{)}{}_{\beta} f^{(A)\alpha}_{\ \ i} f^{(B)\beta}_{\ \ m}. \tag{4.11}$$

5. Torsion and curvature tensor fields in invariant frames

The torsion tensor field of the canonical metrical N-linear connection C is given by:

$$\mathcal{T}(X,Y) = D_X Y - D_Y X - [X,Y], \quad \forall X,Y \in \chi(E) \tag{5.1}$$

In the frames $\mathcal{R}$ this tensor field has some horizontal and vertical components corresponding to D^H , D^{V_1} , D^{V_2} :

$$\begin{cases} h\mathcal{T}\left(\frac{\delta}{\delta s^{(0)\alpha}}, \frac{\delta}{\delta s^{(0)\beta}}\right) &= \underset{(0)}{\bar{T}}{}^{\gamma}{}_{\beta\alpha}\frac{\delta}{\delta s^{(0)\gamma}} \\ v_A\mathcal{T}\left(\frac{\delta}{\delta s^{(0)(\alpha}}, \frac{\delta}{\delta s^{(0)\beta}}\right) &= \underset{(0A)}{\bar{R}}{}^{\gamma}{}_{\beta\alpha}\frac{\delta}{\delta s^{(A)\gamma}} \end{cases} \tag{5.2}$$

$$\begin{cases} h\mathcal{T}\left(\frac{\delta}{\delta s^{(0)\alpha}}, \frac{\delta}{\delta s^{(1)\beta}}\right) &= \underset{(1)}{\bar{K}}{}^{\gamma}{}_{\beta\alpha}\frac{\delta}{\delta s^{(0)\gamma}} \\ v_A\mathcal{T}\left(\frac{\delta}{\delta s^{(0)\alpha}}, \frac{\delta}{\delta s^{(1)\beta}}\right) &= \underset{(A1)}{\bar{P}}{}^{\gamma}{}_{\beta\alpha}\frac{\delta}{\delta s^{(A)\gamma}} \end{cases} \tag{5.3}$$

$$\begin{cases} h\mathcal{T}\left(\frac{\delta}{\delta s^{(0)\alpha}}, \frac{\delta}{\delta s^{(2)\beta}}\right) &= \underset{(2)}{\bar{K}}{}^{\gamma}{}_{\beta\alpha}\frac{\delta}{\delta s^{(0)\gamma}} \\ v_A\mathcal{T}\left(\frac{\delta}{\delta s^{(0)\alpha}}, \frac{\delta}{\delta s^{(2)\beta}}\right) &= \underset{(A2)}{\bar{P}}{}^{\gamma}{}_{\beta\alpha}\frac{\delta}{\delta s^{(A)\gamma}} \end{cases} \tag{5.4}$$

$$\begin{cases} h\mathcal{T}\left(\frac{\delta}{\delta s^{(1)\alpha}}, \frac{\delta}{\delta s^{(1)\beta}}\right) &= 0 \\ v_A\mathcal{T}\left(\frac{\delta}{\delta s^{(1)\alpha}}, \frac{\delta}{\delta s^{(1)\beta}}\right) &= \underset{(A1)}{\bar{Q}}{}^{\gamma}{}_{\beta\alpha}\frac{\delta}{\delta s^{(A)\gamma}} \end{cases} \tag{5.5}$$

$$\begin{cases} v_A\mathcal{T}\left(\frac{\delta}{\delta s^{(1)\alpha}}, \frac{\delta}{\delta s^{(2)\beta}}\right) = \underset{(A2)}{\bar{Q}}{}^{\gamma}{}_{\beta\alpha}\frac{\delta}{\delta s^{(A)\gamma}} \end{cases} \tag{5.6}$$

$$\begin{cases} v_1\mathcal{T}\left(\frac{\delta}{\delta s^{(2)\alpha}}, \frac{\delta}{\delta s^{(2)\beta}}\right) &= 0 \\ v_2\mathcal{T}\left(\frac{\delta}{\delta s^{(2)\alpha}}, \frac{\delta}{\delta s^{(2)\beta}}\right) &= \underset{(2)}{\bar{S}}{}^{\gamma}{}_{\beta\alpha}\frac{\delta}{\delta s^{(2)\gamma}} \end{cases} \tag{5.7}$$

$$(A = 1, 2.)$$

By definition we take:

$$\begin{cases} \underset{(0)}{\bar{T}}{}^{\gamma}{}_{\beta\alpha} &= \overset{(00)}{L}{}^{\gamma}{}_{\beta\alpha} - \overset{(00)}{L}{}^{\gamma}{}_{\alpha\beta} - \underset{(00)}{\overset{(0)}{W}}{}^{\gamma}{}_{\beta\alpha} \\ \underset{(0A)}{\bar{R}}{}^{\gamma}{}_{\beta\alpha} &= \underset{(00)}{\overset{(A)}{W}}{}^{\gamma}{}_{\beta\alpha} \end{cases} \tag{5.8}$$

$$\begin{cases} \underset{(1)}{\bar{K}}{}^{\gamma}{}_{\beta\alpha} &= -\overset{(10)}{C}{}^{\gamma}{}_{\beta\alpha} - \underset{(01)}{\overset{(0)}{W}}{}^{\gamma}{}_{\beta\alpha} \\ \underset{(11)}{\bar{P}}{}^{\gamma}{}_{\beta\alpha} &= \overset{(01)}{L}{}^{\gamma}{}_{\beta\alpha} + \underset{(01)}{\overset{(1)}{W}}{}^{\gamma}{}_{\beta\alpha} \\ \underset{(12)}{\bar{P}}{}^{\gamma}{}_{\beta\alpha} &= \underset{(01)}{\overset{(2)}{W}}{}^{\gamma}{}_{\beta\alpha} \end{cases} \tag{5.9}$$

$$\begin{cases} \underset{(2)}{\bar{K}}{}^{\gamma}{}_{\beta\alpha} &= -\overset{(20)}{C}{}^{\gamma}{}_{\beta\alpha} - \underset{(02)}{\overset{(0)}{W}}{}^{\gamma}{}_{\beta\alpha} \\ \underset{(21)}{\bar{P}}{}^{\gamma}{}_{\beta\alpha} &= \underset{(02)}{\overset{(1)}{W}}{}^{\gamma}{}_{\alpha\beta} - \underset{(01)}{\overset{(1)}{W}}{}^{\gamma}{}_{\beta\alpha} \\ \underset{(22)}{\bar{P}}{}^{\gamma}{}_{\beta\alpha} &= \overset{(02)}{L}{}^{\gamma}{}_{\beta\alpha} + \underset{(02)}{\overset{(2)}{W}}{}^{\gamma}{}_{\beta\alpha} \end{cases} \tag{5.10}$$

$$\begin{cases} \underset{(11)}{\bar{Q}}{}^{\gamma}{}_{\beta\alpha} = \overset{(11)}{C}{}^{\gamma}{}_{\beta\alpha} - \overset{(11)}{C}{}^{\gamma}{}_{\alpha\beta} - \underset{(11)}{\overset{(1)}{W}}{}^{\gamma}{}_{\beta\alpha} \\ \underset{(21)}{\overset{(2)}{\bar{Q}}}{}^{\gamma}{}_{\beta\alpha} = \underset{(11)}{\overset{(2)}{W}}{}^{\gamma}{}_{\beta\alpha} \end{cases} \tag{5.11}$$

$$\begin{cases} \underset{(12)}{\bar{Q}}{}^{\gamma}{}_{\beta\alpha} = \overset{(21)}{C}{}^{\gamma}{}_{\beta\alpha} - \underset{(12)}{\overset{(1)}{W}}{}^{\gamma}{}_{\beta\alpha} \\ \underset{(22)}{\bar{Q}}{}^{\gamma}{}_{\beta\alpha} = -\overset{(12)}{C}{}^{\gamma}{}_{\alpha\beta} - \underset{(12)}{\overset{(2)}{W}}{}^{\gamma}{}_{\beta\alpha} \end{cases} \tag{5.12}$$

$$\left\{ \underset{(2)}{\bar{S}}{}^{\gamma}{}_{\beta\alpha} = \overset{(22)}{C}{}^{\gamma}{}_{\beta\alpha} - \overset{(22)}{C}{}^{\gamma}{}_{\alpha\beta} - \underset{(22)}{\overset{(2)}{W}}{}^{\gamma}{}_{\beta\alpha} \right. \tag{5.13}$$

Theorem 5.1 *The d-tensors defined by 5.8–5.13 represent the invariant components at local changes of coordinates of the components of the d-tensor of torsion of the connection C.*

We observe that all the invariant components of the torsion tensor vanish except:

$$\underset{(1)}{\bar{K}}{}^{\gamma}_{\beta\alpha} = f^{(0)\gamma}{}_{m} e^{(1)i}{}_{\alpha} e^{(0)j}{}_{\beta} \underset{(1)}{C}{}^{m}_{ij}, \quad \underset{(21)}{\bar{P}}{}^{\gamma}_{\beta\alpha} = f^{(1)\gamma}{}_{l} \frac{\delta e^{(1)l}{}_{\beta}}{\delta s^{(0)\alpha}} ;$$

The curvature tensor field of the N-linear connection D on E are given by

$$\mathbb{R}(X,Y) = [\, D_X \,,\, D_Y \,]\, Z - D_{[X,Y]} Z, \quad \forall X,Y,Z \in \chi(E). \tag{5.14}$$

In the frame $\mathcal{R}$ we define the d-tensor fields with the local components:

$$\mathbb{R}\left(\frac{\delta}{\delta s^{(0)\alpha}}, \frac{\delta}{\delta s^{(0)\beta}}\right)\frac{\delta}{\delta s^{(0)\gamma}} = \bar{R}_{\gamma}{}^{\varphi}{}_{\beta\alpha}\frac{\delta}{\delta s^{(0)\varphi}} \tag{5.15}$$

$$\mathbb{R}\left(\frac{\delta}{\delta s^{(0)\alpha}}, \frac{\delta}{\delta s^{(1)\beta}}\right)\frac{\delta}{\delta s^{(0)\gamma}} = \underset{(1)}{\bar{P}}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha}\frac{\delta}{\delta s^{(0)\varphi}} \tag{5.16}$$

$$\mathbb{R}\left(\frac{\delta}{\delta s^{(0)\alpha}}, \frac{\delta}{\delta s^{(2)\beta}}\right)\frac{\delta}{\delta s^{(0)\gamma}} = \underset{(2)}{\bar{P}}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha}\frac{\delta}{\delta s^{(0)\varphi}} \tag{5.17}$$

$$\mathbb{R}\left(\frac{\delta}{\delta s^{(1)\alpha}}, \frac{\delta}{\delta s^{(1)\beta}}\right)\frac{\delta}{\delta s^{(0)\gamma}} = \underset{(11)}{\bar{S}}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha}\frac{\delta}{\delta s^{(0)\varphi}} \tag{5.18}$$

$$\mathbb{R}\left(\frac{\delta}{\delta s^{(1)\alpha}}, \frac{\delta}{\delta s^{(2)\beta}}\right)\frac{\delta}{\delta s^{(0)\gamma}} = \underset{(21)}{\bar{S}}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha}\frac{\delta}{\delta s^{(0)\varphi}} \tag{5.19}$$

$$\mathbb{R}\left(\frac{\delta}{\delta s^{(2)\alpha}}, \frac{\delta}{\delta s^{(2)\beta}}\right)\frac{\delta}{\delta s^{(0)\gamma}} = \underset{(22)}{\bar{S}}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha}\frac{\delta}{\delta s^{(0)\varphi}} \tag{5.20}$$

where:

$$\bar{R}_{\gamma}{}^{\varphi}{}_{\beta\alpha} = \frac{\delta \overset{(00)}{L}{}^{\varphi}{}_{\gamma\beta}}{\delta s^{(0)\alpha}} - \frac{\delta \overset{(00)}{L}{}^{\varphi}{}_{\gamma\alpha}}{\delta s^{(0)\beta}} + \overset{(00)}{L}{}^{\eta}{}_{\gamma\beta} \overset{(00)}{L}{}^{\varphi}{}_{\eta\alpha} - \overset{(00)}{L}{}^{\eta}{}_{\gamma\alpha} \overset{(00)}{L}{}^{\varphi}{}_{\eta\beta} -$$
$$- \underset{(00)}{\overset{0}{W}}{}^{\psi}_{\beta\alpha} \overset{(00)}{L}{}^{\varphi}{}_{\gamma\psi} + \underset{(00)}{\overset{1}{W}}{}^{\psi}_{\beta\alpha} \overset{(10)}{C}{}^{\varphi}{}_{\gamma\psi} + \underset{(00)}{\overset{2}{W}}{}^{\psi}_{\beta\alpha} \overset{(20)}{C}{}^{\varphi}{}_{\gamma\psi} \tag{5.21}$$

$$\underset{(1)}{\bar P}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha} = \frac{\delta \overset{(10)}{C}{}^{\varphi}{}_{\gamma\beta}}{\delta s^{(0)\alpha}} - \frac{\delta \overset{(00)}{L}{}^{\varphi}{}_{\gamma\alpha}}{\delta s^{(1)\beta}} + \overset{(10)}{C}{}^{\eta}{}_{\gamma\beta}\overset{(00)}{L}{}^{\varphi}{}_{\eta\alpha} - \overset{(00)}{L}{}^{\eta}{}_{\gamma\alpha}\overset{(10)}{C}{}^{\varphi}{}_{\eta\beta} - \underset{(01)}{\overset{0}{W}}{}^{\psi}_{\beta\alpha}\overset{(00)}{L}{}^{\varphi}{}_{\gamma\psi} + \underset{(01)}{\overset{1}{W}}{}^{\psi}_{\beta\alpha}\overset{(10)}{C}{}^{\varphi}{}_{\gamma\psi} + \underset{(01)}{\overset{2}{W}}{}^{\psi}_{\beta\alpha}\overset{(20)}{C}{}^{\varphi}{}_{\gamma\psi} \tag{5.22}$$

$$\underset{(2)}{\bar P}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha} = \frac{\delta \overset{(20)}{C}{}^{\varphi}{}_{\gamma\beta}}{\delta s^{(0)\alpha}} - \frac{\delta \overset{(00)}{L}{}^{\varphi}{}_{\gamma\alpha}}{\delta s^{(2)\beta}} + \overset{(20)}{C}{}^{\eta}{}_{\gamma\beta}\overset{(00)}{L}{}^{\varphi}{}_{\eta\alpha} - \overset{(00)}{L}{}^{\eta}{}_{\gamma\alpha}\overset{(20)}{C}{}^{\varphi}{}_{\eta\beta} - \underset{(02)}{\overset{0}{W}}{}^{\psi}_{\beta\alpha}\overset{(00)}{L}{}^{\varphi}{}_{\gamma\psi} + \underset{(02)}{\overset{1}{W}}{}^{\psi}_{\beta\alpha}\overset{(10)}{C}{}^{\varphi}{}_{\gamma\psi} + \underset{(02)}{\overset{2}{W}}{}^{\psi}_{\beta\alpha}\overset{(20)}{C}{}^{\varphi}{}_{\gamma\psi} \tag{5.23}$$

$$\underset{(11)}{\bar S}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha} = \frac{\delta \overset{(10)}{C}{}^{\varphi}{}_{\gamma\beta}}{\delta s^{(1)\alpha}} - \frac{\delta \overset{(10)}{C}{}^{\varphi}{}_{\gamma\alpha}}{\delta s^{(1)\beta}} + \overset{(10)}{C}{}^{\eta}{}_{\gamma\beta}\overset{(10)}{C}{}^{\varphi}{}_{\eta\alpha} - \overset{(10)}{C}{}^{\eta}{}_{\gamma\alpha}\overset{(10)}{C}{}^{\varphi}{}_{\eta\beta} - \underset{(11)}{\overset{1}{W}}{}^{\psi}_{\beta\alpha}\overset{(10)}{C}{}^{\varphi}{}_{\gamma\psi} + \underset{(11)}{\overset{2}{W}}{}^{\psi}_{\beta\alpha}\overset{(20)}{C}{}^{\varphi}{}_{\gamma\psi} \tag{5.24}$$

$$\underset{(21)}{\bar S}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha} = \frac{\delta \overset{(20)}{C}{}^{\varphi}{}_{\gamma\beta}}{\delta s^{(1)\alpha}} - \frac{\delta \overset{(10)}{C}{}^{\varphi}{}_{\gamma\alpha}}{\delta s^{(2)\beta}} + \overset{(20)}{C}{}^{\eta}{}_{\gamma\beta}\overset{(10)}{C}{}^{\varphi}{}_{\eta\alpha} - \overset{(10)}{C}{}^{\eta}{}_{\gamma\alpha}\overset{(20)}{C}{}^{\varphi}{}_{\eta\beta} + \underset{(12)}{\overset{1}{W}}{}^{\psi}_{\beta\alpha}\overset{(10)}{C}{}^{\varphi}{}_{\gamma\psi} + \underset{(12)}{\overset{2}{W}}{}^{\psi}_{\beta\alpha}\overset{(20)}{C}{}^{\varphi}{}_{\gamma\psi} \tag{5.25}$$

$$\underset{(22)}{\bar S}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha} = \frac{\delta \overset{(20)}{C}{}^{\varphi}{}_{\gamma\beta}}{\delta s^{(2)\alpha}} - \frac{\delta \overset{(20)}{C}{}^{\varphi}{}_{\gamma\alpha}}{\delta s^{(2)\beta}} + \overset{(20)}{C}{}^{\eta}{}_{\gamma\beta}\overset{(20)}{C}{}^{\varphi}{}_{\eta\alpha} - \overset{(20)}{C}{}^{\eta}{}_{\gamma\alpha}\overset{(20)}{C}{}^{\varphi}{}_{\eta\beta} + \underset{(22)}{\overset{2}{W}}{}^{\psi}_{\beta\alpha}\overset{(20)}{C}{}^{\varphi}{}_{\gamma\psi} \tag{5.26}$$

Theorem 5.2 *The formulas 5.21–5.26 represent the invariant components at local changes of coordinates of the curvature tensor fields of an N-linear connection D, which adapted basis have the coefficients*
$D\Gamma(N) = \left(L^{i}{}_{jk},\ \underset{(1)}{C}{}^{i}{}_{jk},\ \underset{(2)}{C}{}^{i}{}_{jk}\right)$

Corollary 5.1 *For the canonical metrical N-linear connection C all the component of the curvature tensor fields vanish except* $\underset{(1)}{\bar P}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha}$, $\underset{(11)}{\bar S}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha}$, $\underset{(21)}{\bar S}{}_{\gamma}{}^{\varphi}{}_{\beta\alpha}$

6. Structure equations in invariant frames

Introducing the invariant covariant differential of the vector field X in the shape

$$DX = \left\{dX^{(A)\alpha} + X^{(A)\gamma}\overset{(A)}{\omega}{}^{\alpha}{}_{\gamma}\right\}\frac{\delta}{\delta s^{(A)\alpha}} \tag{6.1}$$

(A=0,1,2; sumation by A)

In order to obtain the structure equations in invariant frames of the canonical metrical N-linear connection C it is necessary to compute the exterior differentials of the 1-forms of connection and of the 1-forms $\delta s^{(A)\alpha}$.

Theorem 6.1 *The exterior differentials of the 1-forms $\delta s^{(A)\alpha}$ (A=0,1,2) depend only on the non-holonomy coefficients of Vranceanu and are given by*

$$d(\delta s^{(0)\gamma}) = \underset{(00)}{\overset{(0)}{W}}{}^{\gamma}_{\beta\alpha}\delta s^{(0)\alpha}\wedge\delta s^{(0)\beta} + \underset{(01)}{\overset{(0)}{W}}{}^{\gamma}_{\beta\alpha}\delta s^{(0)\alpha}\wedge\delta s^{(1)\beta} + \underset{(02)}{\overset{(0)}{W}}{}^{\gamma}_{\beta\alpha}\delta s^{(0)\alpha}\wedge\delta s^{(2)\beta} \tag{6.2}$$

$$d(\delta s^{(1)\gamma}) = \underset{(01)}{\overset{(1)}{W}}{}^{\gamma}_{\beta\alpha}\delta s^{(0)\alpha}\wedge\delta s^{(1)\beta} + \underset{(11)}{\overset{(1)}{W}}{}^{\gamma}_{\beta\alpha}\delta s^{(1)\alpha}\wedge\delta s^{(1)\beta} + \underset{(12)}{\overset{(1)}{W}}{}^{\gamma}_{\beta\alpha}\delta s^{(1)\alpha}\wedge\delta s^{(2)\beta} \tag{6.3}$$

$$d(\delta s^{(2)\gamma}) = \underset{(02)}{\overset{(2)}{W}}{}^{\gamma}_{\beta\alpha}\delta s^{(0)\alpha}\wedge\delta s^{(2)\beta} + \underset{(12)}{\overset{(2)}{W}}{}^{\gamma}_{\beta\alpha}\delta s^{(1)\alpha}\wedge\delta s^{(2)\beta} + \underset{(22)}{\overset{(2)}{W}}{}^{\gamma}_{\beta\alpha}\delta s^{(2)\alpha}\wedge\delta s^{(2)\beta} \tag{6.4}$$

Using the invariant 1-form of connection of C we prove

Theorem 6.2 *The structure equations of the canonical metrical N-linear connection C in invariant frames are given by the relations:*

$$d(\delta s^{(A)\alpha}) \; - \; \delta s^{(A)\beta}\wedge \overset{(A)}{\omega}{}^{\alpha}{}_{\beta} = \; - \; \overset{(A)}{\Omega}{}^{\alpha} \tag{6.5}$$

$$d(\overset{(A)}{\omega}{}^{\alpha}{}_{\beta}) - \overset{(A)}{\omega}{}^{\gamma}{}_{\beta}\wedge\overset{(A)}{\omega}{}^{\alpha}{}_{\gamma} = - \overset{(A)}{\Omega}{}^{\alpha}{}_{\beta} \qquad (A=0,1,2), \tag{6.6}$$

where the 2-forms of torsion $\overset{(A)}{\Omega}{}^{\alpha}$ are given by:

$$\overset{(0)}{\Omega}{}^{\alpha} = \underset{(1)}{K}{}^{\alpha}{}_{\beta\gamma}\delta s^{(0)\beta}\wedge\delta s^{(1)\gamma}; \quad \overset{(1)}{\Omega}{}^{\alpha} = \underset{(21)}{P}{}^{\alpha}{}_{\beta\gamma}\delta s^{(0)\beta}\wedge\delta s^{(2)\gamma}; \quad \overset{(2)}{\Omega}{}^{\alpha} = 0 \tag{6.7}$$

and the 2-forms of curvature $\overset{(A)}{\Omega}{}^{\alpha}{}_{\beta}$, (A=0,1,2), are given by:

$$\overset{(A)}{\Omega}{}^{\alpha}{}_{\beta} = \underset{(1)}{\overset{(A)}{P}}{}_{\beta}{}^{\alpha}{}_{\varphi\psi}\delta s^{(0)\varphi}\wedge\delta s^{(1)\psi} + \underset{(11)}{\overset{(A)}{S}}{}_{\beta}{}^{\alpha}{}_{\varphi\psi}\delta s^{(1)\varphi}\wedge\delta s^{(1)\psi} + \underset{(21)}{\overset{(A)}{S}}{}_{\beta}{}^{\alpha}{}_{\varphi\psi}\delta s^{(1)\varphi}\wedge\delta s^{(2)\psi} \tag{6.8}$$

7. Vrânceanu identities

Let consider the Jacobi identity for three vector fields $X, Y, Z \in \chi(E)$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{7.1}$$

Using the projectors h-, v_1-, v_2- applied on the vector fields X, Y, Z we obtain:

$$[X^A,[Y^B,Z^C]] + [Y^B,[Z^C,X^A]] + [Z^C,[X^A,Y^B]] = 0 \tag{7.2}$$

$$A,B,C = 0,1,2; \quad A \le B \le C; \quad T^0 = hT,\ T^1 = v_1T,\ T^2 = v_2T, \quad T \in \{X,Y,Z\}$$

The action of a field X on a function

$$f \in \mathcal{F}(E), \qquad (x, y^{(1)}, y^{(2)}) \longmapsto f(x, y^{(1)}, y^{(2)}) \in \mathbf{R}$$

is given by the action of the operators $\dfrac{\delta}{\delta s^{(a)\alpha}}$ (a=0,1,2) on f.
Let consider the table

X	U	U	U	U	U	U	V	V	V	W
Y	U	U	U	V	V	W	V	V	W	W
Z	U	V	W	V	W	W	V	W	W	W

where $U = \dfrac{\delta}{\delta s^{(0)\alpha}}$, $V = \dfrac{\delta}{\delta s^{(1)\alpha}}$, $W = \dfrac{\delta}{\delta s^{(2)\alpha}}$.
Using identity (7.2) and the linear independence of the system

$$\left\{ \frac{\delta}{\delta s^{(0)\alpha}}, \ \frac{\delta}{\delta s^{(1)\alpha}}, \ \frac{\delta}{\delta s^{(2)\alpha}} \right\}$$

we can obtain a lot of identities. From them we mention:

Proposition 7.1 *The non-holonomy coefficients of Vranceanu satisfy the fundamental identities:*

$$\sum_{\alpha,\beta,\gamma} \left(\underset{(AA)}{\overset{A}{W}}{}^{\sigma}_{\beta\gamma} \ \underset{(AA)}{\overset{A}{W}}{}^{\eta}_{\alpha\sigma} + \frac{\delta \underset{(AA)}{\overset{A}{W}}{}^{\eta}_{\beta\gamma}}{\delta s^{(A)\alpha}} \right) = 0 \tag{7.3}$$

cyclic sumation on α, β, γ, *A=0,1,2.*

$$\diamondsuit(\alpha,\beta) \left[\underset{(01)}{\overset{0}{W}}{}^{\sigma}_{\beta\gamma} \underset{(00)}{\overset{0}{W}}{}^{\eta}_{\alpha\sigma} + \underset{(01)}{\overset{1}{W}}{}^{\sigma}_{\beta\gamma} \underset{(01)}{\overset{0}{W}}{}^{\eta}_{\alpha\sigma} + \frac{\delta \underset{(01)}{\overset{0}{W}}{}^{\eta}_{\beta\gamma}}{\delta s^{(0)\alpha}} + \frac{1}{2} \underset{(00)}{\overset{0}{W}}{}^{\sigma}_{\alpha\beta} \underset{(01)}{\overset{0}{W}}{}^{\eta}_{\sigma\gamma} + \frac{1}{2} \frac{\delta \underset{(00)}{\overset{0}{W}}{}^{\eta}_{\alpha\beta}}{\delta s^{(1)\gamma}} \right] = 0 \tag{7.4}$$

$$\diamondsuit(\alpha,\beta) \left[\underset{(01)}{\overset{1}{W}}{}^{\sigma}_{\beta\gamma} \underset{(01)}{\overset{1}{W}}{}^{\eta}_{\alpha\sigma} + \frac{\delta \underset{(01)}{\overset{1}{W}}{}^{\eta}_{\beta\gamma}}{\delta s^{(0)\alpha}} + \frac{1}{2} \underset{(00)}{\overset{0}{W}}{}^{\sigma}_{\alpha\beta} \underset{(01)}{\overset{1}{W}}{}^{\eta}_{\sigma\gamma} \right] = 0$$

$$\diamondsuit(\alpha,\beta) \left[\underset{(02)}{\overset{0}{W}}{}^{\sigma}_{\beta\gamma} \underset{(00)}{\overset{0}{W}}{}^{\eta}_{\alpha\sigma} + \underset{(02)}{\overset{2}{W}}{}^{\sigma}_{\beta\gamma} \underset{(02)}{\overset{0}{W}}{}^{\eta}_{\alpha\sigma} + \frac{\delta \underset{(02)}{\overset{0}{W}}{}^{\eta}_{\beta\gamma}}{\delta s^{(0)\alpha}} + \frac{1}{2} \underset{(00)}{\overset{0}{W}}{}^{\sigma}_{\alpha\beta} \underset{(02)}{\overset{0}{W}}{}^{\eta}_{\sigma\gamma} + \frac{1}{2} \frac{\delta \underset{(00)}{\overset{0}{W}}{}^{\eta}_{\alpha\beta}}{\delta s^{(1)\gamma}} + \frac{1}{2} \frac{\delta \underset{(00)}{\overset{0}{W}}{}^{\eta}_{\alpha\beta}}{\delta s^{(2)\gamma}} \right] = 0$$

$$\frac{\delta \underset{(01)}{\overset{0}{W}}{}^{\eta}_{\beta\gamma}}{\delta s^{(0)\alpha}} + \frac{\delta \underset{(12)}{\overset{1}{W}}{}^{\eta}_{\beta\gamma}}{\delta s^{(0)\alpha}} + \underset{(12)}{\overset{1}{W}}{}^{\sigma}_{\beta\gamma} \underset{(01)}{\overset{1}{W}}{}^{\eta}_{\alpha\sigma} + \underset{(02)}{\overset{0}{W}}{}^{\sigma}_{\beta\gamma} \underset{(01)}{\overset{1}{W}}{}^{\eta}_{\alpha\sigma} + \underset{(02)}{\overset{2}{W}}{}^{\sigma}_{\beta\gamma} \underset{(12)}{\overset{1}{W}}{}^{\eta}_{\alpha\sigma} - \underset{(01)}{\overset{1}{W}}{}^{\sigma}_{\alpha\beta} \underset{(12)}{\overset{1}{W}}{}^{\eta}_{\sigma\gamma} = 0 \tag{7.5}$$

$$\frac{\delta \underset{(12)}{\overset{2}{W}}{}^{\eta}_{\beta\gamma}}{\delta s^{(0)\alpha}} + \frac{\delta \underset{(02)}{\overset{2}{W}}{}^{\eta}_{\beta\gamma}}{\delta s^{(0)\alpha}} + \underset{(12)}{\overset{2}{W}}{}^{\sigma}_{\beta\gamma} \underset{(02)}{\overset{2}{W}}{}^{\eta}_{\alpha\sigma} - \underset{(01)}{\overset{0}{W}}{}^{\sigma}_{\alpha\beta} \underset{(02)}{\overset{2}{W}}{}^{\eta}_{\sigma\gamma} - \underset{(01)}{\overset{1}{W}}{}^{\sigma}_{\alpha\beta} \underset{(12)}{\overset{2}{W}}{}^{\eta}_{\sigma\gamma} = 0$$

$\diamondsuit(\alpha,\beta)$ *meaning index permutation and subtracting results,*

8. Maxwell equations in invariant frames

Let

$$Y^{(1)} = y^{(1)i} \frac{\delta}{\delta y^{(1)i}} \ , \quad Y^{(2)} = y^{(2)i} \frac{\delta}{\delta y^{(2)i}}.$$

be a vector field. Its expression in an invariant frame is

$$Y^{(1)} = y^{(1)i} f^{(1)\alpha}_{\ i} \frac{\delta}{\delta s^{(1)\alpha}}; \qquad Y^{(2)} = y^{(2)i} f^{(2)\alpha}_{\ i} \frac{\delta}{\delta s^{(2)\alpha}} \tag{8.1}$$

Define:

$$s^{(1)\alpha} = y^{(1)i} f^{(1)\alpha}_{\ i}; \qquad s^{(2)\alpha} = y^{(2)i} f^{(2)\alpha}_{\ i} \tag{8.2}$$

Denoting:

$$q^{(1)\alpha} = s^{(1)\alpha}; \qquad q^{(2)\alpha} = s^{(2)\alpha} + \frac{1}{2} \underset{(1)}{M}{}^{\alpha}{}_{\beta} s^{(1)\beta} \tag{8.3}$$

Then the Liouville vector fields in invariant frames are:

$$\overset{1}{\Gamma} = q^{(1)\alpha} \frac{\delta}{\delta s^{(2)\beta}} \tag{8.4}$$

$$\overset{2}{\Gamma} = q^{(1)\alpha} \frac{\delta}{\delta s^{(1)\alpha}} + 2 q^{(2)\alpha} \frac{\delta}{\delta s^{(2)\alpha}}$$

Consider the case where the frame has identical components on the three directions. We denote by:

$$e^{(0)i}_{\ \alpha} = e^{(1)i}_{\ \alpha} = e^{(2)i}_{\ \alpha} = e^{i}_{\alpha} \tag{8.5}$$

and for the duals we have:

$$f^{(0)\alpha}_{\ i} = f^{(1)\alpha}_{\ i} = f^{(2)\alpha}_{\ i} = f^{\alpha}_{i} \tag{8.6}$$

Then:

$$\overset{(00)}{L}{}^{\alpha}{}_{\beta\gamma} = \overset{(01)}{L}{}^{\alpha}{}_{\beta\gamma} = \overset{(02)}{L}{}^{\alpha}{}_{\beta\gamma} = \frac{1}{2} \underset{(00)}{\overset{0}{W}}{}^{\alpha}_{\beta\gamma} \tag{8.7}$$

$$\overset{(10)}{C}{}^{\alpha}{}_{\beta\gamma} = \overset{(11)}{C}{}^{\alpha}{}_{\beta\gamma} = \overset{(12)}{C}{}^{\alpha}{}_{\beta\gamma} = \frac{1}{2} \underset{(11)}{\overset{1}{W}}{}^{\alpha}_{\beta\gamma} + f^{\alpha}_{i} e^{j}_{\beta} e^{k}_{\gamma} \underset{(1)}{C}{}^{i}{}_{jk}$$

$$\overset{(20)}{C}{}^{\alpha}{}_{\beta\gamma} = \overset{(21)}{C}{}^{\alpha}{}_{\beta\gamma} = \overset{(22)}{C}{}^{\alpha}{}_{\beta\gamma} = \frac{1}{2} \underset{(22)}{\overset{2}{W}}{}^{\alpha}_{\beta\gamma}$$

We denote this connection by $C\bar{\Gamma}(N)$.

Definition 8.1 *The deflection tensors of $C\bar{\Gamma}(N)$ are:*

$$D^{(A)\alpha}{}_{\beta} = q^{(A)\alpha}{}_{)\beta}; \qquad d^{(AB)\alpha}{}_{\beta} = q^{(A)\alpha} \overset{(B)}{)}{}_{\beta} \qquad (A, B = 1, 2) \tag{8.8}$$

Definition 8.2 *The invariant electromagnetic tensor fields are:*

$$F^{(A)}{}_{\alpha\beta} = \frac{1}{2} \left(\frac{\delta\, q^{(A)}_{\ \alpha}}{\delta s^{(0)\beta}} - \frac{\delta\, q^{(A)}_{\ \beta}}{\delta s^{(0)\alpha}} \right) \tag{8.9}$$

$$f^{(AB)}{}_{\alpha\beta} = \frac{1}{2} \left(\frac{\delta\, q^{(A)}_{\ \alpha}}{\delta s^{(B)\beta}} - \frac{\delta\, q^{(A)}_{\ \beta}}{\delta s^{(B)\alpha}} \right) \qquad (A, B = 1, 2)$$

where $q^{(A)}{}_{\alpha} = \epsilon_{\alpha\beta} q^{(A)\beta}$ and $\epsilon_{\alpha\beta}$ are the invariant components of the metric tensor

Theorem 8.1 *The invariant electromagnetic tensor fields satisfy the following generalized Maxwell equations*

$$\sum_{cicl\ \alpha\beta\gamma} F^{(A)}{}_{\alpha\beta)\gamma} = 0 \tag{8.10}$$

$$\sum_{cicl\ \alpha\beta\gamma} F^{(A)}{}_{\alpha\beta}\overset{(1)}{)}_{\gamma} + \sum_{cicl\ \alpha\beta\gamma} f^{(A1)}{}_{\alpha\beta)\gamma} = \sum_{cicl\ \alpha\beta\gamma} \left\{ q^{(A)\eta}\left(\underset{(1)}{P}_{\eta\beta\alpha\gamma} - \underset{(1)}{P}_{\eta\beta\gamma\alpha}\right) - d^{(A1)}{}_{\beta\eta}\left(\underset{(A)}{\overset{(0)}{P}}{}^{\eta}{}_{\alpha\gamma} - \underset{(A)}{\overset{(0)}{P}}{}^{\eta}{}_{\gamma\alpha}\right)\right\}$$

$$\sum_{cicl\ \alpha\beta\gamma} F^{(A)}{}_{\alpha\beta}\overset{(2)}{)}_{\gamma} + \sum_{cicl\ \alpha\beta\gamma} f^{(A2)}{}_{\alpha\beta)\gamma} = 0$$

$$\sum_{cicl\ \alpha\beta\gamma} f^{(A1)}{}_{\alpha\beta}\overset{(1)}{)}_{\gamma} = \sum_{cicl\ \alpha\beta\gamma} q^{(A)\eta} \underset{(A1)}{S}{}_{\eta\beta\alpha\gamma}$$

$$\sum_{cicl\ \alpha\beta\gamma} f^{(A2)}{}_{\alpha\beta}\overset{(2)}{)}_{\gamma} = 0$$

$$\sum_{cicl\ \alpha\beta\gamma} f^{(A1)}{}_{\alpha\beta}\overset{(2)}{)}_{\gamma} = \sum_{cicl\ \alpha\beta\gamma} \left(d^{(A1)}{}_{\beta\eta} \underset{(21)}{P}{}^{\eta}{}_{\alpha\gamma} - d^{(A1)}{}_{\beta\eta}\left(\underset{(1)}{C}{}^{\eta}{}_{\alpha\gamma} - \underset{(1)}{C}{}^{\eta}{}_{\gamma\alpha}\right)\right) \tag{8.11}$$

$$\sum_{cicl\ \alpha\beta\gamma} f^{(A2)}{}_{\alpha\beta}\overset{(1)}{)}_{\gamma} = \sum_{cicl\ \alpha\beta\gamma} q^{(A)\eta} \underset{(21)}{S}{}_{\eta\beta\alpha\gamma}$$

By direct calculations we see that:

$$\begin{aligned} F^{(A)}{}_{\alpha\beta} &= \frac{1}{2} \underset{(00)}{\overset{0}{W}}{}_{\alpha\varphi\beta} s^{(A)\varphi}, \\ f^{(AB)}{}_{\alpha\beta} &= \frac{1}{2} \underset{(AA)}{\overset{A}{W}}{}_{\alpha\varphi\beta} s^{(B)\varphi} \qquad (A, B = 1, 2) \end{aligned} \tag{8.12}$$

Theorem 8.2 *In a generalized Lagrange space endowed with a Berwald-Moor type metric, w.r.t. the canonical metrical N-linear connection and in normal invariant frames Maxwell equations 8.10 are equivalent to Vranceanu identities 7.3–7.5.*

References

[AN] **Anastasiei, M.**, *Vector bundles. Einstein Equations.*, An. St. Univ. Al. I. Cuza Iaşi, 32, (1986), 17–24.

[AB] **Antonelli, P. L., Bucătaru, I.**, *On Holland's frame for Randers spaces and its applications in physics*, Proc. Col. on Diff Geom, 23–30 july, Debrecen (2000), 39–53.

[AT5] **Atanasiu, Gh.**, *The Equations of structure of an N-linear connection in the bundle of accelerations*, Balkan Journal of Geometry and its Appl., Vol I, No 1 (1996), 11–19.

[BB] **Balan, V., Brinzei, N.,**, *Einstein equations for the Berwald-Moór type Euclidean-locally Minkowsky relativistic model*, 5-th Conf of B.S.G, Mangalia, Romania 27–31 aug 2005.

[IN] **Ingarden, R. S.**, *Differential geometry and physics*, Tensor, N. S. 30, (1976), 201–209.

[M] **Miron, R.**, *A Lagrangeian theory of relativity*, I,II,An. St. Univ. Al. I. Cuza Iaşi, sI a Mat. 32, (1986), 37–62 şi 33, (1987) 137–149.

[M1] **Miron, R.**, *The geometry of higher order Lagrange spaces. Applications to Mechanics and Physics*, Kluwer Acad.Publ. FTPH (1996).

[M2] **Miron, R.**, *Invariant frames in the normal distribution of a non-holonomic manifold of a Riemann space*, Luc. Conf. Nat. Sp. Neolo. Ed. Acad. R.S.R. (1979), 111–119.

[MA] **Miron, R., Atanasiu, Gh.**, *Geometrical Theory of gravitational and Electromagnetic fields in higher order Lagrange spaces*, Tsukuba j. Math. vol 20, No 1, (1996), 137–149.

[MAN] **Miron, R., Anastasiei, M.**, *The geometry of Lagrange spaces. Theory and applications.*, Kluwer Acad. Pub. FTPH 59, (1994).

[MM] **Miron, R., Izumi, H.,**, *Invariant frame in generalized metric spaces*, Tensor, N. S., vol 42 (1985) 272, 282.

[P] **Păun, M.**, *Einstein Equations for a generalized Lagrange space of order two in invariant frames*, Proc Glob An., Diff. Geom. Lie Alg. G.B.P. 4, (1998), 76–82.

[P1] **Păun, M.**, *Structure equations in invariant frames*, Proc Glob An., Diff. Geom. Lie Alg. G.B.P. 3, (1996), 55–61.

[P2] **Păun, M.**, *The concept of invariant geometry of second order*, Balkan J. of Geom. vol 7, no 2, 2002, 57–66.

[P4] **Păun, M.**, *Fundamental identities in invariant frames*, Proc Glob An., Diff. Geom. Lie Alg. G.B.P. 5, (1999), 63–72.

[PA] **Pavlov, D. G.**, *Generalization of scalar product axioms*, Hypercomplex Numbers in Geometry and Physics, Ed. Mozet, Russia, 1, 1 (2004), 5–18.

[PA1] **Pavlov, D. G.**, *Chronometry of the three-dimensional time*, Hypercomplex Numbers in Geometry and Physics, Ed. Mozet, Russia, 1, 1 (2004), 19–30.

[PA2] **Pavlov, D. G.**, *Four-dimensional time*, Hypercomplex Numbers in Geometry and Physics, Ed. Mozet, Russia, 1, 1 (2004), 31–39.

[VR] **Vrănceanu, Gh.**, *Les espaces non holonomes*, Gauthier, Paris, (1936).

[VR1] **Vrănceanu, Gh.**, *Lecons de geometrie differentielle*, volI,II,III, Ed. Acad. Rom. Bucureşti, (1957).

[U] **Udrişte, C.**, *Congruences on the tangent bundle of a differentiabil manifold*, Rev. Rou. Math. Pur. et Appl. 15, (1970), 1079–1096.

[U3] **Udrişte, C.**, *Geometric Dynamics*, Southeast As. Bull. Math. Springer-Verlag,24,1, (2000), 1–11.