GEODETICS, CONNECTIONS AND JACOBI FIELDS FOR BERWALD-MOOR QUARTIC METRICS

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For Finsler spaces (M, F) with quartic metrics $F = \sqrt[4]{G_{ijkl}(x,y)y^i y^j y^k y^l}$, we determine the equations of geodesics and the corresponding arising geometrical objects-canonical spray, nonlinear Cartan connection, Berwald linear connection – in terms of the non-homogenized flag Lagrange metric $h_{ij} = G_{ij00}$. Further, are studied the geodesics and Jacobi fields of the tangent space TM for hv-metric models.

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1 The equations of geodesics in quartic Berwald-Moor spaces

Let (M, F) be an n-dimensional Finsler space. We shall denote by (x, y) the local coordinates on TM and by the signs "," and ";" preceding an index, the partial derivative relative to the corresponding component of x and of the direction y, respectively. Let G_{ijkl} be the local components of the 0-homogeneous 4-metric

$$G_{ijkl}(x,y) = \frac{1}{4!}(F^4)_{;ijkl}. \tag{1.1}$$

We denote by h_{ij} the flag non-homogenized metric

$$h_{ij} = \frac{1}{12}(F^4)_{;ij} \tag{1.2}$$

which coincides with the tensor field $y^{(4)}_{ij}$ from ([9]). We shall further prove that h_{ij} is nondegenerate. The link between the two tensors (1.1) and (1.2) is

$$h_{ij} = G_{ij00}, \quad G_{ijkl} = \frac{1}{2}h_{ij;kl}$$

where the index 0 means transvection by y. We consider the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0 \tag{1.3}$$

and we look for the solutions $c : t \in [0, 1] \to x(t) \in M$, parametrized by arclength, this is, $v(t) = 1, \forall t \in [0, 1]$, where

$$v(t) = F(x(t), y(t)), \quad y(t) = \frac{dx}{dt}(t), \quad \forall t \in [0, 1].$$

Then we have the following

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Proposition 1. *The system (1.3) is equivalent with*

$$\frac{d}{dt}\left(\frac{\partial F^\alpha}{\partial y^i}\right) - \frac{\partial F^\alpha}{\partial x^i} = 0, \ \alpha \neq 0. \tag{1.4}$$

Proof. We have $\frac{\partial F^\alpha}{\partial x^i} = \alpha F^{\alpha-1}\frac{\partial F}{\partial x^i}$, $\frac{\partial F^\alpha}{\partial y^i} = \alpha F^{\alpha-1}\frac{\partial F}{\partial y^i}$, and since c is a unit-speed curve, it follows that $\frac{dv}{dt} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial F^\alpha}{\partial y^i}\right) = \alpha F^{\alpha-1}\frac{d}{dt}(\frac{\partial F}{\partial y^i})$, which lead to the claim. □

Remark. In particular, for $\alpha = 4$, (1.4) leads to

$$\frac{d}{dt}\left(\frac{\partial F^4}{\partial y^i}\right) - \frac{\partial F^4}{\partial x^i} = 0. \tag{1.5}$$

Using $F^4 = G_{mjkl}y^m y^j y^k y^l$, it follows ([5]) $(F^4)_{;i} = 4G_{i000}$, and further,

$$\begin{aligned}\tfrac{d}{dt}\left(\tfrac{\partial F^4}{\partial y^i}\right) &= 4\tfrac{dG_{ijkl}}{dt}y^j y^k y^l + 12G_{ijkl}\tfrac{dy^j}{dt}y^k y^l = \\ &= 12G_{ijkl}\tfrac{dy^j}{dt}y^k y^l + 4\left(\tfrac{\partial G_{ijkl}}{\partial x^m}y^m y^j y^k y^l + G_{ijkl;m}\tfrac{dy^m}{dt}y^j y^k y^l\right).\end{aligned}$$

Since G_{ijkl} is 0-homogeneous, using Euler's relation we infer

$$G_{ijkl;m}\frac{dy^m}{dt}y^j y^k y^l = (G_{imkl;j}y^j)\frac{dy^m}{dt}y^k y^l = 0 \tag{1.6}$$

and hence

$$\frac{d}{dt}\left(\frac{\partial F^4}{\partial y^i}\right) = 12G_{ijkl}\frac{dy^j}{dt}y^k y^l + 4\frac{\partial G_{ijkl}}{\partial x^m}y^m y^j y^k y^l.$$

Replacing (1.6) and the x^i-derivative $(F^4)_{,i} = G_{mjkl,i}y^m y^j y^k y^l$ in the Euler-Lagrange equation (1.5), this rewrites

$$12G_{ijkl}y^k y^l\frac{dy^j}{dt} + (4G_{ijkl,m} - G_{mjkl,i})y^m y^j y^k y^l = 0, \tag{1.7}$$

where $y^i = \frac{dx^i}{dt}$. Using the notation $h_{ij} = y^{(4)}_{ij} = G_{ijkl}y^k y^l$ ([9]), (1.7) becomes

$$h_{ij}\frac{dy^j}{dt} + \frac{1}{12}(4G_{ijkl,m} - G_{jklm,i})y^m y^j y^k y^l = 0. \tag{1.8}$$

Denoting

$$\gamma^i_{jklm} = \frac{1}{12}h^{ip}\gamma_{p\,jklm}, \quad \gamma_{p\,jklm} = (4G_{pjkl,m} - G_{jklm,p}), \tag{1.9}$$

we note that ${\gamma^i}_{jklm}$ is symmetric w.r.t. the first three lower indices and the equations of geodesics can be written as

$$\frac{dy^i}{dt} + {\gamma^i}_{jklm}y^j y^k y^l y^m = 0. \tag{1.10}$$

As well, denoting $\tilde\gamma^i_{jklm} = h^{ip}\tilde\gamma_{p\,jklm}/12$, where

$$\tilde\gamma_{p\,jklm} = G_{pjkl,m} + G_{pmjk,l} + G_{plmj,k} + G_{pklm,j} - G_{mjkl,p},$$

we can easily see that (1.10) can be rewritten as

$$\frac{dy^i}{dt} + {\tilde\gamma^i}_{jklm}y^j y^k y^l y^m = 0. \tag{1.11}$$

Remarks. 1. The tensor with which we have raised the indices is $h_{ij} = y^{(4)}_{ij}$, not $\widetilde{y}^{(4)}_{ij} = F_{;i}F_{;j} - h_{ij}$ (cf. [9]), which is degenerate, as we shall further prove. The equations of geodesics (1.10) can be expressed only in terms of the non-homogenized flag 2-metric $h_{ij} = G_{ij00}$. Having in view that $G_{ijkl,m}y^k y^l = h_{ij,m}$, we rewrite (1.8) as

$$h_{is}\frac{dy^s}{dt} + \frac{1}{12}(4h_{ij,k} - h_{jk,i})y^j y^k = 0, \tag{1.12}$$

or, still

$$\frac{dy^i}{dt} + \frac{h^{is}}{12}(4h_{ij,k} - h_{jk,i})y^j y^k = 0. \tag{1.13}$$

Applying the variational principle to $F^4 = h_{ij}y^i y^j$ one gets the same equations of geodesics (1.12), which are the equations of geodesics of the Lagrange space (M, L) with the Lagrangian $L = F^4 = h_{ij}y^i y^j = G_{ijkl}y^i y^j y^k y^l$.

Unfortunately, the coefficients $\gamma^i_{jm00} = h^{ij}(4h_{ij,m} - h_{mj,i})/12$ can *not* stand for the coefficients of a linear connection on TM.

Last but not least, we point out several considerations regarding the used (0,2) tensor fields. We shall further skip for brevity the symbol ";" in the partials of F w.r.t. y (e.g., $F_i = F_{;i}, F_{ij} = F_{;ij}$, etc). Let $l_i = F^{-1}y_i = F_i$, where $y_i = g_{ij}y^j$ and $g_{ij} = (F^2)_{;ij}/2$ is the *fundamental Finsler metric tensor field.* Then we have:

Proposition 2. *Consider the following family of* $(0,2)$*-tensor fields*

$$\Theta_{ij} = \lambda g_{ij} + \mu l_i l_j, \quad \lambda, \mu \in \mathcal{F}(M), \tag{1.14}$$

Denote by g^{ij} *the dual and by* δ *the determinant of* g_{ij}*. Then*

a) Θ_{ij} *is non-degenerate for* $\lambda(\lambda+\mu) \neq 0$ *on* TM*.*

b) The dual of Θ_{ij} *is*

$$\Theta^{ij} = \frac{1}{\lambda}g^{ij} + \frac{-\mu}{\lambda(\lambda+\mu)F^2}y^i y^j.$$

c) The determinant of Θ_{ij} *is*

$$\Delta = \lambda^{n-1}(\lambda+\mu)\cdot\delta.$$

Proof. From the 1-homogeneity of F follow $F_i y^i = F$, $F_{ij}y^j = 0$, $y_i = FF_i$. The claim follows using these relations and from straightforward calculation using properties of determinants.

Lemma. *Consider the matrix* $\tilde{\Gamma} = (\tilde{\gamma}_{ij})_{i,j\in\overline{1,n}}$, $\tilde{\gamma}_{ij} = \gamma_{ij} + u_i u_j$, *with* $\Gamma = (\gamma_{ij})_{i,j\in\overline{1,n}}$ *non-degenerate. Then:*
a) The inverse of $\tilde{\Gamma}$ *has the coefficients* $\tilde{\gamma}^{ij} = \gamma^{ij} - (1 + u_s u^s)^{-1}u^i u^j$, *where* $u^i = \gamma^{is}u_s$.
b) We have $\det(\tilde{\Gamma}) = \det(\Gamma)\cdot(1 + u_s u^s)$.

Particular cases.

1. Obviously, g_{ij} is part of the pencil (1.14), obtained for $\lambda = 1, \mu = 0$.

2. We note that $g_{ij} = FF_{ij} + F_i F_j$ which infers that

$$\Theta_{ij} = \lambda FF_{ij} + (\lambda+\mu)F_i F_j, \quad \lambda, \mu \in \mathcal{F}(M), \tag{1.15}$$

where both tensor fields F_{ij} and $F_i \cdot F_j$ are degenerate.

3. For $\lambda = 1, \mu = -1$ (1.14) provides *the angular metric*

$$\hat{g}_{ij} = g_{ij} - l_i l_j. \tag{1.16}$$

Its halved version-denoted by $\tilde{y}_{ij}$, is employed in [9, (10)].

4. From (1.15) we respectively obtain the tensor fields used in [9, (13), (20')], as particular cases:

$$\begin{aligned} h_{ij} &= y^{(4)}_{ij} = \tfrac{1}{12}(F^4)_{;ij}, \quad &&\text{for} \;\; \lambda = F^2/3, \mu = 2F^2/3 \\ \tilde{y}^{(4)}_{ij} &= y_i y_j - y^{(4)}_{ij}, &&\text{for} \;\; \lambda = -\mu = -F^2/3. \end{aligned} \tag{1.17}$$

We emphasize that the $(0,2)$-tensor field $\tilde{y}^{(4)}_{ij}$ satisfies the following equalities

$$\tilde{y}^{(4)}_{ij} = -F^3 F_{ij}/3 = -F^2 \hat{g}_{ij}/3,$$

and hence has the property of $\hat{g}_{ij}$ of being degenerate.

We note that the proposition above provides for $\lambda = 1 + \alpha, \mu = -\alpha \in \mathbb{R}$ the following

Corollary ([1]). *The following* $(0,2)$ *Finsler tensor fields are 0-homogeneous and non-degenerate:*

$$g_{ij} + \alpha \hat{g}_{ij}, \alpha \in \mathbb{R}. \tag{1.18}$$

Regarding h_{ij}, this can be homogenized by dividing to F^2. According to the Corollary, the resulting (Generalized Lagrange) homogeneous metric is included in the family of metrics (1.18). More exactly, we have

$$\frac{h_{ij}}{F^2} = \frac{1}{12F^2}\left[2F^2(F^2)_{;i}\right]_{;j} = \frac{1}{6}\left[(F^2)_{;ij} + 4F^{-2} y_i y_j\right] = g_{ij} + \alpha \hat{g}_{ij}, \;\; \alpha = -2/3.$$

Definition 1. We call *generalized 4-index angular metric tensor*, the tensor field

$$\omega_{ijkl} \equiv G_{ijkl} - l_i l_j l_k l_l. \tag{1.19}$$

This definition may be easily extended to any number of indices. In analogy with [1] we have the following

Proposition 3. *The tensors of form* $\tilde{G}_{ijkl} = G_{ijkl} + \alpha \omega_{ijkl}$, $\alpha \in \mathbb{R}$ *are generalized metric tensors which share the same energy* F^4.

Proof. Using that $l_i y^i = F^{-1} y_i y^i = F$, we get $\tilde{G}_{0000} = G_{0000} + \alpha(G_{0000} - (l_s y^s)^4) = F^4$, whence the claim follows.

We should note as well the relation

$$\omega_{ij00} = G_{ij00} - F^2 l_i l_j = h_{ij} - F^2 F_i F_j = F^2(F_i F_j + F F_{ij}/3) - F^2 F_i F_j = -\tilde{y}^{(4)}_{ij}.$$

2 The nonlinear connection

Consider the semispray given by the second term in the equations of geodesics (1.13)

$$2G^i = \frac{h^{ip}}{12}(4G_{pjkl,m} - G_{mjkl,p}) y^m y^j y^k y^l.$$

By taking into account (1.1) and the 1-homogeneity of F, we get $G_{mjkl} y^m y^j y^k y^l = F^4$, $G_{pjkl} y^j y^k y^l = G_{p000} = \frac{1}{4}(F^4)_{;p}$, and hence G^i can be written as

$$2G^i = \frac{h^{ip}}{12}\left(\frac{\partial^2 F^4}{\partial x^m \partial y^p} y^m - \frac{\partial F^4}{\partial x^j}\right). \tag{2.1}$$

Within the Lagrange structure $(M, L = F^4/6)$, where the classical Lagrange metric induced by L is $h_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$, (2.1) is exactly the Kern canonical semi-spray of L ([8], [11, Theorem 7.4.1, p. 113]),

$$G^i = \frac{h^{ip}}{4}\left(\frac{\partial^2 L}{\partial x^m \partial y^p} y^m - \frac{\partial L}{\partial x^j}\right). \tag{2.2}$$

and $N^i{}_j = \frac{\partial G^i}{\partial y^j}$ are the Kern coefficients of the canonical nonlinear connection attached to L on TM. Its autoparallel curves described by (1.13) are exactly the geodesics determined by L. Then the equations (1.10) can be written as

$$\frac{d^2x^i}{dt^2} + 2G^i = 0 \Leftrightarrow \frac{d^2x^i}{dt^2} + N^i{}_j y^j = 0,$$

or, denoting $\delta y^i = dy^i + N^i_s dx^s$,

$$\frac{\delta y^i}{dt} = 0.$$

Aiming to obtain a normal linear connection $(L^i{}_{jk}, C^i{}_{jk})$ on TM, one possible choice is, for example, $L^i{}_{jk} = \frac{\partial N^i{}_j}{\partial y^k}$ and $C^i_{jk} = 0$. Then the equations of geodesics rewrite

$$\frac{d^2x^i}{dt^2} + L^i{}_{jk} y^j y^k = 0.$$

Remark. The candidates for a nonlinear connection

$$\tilde{N}^i{}_l = \frac{h^{ip}}{12}\left(4\frac{\partial G_{pjkl}}{\partial x^m} - \frac{\partial G_{mjkl}}{\partial x^p}\right) y^m y^j y^k = \gamma^i{}_{j000},$$

i.e., the coefficients of y^l from the equations of geodesics from (1.10), do not obey the specific component changes; hence they do *not* define a nonlinear connection.

3 Geodesics in the (h, v)-metric context

Let TM be endowed with: a nonlinear connection N, a metric structure

$$G = \underset{(0)}{g}{}_{ij} dx^i \otimes dx^j + \underset{(1)}{g}{}_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j},$$

where the metrics $\underset{(0)}{g}$ and $\underset{(1)}{g}$ can be specified as in the previous sections. Consider as well a metrical normal linear d-connection D, $D\Gamma(N) = (L^i{}_{jk}, C^i{}_{jk})$ ([11]). Then N induces a local adapted basis $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$, and the dual adapted basis, $\{dx^i, \delta y^i\}$. We denote by $\langle\ ,\ \rangle$ the scalar product defined on TM by G, by $\underset{(\beta\alpha)}{\overset{(\gamma)}{T}}{}^i{}_{jk}$ the components of the torsion tensor $T(\delta_{\alpha k}, \delta_{\beta j}) = \underset{(\beta\alpha)}{\overset{(\gamma)}{T}}{}^i{}_{jk} \delta_{\gamma i}$, and by $\underset{(\alpha\beta\gamma)}{\overset{(\alpha)}{R}}{}^i{}_{jkl}$ the components of the curvature tensor $R(\delta_{\gamma l}, \delta_{\beta k})\delta_{\alpha j} = \underset{(\alpha\beta\gamma)}{\overset{(\alpha)}{R}}{}^i{}_{jkl} \delta_{\alpha i}$, where $\delta_{0i} = \frac{\delta}{\delta x^i}$, $\delta_{1i} = \frac{\partial}{\partial y^i}$.

For a curve $c : [0, 1] \to TM,\ t \mapsto c(t) = (x^i(t), y^{(1)i}(t))$, we consider its velocity $V := V(t) = \dot{c} = V^{(\alpha)i}\delta_{\alpha i}$, where

$$V^{(0)i} = \frac{dx^i}{dt}, \quad V^{(1)i} = \frac{\delta y^{(1)i}}{dt}.$$

The energy of c is

$$E(c) = \int_0^1 \langle \dot{c}, \dot{c} \rangle dt = \int_0^1 \langle V, V \rangle dt = \int_0^1 \underset{(0)}{g}_{ij} V^{(0)i} V^{(0)j} + \underset{(1)}{g}_{ij} V^{(1)i} V^{(1)j} dt.$$

Theorem 1 (The first variation of energy). *If $c : [0,1] \to TM$, $\alpha : (-\varepsilon, \varepsilon) \times [0,1] \to TM$ is a variation of c by piecewise smooth curves with fixed ends, and $W = \frac{\partial \alpha}{\partial u}(0,t)$ is the associated deviation vector field, then the first variation of energy is given by*

$$\frac{1}{2} \left. \frac{dE(\bar{\alpha}(u))}{du} \right|_{u=0} = -\sum_{i=0}^{k-1} \langle W, \Delta_{t_i} V \rangle + \int_0^1 \langle T(W,V), V \rangle - \langle W, A \rangle dt,$$

where A is the acceleration vector field

$$A = D_{\dot{c}} V = \frac{DV}{dt} = A^{(0)i} \delta_{0i} + A^{(1)i} \delta_{1i}$$

and $\Delta_t X$ is the jump

$$\Delta_t X = X(t_+) - X(t_-), \quad t \in [0,1], \ X \in \mathcal{X}(TM).$$

We note that $\langle T(\,\cdot\,, V), V \rangle$ defines a 1-form. Hence there exists a vector field F on TM such that $\langle T(W,V), V \rangle = \langle F, W \rangle$. Then, denoting

$$V = V^{(\alpha)i} \delta_{\alpha i}, \quad W = W^{(\beta)j} \delta_{\beta j}, \quad F = \sum_{\alpha=0}^{1} F^{(\alpha)i} \delta_{\alpha i}$$

we have $\langle T(W,V), V \rangle = \sum\limits_{\beta=0}^{1} \underset{(\beta)}{g}_{jh} F^{(\beta)h} W^{(\beta)j}$, and the components of the field F are given by

$$F^{(\alpha)i} = \sum_{\beta,\gamma=0}^{1} \underset{(\alpha)}{g}^{il} \underset{(\gamma)}{g}_{kh} \overset{(\gamma)}{\underset{(\beta\alpha)}{T}}{}^{k}{}_{jl} V^{(\beta)j} V^{(\gamma)h}, \quad \alpha = \overline{0,1}.$$

Remark. The vector field F does not depend on the chosen variation with fixed endpoints of c.

By replacing F into the expression of the first variation of energy, we get

$$\frac{1}{2} \left. \frac{dE(\bar{\alpha}(u))}{du} \right|_{u=0} = -\sum_{i=0}^{k-1} \langle W, \Delta_{t_i} V \rangle + \int_0^1 \langle W, F - A \rangle dt.$$

For a smooth curve c on the whole $[0,1]$ the jumps in the sum cancel and we have

$$\frac{1}{2} \left. \frac{dE(\bar{\alpha}(u))}{du} \right|_{u=0} = \int_0^1 \langle W, F - A \rangle dt,$$

which means that $u = 0$ is a critical point of E if and only if, along $c = \bar{a}(0)$, we have $F = A$. Consequently we state the following

Theorem 2. *Any geodesic* $c : [0,1] \to TM$, $t \to (x^i(t), y^{(1)i}(t))$ *of* (TM, G) *satisfies*

$$\frac{D}{dt}\frac{dc}{dt} = F.$$

Then, the smooth curve $c : [0,1] \to TM$, $t \to (x^i(t), y^{(1)i}(t))$ is a geodesic of TM iff

$$\frac{DV^{(0)i}}{dt} = F^{(0)i}, \quad \frac{DV^{(1)i}}{dt} = F^{(1)i}, \tag{3.1}$$

which rewrites explicitely as

$$\begin{aligned} &\tfrac{dV^{(0)i}}{dt} + L^i{}_{jk}V^{(0)k}V^{(0)j} + C^i{}_{jk}V^{(1)k}V^{(0)j} = \sum_{\beta,\gamma=0}^{1} \underset{(0)}{g}{}^{il}\, \underset{(\gamma)}{g}{}_{kh}\, \underset{(\beta 0)}{\overset{(\gamma)}{T}}{}^k{}_{jl} V^{(\beta)j}V^{(\gamma)h} \\ &\tfrac{dV^{(1)i}}{dt} + L^i{}_{jk}V^{(0)k}V^{(1)j} + C^i{}_{jk}V^{(1)k}V^{(1)j} = \sum_{\beta,\gamma=0}^{1} \underset{(1)}{g}{}^{il}\, \underset{(\gamma)}{g}{}_{kh}\, \underset{(\beta 1)}{\overset{(\gamma)}{T}}{}^k{}_{jl} V^{(\beta)j}V^{(\gamma)h}. \end{aligned} \tag{3.2}$$

Example. In particular, in a Finsler space (M, F), for $g_{ij} = \underset{(0)}{g}{}_{ij} = \underset{(1)}{g}{}_{ij} = \frac{1}{2}F^2_{,y^iy^j}$ considering the Cartan connection ([11]), we infer that (3.2) rewrite

$$\begin{cases} \dfrac{d^2x^i}{dt} + L^i{}_{jk}V^{(0)k}V^{(0)j} + C^i{}_{jk}V^{(1)k}V^{(0)j} = g^{il}g_{kh}\big(R^k{}_{jl}V^{(0)j}V^{(1)h} - P^k_{lj}V^{(1)h}V^{(1)j} - C^k_{lj}V^{(0)h}V^{(1)j}\big) \\ \dfrac{dV^{(1)i}}{dt} + L^i{}_{jk}V^{(0)k}V^{(1)j} + C^i{}_{jk}V^{(1)k}V^{(1)j} = g^{il}g_{kh}\big(P^k{}_{jl}V^{(0)j}V^{(1)h} + C^k{}_{jl}V^{(0)j}V^{(0)h}\big) \end{cases}$$

Remark. If we consider, instead of a normal linear d-connection $(L^i{}_{jk}, C^i{}_{jk})$, a (simple) d-connection $(L^i{}_{jk},\ L^a{}_{bk},\ C^i{}_{jc}\ ,C^a{}_{bc})$, then the above equations become

$$\begin{cases} \frac{dV^{(0)i}}{dt} + L^i{}_{jk}V^{(0)k}V^{(0)j} + C^i{}_{jc}V^{(1)c}V^{(0)j} = F^{(0)i} \\ \frac{dV^{(1)a}}{dt} + L^a{}_{bk}V^{(0)k}V^{(1)b} + C^a{}_{bc}V^{(1)c}V^{(1)b} = F^{(1)a}. \end{cases}$$

4 The second variation of energy. Deviations of geodesics on TM

Consider as well TM endowed with a nonlinear connection N, a metric structure

$$G = \underset{(0)}{g}{}_{ij}dx^i \otimes dx^j + \underset{(1)}{g}{}_{ij}\delta y^{(1)i} \otimes \delta y^{(1)j}$$

and a normal metrical linear d-connection D, $D\Gamma(N) = (L^i{}_{jk}, C^i{}_{jk})$.

Let $c : [0,1] \to TM$, $t \mapsto (x^i(t), y^i(t))$ be a geodesic, i.e., c is $\mathcal{C}^\infty$ on the whole $[0,1]$ and c is a critical point of the energy

$$E = \int_0^1 \langle \dot c, \dot c\rangle dt. \tag{4.1}$$

Let $\alpha : U \times [0,1] \to TM$ be a 2-parameter variation with fixed endpoints of c by smooth curves on $[0,1]$, U being a neighbourhood of $(0,0) \in \mathbb{R}^2$. We have $\alpha(0,0,t) = c(t)$, $\forall t \in [0,1]$. Let W_1, W_2 be the induced deviation vector fields

$$W_1(t) = \frac{\partial\alpha}{\partial u_1}(0,0,t), \quad W_2(t) = \frac{\partial\alpha}{\partial u_2}(0,0,t),$$

and let $\bar{\alpha}$ be the mapping defined on $\bar{U}$ by

$$\bar{\alpha}(u_{1,}, u_1)(t) = \alpha(u_1, u_2, t),\ (u_1, u_2, t) \in U \times [0,1].$$

The Hessian E_{**} of the energy (4.1) is

$$E_{**}(W_1, W_2) = \left.\frac{\partial^2 E(\bar{\alpha}(u_{1,}, u_2))}{\partial u_1 \partial u_2}\right|_{(0,0)}.$$

Let $\mathcal{F} = \mathcal{F}^{(\alpha)i}\delta_{\alpha i}$ be the vector field defined by

$$\left\langle T\left(\frac{\partial \alpha}{\partial u_2}, \frac{\partial \alpha}{\partial t}\right), \frac{\partial \alpha}{\partial t}\right\rangle = \left\langle \mathcal{F}, \frac{\partial \alpha}{\partial u_2}\right\rangle,$$

having the local coefficients

$$\mathcal{F}^{(\alpha)i} = \sum_{\beta,\gamma=0}^{1} \underset{(\alpha)}{g}{}^{il} \underset{(\gamma)}{g}{}_{kh} \underset{(\beta\alpha)}{\overset{(\gamma)}{T}}{}^{k}_{jl} \frac{\partial \alpha^{(\beta)j}}{\partial t} \frac{\partial \alpha^{(\gamma)h}}{\partial t} \rfloor_{(u_1,u_2,t)},\ \alpha = \overline{0,1}. \tag{4.2}$$

Extending the results obtained in the Finslerian framework ([4], [7]) to the case of (h, v)-metrics (e.g., as in [13], [6]), we further state the following

Theorem 3 (The second variation of energy). *If $c : [0,1] \to TM$ is a geodesic and $\alpha : U \times [0,1] \to TM$ (where $\varepsilon > 0$) is a variation with fixed endpoints of c by piecewise smooth curves, then the Hessian E_{**} is given by:*

$$E_{**}(W_1, W_2) = -\sum_{i=1}^{k-1} \left\langle W_2, \Delta_{t_i}\left(T(W_1, V) + \tfrac{DW_1}{dt}\right)\right\rangle +$$

$$+\int_0^1 \left\langle W_2, \left.\frac{D\mathcal{F}}{\partial u_1}\right|_{u_1=u_2=0} + R(V, W_1)V - \frac{D}{dt}T(W_1, V) - \frac{D^2 W_1}{dt^2}\right\rangle dt,$$

where $0 = t_0 < t_1 < ... < t_k = 1$ is a division of $[0,1]$ such that α be smooth on each $U \times (t_{i-1}, t_i)$, $i = \overline{1,k}$.

As consequence, if $c : [0,1] \to TM$ is a smooth geodesic and $\alpha : (-\varepsilon, \varepsilon) \times [0,1] \to TM$ $(\varepsilon > 0)$ is a variation of c through smooth geodesics, then the deviation vector fields - called also *generalized Jacobi fields*, $W = W^{(\alpha)i}\delta_{\alpha i}$ are given by

$$\frac{D^2 W^{(\alpha)i}}{dt^2} + \frac{D\overset{(\alpha)}{T}{}^i}{dt} = \left.\frac{D\mathcal{F}^{(\alpha)i}}{du}\right|_{u=0} + \overset{(\alpha)}{R}{}^i, \quad \alpha = \overline{0,1}, i = \overline{1,n},$$

where $\mathcal{F}^{(\alpha)i}$ are given by (4.2) and we denoted

$$\begin{cases} \overset{(\alpha)}{T}{}^i = \sum\limits_{\beta,\gamma=0}^{1} V^{(\beta)j} W^{(\gamma)k} \underset{(\beta\gamma)}{\overset{(\alpha)}{T}}{}^i{}_{jk} \\ \overset{(\alpha)}{R}{}^i := -\sum\limits_{\beta,\gamma=0}^{1} V^{(\alpha)h} V^{(\beta)j} W^{(\gamma)k} \underset{(\alpha\beta\gamma)}{\overset{(\alpha)}{R}}{}^i{}_{hjk}. \end{cases}$$

5 Projectability of horizontal geodesics of TM

Let N be an arbitrary nonlinear connection and let $(L^i{}_{jk}, C^i{}_{jk})$ be the coefficients of an arbitrary metrical normal linear d-connection. A curve $c : [0,1] \to TM$, $t \to (x^i(t), y^i(t))$ is a *horizontal geodesic* of TM iff

$$\begin{cases} V^{(1)i} \equiv \dfrac{dy^i}{dt} + N^i{}_j y^j = 0 \\ \dfrac{dV^{(0)i}}{dt} + L^i{}_{jk} V^{(0)j} V^{(0)k} = \underset{(0)}{g}{}^{il} \underset{(0)}{g}{}_{mh} \left(L^m{}_{jl} - L^m{}_{lj} \right) V^{(0)j} V^{(0)h} \\ \underset{(1)}{g}{}^{il} \underset{(0)}{g}{}_{mh} C^m{}_{jl} V^{(0)j} V^{(0)h} = 0. \end{cases} \tag{5.1}$$

The last two equations in (5.1) are obtained from (3.2), in which we have used the relations

$$\underset{(00)}{\overset{(0)}{T}}{}^m{}_{jl} = \left(L^m{}_{jl} - L^m{}_{lj} \right), \quad \underset{(01)}{\overset{(0)}{T}}{}^m{}_{jl} = C^m{}_{jl}.$$

We note that we take into account only curves $c : [0,1] \to TM$ with $y^i = \frac{dx^i}{dt} = V^{(0)i}$, i.e., *extensions* to TM of curves $t \mapsto x^i(t)$ on M, and we look for conditions for such horizontal geodesics to project to geodesics of M. For any curve on TM, we have $V^{(0)i} = \dfrac{dx^i}{dt}$, and hence from (5.1), we infer that the h-geodesics of TM which are extensions of curves of M are locally characterized by

$$\begin{cases} \dfrac{dy^i}{dt} + N^i{}_j y^j = 0 \\ \dfrac{dy^i}{dt} + L^i{}_{jk} y^j y^k = \underset{(0)}{g}{}^{il} \underset{(0)}{g}{}_{mh} \left(L^m{}_{jl} - L^m{}_{lj} \right) y^j y^h \\ \underset{(1)}{g}{}^{il} \underset{(0)}{g}{}_{mh} C^m{}_{jl} y^j y^h = 0. \end{cases} \tag{5.2}$$

We further obtain:

Proposition 4. *Let G^i be the coefficients of the Kern canonical semispray (2.2) of the Lagrangian $L = \underset{(0)}{g}{}_{ij} y^i y^j$. If one of the two following relations holds along any curve $t \to (x^i(t))$ of M:*

1. $2G^i\left(x, \dfrac{dx}{dt}\right) = \left(L^i{}_{jh} - \underset{(0)}{g}{}^{il} \underset{(0)}{g}{}_{mh} \left(L^m{}_{jl} - L^m{}_{lj} \right) \right) \dfrac{dx^j}{dt} \dfrac{dx^h}{dt}$;

2. $2G^i\left(x, \dfrac{dx}{dt}\right) = N^i{}_j y^j$;

then any horizontal geodesic of TM projects onto a geodesic of M.

Example. If F is a Finsler metric on M and N is the canonical (Cartan) nonlinear connection of F^2, given by $N^i{}_j = \dfrac{\partial G^i}{\partial y^j}$, then any horizontal curve (including the case of a horizontal geodesic) of TM is projected onto a geodesic of M.

In particular, for $\underset{(0)}{g} = \underset{(1)}{g}$ and $(N^i{}_j, L^i{}_{jk}, C^i{}_{jk})$ the Cartan connection, both the conditions 1) and 2) in the above Proposition are satisfied. Moreover, the third set of equations (5.2) is satisfied by any curve, and the first and the second one are both equivalent with the equations of geodesics of M. Then in this case, there holds:

Corollary 1 ([1]). *For the canonical Cartan connection and a given extension Γ on TM, we have:*

a) If Γ is a horizontal curve then Γ is a horizontal geodesic;
b) Γ is a horizontal curve iff Γ is projectable onto a geodesic of M.

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