CMC AND MINIMAL SURFACES IN BERWALD-MOOR SPACES

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For Randers and Kropina Finsler spaces are described the extended equations of minimal and CMC hypersurfaces. For the Berwald-Moor type Finsler metric are then considered different types of symmetric polynomials generating the fundamental function and classes of CMC surfaces are evidentiated. Maple 9.5 representations of indicatrices point out structural differences among Berwald-Moor fundamental functions of different order, leading to different CMC approaches.

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1 Introduction

Recently Z. Shen ([16]), and further M. Souza and K. Tenenblat ([17]) have investigated minimal surfaces immersed in Finsler spaces from differential geometric point of view. Still, earlier rigorous attempts using functional analysis exist in the works of G. Bellettini and M. Paolini (after 1995, e.g., [7–9]). In 1998, based on the notion of Hausdorff measure, Z. Shen ([16]) has introduced the notion of mean curvature on submanifolds of Finsler spaces as follows.

If $(\tilde{M}, \tilde{F})$ is a Finsler structure, and $\varphi : (M, F) \to (\tilde{M}, \tilde{F})$ is an isometric immersion (hence F is induced by $\tilde{F}$), then the mean curvature of M is given by ([16, (57), p. 563])

$$H_\varphi(X) = \frac{1}{G}\left(G_{;x^i} - G_{;z^i_a z^j_b}\varphi^j_{;u^a u^b} - G_{;x^j z^i_a}\varphi^j_{;u^a}\right) X^i,$$

where lower indices stand for corresponding partial derivatives and:
- $(u^a, v^b)_{a,b\in\overline{1,n}}$ are local coordinates in TM ($\dim M = n$);
- $(x^i, y^j)_{i,j\in\overline{1,m}}$ are local coordinates in $T\tilde{M}$ ($\dim \tilde{M} = m$);
- z^i_a are the entries of the Jacobian matrix $[J(\varphi)] = (\partial\varphi^i/\partial u^a)_{a=\overline{1,n},\, i=\overline{1,m}}$;
- $\varphi_t : M \to \tilde{M}$, $t \in (-\varepsilon, \varepsilon)$, $\varphi_0 = \varphi$, is the variation of the surface;
- X is the vector field $X_x = \frac{\partial\varphi_t}{\partial t}\mid_{t=0}(x)$ induced along φ attached to the variation;
- G is the Finsler induced volume form

$$G_{\tilde{e}}(z) = \frac{vol[B^n]}{vol\{(v^a) \in \mathbb{R}^n \mid \tilde{F}(v^a z^i_a \tilde{e}_i) \le 1\}}, \tag{1.1}$$

where $z = (z^i_a)_{a=\overline{1,n},\, i=\overline{1,m}} \in GL_{m\times n}(\mathbb{R})$, $\tilde{e} = \{\tilde{e}_i\}_{i=\overline{1,m}}$ is an arbitrary basis in $\mathbb{R}^m$ and $B^n \subset \mathbb{R}^n$ is the standard Euclidean ball.

It was proved that the variation of the volume in M reaches a minimum for $H_\varphi = 0$ ([16]). Recent advances in constructing minimal surfaces ($n = 2$) based on (1.1) were provided in ([17]), by characterizing the minimal surfaces of revolution in Randers spaces $(\tilde{M} = R^3, \tilde{F})$ with the Finsler (α, β)-fundamental function

$$\tilde{F}(x, y) = \alpha(x, y) + \beta(x, y), \quad \alpha(x, y) = \sqrt{a_{ij}(x)^i y^j}, \quad \beta(x, y) = b_i(x)y^i,$$

for the particular case when $a_{ij} = \delta_{ij}$ (the Euclidean metric) and $\beta = b \cdot \mathrm{d}x^3$, with $b \in [0, 1)$.

We further consider a real smooth manifold $\tilde{M}$ of dimension $n+1$ endowed with a positive 1-homogeneous locally Minkowski Finsler fundamental function $F : T\tilde{M} \to \mathbb{R}$ ([MA]).

2 Generalized Randers-Kropina hypersurfaces ([4])

Let $H = Im\ \varphi$, $\varphi : D \subset \mathbb{R}^n \to \tilde{M} = \mathbb{R}^{n+1}$ be a simple hypersurface. We denote $z^i_\alpha = \frac{\partial \varphi^i}{\partial u^\alpha}$, $u = (u^1, \ldots, u^n) \in D$. We shall further determine the volume of the body $Q \subset T_{\varphi(u)}H$ bounded by the induced on $T_{\varphi(u)}H$ indicatrix from $\tilde{M}$

$$\Sigma_* = T_{\varphi(u)}H \cap \{y \in T_{\varphi(u)}\mathbb{R}^{n+1} | F(y) = 1\}.$$

If $v = v^\alpha \frac{\partial}{\partial u^\alpha} \in T_u D$, then $\varphi_{*,u}(v) = z^i_\alpha v^\alpha \left.\frac{\partial}{\partial y^i}\right|_{\varphi(u)} \in T_{\varphi(u)}H$ and hence at some fixed point $u \in D$, Q is given by

$$Q = \{v \in T_u D \mid F(\varphi(u), \varphi_{*,u}(v)) \leq 1\}.$$

We have the following:

Theorem 1. *If the body Q is given by*

$$Q : \sum_{i=1}^{n} (z^i_\alpha v^\alpha)^2 + \mu(z^{n+1}_\alpha v^\alpha)^2 + 2\nu z^{n+1}_\alpha v^\alpha + \rho \leq 0, \tag{2.1}$$

where $\mu, \nu, \rho \in \mathbb{R}$, then

$$Vol(Q) = \begin{cases} \dfrac{Vol(B_n)}{\sqrt{\delta} \cdot (1+\tau)^{(n+1)/2}} \cdot \left(\dfrac{\nu^2 \tau}{\mu - 1} - \rho(1+\tau)\right)^{n/2}, & for\ \mu \neq 1 \\ \dfrac{Vol(B_n) \cdot (-\rho + \nu^2 z^{n+1}_a z^{n+1}_b h^{ab})^{n/2}}{\sqrt{\delta}}, & for\ \mu = 1, \end{cases} \tag{2.2}$$

where τ and δ are given by (2.3).

$$\tau = (\mu - 1) z^{n+1}_a z^{n+1}_b h^{ab}, \quad \delta = \det(h_{ab})_{a,b=\overline{1,n}}, \tag{2.3}$$

and $B_n \subset \mathbb{R}^n$ is the standard n-dimensional ball and h^{ab} is the dual of h_{ab} ($h^{as}h_{sb} = \delta^a_b$).

In particular, we obtain the following result:

Corollary 1. *a) In the Randers case*

$$F(x, y) = \sqrt{\sum_{i=1}^{n+1} (y^i)^2} + by^{n+1}, \quad b \in [0, 1), \tag{2.4}$$

we obtain the known result ([17, (5), p. 627]),

$$Vol(Q_R) = \frac{Vol(B_n)}{\sqrt{\delta}(1 - b^2 z^{n+1}_a z^{n+1}_b h^{ab})^{(n+1)/2}}.$$

b) In the Kropina case

$$F(x, y) = (by^{n+1})^{-1} \cdot \sum_{i=1}^{n+1} (y^i)^2, \quad b \in [0, 1), \tag{2.5}$$

we have

$$Vol(Q_K) = \frac{Vol(B_n)\left(\frac{b^2}{4} z^{n+1}_a z^{n+1}_b h^{ab}\right)^{n/2}}{\sqrt{\delta}}.$$

Remarks. In the Kropina case, the function G in (1.1) has the expression

$$G = \frac{Vol(B_n)}{Vol(Q_K)} = \frac{\sqrt{\delta}}{(z_a^{n+1} z_b^{n+1} h^{ab} \cdot b^2/4)^{n/2}} = 2^n \cdot C B^{-n/2},$$

where we have used the notations from [17], $B = b^2 z_a^{n+1} z_b^{n+1} h^{ab}$, $C = \sqrt{\delta}$. Then the mean curvature vector field has the components

$$\bar{H}_i = \frac{1}{G}\left(\frac{\partial^2 G}{\partial z_\varepsilon^i z_\eta^j} \cdot \frac{\partial^2 \varphi}{\partial u^\varepsilon \partial u^\eta}\right), \quad i = \overline{1, n+1},$$

and the volume form of the hypersurface H is

$$dV_F = \frac{\sqrt{\delta}}{\left(\frac{b^2}{4} z_a^{n+1} z_b^{n+1} h^{ab}\right)^{n/2}} du^1 \wedge \cdots \wedge du^n.$$

Theorem 2. *The mean curvature vector field of the hypersurface M in the Kropina space $\tilde{M} = \mathbb{R}^{n+1}$ with the fundamental function (2.5) has the following expression in terms of B and C*

$$\begin{aligned} H_i &= 2^n B^{-(n+4)/2}\left[\tfrac{\partial^2 C}{\partial z_\varepsilon^i \partial z_\eta^j} B^2 + \tfrac{n(n+2)}{4} C \tfrac{\partial B}{\partial z_\varepsilon^i}\tfrac{\partial B}{\partial z_\eta^j} - \right. \\ &\left. - \tfrac{nB}{2}\left(\tfrac{\partial C}{\partial z_\varepsilon^i}\tfrac{\partial B}{\partial z_\eta^j} + \tfrac{\partial C}{\partial z_\eta^j}\tfrac{\partial B}{\partial z_\varepsilon^i} + C\tfrac{\partial^2 B}{\partial z_\varepsilon^i \partial z_\eta^j}\right)\right] \tfrac{\partial^2 \varphi}{\partial u^\varepsilon \partial u^\eta}, \quad i = \overline{1, n+1}. \end{aligned}$$

Corollary 2. *The mean curvature vector field of the surface M in the Kropina space $\tilde{M} = \mathbb{R}^{n+1}$ with the fundamental function (2.5) has the following expression*

$$\begin{aligned} H_i &= \tfrac{4C}{E^3}\left[6E^2 \tfrac{\partial C}{\partial z_\varepsilon^i}\tfrac{\partial C}{\partial z_\eta^j} + 2C^2 \tfrac{\partial E}{\partial z_\varepsilon^i}\tfrac{\partial E}{\partial z_\eta^j} - C^2 E \tfrac{\partial^2 E}{\partial z_\varepsilon^i \partial z_\eta^j} - \right. \\ &\left. -3CE\left(\tfrac{\partial E}{\partial z_\varepsilon^i}\tfrac{\partial C}{\partial z_\eta^j} + \tfrac{\partial E}{\partial z_\eta^j}\tfrac{\partial C}{\partial z_\varepsilon^i}\right) + 3E^2 C \tfrac{\partial^2 C}{\partial z_\varepsilon^i \partial z_\eta^j}\right] \tfrac{\partial^2 \varphi}{\partial u^\varepsilon \partial u^\eta}, \quad i = \overline{1, n+1}, \end{aligned} \tag{2.6}$$

where

$$E = b^2 \sum_{k=1}^{3} \sum_{\alpha,\beta=1}^{2} (-1)^{\alpha+\beta} z_{\tilde{\alpha}}^k z_{\tilde{\beta}}^k z_\alpha^3 z_\beta^3, \quad \tilde{\alpha} = 3 - \alpha.$$

Corollary 3. *The mean curvature of the surface M in the Kropina space (2.5) is $H_* = H_i X^i$, where H_i are given by (2.6),*

$$X \in Ker(G_* Z^1) \cap Ker(G_* Z^2) \cap \{y \in T_{\varphi(u)}\tilde{M} \mid F(y) = 1\},$$

$Z^1 = (z_1^1, z_1^2, z_1^3)$, $Z^2 = (z_2^1, z_2^2, z_2^3)$, and $G_* v$ is defined by the equality $(G_* v)(v') = \langle v, v' \rangle_F = \frac{1}{2}\frac{\partial F^2}{\partial y^i \partial y^j} v^i v'^j$.

Corollary 4. *Let $M = \Sigma = Im\,\varphi$ be a surface of revolution described by*

$$\varphi(t, \theta) = (f(t)\cos\theta, f(t)\sin\theta, t), \quad (t, \theta) \in D = \mathbb{R} \times [0, 2\pi).$$

Then M is minimal iff the function f satisfies the ODE

$$1 + f'^2 = 3ff''(1 + 2f'^2).$$

3 The Berwald-Moor Finsler case

We shall further point out the obstructions present in the case of a Berwald-Moor Finsler metric and evidentiate the means of construction of spatial and temporal CMC and minimal surfaces. The substantial difference between the Randers-Kropina framework and the Berwald-Moor Finsler metric relies in the fact that the indicatrix $\Sigma : F(x,y) = 1, x \in M$ is in general noncompact for all values of x. Hence, one may not talk about the volume contained inside this hypersurface Σ, which in the latter case extends to infinity and the volume is provided by a divergent integral.

However, specializing to certain temporal or spatial slices, one may define within them CMC or minimal submanifolds of codimension 1, in particular surfaces.First we note that in the case of Minkowski Finsler metrics of Berwald Moor type

$$F(x,y) = \sqrt[k]{P_k(y^1,\dots,y^n)}, \quad (\dim M = n \geq 3),$$

provided by appropriate order square roots of homogeneous polynomials P_k, even in the case when the indicatrix Σ is compact and strongly convex, e.g.,

$$F(x,y) = \sqrt[2k]{(y^1)^{2k} + \dots + (y^n)^{2k}}, \quad (k \geq 2), \tag{3.1}$$

the task of computing the volume bounded by Σ becomes difficult for higher orders k (see Appendix I). This points out once more that from technical point of view choosing an appropriate submanifold which would decrease the dimension, is a desirable attempt.

We shall discuss further several cases of nonpositive signature of the Finsler metric tensor field given by the halved y-Hessian of F^2.

1. The $H(4) \sim \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ - type Berwald-Moor Minkowski - Finsler metrics [11]:

$$\begin{aligned} F_2(y) &= \sqrt{|(a+b)(c+d) + ab + cd|}, \\ F_3(y) &= \sqrt[3]{|ab(c+d) + cd(a+b)|}, \\ F_4(y) &= \sqrt[4]{|abcd|}, \end{aligned} \tag{3.2}$$

where $n = 4$, $y = (y^1, y^2, y^3, y^4) = (a,b,c,d) \in T_p(\mathbb{R}^4)$. After performing the Hadamard change of basis of matrix $C = \frac{1}{4}\begin{pmatrix} A & A \\ A & -A \end{pmatrix}$, with $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ given by $y^T = C\hat{y}^T$, $\hat{y} = (t,x,y,z)$, the functions (3.2) transform into

$$\begin{aligned} \hat{F}_2(\hat{y}) &= \sqrt{|6t^2 - 2(x^2+y^2+z^2)|}, \\ \hat{F}_3(\hat{y}) &= \sqrt[3]{|8xyz + 4t(t^2 - x^2 - y^2 - z^2)|}, \\ \hat{F}_4(\hat{y}) &= \sqrt[4]{|x^4 + y^4 + z^4 + t^4 + 8txyz - 2[(x^2+y^2)(z^2+t^2) + x^2y^2 + z^2t^2]|}. \end{aligned}$$

Hence for $\hat{F}_2$ one might consider the slice submanifold $\hat{y}^1 \equiv t = const$ where the CMC imbedded surfaces are the Euclidean ones.

For $\tilde{F}_4$, besides considering the spatial slices $v^i = const$, $(i \in \overline{1,4})$ one might look for subclasses of CMC surfaces which satisfy additional PDEs, by reformulating the energy-minimizing problem using Lagrange multipliers imposed, e.g., by

$$(v^1)^4 + \dots + (v^n)^4 \equiv \tilde{F}_4(\hat{y})|_{\hat{y}^i = (C^{-1})^i_j z^j_\alpha v^\alpha,\ i=\overline{1,4}},$$

or, for the initial basis,

$$(v^1)^4 + \cdots + (v^n)^4 \equiv (z^1_\alpha v^\alpha)(z^2_\beta v^\beta)(z^3_\gamma v^\gamma)(z^4_\delta v^\delta),$$

where the Greek indices run through $\overline{1,n}$, with $n \geq 1$.

2. In general, for $m \geq 3$ and Berwald-Moor metrics of type (3.1), valid additional PDEs which impose the change of energy to provide surface-like CMC surfaces are

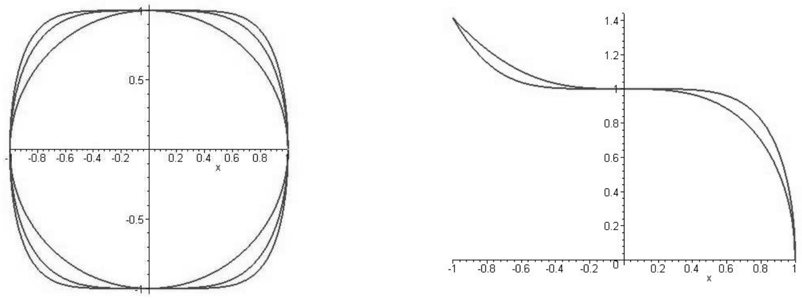
$$(v^1)^{2k} + \cdots + (v^n)^{2k} \equiv F(y)|_{y^i = z^i_\alpha v^\alpha,\ i=\overline{1,m}}\,,$$

with the same conventions as above.

3. A notable difference exhibited by Berwald-Moor Finsler fundamental functions

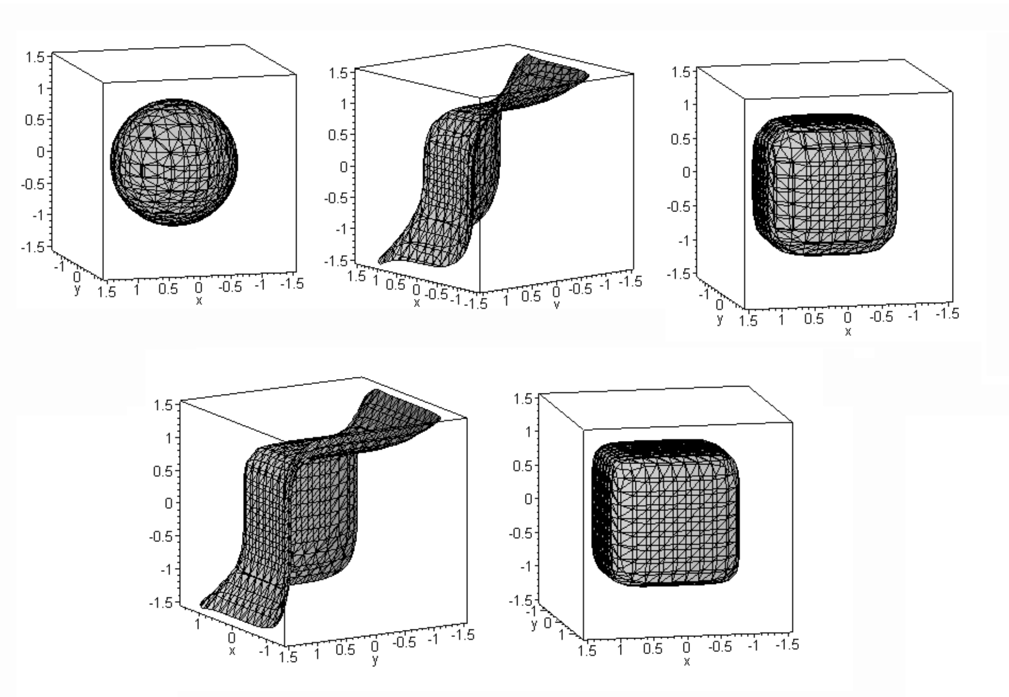
$$F(v) = \sqrt[k]{(v^1)^k + \ldots + (v^n)^k}, \quad n \in \{2, 3\} \tag{3.3}$$

and hence, by their indicatrices, is the dependence of the topologic properties on the index k. For k even, the indicatrices are compact and have a strictly convex interior set, while for k odd, the indicatrices are unbounded and define no finite volume. This is illustrated for $m = 2$ by the following Maple plots of indicatrices $F(v) = 1$ with F provided by (3.3):



Berwald-Moor indicatrices ($m = 2$; even ($k \in \{2, 4, 6\}$) and odd ($k \in \{3, 5\}$) root index).

In higher dimensions (e.g. for $m = 3$) the topology strongly differs as well:



Berwald-Moor indicatrices for $k \in \{2, \ldots, 6\}$ ($m = 3$)

Moreover, even for small even values of k, to compute the encompassed volume inside a bounded indicatrix implies the usage of special functions. Though the case $k = 2$ is calssical, providing volumes of (hyper)-spheres ($Vol(Q)_{m=2,k=2} = \pi$, $Vol(Q)_{m=2,k=2} = 4\pi/3$, etc), for larger values of k we get results as:

$$Vol(Q)_{m=2,k=4} = \frac{1}{4}B\left(\frac{1}{4},\frac{3}{2}\right), \quad Vol(Q)_{m=2,k=6} = \frac{1}{6}B\left(\frac{1}{6},\frac{3}{2}\right),$$

where $B(\,\cdot\,,\,\cdot\,)$ is the Bessel function.

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References

[1] J. C. Álvarez Paiva, A. C. Thompson, *Volumes on normed and Finsler spaces*, In: Riemann-Finsler Geometry, MSRI Publications, 49 (2004), 1–46.

[2] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Geometry*, D.Reidel, Dordrecht, 1985.

[3] V. Balan, *DPW method for CMC surfaces in Randers spaces*, in: "Modern Trends in Geometry and Topology" (Eds: D. Andrica, P. A. Blaga, S. Moroianu), Proc. of The Seventh International Workshop on Differential Geometry and Its Applications, 5–11 September 2005, Deva, Romania, Cluj University Press 2006, 79–86.

[4] V. Balan, *Minimal surfaces in special Finsler spaces*, Proc. of The National Seminar of Finsler, Lagrange and Hamilton Spaces, 22–23 September **2006**, Braşov, Romania, to appear.

[5] D. Bao, S.-S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, 2000.

[6] A. Bejancu, *Finsler Geometry and Applications*, E. Horwood, 1990.

[7] G. Belletini, M. Paolini, *Anisotropic motion by mean curvature in the context of Finsler geometry*, Hokkaido Math. Jour. 25 (1995), 537–566.

[8] G. Belletini, M. Paolini, *Some aspects of mean curvature flow in presence of nonsmooth anisotropies*, Progress in Mathematics, Birkhäuser 202 (2001), 245–253.

[9] G. Belletini, M. Paolini, S. Venturini, *Some results on surface measures in claculus of variations*, Ann. di Mat. Pura ed Appl., IV vol. CLXX (1996), 329–259.

[10] G. Breck, *Minimalité des sous-variétés totalement géodésiques en géométrie Finslerienne*, arXiv:math.DG/0409320, v1, 18 Sept. 2004.

[11] V. M. Chernov, *On defining equations for the elements of associative and commutative algebras and on associated metric forms*, Space-Time Structure (Ed: D. Pavlov), Tetru Eds., 2006, 227–249.

[12] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry* I, II, Interscience Publishers, New York, (1963), (1969).

[13] R. Miron, M. Anastasiei, *The Geometry of Vector Bundles. Theory and Applications*, Kluwer Acad. Publishers, FTPH, no. 59, 1994.

[14] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, 1959.

[15] Z. Shen, *Differental Geometry of Sprays and Finsler Spaces*, Kluwer Acad. Publishers, 2001.

[16] Z. Shen, *On Finsler geometry of submanifolds*, Mathematische Annalen, Springer-Verlag 311 (1998), 549–576.

[17] M. Souza, K. Tenenblat, *Minimal surface of rotation in Finsler space with a Randers metric*, Math. Ann. 325 (2003), 625–642.