THE HORIZONTAL AND VERTICAL SEMISYMMETRIC METRICAL D-CONNECTIONS IN THE RELATIVITY THEORY

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Let E be the $(m+n)$-dimensional total space of a vector bundle (E,p,M), $dim\ M = n$, a given fixed nonlinear connection N on E and a given (h,v)-metrical structure $G \in \mathcal{T}_2^0(E)$. In the paper, we determine the Einstein equations of an h- and v-semisymmetric metrical distinguished connection on $E = TM$, if $n = 4$, for a Riemann – local Minkowski model.

1 Vector bundles. Distinguished linear connections ([11])

Let $\xi = (E,p,M)$ be a vector bundle with $\dim E = m+n,\ p : E \to M$, where M is a n-dimensional smooth differentiable manifold. If N is a nonlinear connection on E and V is a complementary vertical distribution of N then,

$$T_uE = H_uE \oplus V_uE, \quad \forall u \in E. \tag{1.1}$$

Definition 1.1 *A linear connection D on E is called distinguished linear connection or d-connection if the linear connection D preserves by parallelism the horizontal and vertical distributions:*

$$D_ZX \in HE,\ \ D_ZY \in VE,\ \ \forall X \in HE,\ \ Y \in VE,\ \ Z \in \mathcal{X}(E). \tag{1.2}$$

For a d-connection D we have the unique decomposition

$$D = D^H + D^V. \tag{1.3}$$

where D^H and D^V are the h- and v-covariant derivatives on $\mathcal{X}(E)$

We denote by $X^H\left(X^V\right)$ and $\omega^H\left(\omega^V\right)$, the horizontal (vertical) components of $X \in \mathcal{X}(E)$ respectively $\omega \in \mathcal{X}^*(E)$

In the local coordinates (x^i, y^a) of point $u(u^\alpha) \in E,\ \alpha = \overline{1,m+n},\ i = \overline{1,n},\ a = \overline{1,m}$, we have $\left(\delta_i, \dot\partial_a\right)$, $(dx^i, \delta y^a)$ the adapted frames to $N\left(N^a{}_i(x,y)\right)$:

$$\begin{gathered}\delta_i \ =\ \partial_i - N^a{}_i(x,y)\,\dot\partial_a, \quad \delta y^a = dy^a + N^a{}_i(x,y)\,dx^i, \\ \left(\delta_i = \delta/\delta x^i,\ \partial_i = \partial/\partial x^i,\ \dot\partial_a = \partial/\partial y^a\right).\end{gathered} \tag{1.4}$$

Then, $\left(L^i{}_{jk}(x,y),\ L^a{}_{bk}(x,y),\ C^i{}_{jc}(x,y),\ C^a{}_{bc}(x,y)\right)$ are the local components of a d-connection $D\Gamma(N)$.

$$\begin{aligned} D_{\delta_k}\delta_j \ &=\ L^i{}_{jk}(x,y)\,\delta_i, \quad D_{\dot\partial_c}\delta_j = C^i{}_{jc}\delta_i, \\ D_{\delta_k}\dot\partial_b \ &=\ L^a{}_{bk}(x,y)\,\dot\partial_a, \quad D_{\dot\partial_c}\dot\partial_b = C^a{}_{bc}\dot\partial_a. \end{aligned} \tag{1.5}$$

Also, we denote by: $T^i{}_{jk}, R^a{}_{jk}, P^a{}_{jc}, C^i{}_{jc}, S^a{}_{bc}$, the local components of five d-tensor fields of torsion of d-connection $D\Gamma(N)$, (1.4) and with: $R_j{}^i{}_{kh}, R_b{}^a{}_{jk}, P_j{}^i{}_{kd}, P_b{}^a{}_{kd}, S_j{}^i{}_{cd}, S_b{}^a{}_{cd}$, the local component of six d-tensors fields of curvature of d-connection $D\Gamma(N)$, (1.4).

The Algebra of d-tensor fields on E is locally generated by $\left\{1, \delta_i\dot\partial_a\right\}$ over the differentiable functions $\mathcal{F}(E)$.

2 Metrical structures and metrical d-connections on E ([11])

We will consider a given fixed nonlinear connection N on E with the local components $N^a{}_i(x,y)$ and a given (h,v) −metrical structure $\mathbb{G} \in \tau_2^0(E)$:

$$\mathbb{G} = g_{ij}(x,y)\,dx^i \otimes dx^j + h_{ab}(x,y)\,\delta y^a \otimes \delta y^b, \tag{2.1}$$

where

$$\begin{aligned} g_{ij}(x,y) &= g_{ji}(x,y), \quad rank\,\|g_{ij}(x,y)\| = n, \\ h_{ab}(x,y) &= h_{ba}(x,y), \quad rank\,\|h_{ab}(x,y)\| = m \end{aligned} \tag{2.2}$$

Obviously, we have

$$\mathbb{G}\left(X^H,\ Y^V\right) = 0, \quad \forall X \in HE, \quad Y \in VE, \tag{2.3}$$

in other words, the distributions H and V are orthogonal with respect to $\mathbb{G}$ given by (2.1).

Remark If $E = TM$, there exist metrics of type (2.1) wich satisfy (2.2). Indeed, we shall consider a Lagrange (Finsler) structure $g_{ij}(x,y)$ on TM and by Sasaki-Matsumoto lift:

$$\mathbb{G} = g_{ij}(x,y)\,dx^i \otimes dx^j + g_{ij}(x,y)\,\delta y^i \otimes \delta y^j \tag{2.4}$$

is obtained a metric of type (2.1) wich satisfy the relation (2.2).

Conversely, if $\mathbb{G} \in \tau_2^0(E)$ is a metric on E, then there exists a nonlinear connection $N\left(N^a{}_i(x,y)\right)$ given by $\mathbb{G}\left(X^H, Y^V\right) = 0$.

Definition 2.1 *A d-connexion D on E is called a* **metrical d-connexion** with *respect to $\mathbb{G} \in \tau_2^0(E)$ given by (2.1), if* $D_X\mathbb{G} = 0$, $\forall X \in \mathcal{X}(E)$.

Proposition 2.1 *A d-connexion D on E* it *is metrical if and only if*

$$D_X^H\mathbb{G}^H = 0, \quad D_X^H\mathbb{G}^V = 0, \quad D_X^V\mathbb{G}^H = 0, \quad D_X^V\mathbb{G}^V = 0, \quad \forall X \in \mathcal{X}(E), \tag{2.5}$$

where $\mathbb{G}^H = g_{ij}(x,y)\,dx^i \otimes dx^j$ is the horizontal part and $\mathbb{G}^V = h_{ab}(x,y)\,\delta y^a \otimes \delta y^b$ is the vertical part of $\mathbb{G}$ given by (2.1).

Proposition 2.2 *There exists a metrical $d-$connection on E which dependes only $N^a{}_i(x,y)$, $g_{ij}(x,y)$ and $h_{ab}(x,y)$. This is given by*

$$\begin{aligned} \overset{M}{L}{}^i{}_{jk}(x,y) &= 1/2g^{ih}\left(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}\right), \\ \overset{M}{L}{}^a{}_{bk}(x,y) &= \dot{\partial}_b N^a{}_k + 1/2h^{ad}\left(\delta_k h_{bd} - h_{bc}\dot{\partial}_d N^c{}_k - h_{cd}\dot{\partial}_b N^c{}_k\right), \\ \overset{M}{C}{}^i{}_{jc}(x,y) &= 1/2g^{ih}\dot{\partial}_c g_{jh}, \\ \overset{M}{C}{}^a{}_{bc}(x,y) &= 1/2h^{ad}\left(\dot{\partial}_b h_{dc} + \dot{\partial}_c h_{bd} - \dot{\partial}_d h_{bc}\right), \end{aligned} \tag{2.6}$$

where $\|g^{ij}\| = \|g_{ij}\|^{-1}$, $\left\|h^{ab}\right\| = \|h_{ab}\|^{-1}$.

The distinguished metrical d-connection (2.4) is said to be **Miron connection** of $\mathbb{G}$ and it will denoted by $MD\Gamma(N)$.

Proposition 2.3 *There exists an unique metrical $d-$connection* $D\Gamma(N) = \left(L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\right)$ *on E for which:*

$$L^a{}_{bk}(x,y) = \overset{M}{L}{}^a{}_{bk}(x,y), \quad C^i{}_{jc}(x,y) = \overset{M}{C}{}^i{}_{jc}(x,y) \tag{2.7}$$

and the d-tensor fields $T^{i}{}_{jk}, S^{a}{}_{bc}$ *are prescribed. This connection is given by (2.5) and*

$$
\begin{aligned}
L^{i}{}_{jk}(x,y) &= \overset{M}{L}{}^{i}{}_{jk}(x,y) + 1/2 g^{ir}\left(g_{rh}T^{h}{}_{jk} - g_{jh}T^{h}{}_{rk} + g_{kh}T^{h}{}_{jr}\right), \\
C^{a}{}_{bc}(x,y) &= \overset{M}{C}{}^{a}{}_{bc}(x,y) + 1/2 h^{ad}\left(h_{df}S^{f}{}_{bc} - h_{bf}S^{f}{}_{dc} + h_{cf}S^{f}{}_{bd}\right).
\end{aligned}
\tag{2.8}
$$

The metrical distinguished connection given by (2.5) and (2.6) will be called **generalized Miron connection** of the metric $\mathbb{G}$ given by (2.1) and it will denoted by $GMD\Gamma(N)$.

We note

$$\varepsilon(x,y) = < y, y > = h_{ab}(x,y)\, y^{a} y^{b} \tag{2.9}$$

the **absolut energy** of vertical part G^{V} and

$$h^{*}_{ab}(x,y) = \frac{1}{2}\frac{\partial^{2}\varepsilon}{\partial y^{a}\partial y^{b}}. \tag{2.10}$$

Definition 2.2 *The d-tensor field* $h_{ab}(x,y)\,\delta y^{a}\otimes\delta y^{b}$ *is said to be* **weakly regular** *if the d-tensor field with components* $h^{*}_{ab}(x,y)$ *given by (2.8) is nondegenerate, i.e.* $\det\|h^{*}_{ab}(x,y)\| \neq 0$, *where* $E = TM$.

Theorem 2.1 (R. Miron, [10]; *see also* [11] *pg. 127 and* [12]*) If* $h_{ab}(x,y)\,\delta y^{a}\otimes\delta y^{b}$ *is a weakly regular v-metric on* $E = TM$ *then the functions*

$$N^{a}{}_{i}(x,y) = \dot{\partial}_{b}G^{a}(x,y)\,\delta^{b}_{i}, \quad G^{a} = \frac{1}{2}h^{*ab}\left[\left(\dot{\partial}_{b}\partial_{k}\varepsilon\right)\delta^{k}_{c}y^{c} - (\partial_{k}\varepsilon)\,\delta^{k}_{b}\right], \tag{2.11}$$

are the coefficients of a nonlinear connection completely determined by $h_{ab}(x,y)$.

3 h- and v-semisymmetric metrical d-connections and their transformations

Definition 3.1 *A metrical d-connection on* E *is said to be* **h-semisymmetric** *if*

$$T^{i}{}_{jk} = \sigma_{j}\delta^{i}_{k} - \sigma_{k}\delta^{i}_{j}, \tag{3.1}$$

and **v-semisymmetric** *if*

$$S^{a}{}_{bc} = \tau_{b}\delta^{a}_{c} - \tau_{c}\delta^{a}_{b}, \tag{3.2}$$

where σ_{i}, τ_{a} *are d-covector fields on* E.

Theorem 3.1 *There exists on* E *an unique metrical d-connection* both *h-and v-semisymmetric,* $D\Gamma(N) = \left(L^{i}{}_{jk}, L^{a}{}_{bk}, C^{i}{}_{jc}, C^{a}{}_{bc}\right)$, *with prescribed d-covector fields* σ_{i}, τ_{a}. *That d-connection is given by (2.5) and*

$$
\begin{aligned}
L^{i}{}_{jk} &= \tfrac{1}{2}g^{ih}\left(\delta_{j}g_{hk} + \delta_{k}g_{jh} - \delta_{h}g_{jk}\right) + \sigma_{j}\delta^{i}_{k} - g_{jk}\sigma^{i}, \\
C^{a}{}_{bc} &= \tfrac{1}{2}h^{ad}\left(\dot{\partial}_{b}h_{dc} + \dot{\partial}_{c}h_{bd} - \dot{\partial}_{d}h_{bc}\right) + \tau_{b}\delta^{a}_{c} - h_{bc}\tau^{a},
\end{aligned}
\tag{3.3}
$$

where $\sigma^{i} = g^{ij}\sigma_{j}$ *and* $\tau^{a} = h^{ab}\tau_{b}$.

Now, we have the following interesting transformations of h- and v-semisymmetric metrical d-connections.

Theorem 3.2 *The transformations of h-and v-semisymmetric metrical d-connections, which preserve the nonlinear connection* $N, D\Gamma(N) \longrightarrow D\bar{\Gamma}(N)$, *are given by*

$$\begin{aligned} &\bar{L}^i{}_{jk} = L^i{}_{jk} + p_j\delta^i_k - g_{jk}p^i, \\ &\bar{L}^a{}_{bk} = L^a{}_{bk}, \\ &\bar{C}^i{}_{jc} = C^i{}_{jc}, \\ &\bar{C}^a{}_{bc} = C^a{}_{bc} + q_b\delta^a_c - h_{bc}q^a, \end{aligned} \tag{3.4}$$

where $p^i = g^{ij}p_j, q^a = h^{ab}q_b$ *and* p_i, q_a *are arbitrary d-covector fields on* E.

We shall denote these transformations by $t(p,q)$.

Theorem 3.3 *The set of all transformations* $t(p,q)$ *given by* (3.4) *is a transformations group* $\mathcal{G}_N$ *of the set of all h- and v-semisymmetric metrical d-connections, with respect to* (2.1), *together with the mapping product*

$$t(p',q') \circ t(p,q) = t(p+p', q+q').$$

This group $\mathcal{G}_N$ *is an Abelian group and acts on the set of all h-and v-semisymmetric metrical d-connections, having the same nonlinear connection, transitively.*

If we investigate the influences for the torsion and curvature tensor fields, we have

Theorem 3.4 *The following d-tensor fields*

$$\begin{gathered} R^a{}_{jk},\ P^a{}_{jc},\ C^i{}_{jc} \\ T^i{}_{jk} - \tfrac{1}{n-1}\left(T_j\delta^i_k - T_k\delta^i_j\right), \quad S^a{}_{bc} - \tfrac{1}{m-1}\left(S_b\delta^a_c - S_c\delta^a_b\right), \\ \left(T_j = T^k{}_{jk},\ S_b = S^c{}_{bc}\right), \end{gathered} \tag{3.5}$$

are invariants with respect to transformations of the group $\mathcal{G}_N$.

Theorem 3.5 *For* $n > 2,\ m > 2$, *the following d-tensor fields* $H_j{}^i{}_{kl}, M_b{}^a{}_{cd}$ *of h- and v-semisymmetric metrical d-connections, are invariants of the group* $\mathcal{G}_N$:

$$H_j{}^i{}_{kl} = R_j{}^i{}_{kl} + 2\underset{(k,l)}{\mathcal{A}}\left\{\underset{1}{\Omega}{}^{si}_{jk}\left[R_{sl} - Rg_{sl}/2(n-1)\right]\right\}/(n-2), \tag{3.6}$$

$$M_b{}^a{}_{cd} = S_b{}^a{}_{cd} + 2\underset{(c,d)}{\mathcal{A}}\left\{\underset{1}{\wedge}{}^{cd}_{bc}\left[S_{ed} - Sh_{ed}/2(m-1)\right]\right\}/(m-2), \tag{3.7}$$

where we denoted the alternation operator by $\mathcal{A}$, *the Obata operators* $\underset{1}{\Omega}$ *and* $\underset{1}{\wedge}$ *of* g_{ij} *and* h_{ab} *respectively, by:*

$$\begin{aligned} \underset{1}{\Omega}{}^{ij}_{kl} &= \frac{1}{2}\left(\delta^i_k\delta^j_l - g_{kl}g^{ij}\right), \quad \underset{1}{\wedge}{}^{ab}_{cd} = \frac{1}{2}\left(\delta^a_c\delta^b_d - h_{cd}h^{ab}\right), \\ &\textit{and} \\ R_{jk} &= R_j{}^l{}_{kl}, \quad S_{bc} = S_b{}^d{}_{cd}, \quad R = g^{ij}R_{ij}, \quad S = h^{ab}S_{ab}. \end{aligned}$$

Theorem 3.6 *We have*

$$H_j{}^i{}_{kl} = \overset{M}{H}_j{}^i{}_{kl}, \quad M_b{}^a{}_{cd} = \overset{M}{M}_b{}^a{}_{cd}, \tag{3.8}$$

where $\overset{M}{H}_j{}^i{}_{kl}$, $\overset{M}{M}_b{}^a{}_{cd}$ *are construct by means of the Miron connection of* $\mathbb{G}$, $MD\Gamma(N)$, *given by* (2.4).

Proof. We consider (3.3) as a transformation of h- and v-semisymmetric metrical d-connections $MD\Gamma(N) \longrightarrow D\Gamma(N)$ and we obtain (3.8), with respect to $(3.6), (3.7)$

By straightforward calculus, we get:

Theorem 3.7 *If the Miron connection, $MD\Gamma(N)$, (2.4), has the properties of $h-$ and v-isotropie:*

$$\overset{M}{R}{}_j{}^i{}_{kl} = h(x,y)\left(g_{jk}\delta^i_l - g_{jl}\delta^i_k\right), \quad \overset{M}{S}{}_b{}^a{}_{cd} = v(x,y)\left(h_{bc}\delta^a_d - h_{bd}\delta^a_c\right) \tag{3.9}$$

then, we have

$$H_j{}^i{}_{kl} = 0, \quad M_b{}^a{}_{cd} = 0 \tag{3.10}$$

4 The Riemann-local Minkowski model of relativity with h- and v-semisymmetric torsions

In this Section, we consider $E = TM$, $\dim M = n$.

If $h_{ab}(x,y) = h_{ab}(y)$, the metric $\mathbb{G}$ given by (2.1) is called **v-local Minkowski**

We have

Theorem 4.1 *If the metric structure $\mathbb{G}$ given by (2.1) is h-Riemannian, v-locally Minkowski and $h_{ab}(y)$ is weakly regular, then:*

I) The h- and v-semisymmetric metrical d-connection, compatible with respect to $\mathbb{G}$, that corresponds to the 1-forms $\sigma_i(x,y) = \sigma_i(x)$, $\tau_a(x,y) = \tau_a(y)$ has the coefficients given by

$$\begin{aligned} \hat{L}^i{}_{jk} &= \gamma^i_{jk} + \sigma_j\delta^i_k - g_{jk}\sigma^i, \\ \hat{L}^a{}_{bk} &= 0, \\ \hat{C}^a{}_{jc} &= 0, \\ \hat{C}^a{}_{bc} &= \gamma^a_{bc} + \tau_b\delta^a_c - h_{bc}\tau^a, \end{aligned} \tag{4.1}$$

here γ^i_{jk} and γ^a_{bc} are the Levi-Civita connections corresponding to the $g_{ij}(x)$ and $h_{ab}(y)$, respectively.

II) d-tensor fields of (4.1) are

$$\begin{aligned} \hat{T}^i{}_{jk} &= \sigma_j\delta^i_k - \sigma_k\delta^i_j, \\ \hat{R}^a{}_{jk} &= 0, \quad \hat{C}^i{}_{jc} = 0, \quad \hat{P}^a{}_{jc} = 0, \\ \hat{S}^a{}_{bc} &= \tau_b\delta^a_c - \tau_c\delta^a_b. \end{aligned} \tag{4.2}$$

III) d-curvature fields of (4.1) are

$$\begin{aligned} \hat{R}_j{}^i{}_{kl} &= r_j{}^i{}_{kl} + 2\underset{(k,l)}{\mathcal{A}}\left\{\underset{1}{\Omega}{}^{si}_{jk}\sigma_{sl}\right\}, \\ \hat{R}_b{}^a{}_{kl} &= 0,\ \hat{P}_j{}^i{}_{kd} = 0, \quad \hat{P}_b{}^a{}_{kd} = 0, \quad \hat{S}_j{}^i{}_{cd} = 0, \\ \hat{S}_b{}^a{}_{cd} &= s_b{}^a{}_{cd} + 2\underset{(c,d)}{\mathcal{A}}\left\{\underset{1}{\wedge}{}^{fa}_{bc}\tau_{fd}\right\}, \end{aligned} \tag{4.3}$$

where we denoted $\mathcal{A}$, $\underset{1}{\Omega}$, $\underset{1}{\wedge}$, as in Theorem 3.6, by $r_j{}^i{}_{kl}$, $s_b{}^a{}_{cd}$ the tensor fields of curvatures of $\gamma^i_{jk}, \gamma^a_{bc}$ *respectively, and*

$$\begin{aligned} \sigma_{ij} &= \sigma_{i\,\overset{\frown}{|}\,j} - 2\sigma_i\sigma_j + g_{ij}\alpha, \quad 2\alpha = g^{ij}\sigma_i\sigma_j, \\ \tau_{ab} &= \tau_{a\,\overset{\frown}{|}\,b} - 2\tau_a\tau_b + h_{ab}\beta, \quad 2\beta = h^{ab}\tau_a\tau_b; \end{aligned} \tag{4.4}$$

(*here* $\hat{\shortmid}$ *and* $\hat{|}$ *denote the h- and v-covariante derivatives with respect to* $D\hat{\Gamma}$, (3.4)).

Remark 4.1 For d-connection (4.1), $h\,(h)$-torsion and $h\,(hh)$-curvature are internal, only and $v\,(v)$-torsion and $v\,(vv)$-curvature are external, only.

Let $\mathbb{G}$ be a metrical h-Riemannian, v-locally Minkowski on $E = TM$, v-weakly regular (Theorem 2.1) and we denote $r_{ij} = r_i{}^k{}_{jk}$, $r = g^{ij} r_{ij}$, $s_{ab} = s_a{}^c{}_{bc}$, $s = h^{ab} s_{ab}$, etc.

Taking into account the results of [1] and [2] (see, also [5] and [11] , pg.83), we obtain

Theorem 4.2 *The Einstein equations of d-connection* $D\hat{\Gamma}$, *(4.1) of Riemann-local Minkowski metric* $\mathbb{G}$, *(2.1), are given by*

$$\begin{aligned} &r_{jk} - \tfrac{1}{2}(r+s)\,g_{jk} - (n-2)\left(\sigma_{jk} - \tfrac{1}{2}\sigma g_{jk}\right) + \tfrac{1}{2}(m-1)\,\tau g_{jk} = \kappa \overset{1}{\mathcal{T}}_{jk}, \\ &s_{bc} - \tfrac{1}{2}(s+r)\,h_{bc} - (m-2)\left(\tau_{bc} - \tfrac{1}{2}\tau h_{bc}\right) + \tfrac{1}{2}(n-1)\,\sigma h_{bc} = \varkappa \overset{4}{\mathcal{T}}_{bc}, \end{aligned} \tag{4.5}$$

where κ *is constant,* $\overset{1}{\mathcal{T}}_{ij}, \overset{2}{\mathcal{T}}_{ij} = 0, \overset{3}{\mathcal{T}}_{ib} = 0, \overset{4}{\mathcal{T}}_{ab}$ *are the components in the adapted basis of the energy-momentum tensor field*

$$\mathcal{T} = \overset{1}{\mathcal{T}}_{ij} dx^i \otimes dx^j + \overset{4}{\mathcal{T}}_{ab} \delta y^a \otimes \delta y^b, \tag{4.6}$$

$$\sigma = 2g^{ij}\sigma_{ij}, \quad \tau = 2h^{ab}\tau_{ab}. \tag{4.7}$$

Theorem 4.3 *The conservation law in this model is given by*

$$\begin{aligned} &\left[r^i_j - \frac{1}{2} r \delta^i_j - (n-2)\left(\sigma^i_j - \frac{1}{2}\sigma\delta^i_j\right)\right]_{\hat{\shortmid} i} = 0, \\ &\left[s^a_b - \frac{1}{2} s \delta^a_b - (m-2)\left(\tau^a_b - \frac{1}{2}\tau\delta^a_b\right)\right]_{\hat{|} a} = 0 \end{aligned} \tag{4.8}$$

where

$$r^i_j = g^{ik} r_{kj}, \quad \sigma^i_j = g^{ik}\sigma_{kj}, \quad s^a_b = h^{ac} s_{cb}, \quad \tau^a_b = h^{ac}\tau_{cb}. \tag{4.9}$$

Theorem 4.4 *The divergence of energy-momentum tensor is as follows*

$$\left(Div\overset{1}{\mathcal{T}}\right)_j = \frac{1}{\kappa} U_j = 0, \quad \left(Div\overset{4}{\mathcal{T}}\right)_b = \frac{1}{\kappa} U_b = 0, \tag{4.10}$$

where

$$\left(Div\overset{1}{\mathcal{T}}\right)_j = \overset{1}{\mathcal{T}}{}^i_{j\hat{\shortmid} i}, \quad \left(Div\overset{4}{\mathcal{T}}\right)_b = \overset{4}{\mathcal{T}}{}^a_{b\hat{|} a}$$

and

$$\begin{aligned} U_j &= \frac{1}{2}\sigma_i \left(r^i_j - \frac{1}{2}\sigma\delta^i_j\right) + (n-2)\left[\sigma^i\left(\partial_i\sigma_j - \partial_j\sigma_i\right) - \frac{1}{2}\left(\partial_j\alpha - 3\alpha\sigma_j\right)\right], \\ U_b &= \frac{1}{2}\tau_a \left(s^a_b - \frac{1}{2}\tau\delta^a_b\right) + (m-2)\left[\tau^a\left(\dot{\partial}_a\tau_b - \dot{\partial}_b\tau_a\right) - \frac{1}{2}\left(\dot{\partial}_b\beta - 3\beta\tau_b\right)\right]. \end{aligned} \tag{4.11}$$

Generally, the equations (4.4) are not identically satisfied. Therefore, we need to find the conditions for 1-forms σ_i and τ_a, such that the conservation law to be satisfied.

In this aim, if we denote by $_{\shortparallel}$ the covariant derivative with respect to Levi-Civita connection γ^i_{jk} of $g_{ij}(x)$ and with $\|$ the covariant derivative with respect to Levi-Civita connection γ^a_{bc} of $h_{ab}(y)$, we obtain

Theorem 4.5 *The conservation law in the Riemann-local Minkowski model with h- and v-semisymmetric torsions is satisfied, if and only if the fields of* 1*-forms* σ_i *and* τ_a *satisfies the equations*

$$\begin{aligned}&\left(r^i_j - \tfrac{1}{2} r \delta^i_j\right)\sigma_i + (n-2)\left[\sigma_{j \shortmid\shortmid i}\sigma^i + \sigma\sigma_j + (n-4)\,\partial_j \alpha - 3\,(n-3)\,\sigma_j \alpha\right] = 0,\\ &\left(s^a_b - \tfrac{1}{2} s \delta^a_b\right)\tau_a + (m-2)\left[\tau_{b\|a}\tau^a + \tau\tau_b + (m-4)\,\dot{\partial}_b \beta - 3\,(m-3)\,\tau_b \beta\right] = 0.\end{aligned} \tag{4.12}$$

Now, we consider $\dim M = 4$. We have, also $m = 4$.

Taking into account the above notations, we obtain:

Theorem 4.6 Let $\mathbb{G}$ be a Riemannian-locally Minkowski structure on $E = TM$, $\dim M = 4$, v-weakly regular. Then:

(i) The Einstein equations of the d-connection (4.1) are given by:

$$\begin{aligned} r_{jk} - \frac{1}{2}\,(r + s - 2\sigma - 3\tau)\, g_{jk} - 2\sigma_{jk} &= \varkappa \overset{1}{T}_{jk},\\ s_{bc} - \frac{1}{2}\,(r + s - 2\tau - 3\sigma)\, h_{bc} - 2\tau_{bc} &= \varkappa \overset{4}{T}_{bc}. \end{aligned} \tag{4.13}$$

(ii) The conservation law is given by:

$$\begin{aligned} \left[r^i_j - \frac{1}{2}\,(r - 2\sigma)\,\delta^i_j - 2\sigma^i_j\right]_{\widehat{\shortmid}\, i} &= 0,\\ \left[s^a_b - \frac{1}{2}\,(s - 2\tau)\,\delta^a_b - 2\tau^a_b\right]_{\widehat{\shortmid}\, a} &= 0. \end{aligned} \tag{4.14}$$

(iii) The conservation law is satisfied if and only if the fields of 1*-forms* $\sigma_i(x)$ *and* $\tau_a(y)$ *satisfies the equations*

$$\begin{aligned} \left(r^i_j - \frac{1}{2} r \delta^i_j\right)\sigma_i + 2\left[\sigma_{j \shortmid\shortmid i}\sigma^i + (\sigma - 3\alpha)\,\sigma_j\right] &= 0,\\ \left(s^a_b - \frac{1}{2} s \delta^a_b\right)\tau_a + 2\left[\tau_{b\,\|\,a}\tau^a + (\tau - 3\beta)\,\tau_b\right] &= 0. \end{aligned} \tag{4.15}$$

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