FINSLER SPACES WITH POLYNOMIAL METRIC

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1. Introduction

Let M be a paracompact n-dimensional manifold and a an r-form, that is a covariant tensor $a \in \mathcal{T}_r^\circ$ of type $(0, r)$ on M with components $a_{i_1 \dots i_r}(x)$, $1 \le i_1, \dots, i_r \le n$ is a local coordinate system (x). Then $a_{i_1 \dots i_r}(x) y^{i_1} \dots, y^{i_r}$, $y \in T_x M$ (summation over $1 \le i_1, \dots, i_r \le n$) is a homogeneous polynomial in $T_x M(y)$. We suppose that

$$a_{i_i \dots i_r}(x) y^{i_1} \dots y^{i_r} = 1$$

is a star-shaped convex hypersurface in $T_x M(y)$. Then $F^n = (M, \mathcal{F})$ with the Finsler metric

$$\mathcal{F}^r(x, y) = a_{i_1 \dots i_r}(x) y^{i_1}, \dots, y^{i_r} \tag{1}$$

is a Finsler space with *polynomial metric*. Such F^n are generalizations of the Berwald-Moór metric (see [12] p. 53 or [20], [21], [16], [17], [18]). F^n with polynomial metric were recently investigated by several authors, such as V. Balan, N. Brinzei, S. Lebedev, D. G. Pavlov etc. in [2], [3], [13], [15], [19]. They considered these spaces endowed with linear metrical connections acting in the vector bundle

$$TM \times_M TM = \mathcal{V}TM = (VTM, \pi, \mathcal{V}^n)$$
$$\pi^{-1}(x, y) = \mathcal{V}^n = \{\xi(x, y)\},$$

where $\mathcal{V}^n$ is an n-dimensional real vector space, and (x, y) is a line-element. $\mathcal{V}TM$ is no tangent bundle, for $\dim TM = 2n \ne n = \dim \mathcal{V}^n$. In $\mathcal{V}TM$ there exist linear metrical connections (e.g. Cartan connection), which allow to develop a curvature theory, etc. in a way similar to that of Riemannian geometry. But using this bundle and line-elements (x, y) has some disadvantages too. The theory becomes more complicated, and the difference between the dimensions of the base space TM and the fiber $\mathcal{V}^n$ is sometimes incovenient, especially in physics. A linear connection acting in the bundle $\tau_M = (TM, \pi, M, \mathcal{V}^n)$ is more simple, but in a Finsler space it cannot be metrical in general. Nevertheless there are many Finsler spaces which allow linear metrical connections in the tangent bundle. Such are the Riemannian space V^n, Minkowski spaces $\mathcal{M}^n$, locally Minkowski spaces $\ell\mathcal{M}^n$, and also the affine deformations $\mathcal{A}\ell\mathcal{M}^n$ of locally Minkowski spaces ([23], [24]), the Finsler spaces with 1-form metric ([15], [16]), the space modelled on Minkowski spaces ([11], [12]). Some of these spaces may not exist on every paracompact manifold ([4], [24]). There are also Finsler spaces admitting metrical connections in τ_M which are only near to linear connections [25] or which are homogeneous only [13].

In this paper we want to show that Finsler spaces with polynomial metric allow metrical tensorial connections (linear for a given type of tensors). Many of them induce, in a natural way, metrical non-linear connections in τ_M.

2. Tensorial connection

Let us consider the tensors t of type $(r,0)$, $t \in \mathcal{T}_0^r$ over the n-dimensional manifold M. $\mathcal{T}_0^r$ is a real vector space $\mathcal{V}^N$ of dimension $N = n^r$. Thus t^A, $A = 1, 2, \ldots, N$ can mean the components of t.

$$\hat{\mathcal{E}} = (\hat{E}, \pi, M, \mathcal{V}^N), \quad \pi : \hat{E} \to M, \quad \pi^{-1}(p) \approx \mathcal{V}^N, \qquad p \in M \tag{2}$$

is a tensor bundle, that is a vector bundle of rank N over M. A linear connection γ acting in $\hat{\mathcal{E}}$ is called *tensorial connection.* In a local coordinate system (x) it is given by the connection coefficients

$$\gamma_A{}^B{}_k(x), \qquad A, B = 1, 2, \ldots, N, \quad k = 1, 2, \ldots, n$$

and the parallel translated $\mathcal{P}^\gamma_{x(\tau)} t_0$ of $t_0 \in \pi^{-1}(x(\tau_0))$ along a curve $x(\tau)$ according to γ is defined by the solution $t(\tau)$ of the ODE system

$$\frac{Dt^B}{d\tau} \equiv \frac{dt^B}{d\tau} + \gamma_A{}^B{}_k(x(\tau)) t^A \frac{dx^k}{d\tau} \tag{3}$$

with initial value $t(0) = t_0$. With an appropriate γ one can realize any linear mapping between $\pi^{-1}(x(\tau_0)) \approx \mathcal{V}^N$ and $\pi^{-1}(x(\tau)) \approx \mathcal{V}^N$. – An affine connection Γ with coeffients $\Gamma_j{}^i{}_k(x)$ also induces certain (linear) mappings between the above fibers. These mappings are realized by special tensorial connections. In case of $r = 2$ the corresponding γ has the coefficients

$$\gamma_A{}^B{}_k(x) \equiv \gamma_{rs}{}^{ij}{}_k(x) = \Gamma_r{}^i{}_k(x)\delta^i_s + \delta^i_r \Gamma_s{}^j{}_k(x). \tag{4}$$

Clearly γ-s of this form are special ones, and they do not generate all linear mappings between $\pi^{-1}(x(\tau_0))$ and $\pi^{-1}(x(\tau))$. Also conversely, if a $\gamma_A{}^B{}_k$ can be represented in the form (4), then the tensorial connection γ reduces to the affine connection Γ.

The tensorial connection given by (3) is linear in $t \in \mathcal{T}_0^r$, and the operator $\frac{D}{d\tau}$ of (3) can be extended to the tensor algebra of tensors of type $(\lambda r, \mu r)$, where λ and μ can be arbitrary no-negative integers. Tensorial connection was introduced by E. Bompiani [9], and investigated by A. Cossu [10], L.Tamássy [22], M. Kucharzewski [14], and others.

Let $\overline{M}$ be an $N = n^r$ dimensional manifold with local coordinates $\overline{x}$, such that $M \subset \overline{M}$, and let $\overline{\gamma}(\overline{x})$ be a C° extension of γ, such that its restriction to M yields $\gamma : \overline{\gamma}(\overline{x}) \restriction_M = \gamma(x)$. Then $(\overline{M}, \overline{\gamma})$ is an (ordinary) affine connection in the tangent bundle $\tau_{\overline{M}} = (T\overline{M}, \pi, \overline{M}, \mathcal{V}^N)$. So we obtain the

Proposition 1 *Any tensorial connection $(M^n(x), \gamma(x))$ is the restriction of an affinely connected space $(\overline{M}^N(\overline{x}), \overline{\gamma}(\overline{x}))$ in the form*

$$(M^n, \gamma(x)) = (\overline{M}^N, \overline{\gamma}(\overline{x})) \restriction M, \qquad N = n^r.$$

Here the restriction happens in the base manifold $\overline{M}$. This is in analogy to the fact that any Finsler space F^n can be considered as the restriction of a Riemannian space $V^{2n} = (TM, \mathcal{G})$, where $\mathcal{G}$ is the Sasakian type metric of F^n. Here the restriction happens in the fiber. The tangent space TTM of V^{2n} is restricted to the vertical bundle $\mathcal{V}TM$ of the Finsler space.

A tensorial connection γ has two curvatures $\mathcal{A}_E{}^C{}_i$; $\mathbb{R}_j{}^i{}_{k\ell}$, and a torsion tensor $\mathcal{S}_j{}^i{}_k$. Vanishing of $\mathcal{A}$ characterizes the reduction of γ to Γ. In this case also $\mathbb{R}$ and $\mathcal{S}$ reduce to curvature R^Γ and torsion $\mathcal{S}^\Gamma$ of Γ ([22]).

3. Tensorial connections in case of polynomial metric

The $a(x) \in \mathcal{T}_0^r$ appearing in (1) is *parallel* along $x(\tau)$ according to γ, if

$$\frac{da_A}{d\tau} = \gamma_A{}^B{}_k(x(\tau))a_B\frac{dx^k}{d\tau},$$

and $a(x)$ is an *absolute parallel* tensor field on M (or on a domain of it), if

$$\nabla_k a_A = 0, \tag{5}$$

that is

$$\frac{\partial a_A}{\partial x^k} = \gamma_A{}^B{}_k(x)a_B. \tag{5'}$$

The Finsler norm $\|y\|_F$ of a vector $y \in T_xM$ in our F^n with polynomial metric is $\|y\|_F^r = \mathcal{F}^r(x,y) = a_A b^A$, and we define the Finsler norm $\|t\|_F$ of tensor $t \in \mathcal{T}_0^r$ in our F^n by

$$\|t\|_F : a_A(x)t^A(x). \tag{6}$$

Thus

$$\|y\|_F^r = \mathcal{F}(x,y) = \|b\|_F. \tag{7}$$

The tensorial connection is called *metrical* if

$$\|\mathcal{P}^\gamma_{x(\tau)}t_0\|_F = \|t(\tau)\|_F = \text{const.}, \qquad \forall\, x(\tau) \subset N, \quad t_0 \in \mathcal{T}^c, \tag{8}$$

and thus

$$\frac{d}{d\tau}\|t(\tau)\|_F = \frac{D}{d\tau}\big(a_A(x(\tau))t^A(\tau)\big) = \left[(\nabla_k a_A)\frac{dx^k}{d\tau}\right]t^A + a_A\frac{Dt^A}{d\tau} = \frac{d}{d\tau}\,\text{const} = 0 \tag{8'}$$

for any $t(\tau)$ parallel along any $x(\tau)$. Since for parallel $t(\tau)$ $\frac{Dt^A}{d\tau} = 0$ and for an appropriate $x(\tau)$ we can obtain every x_0 and $\dot{x}_0$, (8) is equivalent to (5').

For given $a(x)$ (5') is a linear equation system at any point x for the unknowns $\gamma_A{}^B{}_k(x_0)$. The equations of (5') are independent in the sense that each $\gamma_A{}^B{}_k(x_0)$ appears in a single equation only. Hence (5') is solvable for $\gamma_A{}^B{}_k(x)$. Thus we obtain

Theorem 1 *Any Finsler space with polynomial metric* (1) *has metrical tensorial connections.*

(5') consist of Nn equations, and in each of them (for fix A and k) appear N unknowns $\gamma_A{}^B{}_k$, of which $N-1$ can arbitrarily be choosen. Thus in the solution of (5') $Nn(N-1) = (N^2-N)n$ of the $\gamma_A{}^B{}_k$ remain arbitrary.

The upper script indices of a totally symmetric tensor $t^{i_1\ldots i_r} \in \mathcal{T}_0^r$ are the multiple combinations of order r from the elements $1,2,\ldots,n$. These tensors form a linear subspace ${}^s\mathcal{T}_0^r$ of $\mathcal{T}_0^r$. The dimension of ${}^s\mathcal{T}_0^r$ is $C^m_{r,n} = \frac{(n-1+r)!}{(n-1)!r!} = C$, the number of the multiple combinations of order r from n elements $1,2,\ldots,n$. The components of such a tensor will be denoted by t^α, $\alpha = 1,2,\ldots,C$. Also $y^{i_1}\ldots y^{i_r} = b^{i_1\ldots i_r} = b^\alpha \in {}^s\mathcal{T}_0^r$. If in (1) we draw together those $a_{i_1\ldots i_r}$ in which the same $i_1,i_2,\ldots i_r$ appear (independently from the order), and denote their sum by g_α, then with respect to (6), (1) gets the form

$$\mathcal{F}^r(x,y) = g_\alpha b^\alpha = \|b\|_\alpha, \qquad \alpha = 1,2,\ldots,C. \tag{1'}$$

b is decomposable. It is an r-times tensor product of $y \in \mathcal{T}_x M$:

$$b = \overset{1}{y} \otimes \cdots \otimes \overset{r}{y}.$$

Thus

$$\phi := \{b\}$$

is a cone in ${}^s\mathcal{T}_0^r$. Its parameter representation is

$$b^\alpha = f^\alpha(y', \dots, y^n) := y^{i_1} \dots y^{i_r}, \quad \alpha = i_1 \dots i_r. \tag{9}$$

The correspondence between $(y^1, \dots, y^n) \in \mathcal{V}^n(y)$ and $b \in \phi \subset {}^s\mathcal{T}_0^r$ is $1:1$. Thus $\dim \phi = n$. (9) is independent of $x \in M$. Thus ϕ has the same form in each fiber $\mathcal{V}^C \approx {}^s\mathcal{T}_0^r \subset \mathcal{T}_0^r \approx \pi^{-1}(x)$ of the bundle $\tilde{\mathcal{E}} = (\tilde{E}, \pi, M, \mathcal{V}^C)$.

One can see that

$$\mathcal{P}^\gamma_{x(\tau)} b_0 = b(x(\tau)) \equiv b(\tau) \in \phi(x(\tau)), \tag{10}$$

or in another form

$$\mathcal{P}^\gamma_{x(\tau)} \phi(x_0) = \phi(x(\tau)) \tag{10'}$$

does not hold in every tensorial connection γ. We want to obtain necessary and sufficient conditions for (10) to hold. We suppose that $b(x) = b(\tau) \in \phi(x(\tau)) = \phi(x)$, where $\phi(x)$ is independent of x. Hence every $b(x(\tau)) = b(x)$ can be considered as a point of a single representative ϕ of the $\phi(x)$-s. Thus in case of (10) every $\frac{\partial b^\alpha}{\partial x^k}$ is a tangent of this ϕ:

$$\frac{\partial b^\alpha}{\partial x^k} \in T_b \phi. \tag{11}$$

But also conversely, if (11) is satisfied, then so is (10).

On the other hand $b(\tau)$ of (10) is a solution of

$$\frac{db^\alpha}{d\tau} = \frac{\partial b^\alpha}{\partial x^k}\frac{dx^k}{d\tau} = \gamma_\beta{}^\alpha{}_k(x(\tau)) b^\beta \frac{dx^k}{d\tau}, \quad \alpha, \beta = 1, 2, \dots, C, \quad \forall\, x, \dot{x}.$$

Thus $\gamma_\beta{}^\alpha{}_k$ must satisfy the relation

$$\frac{\partial b^\alpha}{\partial x^k}(y) = \gamma_\beta{}^\alpha{}_k(x) b^\beta(y). \tag{12}$$

Any tangent of ϕ is a linear combination of $\frac{\partial f^\alpha}{\partial y^j} \equiv \frac{\partial b^\alpha}{\partial y^j}$ at y. Thus the required necessary and sufficient condition (11) gets the form

$$c_k^j(y)\frac{\partial b^\alpha}{\partial y^j}(y) = \gamma_\beta{}^\alpha{}_k(x) b^\beta(y). \tag{13}$$

This must be satisfied identically in y.

(13) can be considered as a linear equation system for $\gamma_\beta{}^\alpha{}_k$ and c_k^j. We show that (13) has a solution, while many of the unknowns $\gamma_\beta{}^\alpha{}_k$ and c_k^j remain undetermined (free).

$b^\beta(y)$ is a homogeneous polynomial of order r in y. $\frac{\partial b^\alpha}{\partial y^j}$ is also a homogeneous polynomial of order $r-1$. Thus c_k^j must be a homogeneous polynomial of order $1 : c_k^j(y) = {}_s c_k^j y^s$. So (13) gets the form

$$ {}_s c_k^j y^s \frac{\partial b^\alpha}{\partial y^j}(y) = \gamma_\beta{}^\alpha{}_k(x) b^\beta(y). \tag{13'}$$

This is a special, very simple equation system. For any fixed k_0 we obtain a subsystem

$$ {}_sc^j y^s \frac{\partial b^\alpha}{\partial y^j}(y) = \gamma_\beta{}^\alpha(x) b^\beta(y), \quad {}_sc^j = {}_sc^j_{k_0}, \ \gamma_\beta{}^\alpha = \gamma_\beta{}^\alpha{}_{k_0}. \tag{14} $$

The unknowns ${}_sc^j_{k_0}$ and $\gamma_\beta{}^\alpha{}_{k_0}$ appear in one single subsystem only. Since every subsystem has the same structure, we have only to solve (14). Let us fix $\alpha = \alpha_0$. Then on both sides of (14) there is a homogeneous polynomial of order r in y, and (14) must hold identically. Thus the coefficients of $y^{i_1} \dots y^{i_r}$ consisting of the different ${}_sc^j$ and $\gamma_\beta{}^\alpha$ must be equal on the two sides. These yield homogeneous linear equations, C in number, for ${}_sc^j$ and $\gamma_\beta{}^\alpha$. The number of the unknowns ${}_sc^j$ and $\gamma_\beta{}^\alpha$ is $n^2 + C^2$. For the different α-s (14) consists of C equations. So the number of the equations for ${}_sc^j$ and $\gamma_\beta{}^\alpha$ stemming from (14) is C^2, and the number of the unknowns remains $n^2 + C^2$. (13') consists of n subsystems for the different k_0 with new unknows in each. Thus (13') yields, as identities in y^s, C^2n equations with $n^3 + C^2n$ unknowns. So we obtain

Proposition 2 *There are many tensorial connections γ taking by parallel translation any decomposable tensor $b = \overset{i}{y} \otimes \dots \otimes \overset{r}{y}$ into a similar one: $\mathcal{P}^\gamma_{x(\tau)} b_0 = b(\tau)$.*

4. Induced non-linear connection in τ_M

A tensorial connection γ for which $\mathcal{P}^\gamma_{x(\tau)} b_0 \overset{(10)}{=} b(x(\tau))$, or in another form $\mathcal{P}^\gamma_{x(\tau)} \phi(x_0) \overset{(10')}{=} \phi(x(\tau)) \approx \phi$ holds, induces a non-linear connection in τ_M. Namely, as also the diagram

$$ \begin{array}{ccc} b_0 \in \phi(x_0) & \xrightarrow{\mathcal{P}^\gamma_{x(\tau)}} & b(\tau) \in \phi(x(\tau)) \\ \uparrow f & & \downarrow f^{-1} \\ y_0 \in T_{x_0}M & \overset{\mathcal{N}}{\dashrightarrow} & y(\tau) \in T_{x(\tau)}M \end{array} $$

shows (f^α from (9))

$$ \mathcal{N} := (f^\alpha)^{-1} \circ \mathcal{P}^\gamma_{x(\tau)} \circ f^\alpha \tag{15} $$

takes any $y_0 \in T_{x_0}M$ into a $y(\tau) \in T_{x(\tau)}M$. Thus

$$ \mathcal{P}^{\mathcal{N}}_{x(\tau)} y_0 = y(\tau). \tag{16} $$

$\mathcal{N}$ is non-linear in y, for $\mathcal{P}^\gamma$ is so in b. Thus we obtain

Theorem 2 *Any tensorial connection, which takes tensors $b = \overset{1}{y} \otimes \dots \otimes \overset{r}{y}$ into similar ones determines in τ_M among the vectors $y \in T_xM$ a non-linear connection $\mathcal{N}$ in a natural way.*

We want to investigate *metrical* tensorial connections γ of a Finsler space with polynomial metric, which induce non-linear connections $\mathcal{N}$ in τ_M. Then γ satisfies (13'), and it is metrical. A tensorial connection is metrical, if (5') or, in view of the symmetry of a_A,

$$ \frac{\partial g_\alpha}{\partial x^k} = \gamma_\alpha{}^\beta{}_k(x) g_\beta \tag{17} $$

holds. At a given (x) (17) means Cn linear equations for the unknowns $\gamma_\alpha{}^\beta{}_k$. So (13') (respectively the equations stemming from the fact that the equations of (13') must be identities in y^s) combined with (17) consists of $Cn + C^2n$ (simple) linear equations, and the number of the unknowns ${}_sc^j_k$ and $\gamma_\beta{}^\alpha{}_k$ remains $C^2n + n^3$. The rank of the combined system is maximal. If the number of the unknowns is not less than the number of the equations, that is if $C^2n + n^3 \geq C^2n + Cn$, or

$$n^2 \geq C^m_{n,r}, \tag{18}$$

then the combined system is solvable. Since γ is metrical, in this case we have

$$\|\mathcal{P}^{\gamma}_{\alpha(\tau)} b_0\|_F \overset{(7)}{=} \|b(\tau)\|_F = \text{const} \overset{(6)}{=} \mathcal{F}(x, y(\tau)) = \|y(\tau)\|^r_F = \|\mathcal{P}^{\mathcal{N}}_{x(\tau)} y_0\|^r_F.$$

Thus $\|\mathcal{P}^{\mathcal{N}}_{x(\tau)} y_0\|_F = \text{const}$. This yields

Theorem 3 *If γ is metrical (satisfies* (17)*), and takes every $b = \overset{1}{y}\otimes, \dots, \overset{r}{y}$ into a similar tensor (which satisfies* (13')*), then also the induced non-linear connection $\mathcal{N}$ is metrical with respect to the F^n with polynomial metric.*

The condition of the solvability of the combined system is (18). For which n and r will it be satisfied? It is clear from the notion of multiple combination that $C^m_{n,r}$ is monotone increasing in r for every fix n, and also in n for every fix r. Therefore there exists a minimal r for every n for which $n^2 \geq C^m_{n,r}$. We denote this r by r_n. Then we obtain

Proposition 3 (18) *holds iff $r < r_n$. In this case the combined system* (13') *and* (17) *is solvable, and the induced non-linear connection $\mathcal{N}$ is metrical.*

In case of $r = 2$ we have $C^m_{n,r} = \frac{n(n+1)}{2} < n^2$. Thus (18) holds for $\forall\, n$, and so we have tensorial connections γ inducing metrical non-linear connections $\mathcal{N}$ in τ_M. In this case $\mathcal{F}^2(x, y) = a_\alpha(x) b^a = a_{ij}(x) y^i y^j$. This means that for $r = 2$ the Finsler space with polynomial metric is a Riemann space: $F^n = V^n$. Then $\gamma_\alpha{}^\beta{}_k(x) = \gamma_{ij}{}^{rs}{}_k(x) = \Gamma_i{}^r{}_k(x)\delta^s_j + \delta^r_i \Gamma_j{}^s{}_k(x)$. This γ is constructed from the symmetric (torsion free) or non-symmetric Christoffel symbols of V^n. This γ yields a metrical tensorial connection, and the metrical connection $\mathcal{N}$ in τ_M becomes linear with coefficients $\Gamma_j{}^i{}_k(x)$.

In case of $r = 3$ (18) reads as $C^m_{n,3} = \frac{n(n+1)(n+2)}{6} \leq n^2$ or equivalently $n^2 + 1 \leq 3n$. This holds for $n = 2$, but for $r = 3$ and $n = 3$ (18) is not yet true. For $n_0 \geq 3$, $r \geq 3$ we have $n_0^2 < C^m_{n_0,2} < C^m_{n_0,r}$, since $C^m_{n_0,r}$ is increasing in r. Thus for $n \geq 3$, $r \geq 3$ (18) does not hold. For $n = 2$ $C^m_{2,r} = r + 1$. Thus (18) holds for $n = 2$, $r = 3$: $2^2 = C^m_{2,3}$ (as we have already seen), but $C^m_{2,3} < C^m_{2,r}$, $r > 3$, since $C^m_{2,r}$ is inceasing in r. So we have $n^2 = 4 = C^m_{2,3} < C^m_{2,r}$, that is (18) holds neither for $n = 2$, $r > 3$.

But there may exist special $g_\alpha(x)$ for which the number of the independent equations of (17) is smaller than Cn, and thus the combined system (13') and (17) still has a solution, for example if $\frac{\partial g_{\alpha_1}}{\partial \alpha^k} + \frac{\partial g_{\alpha_2}}{\partial x^k} = \frac{\partial g_{\alpha_3}}{\partial x^k}$ for certain (or several) k. The number of the dependent equations of (17) may run from C to zero. If the curvature $R_\alpha{}^\beta{}_{ij}(x)$ of the tensorial connection γ vanishes, then there exists $g_\alpha(x)$, such that (17) yields identities: presents no new equation for $\gamma_\alpha{}^\beta{}_k$.

Theorem 4 *If γ is metrical (satisfies* (17)*), and takes every $b = \overset{1}{y} \otimes \cdots \otimes \overset{r}{y}$ into a similar tensor (satisfies* (13')*), then the induced non-linear connection $\mathcal{N}$ is also metrical with respect to the F^n with polynomial metric. The condition for this is $n^2 \geq C^m_{n,r}$.*

Such γ exists for any Finsler space with polynomial metric only if $r = 2$ (in this case the Finsler space is a Riemannian space) or in case of $r = n = 3$. Such γ exists also for arbitrary r and n, but not for every polynomial metric.

Finally we make two remarks:

Remark 1 *$a_A(x)$ of* (1) *may have the form*

$$a_{ijk\ell}(x) = g_{ij}(x)h_{m\ell}(x),$$

where $g_{ij}(x)$ and $h_{m\ell}(x)$ are metric tensors of two Riemannian spaces V_1^n and V_2^n on M. Then

$$\mathcal{F}^4(x,y) = \|y\|_F^4 = \|y\|_{V_1}^2 \|y\|_{V_2}^2.$$

This may have a mathematical inteterest, but $\|y\|_{V_1}$ and $\|y\|_{V_2}$ could also mean two different impacts of a physical phenomenon.

Remark 2 *A Randers space $R^n = (M, \mathbb{R}(x,y))$ is a special Finsler space ([7], [16]), where*

$$\mathbb{R}(x,y) = (g_{ij}(x)y^i y^j)^{1/2} + b_i(x)y^i$$

in place of $\mathcal{F}(x,y)$ means the Randers metric. In a degenerate case we may have $\mathbb{R}(x,y) = b_i(x)y^i$. If we endow in the vector bundle $\hat{\mathcal{E}}$ (see (2)*) of rank N each fiber $\pi^{-1}(x) \approx \mathcal{V}^N$ with the metric $\mathbb{R}(x,y) = a_A(x)b^A$, then we obtain a degenerate Randers vector bundle R_N^n. Thus any Finsler space with polynomial metric* (1) *can be considered as a degenerate Randers vector bundle. – It could have some interest to consider a Finsler space with polynomial metric as a degenerate Randers vector bundle.*

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