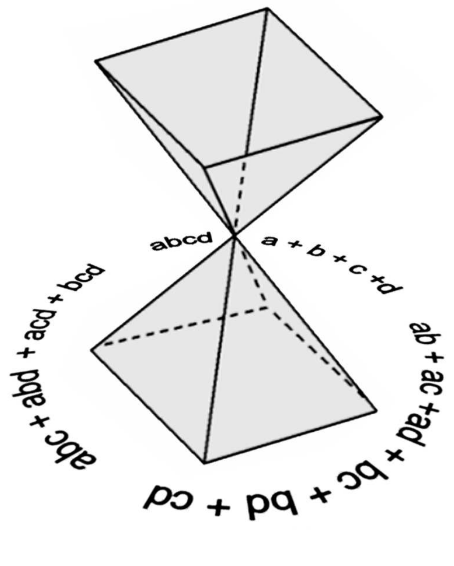
Space-Time Structure. Algebra and Geometry

D. G. Pavlov, Gh. Atanasiu, V. Balan
Editors



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The present volume contains selected articles published in the scientific journal "Hypercomplex Numbers in Geometry and Physics" (numbers 1–6). The topics covered by the included contributions range from hypercomplex algebras to associated Finsler geometries, focusing on the commutative-associative algebras and on Berwald-Moor metric structures, applications of polynumber algebras in Physics, and Finslerian extensions of Relativity Theory.

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Preface

A historical perspective

Finsler geometry is a natural extension of the Riemannian geometry. As specific feature, its metric encompasses information about direction – hence it provides reliable models for both isotropic and anisotropic phenomena. Moreover, the Finsler energy – the square of the fundamental function, does not reduce in general to a quadratic form in the directional coordinates – as it does in the Riemannian subcase.

The historical traces of Finsler geometry go back to 1854, when the first Finsler metric was pointed out by B. Riemann in his famous lecture *On Hypotheses Lying at the Basis of Geometry*[1]. Much later, in 1918 Paul Finsler systematically studied in his Ph. D. Thesis – having as advisor C. Caratheodory, the spaces which later received his name.

The Finslerian framework was intensively developed by J. L. Synge, V. Wagner, L. Berwald, E. Cartan, H. Busemann, H. Rund, M. Matsumoto, S. S. Chern and others. Finsler geometry has become a separate branch of differential geometry, being included in the AMS mathematics subject classification (53B40 & 53C60). The Finsler-type structures have been further extended to fiber spaces (e.g., higher order tangent and cotangent, osculating and jet spaces).

The applications of Finsler geometry in various branches of theoretical physics flourished as well, especially in the last two decades. Apart from such conventional fields as the theory of anisotropic media and Lagrangian mechanics, classical Finsler geometry and its generalizations have found extensive applications in solving optimization problems, in describing systems with chaos, in statistical physics and thermodynamics, in ecology and in the theory of evolution of biological systems, in describing the internal symmetry of hadrons, in the theory of space-time and gravitation as well as in unified gauge field theories.

More recently, Finsler geometry proved to successfully apply to the theory of deformations of crystalline media, seismic phenomena, Zermelo navigation problems and interfaces in thermodynamic systems.

It should be noted that between two historical alternative approaches to Finsler geometry due to Cartan and Busemann, the first one provides efficient tools in dealing with theories of Kaluza-Klein type, offers new structural possibilities and identifies new (comparing with Riemann geometry) elements of structure with physical observables. The physical fields in a Cartan Finsler space, apart from space-time coordinates, turn out to be functions of supplementary so-called internal variables.

On the other hand, there is a close relation between notion of Number and fundamental physical categories as Space, Time, matter and field. Usually, this

[1] B. Riemann, *Uber die Hypothesen, welche der Geometrie zu Grunde liegen*, habilitation address, 1854, translated in: M. Spivak, *Differential Geometry*, vol. II, Publish or Perish, 1970.

relation is associated with particular numbers as the Real and Complex numbers, and sometimes quaternions. Apart from denying the fundamental role of these numbers, the authors of this book point out that there exist others generalizations of numbers, which have substantial physical and geometrical interpretation. Emerging from the fact that the progress in physics often was stimulated by novel geometrical points of view, a high expectation emerges that the proper scientific description of the geometrical background might lead to new qualitative consequences in Physics.

It is a known fact that while creating the Relativity theory, Einstein was forced to get out of the classic Euclidean geometry, replacing it with the Riemann one. Hence it is natural to assume that the future development of Physics will also need a certain new geometry, which might be Finsler geometry – which naturally extends Minkowski geometry as well. It is fundamentally important, that the points of Finsler spaces in some cases may be expressed in terms of hypercomplex numbers, which are algebras with special properties.

The wide background Finsler spaces provides simple and wonderful particular cases linked with hypercomplex numbers – which possess the usual associativity and commutativity properties. Unfortunately, today there exist few attempts of classifying Finsler Spaces emerging from this perspective. Such Finsler related algebras – having their roots in the applied aspects of geometry with modern physics are, e.g., the algebras of the quaternions over the field of complex numbers (biquaternions) and over the ring of duals (diquaternions), and also algebras of complex numbers over complex numbers (bicomplex numbers) and dual numbers above dual numbers (quadranumbers). All this spaces possess multiplicative norms of fourth order, and prove to be tightly connected with the fundamental for physicists Lorenz group.

In Finsler geometry the class of new and classic invariants is much wider than in the Riemannian case, which makes it very tempting to use this type of geometry for modeling different physical phenomena, instead of using habitual Riemannian geometry. The study of Finslerian invariants reveals the existence of interesting special nonlinear transformations, whose Euclidean analogues are the conformal transformations. Moreover, in Finsler spaces with fundamental form of higher order than quadratic, due to the existence of new metric properties (besides conservation of lengths and angles), such new invariants to (usually nonlinear) equiform-type transformations are numerous and play a fundamental role.

The development of Finsler Geometry in parallel with Algebra of numbers had led to a notable benefit to both these fields of mathematics. Thus, for the first time has appeared the possibility to solve one of the key problems of the Geometry – to find natural and simple means to generalize the concept of angle, based not on the classic scalar product, but on a symmetric polyscalar form. This approach shows that the concept of Finsler metric tensor – as introduced by Cartan, lacks from being completely effective in applied models and requires its replacement by another tensor with rank grater than two.

From a physical perspective, as a characteristic feature, Finsler geometry is able to provide the formal grounds for posing the problem of local anisotropy of the real space-time, sustained by the fact that within the framework of the model of locally isotropic (Riemann) space-time it is, in principle, impossible to embody the Mach principle for a test body. Physical evidence leads to the conclusion that the inertial body mass, should be a tensor rather than a scalar. Thus the discovery of an anisotropy of inertia would be a direct evidence for a local anisotropy of space. At the same time it has long been pointed out that the conventional experimental estimate of 3D anisotropy at the level 10^{-22} is not correct (S. T. Epstein and G. Yu. Bogoslovsky), and a reliable upper bound of anisotropy should be reconsidered. The experimental findings of breaking of discrete space-time symmetries, anisotropy of background radiation and the absence of the GZK effect have revived interest in the problem of anisotropy of space-time.

On the other hand, very recent measurements of the Compton edge (made by French, Russian, Italian and Armenian scientists) of the scattered electrons in GRAAL facility in European Synchrotron Radiation Facility (ESRF) in Grenoble with respect to the Cosmic Microwave Background dipole has revealed up to 10 σ variations larger than the statistical errors. It was shown[2] that the variations were not due to the frequency variations of the accelerator, and since the nature of Compton edge variations remains unclear, follows the imperative of dedicated studies of light speed anisotropy. The team of scientists has conjectured that the variations could be an effect of existence in the Universe of vector fields which provide a certain vector distribution.

As well, recent experiments developed in Canada have shown[3] that the Hubble constant is affected by anisotropy, reinforcing thus the idea of having a direction dependent Finsler encompassing mathematical model for General Relativity.

Recent investigations on the Finslerian generalization of relativity aim to find a relativistically invariant Finsler metric with partially or even entirely broken 3D isotropy (e.g., G. Yu. Bogoslovsky and H. F. Goenner). All three relativistically invariant metrics, i.e. the isotropic Riemann and two Finsler metrics (with a partial and full anisotropy), satisfy the correspondence principle, which leads to a "hybrid" geometric model in terms of which spacetime may be not only in a state which is described by Riemann geometry but also in states which are described by Finsler geometry.

The mathematical Finslerian framework has a well-established wide basic background of monographs and a significant variety of both theoretical and applicative research papers. Significant trends in Finslerian geometry have been developed in

[2] V. G. Gurzadyan, J.-P. Bocquet, A. Kashin, A. Margarian..., *On the light speed anisotropy vs cosmic microwave background dipole: European synchrotron radiation facility measurements*; WSPC – Proceedings Compton MG11-4, arXiv:astro-ph/0701127 v1, 5 Jan 2007.

[3] M. L. McClure, C. C. Dyer, *Anisotropy in the Hubble constant as observed in the HST extragalactic distance. Scale key project results*, arXiv:astro-ph/0703556 v1, 21 Mar 2007.

recent years in many countries[4].

A new significant trend in modern Finslerian framework emerges from the works of P. K. Rashevski dated 1935–1936, who stated the problem of developing such a geometry, whose emerging objects would include not only the analogues of the basic notions of point and straight line, but planes and n-planes as well (his hypothetical geometry was called at the time poly-metrical). In the last decade, the goal of developing this theory at a superior level within the framework of the relativistic Berwald-Moor Finslerian model has been proposed by the Russian specialists, who have joined their efforts around Dr. D. G. Pavlov to organize two dedicated to the subject Conferences in Cairo (2005 and 2006) and one in Moscow (2007), and have issued the scientific journal "Hypercomplex Numbers in Geometry and Physics" (Moscow, Russia) – whose submission topics confine to the goal of founding a new Finslerian-based model of Relativity.

The layout of the book

The present volume outlines achievements adjacent to this concern, and embraces papers written by Russian and Romanian physicists and mathematicians.

The first three papers, *"Generalization of scalar product axioms", "Chronometry of three-dimensional Time"* and *"Four-dimensional time"* are authored by D. G. Pavlov. The first article describes the generalization of the classical (pseudo-) scalar products, by introducing the "poly-linear" product associated to a given Finsler structure; this leads to the new notions of related fundamental metric polyform and various generalized geometric objects as vector length, angle between vectors and orthogonality. The second paper proposes a Finslerian model, switching thus from the traditional quadratic (Riemannian) metrics to the Finslerian ones; the discussed topics are: light pyramids, specific groups of transformations, planes of relative simultaneity, physical distance and speed. It is emphasized that neither Euclidean, nor pseudo-Euclidean spaces with three or more dimensions do possess analogous qualities to those exhibited in the studied Finslerian models. The third paper specializes the study to the main properties of the Finsler metric space with four-dimensional time, based on the Berwald-Moor Finsler metric function; among the basic introduced physical objects, we mention: event, world lines, reference frames, set of multiple relatively simultaneous events, proper time, three-dimensional distance, speed, etc.

The paper *"Properties of spaces connected with commutative-associative H_3 and H_4 algebras"* (S. V. Lebedev) provides physical significance to the algebraic specific structure H_3, discussing further world lines of resting particles and properties of

[4] E. g., Russia: G. S. Asanov, G. Bogoslovski, G. I. Garas'ko, S. V. Lebedev, D. G. Pavlov; Romania: R. Miron, Gh. Atanasiu, A. Bejancu; USA: D. Bao, R. G. Beil, R. L. Bryant, S. S. Chern, Z. Shen; Canada: P. L. Antonelli; Germany: H.-B. Rademacher; Italy: G. Bellettini, M. Paolini; Japan: T. Aikou, S. Ikeda, T. Kawaguchi, M. Matsumoto, H. Shimada; Hungary: L. Tamassy, L. Kozma; Serbia and Montenegro: I. Comic, etc.

the surface of simultaneity which to defines the distance between the real axis and a parallel to it, line of universe; as well are described Lorentz transformations attached to the structure.

Further, in *"Generalized-analytical functions of poly-number variable"* (G. I. Garas'ko) the reader will find details on the new notion of generalized-analytic function of polynumber variable, a primary step towards constructing a relevant theory able to develop theoretical-physical models.

Though the next paper, *"On some questions of four dimensional topology. A survey of modern research"* (R. V. Mikhailov), has a rather abstract algebraic character, it aims to justify the fact that 4-dimensionality plays special role in almost all modern physical theories and provides a brief survey of some problems of 4-dimensional topology.

In *"Normal conjugation on the polynumber manifold"* (G. I. Garas'ko and D. G. Pavlov) is defined the normal conjugation on the manifold of non-degenerated n-numbers. In this context, the polynumber space endowed with the introduced specific $(n-1)$-ary operation appears as an ideal support for multi-dimensional fractal sets, one of the perspective directions of applying multi-linear geometries.

Several properties of the generalized-analytic functions of polynumber variable are studied in *"Generalized analytical functions and the congruence of geodesics"* (G. I. Garas'ko). It is shown that each such class of functions is naturally associated to a space of congruences of geodesics.

In *"The notions of distance and velocity modulus in the linear Finsler spaces"* (G. I. Garas'ko and D. G. Pavlov), the authors determine in 4-dimensional spaces with Berwald-Moor metrics, formulas for both the 3-dimensional distance – by means of the surface of relative simultaneity, and for the velocity modulus – which coincides with the corresponding expression of the Galilean space for small (non-relativistic) velocities, while at maximal velocities (i.e., for world lines lying on the surface of the cone), it equals unity. They obtain as well the expressions for the transformations which play the same role as the Lorentz transformations in the Minkowski space.

A generalization of conformal transformations to the case of Finsler spaces is provided in *"Generalization of conformal transformations"* (G. I. Garas'ko); examples are provided as well for the case of complex and hypercomplex numbers H_4.

An adjacent issue to the main flow, is the study of the commutative algebra of bi-complex numbers endowed with a metric of signature $(+,-,-,+)$, performed in *"Some properties of bicomplex numbers"* (A. V. Smirnov).

Further, in *"Philosophical and mathematical reasons for Finsler extensions of Relativity Theory"* (D. G. Pavlov), the author provides an overview of arguments to show that Finsler Geometry (and in particular the 4-dimensional quadratic Berwald-Moore Finsler model) is by far the best candidate to provide relevant models for the further development of Physics (in general) and Relativity Theory (in particular). The associated commutative and associative algebra (*quad-numbers* algebra) is essentially described, and it is suggested to generalize the notion of "symmetry" and to widen the classic case, based on isometric and conformal transformations, by introducing generalized conformal transformations.

Elementary generalized conformal transformations in the space of nondegenerate poly-numbers and generalized analytical functions of the same polynumber variable are further provided, with concrete examples for complex and hypercomplex numbers H_4, in *"The relation of elementary generalized conformal transformations with generalized analytic functions in the polynumber space"* (G. I. Garas'ko).

In *"4-momentum of a particle and the mass shell equation in the entirely anisotropic Space-Time"* (G. Yu. Bogoslovsky), the author motivates the use of Finsler geometry models in physics, especially of those, whose metric is of Berwald-Moor type, and studies a certain Finslerian model which involves a preffered direction in the 3D isotropy space, and a dimensionless parameter r which determines the deviation of the metric from the metric of isotropic Minkowski space. Further it is investigated the model for the entirely anisotropic flat space-time, which generalizes of the Finslerian Berwald-Moor one. The variational principle is used to obtain the formulas that relate the 4-momentum of a particle to the 3-velocity of the latter, and the invariants with respect to the relativistic symmetry group of the entirely anisotropic space-time, are determined.

The way in which a Finslerian metric function provides a 3- and 4-rank generalized Finslerian metric tensors is described further in *"The generalized Finslerian metric tensors"* (S. V. Lebedev). For these tensors the author determines the generalized rank five Christoffel symbols and the generalized differential equations of Finsler geodesics.

In *"Hamilton canonical equations and the Berwald-Moor metric (on the formalism of physical theories)"* S. V. Siparov presents an overview of different approaches used to investigate a Finsler space: the pure mathematical approach, the approach which belongs to theoretical Physics and the approach characteristic for philosophy or for some meta-theory. As effective results, the Hamilton canonical equations are obtained, on the base of the function related to the Berwald-Moor metric, and it is pointed out that these equations can be used to construct the physical theory in a Finsler space.

The theory of finite-dimensional algebras and its methods can be successfully employed in geometry, physics and computer science. This fact is illustrated by the paper *"On defining equations for the elements of associative and commutative algebras and on associated metric forms"* (V. M. Chernov), where the author studies the three non-isomorphic $4D$ algebras $H_4, H_2 \oplus C, C \oplus C$, their automorphisms and metric forms, and provides a generalization for algebras of higher dimension $2d$, constructed using the Grassmann-Clifford algebra, emphasizing that the coefficients of the defining equations of an automorphism of algebras are associated with the Minkowski and Berwald-Moor metrics. Authored by V. M. Chernov as well, the paper *"Generalized n-ary composition laws in the algebra H_4 and their* *relation to associated metric forms"* examines the problem of poly-linearization of norms in the algebra H_4. It is shown that the quadratic Minkowski element norm, the Berwald-Moor norm associated to the 4-th order form, and the cubic norm examined by the author in the previous paper respectively coincide with the introduced composition laws of order 2, 3, and 4 of Zassenhaus type, for an appropriate choice of the Zassenhaus co-factors.

In *"The prolongations of a Finsler metric to the tangent bundle T^kM $(k > 1)$ of the higher order accelerations",* Gh. Atanasiu studies the prolongation of Finsler structures from a differentiable manifold M to the bundle of k-jets T^kM $(k > 1)$; the author introduces a new type of prolongation, which is 0-homogeneous; the introduced almost $(k-1)n$-contact structure is shown to be homogeneous and metrical w.r.t. the prolonged metric, providing thus a geometrical model for the basic subiacent Finsler structure. Further, as an application of previous results obtained by the first author, the paper *"The Berwald-Moor metric in the tangent bundle of the second order"* (Gh. Atanasiu and N. Brinzei) develop the geometry of the second order tangent bundle T^2M endowed with two special types of metrics compatible with the 2-contact structures. Then, in *"The 2-cotangent bundle with Berwald-Moor metric",* Gh. Atanasiu and V. Balan develop the d-geometry on the total space of the dual bundle $(T^{2*}M, \pi^{2*}, M)$ of the 2-tangent bundle (T^2M, π^2, M), study the nonlinear connection existence, distinguished tensor fields, almost contact structure, Riemannian structures, N-linear connections and associated convariant derivations. The Ricci identities are derived, and the local expressions of the d-tensors of torsion and curvature are provided. The metric structures and the metric N-linear connections are studied, and the results are specialized to the case when the metric tensor field is of Berwald-Moor type.

The paper *"Berwald-Moor - type (h, v)-metric physical models"* (V. Balan and N. Brinzei) provides several physical (h, v)-models for relativity, where the vertical part is provided by the flag-Finsler Berwald-Moor metric, while the horizontal part is specialized to the conformal and to Synge-relativistic optics metrics; basic properties of the models are described and the extended Einstein equations are determined. Further, in *"The horizontal and vertical semisymmetric metrical d-connections in the Relativity Theory"*, Gh. Atanasiu and E. Stoica determine the Einstein equations of an h- and v-semisymmetric metrical distinguished connection on the tangent space of a differentiable manifold endowed with a (h, v) Riemann - local Minkowski metric structure.

Within the dual framework, in the paper *"The Pavlov's 4-polyform of momenta $K(p) = \sqrt[4]{p_1p_2p_3p_4}$ and its applications in Hamilton geometry"* authored by Gh. Atanasiu, V. Balan and M. Neagu, a generalized Hamilton space is associated to a 4-pseudoscalar product, which are given in terms of the Cartan metrical fundamental d-tensor; for the function $K(p) = \sqrt[4]{p_1p_2p_3p_4}$, the components of the v-covariant derivation of this generalized Hamilton space are derived.

After providing a brief recall of the known results within the study of gauge field theory in terms of complex Finsler geometry on the total space of a G-complex vector bundle E, in *"The Lagrangian-Hamiltonian formalism in gauge complex field theories"*, Gh. Munteanu develops a similar theory on the dual bundle E^*, using the complex Legendre transformation (the L-dual process). The complex field equations are determined with respect to a gauge complex vertical connections. The complex Hamilton equations are written for the general L-dual Hamiltonian obtained as a sum of particle Hamiltonians, Yang-Mills and Hilbert-Einstein Hamiltonians.

The paper *"Geodesics, connections and Jacobi fields for Berwald-Moor quartic metrics"* (V. Balan, N. Brinzei and S. Lebedev) determines – for Finsler spaces (M, F) with quartic metrics, the equations of geodesics and the corresponding arising geometrical objects: canonical spray, nonlinear Cartan connection, Berwald linear connection; as well, are studied the geodesics and the Jacobi fields for certain (h, v)- metric models.

In *"Finsler spaces with polynomial metric"* (L. Tamassy), the author proves the existence in such spaces of metrical tensorial connections (i.e., which are linear for a given type of tensors); it is shown as well, that many of these connections induce in a natural way, metrical non-linear connections on the considered manifold. The existence of connections which are compatible with a given pair of metrical Finsler metrics is studied in *"Pairs of metrical Finsler structures and Finsler connections compatible to them"* (Gh. Atanasiu); the results are extended to the case when one of the two structures is degenerate.

The state-of-art on the geometry of constant mean curvature (CMC) surfaces in Finsler spaces is briefly presented in *"CMC and minimal surfaces in Berwald-Moor spaces"* (V. Balan), where it is shown that for the Berwald-Moor type Finsler metric there exist structural differences among Berwald-Moor fundamental functions of various orders, leading to different CMC approaches.

A new definition of simultaneous events using the signal method in Finsler Space-Time, is investigated in *"The definition of a simultaneity in Finsler Space-Time"* (R. G. Zaripov). He obtains general transformations which preserve the metric function of the considered projective space and, using the Hamiltonian formalism, are discussed the relations for energy and impulse of a particle and their transformations.

Using the concept of "world function", the paper *"On the world function and the relation between geometries"* (G. I. Garas'ko) gives a detailed motivation to the fact that the Minkowskian space and the polynumber space correspond to the same Physical World.

The construction of the metric tensor of a 4-dimensional pseudo Riemannian space (Space-Time) emerging from the 4-contravariant tensor of the tangent indicatrix equation of the Berwald-Moor space and the World function is provided in *"Construction of the pseudo-Riemannian geometry on the base of Berwald-Moor* *geometry"* (G. I. Garas'ko and D. G. Pavlov). It is emphasized that the algebra of commutative and associative hypercomplex numbers, related to the direct sum of the 4-real algebra denoted by H_4 and the corresponding Finsler geometry can be used as a mathematical model of the real Space-Time, more productive than the pseudo-Riemannian constructions prevailing in Physics now.

In the paper *"On Field Theory and some Finsler spaces",* G. I. Garas'ko discusses the construction of Lagrangians depending on fields, based solely on the metric function of a Finsler space. In spaces which are conformally connected to Minkowski spaces, under supplementary assumptions, the cosmological equation is written for the field describing the Universe (within the geometry connected to the polynumbers H_4, related to Berwald-Moore metrics), yielding the Hubble law for a small neighborhood of the origin.

A review of the actual research on the algebraic, geometric and differential properties of the quaternionic (Q-)numbers and their applications, is presented in *"Quaternions: Algebra, Geometry and Physical Theories"* (A.P. Yefremov).

Based on the fact that in field theories endowed with twistor structure, one may identify particles with caustics of null geodesic congruences defined by the twistor field, in *"The algebrodynamics: primodial light, particles-caustics and flow of time",* V. V. Kassandrov considers as a realization the "algebro-dynamical" approach, which is based on the field equations which originate from the noncommutative analysis over the algebra of biquaternions. The author discusses related concepts of generating the "World Function" and of multivalued physical fields, while the picture of the Lorentz invariant light-formed aether and of matter born from light is shown to arise naturally; as well, the notion of Time Flow is introduced and studied. Further, in *"Quaternionic analysis and the algebrodynamics",* the same author describes the "algebrodynamical" approach to field-particle theory, which is based a nonlinear generalization of the Cauchy-Riemann conditions to non-commutative algebras of quaternion-like type. It is shown that for complex quaternions, the theory is Lorentz invariant, and naturally carries certain gauge and twistor structures. A novel "causal Minkowski geometry with additional phase" is induced by means of the structure of biquaternion algebra, serving as a background for the self-consistent algebraic dynamics of singularities.

Main results of the geometry of Finslerian 4-spinors are stated in *"Finslerian 4-spinors as a generalization of twistors"* (A. V. Solovyov). It is shown that R. Penrose's twistors form a special case of Finslerian 4-spinors of the 16-dimensional vector space equipped with a metric form, and can be associated with Finsler geometry. Also, is formulated the procedure of dimensional reduction which allows to rewrite the expression of the Finslerian length of a 16-vector in terms of 4-dimensional geometric objects, and is described the corresponding isometry group.

The next paper, *"On quartic geometries"* (P. D. Suharevsky) provides arguments for employing quartic symmetric forms as metric tensors. To this aim, a non-associative algebra of anti-commuting 4-order matrices is built, are determined the associated equations of motion, which are quartic analogues of the Dirac equations, and is derived the corresponding Lagrangian. As well, the author yields the infinite-dimensional extension of quaternions and their matrix representation, a prerequisite for solving problems in the multilinear framework.

In the paper, *"Theory of the zero order effect suitable to investigate the Space-Time geometrical properties"*, S. V. Siparov studies the applicability of Einstein's relativity theory at galactic scale and the role of geometry for solving the problems of observational astrophysics are discussed, and is described the theory of the zero order effect.

In *"Experimental investigation of spinning massive body influence on fine structure of distribution functions of α-decay rate fluctuations"* (V. A. Panchelyuga, S. E. Shnoll), the authors present a short review of the phenomenology of the macroscopic fluctuation effect and describe a method used for experimental data processing within the study of the influence of the rapidly spinning massive body on the distribution function of the α-decay rate fluctuations.

The paper *"Local time effect on small Space-Time scale"* (V. A. Panchelyuga, V. A. Kolombet, M. S. Panchelyuga, S. E. Shnoll) studies the existence of the local time effect for relatively small distances between the places of measurements, emphasizing the distribution of time intervals in the neighborhood of the local time peak, and the peak splitting.

The volume addresses graduate students and researchers in Mathematics, Physics and related fields. The contents is written in a clear, discursive (though rigorous) manner, aiming to introduce the methods and basic ideas of applied Finsler Berwald-Moor geometry and hypercomplex algebra. Each article includes exhaustive bibliography which permits interested reader to trace the information.

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Generalization of Scalar Product Axioms

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Emerging from the basic idea that the concept of scalar product is vital in studying the properties of either Euclidean or pseudo-Euclidean spaces, author proposes a generalization of the classical (pseudo-)scalar products, by introducing the "poly-linear" product associated to a given Finsler structure. This permits to axiomatically introduce the notion of related fundamental metric polyform, and further the definition of various generalized geometric objects as vector length, angle between vectors and orthogonality. After a brief presentation of the classic framework of (pseudo-)scalar products, the notion of scalar polyproduct is introduced, and its properties are studied. Further, the article illustrates several main peculiarities of the geometry of the four-dimensional linear Finslerian space for the studied polyform, which plays a special role within the more general study of algebraic commutative-associative hypercomplex (called *quadranumerical*) numbers.

MSC2000: 53B40, 53C60, 15A69, 15A99, 83D05.

1. The scalar product of the Euclidean spaces

For the last two thousand years that have past since the appearance of the famous "Beginnings" mathematics have tried a number of methods of describing the Euclidean spaces. The axiom systems by Euclid and Gilbert are the best well-known ones. But taking into consideration the modern attitude, the system of axioms that uses the ideas of the real number, the linear space, and the scalar product [1] is considered to be the most convenient. At the same time a few know that the latter case owes its appearance in geometry to a discovery of the non-commutative algebra of four-component hypercomplex numbers discovered in 1843 by William Hamilton, he called it the algebra of quaternions [2]. The discovery was preceded by several years of attempts to find three-component numbers, the triplets, that could be confronted to the vectors of the common space the same way as the complex numbers are confronted to the vectors of the Euclidean Plane. The solution was found when Hamilton rejected the commutative multiplication and in place of the triplets limited himself to the four-component numbers.

By definition a quaternion is a hypercomplex number, that can be presented as a linear combination:

$$X = x_0 + i \cdot x_1 + j \cdot x_2 + k \cdot x_3,$$

where x_i are real numbers, and i, j, k are pair-wisely different imaginary units, so that $i^2 = j^2 = k^2 = -1$ and $ij + ji = jk + kj = ki + ik = 0$. These rules including the rule of multiplication on the common real unit, sometimes are set into the so called table of multiplication of hypercomplex numbers, that in the case of quaternions looks the following way:

$$\begin{array}{c|cccc} & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & -1 & k & -j \\ j & j & -k & -1 & i \\ k & k & j & -i & -1 \end{array}\ .$$

Hamilton suggested that in the quaternion we should distinguish the scalar part x_0 from the vector part $\mathbf{V}_x = \mathbf{i}\cdot x_1 + \mathbf{j}\cdot x_2 + \mathbf{k}\cdot x_3$. In this case, as it is easy to check, the product of 2 vector quaternions is a common quaternion:

$$\mathbf{V}_x\mathbf{V}_y = (-x_1y_1 - x_2y_2 - x_3y_3) + [\mathbf{i}(x_2y_3 - x_3y_2) + \mathbf{j}(x_3y_1 - x_1y_3) + \mathbf{k}(x_1y_2 - x_2y_1)],$$

whose scalar part has a symmetric bilinear form, and the vector part looks like a conventional vector multiplication. As a matter of fact, the term of scalar and vector product appeared right from here, and for the first time were introduced by Hamilton.

The first explorers of the quaternions were looking at them mainly as at an opportunity of using algebraic methods while operating with points and vectors of common space, though it is more natural to correspond these hypercomplex numbers with the four-dimensional space. Hamilton himself knew about this, he thought that this circumstance once would be used to describe the time. In this case quaternions would become a natural instrument not only in geometry, but also in physics.

Unfortunately, nowadays only some specialists know quaternions. It is explained by the fact that the idea of scalar product that originates from the quaternion algebra was very convenient and soon became an independent geometrical category, and practically stamped the hypercomplex numbers that had given birth to it. There began a debate among physics and mathematicians between the adherents of the quaternion algebra and of the arising vector calculus. As is well-known, the vector approach won, this fact to a certain extent owes to objective difficulties of quaternion diffusion into algebra and the function of the complex variable, that is conditioned to the peculiarities of non-commutative multiplication.

The scalar product that is connected with the quaternion can be applied only to the three-dimensional vectors. But if we separate the idea of scalar product from concrete numbers and generalize it to the field of arbitrary dimensionality, the advantages of the concept (the opportunity to define the length of vectors and angles between them mathematically) will still be preserved. For this we should postulate a symmetrical bilinear form of two vectors $(\mathbf{A},\mathbf{B}) = \alpha_{ij}a_ib_j$ in the affine m-dimensional space. Reciprocally corresponding quadratic form $(\mathbf{A},\mathbf{A})$ must be not negative. Then by definition we accept that the affine map that maps the vector $\mathbf{A}$ onto $\mathbf{A}'$ is congruent if it leaves the form invariant:

$$(\mathbf{A}, \mathbf{A}) = (\mathbf{A}', \mathbf{A}').$$

Two figures that can be mapped one onto another by a congruent reflection are congruent. By this fact the idea of congruence is defined in the axiomatic construction of the Euclidean geometry. For a congruent map takes place not only invariance of the quadratic form but also the invariance o the bilinear form:

$$(\mathbf{A}, \mathbf{B}) = (\mathbf{A}', \mathbf{B}').$$

For the vectors $\mathbf{A}$ and $\mathbf{A}'$ are congruent if and only if:

$$(\mathbf{A}, \mathbf{A}) = (\mathbf{A}', \mathbf{A}'),$$

it is possible to introduce the $(\mathbf{A}, \mathbf{A})$ as a numerical characteristic of the vector $\mathbf{A}$. But still it is more traditional to use the value of the positive square root of $(\mathbf{A}, \mathbf{A})$, that by definition is called the length of the vector $\mathbf{A}$ and usually is defined as

$$|\mathbf{A}| = (\mathbf{A}, \mathbf{A})^{1/2}.$$

Such definition lets us introduce the definition of the unit vector. Its relationship with common vectors is revealed in the following relation:

$$\mathbf{a} = \mathbf{A}/|\mathbf{A}|.$$

If $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{a}'$ and $\mathbf{b}'$, are two pairs of unit length vectors, then the figure, built by the first two vectors, is congruent to the figure, constructed by the two latter ones, only when the equality

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}')$$

is held true. The angle is considered to be the representative of congruency in the Euclidean spaces. But the mere numerical characteristic is related not to bilinear form of unit vectors, but to transcendental function of its inverse cosine

$$\phi = \arccos(\mathbf{a}, \mathbf{b}).$$

This definition of the angle is equivalent to the statement that the length of the arc on the unit sphere between the ends of the vectors $\mathbf{a}$ and $\mathbf{b}$ is the angle. Such complication of the numerical angle measure is compensated by the obtained property of additivity. When composing two angles laying on the same plane their value is summed up.

The property of perpendicularity of directions is a particular consequence of the idea of the angle. The perpendicular condition of two vectors consists in equality to 0 of the value of their bilinear form. The particular status of the perpendicular directions is accounted for many reasons, for example, for example by the simplification of the form of the quadratic metric function, presented in the basis all vectors of which are reciprocally perpendicular.

Two-dimensional case stands out among all the Euclidean spaces with quadratic metric function. This peculiarity is reflected in the Liouville theorem, that proves that in the three- or more-dimensional Euclidean (or pseudo-Euclidean) spaces the conformal transformations are limited to inversions, dilations, translations and rotations [3]. In other words, there are essentially more transformations that are related to conformal in the two-dimensional case. Mathematically this fact is reflected in the vast majority of analytical functions of the complex variable. To each of them a certain conformal reflection of the Euclidean plane is related.

2. The scalar product of the pseudo-Euclidean spaces

It is well-known that if a symmetrical bilinear form postulated over the affine space creates an alternating-sign quadratic form, then the geometry assigned by it becomes being of not Euclidean but Pseudo-Euclidean type [4]. We can unify both types of geometries by surrendering the claim about the positivity of the quadratic form. This unified system, in particular, can be presented with the following set:

(a) every 2 vectors $\mathbf{A}$ and $\mathbf{B}$ of the linear space are associated with certain real number labeled by

$$k = (\mathbf{A}, \mathbf{B})$$

and called (as well as in the Euclidean case) the scalar product of these vectors;

(b) the scalar product is commutative regarding the permutation of vectors

$$(\mathbf{A}, \mathbf{B}) = (\mathbf{B}, \mathbf{A});$$

(c) the scalar product is distributive regarding the composition of vectors

$$(\mathbf{A} + \mathbf{C}, \mathbf{B}) = (\mathbf{A}, \mathbf{B}) + (\mathbf{C}, \mathbf{B});$$

(d) the real multiplier can be isolated from the scalar product

$$(k\mathbf{A}, \mathbf{B}) = k(\mathbf{A}, \mathbf{B}).$$

The methods of defining the metric characteristics of pseudo-Euclidean spaces, which are the generalizing of corresponding Euclidean parameters, do not change considerably, that enables us to save their names. So, transformations that leave the quadratic form module of all the vectors invariant are of congruent nature:

$$|(\mathbf{A}, \mathbf{A})| = |(\mathbf{A}', \mathbf{A})'|.$$

The vector length is defined as a positive value of the square root of the module of the quadratic form:

$$|\mathbf{A}| = |(\mathbf{A}, \mathbf{A})|^{1/2}.$$

But in this case there appear the so called isotropic and imaginary vectors. In the first case the length equals 0 even at nonzero components, and in the second case the quadratic form is negative. The angle between two directions, as well as in the Euclidean case, is defined by congruence of the figure formed by two unit vectors, and by definition is treated as equal to the special function of their bilinear form:

$$\phi = \operatorname{arcch}(\mathbf{a}, \mathbf{b}),$$

which ensures the additivity of the parameter under plane rotations. So, the angle equals the arc length between a pair of points on the unit sphere. But now, when calculating the angle, it is important to take into consideration the area in which the driving vector that is relative to the isotropic cone is lying, as the indicatrix stops being simply connected.

Also the perpendicular property of vectors is generalized in the pseudo-Euclidean spaces. In this case their scalar product must equal 0. It is customary to call such vectors orthogonal.

The pseudo-Euclidean spaces also admit the generalizing of the idea of a congruent reflection, which is defined as a transformation that saves similarity of infinitesimal forms. Let us note that, as well as in the Euclidean case, the 2-dimensional case, where conformal maps are wider than in higher dimensions, is distinguished in the pseudo-Euclidean space. Let us note another coincidence: The pseudo-Euclidean plane, as well as the Euclidean one, has an algebraic analogue called *double numbers* which differ from the complex by the fact that their square equals not -1, but $+1$. Such numbers along with the complex ones admit the idea of analytical functions where a correspondence of a conformal reflection of the pseudo-Euclidean plane [5] to each of them can be established. These peculiarities of 2-dimensional spaces demonstrate the relationship between the geometries and commutative-associative algebras, for example, the algebras of complex and double numbers.

Apart from the pseudo-Euclidean case, other approaches towards generalizing of the conception of the scalar product are known in geometry. The system of axioms for the so called unitary, where the metric function is set in the field of complex and not real numbers, and symplectic spaces where antisymmetric bilinear form [4, 6] is postulated in place of the symmetric, – are sequent to the scalar product.

Analyzing the above examined examples of the usage of the concept of scalar product and its generalization we can note that they are unified by connection with one or another bilinear form. But such a form is just a special case of the polylinear form. Then there emerges a question whether it is possible to obtain a substantial geometry if we postulate the three-, four-, and so on up to polylinear symmetric form in place of the bilinear one?

3. The scalar polyproduct

Let us try to preserve all the axioms of the real number and m-dimensional affine spaces as the basis and add the following:

(a): to every of n vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}, ..., \mathbf{Z}$ we will associate real number denoted by

$$k = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z}),$$

which we will call the scalar polyproduct;

(b): let us try to make it the way that the scalar polyproduct would be commutative with respect to permutation of any including vectors

$$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z}) = (\mathbf{B}, \mathbf{A}, \mathbf{C}, \ldots, \mathbf{Z}) = (\mathbf{C}, \mathbf{B}, \mathbf{A}, \ldots, \mathbf{Z}) = \cdots = (\mathbf{Z}, \mathbf{C}, \mathbf{B}, \ldots, \mathbf{A});$$

(c): distributive to their composing

$$(\mathbf{A}, \mathbf{B}, \mathbf{C} + \mathbf{E}, \ldots, \mathbf{Z}) = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z}) + (\mathbf{A}, \mathbf{B}, \mathbf{E}, \ldots, \mathbf{Z});$$

(d): a real multiplier at any vector could be taken outside scalar polyproduct:

$$(k\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z}) = k(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z}).$$

These axioms just in a way differ from the corresponding axioms of the scalar product. Besides they can be unified into a concept of the symmetric polylinear form, and that is why we will call the space, endowed with one of the forms, *polylinear*. The above examined Euclidean and pseudo-Euclidean spaces, according to their primary definitions, are special cases of the polylinear spaces, in other words they comply to the above given axiom system when $n = 2$, that enables us to call them *bilinear*.

We will call the scalar polyproduct of the same vector, $\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}$, by analogy with the quadratic form of the bilinear spaces, the *fundamental metric form* of the polylinear space, or simply *n-polyform* of the vector $\mathbf{A}$.

We will call the affine reflections of the polylinear space, that shift the vectors $\mathbf{A}$ into $\mathbf{A}'$, the *congruent* if they leave the module of the fundamental metric form invariant:

$$|(\mathbf{A}, \mathbf{A}, \mathbf{A}, \ldots, \mathbf{A})| = |(\mathbf{A}', \mathbf{A}', \mathbf{A}', \ldots, \mathbf{A}')|. \tag{1}$$

It is in our axiomatic construction of the polylinear space where the idea of congruence, and then of other metric notions, will be defined.

If there is a set of objects over which the axioms of the affine space are held true, we can choose any symmetric polylinear form in it and, therefore, the unambiguously connected n-polyform, and "assign" make the latter to be the fundamental metric form and on its basis define the conception of congruence as it has been done above. Then we a metrics gets introduced into the affine space with the help of the form, and it becomes a correct metric geometry. Such construction is not related neither to number of dimensions in the space nor to the specific number of dimensions in the fundamental form, nor with the type of the latter case.

It follows from the properties of the symmetry and from the linearity of the form $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots, \mathbf{Z})$ where correlations, that are more general than (1), are held true for the congruent reflection of the polylinear space:

$$(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}, \mathbf{B}) = (\mathbf{A}', \mathbf{A}', \ldots, \mathbf{A}', \mathbf{B}'),$$
$$(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{B}, \mathbf{B}) = (\mathbf{A}', \mathbf{A}', \ldots, \mathbf{B}', \mathbf{B}'),$$
$$\cdots\cdots\cdots$$
$$(\mathbf{A}, \mathbf{B}, \ldots, \mathbf{C}, \mathbf{Z}) = (\mathbf{A}', \mathbf{B}', \ldots, \mathbf{C}', \mathbf{Z}').$$

In other words the congruent reflections of the polylinear spaces leave the polyforms invariant where the vectors are present in different combinations.

We will say that the two vectors of the polylinear space $\mathbf{A}$ and $\mathbf{A}'$ are congruent if the modules of the corresponding n-polyforms are equal and are nonzero:

$$|(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}, \mathbf{A})| = |(\mathbf{A}', \mathbf{A}', \ldots, \mathbf{A}', \mathbf{A}')| \neq 0.$$

By definition it is possible to regard a n-polyform as a numerical parameter of the vector $\mathbf{A}$. But in place of this, as well as in the bilinear spaces, striving for additivity and unambiguity of the properties, we will use the positive root of the n-degree of the absolute value $(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}, \mathbf{A})$, calling it the vector length $\mathbf{A}$:

$$|\mathbf{A}| = |(\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A}, \mathbf{A})|^{1/n}.$$

Then the length of the sum of two codirected vectors equals the sum of their length. It is worth noting that this is not the only way of introducing the idea of length with additive properties, but in this approach the length is defined for the maximum number of directions coming from the affine space.

Now it becomes clear to which type of space we should relate the ones we try to construct with the help of the given above axioms or the scalar polyproduct. Firstly, these spaces are *Finslerian* [7,8] as their metric function is not limited by quadratic forms. Secondly they belong to the class known in the Finslerian geometry under the name of Minkowskian space [9], with which it is customary to associate the manifold where the indicatrices do not depend on the point. [The space of the Special theory of Relativity is a specific case of such spaces.] But the examined class of spaces is even smaller, as it is related to a strict idea of polylinear symmetric form. The latter case has a great significance as it becomes possible to introduce characteristics, that generalize such fundamental categories of geometry as the length, the angle, the orthogonality, the conformal reflection, etc. Let us conventionally call such spaces the *polylinear Finslerian spaces* (till the appearance of a more specific name let).

If $\mathbf{a}$ and $\mathbf{b}$, and also $\mathbf{a}'$ and $\mathbf{b}'$, are two pairs of unit vectors, then the figure, constructed with the first two vectors, will be congruent to the figure, constructed with the latter two, if a transformation mapping one figure onto the other there will be found. From the above examined properties of the polylinear forms it follows that such transformation can be found only if

$$\begin{gathered}(\mathbf{a},\mathbf{a},\dots,\mathbf{b}) = (\mathbf{a}',\mathbf{a}',\dots,\mathbf{b}'),\\ (\mathbf{a},\mathbf{a},\dots,\mathbf{b},\mathbf{b}) = (\mathbf{a}',\mathbf{a}',\dots,\mathbf{b}',\mathbf{b}'),\\ \dots\dots\dots\\ (\mathbf{a},\mathbf{b},\dots,\mathbf{b}) = (\mathbf{a}',\mathbf{b}',\dots,\mathbf{b}').\end{gathered} \tag{2}$$

This, in particular, entails that in the bilinear spaces the congruence of the pair of two unit vectors is related to the equality of only one form:

$$(\mathbf{a},\mathbf{b}) = (\mathbf{a}',\mathbf{b}'), \tag{3}$$

which sets the idea of the angle as the parameter that characterizes the difference between two directions. The equality (3) along with the definition of the unit vector are tantamount to the axiom of the triangle congruence from the Hilbert system of axioms of the Euclidean space. Two triangles are congruent in the Euclidean space if the lengths of corresponding sides and angles between them are equal. One may can formulate analogous axioms also for the pseudo-Euclidean spaces. But it follows from the definition (2) that in the polylinear space with the dimension of the form of more than two the congruence of figures constructed of two unit vectors is defined by more than one circumstance. In the spaces with the three-linear form $(\mathbf{a},\mathbf{b},\mathbf{c})$, the two forms must be equal to ensure that the figures would be congruent:

$$(\mathbf{a},\mathbf{a},\mathbf{b}) = (\mathbf{a}',\mathbf{a}',\mathbf{b}'), \qquad (\mathbf{a},\mathbf{b},\mathbf{b}) = (\mathbf{a}',\mathbf{b}',\mathbf{b}').$$

This seeming paradox has a very simple explanation. Usually speaking about a spatial figure, constructed on two vectors, it is thought as of a plain element held among sides, which are the driving vectors. But this is justified only in spaces with the bilinear form. In the spaces with the arbitrary polylinear form, the two vectors are now connected not with a plane but with a special cone-shaped surface, which configuration depends on the metric properties of the surrounding space. There can be more than one parameter, that defines the congruence of such fan-shaped figures, limited in the edges by unit vectors, that in particular is observed in spaces with three-linear symmetric form with two corresponding values.

On the basis of the above given brief analysis it becomes clear that polylinear spaces admit an introduction of analogous of the idea of the angle attributed to bilinear spaces. But we should take into account that the angle as the parameter in the bilinear spaces unifies simultaneously two properties: on the one hand, it serves as a characteristic of the difference between two directions, and on the other hand, is the parameter of one of types of congruent transformations called rotations. In the general case of the polylinear space each of the properties should be characterized by a proper value. It is meaningful to use the negative value of the n- polyform of the difference as the basis to getting the numerical parameter that would characterize the difference of directions of unit vectors: **a** and **b**, to be more specific:

$$(\mathbf{a}-\mathbf{b},\mathbf{a}-\mathbf{b},\dots,\mathbf{a}-\mathbf{b}) = (\mathbf{a},\mathbf{a},\dots,\mathbf{a}) - C_n^1(\mathbf{a},\mathbf{a},\dots,\mathbf{a},\mathbf{b}) + \dots \\ \pm \dots (-1)^{n-1}C_n^{n-1}(\mathbf{a},\mathbf{b},\dots,\mathbf{b}) + (-1)^n(\mathbf{b},\mathbf{b},\dots,\mathbf{b}),$$

where C_i^j are binomial coefficients. Consequently the scalar form of two unit vectors **a** and **b** reads

$$S(\mathbf{a},\mathbf{b}) = -C_n^1(\mathbf{a},\mathbf{a},\dots,\mathbf{a},\mathbf{b}) \pm \dots (-1)^{n-1}C_n^{n-1}(\mathbf{a},\mathbf{b},\dots,\mathbf{b}) \tag{4}$$

or its function can play the role of a numerical parameter that defines the required property. Let us note that if the polylinear space is a two-bilinear one the expression (4) to the constant factor coincides with the definition of the common scalar product of two unit vectors. The value (4) can be called the *scalar product of two vectors* of the polylinear space. But may be it is even justified to divide the scalar product into items symmetrized in pairs:

$$S(\mathbf{a},\mathbf{b}) = C_n^1(-(\mathbf{a},\mathbf{a},\dots,\mathbf{a},\mathbf{b}) + (-1)^{n-1}(\mathbf{a},\mathbf{b},\mathbf{b},\dots,\mathbf{b})) \\ + C_n^2((\mathbf{a},\mathbf{a},\dots,\mathbf{a},\mathbf{b},\mathbf{b}) + (-1)^{n-2}(\mathbf{a},\mathbf{a},\mathbf{b}\dots,\mathbf{b})) \pm \dots = \\ = S_1(\mathbf{a},\mathbf{b}) + S_2(\mathbf{a},\mathbf{b}) + \dots, \tag{5}$$

where every term $S_i(\mathbf{a},\mathbf{b})$ receives its proper value.

In the polylinear spaces there are pairs of vectors with definite ability of positional relationship similar to orthogonal vectors in the bilinear spaces. In the Finslerian space theory the corresponding idea is called the transversality. Let us call the vector **A** *transversal* to the vector **B**, if $(\mathbf{A},\mathbf{A},\dots,\mathbf{A},\mathbf{B}) = 0$. It is seen here that the transversality is not commutative, that is, the vanishing $(\mathbf{A},\mathbf{A},\dots,\mathbf{A},\mathbf{B}) = 0$ does not entail $(\mathbf{B},\mathbf{B},\dots,\mathbf{B},\mathbf{A}) = 0$. But if we use the symmetrized forms (5), then the transversality, assigned by them, will have commutative properties. By definition, we will consider **A** and **B** *mutually transversal of the first degree*, when $S_1(\mathbf{A},\mathbf{B}) = \mathbf{0}$; and of the *second degree*, if $S_2(\mathbf{A},\mathbf{B}) = \mathbf{0}$, and so on up to $n/2$ or $(n-1)/2$ degree. Such differentiation of transversality demonstrates the ability of vectors of the linear Finslerian spaces to form pairs with a multitude of characteristic connection with the direction, – that generalizes the conception of orthogonality.

Apart from the quantities defined by the forms (4) it is meaningful to introduce one more "angle-like" characteristic in some polylinear spaces that have continuous congruent transformations like rotations. We will relate its value with the arc length in the unit sphere outlined by a ray simultaneously with a continuous one-parameter rotation. So generalized conception includes the property of the common angle – to be the additive measure that follows from the additivity of the length.

Not only pairs can be included into polyforms, but also three-, four-, etc., up to n different vectors. It is difficult to say to which quality consequences must lead this circumstance in the area of simple figures. Only one thing is clear: this property of polylinear spaces exists objectively that means that it should be as well taken into account.

There are such spaces among the polylinear ones where in one of the bases all the forms are nullified but for the ones that include only different vectors. For such spaces the fundamental metric forms take the following structure in the special basis:

$$(\mathbf{A},\mathbf{A},\ldots,\mathbf{A}) = \pm a_1a_2\ldots a_m \pm a_1a_2\ldots a_{m-1}a_{m+1} \\ \pm\cdots\pm a_2a_3\ldots a_ma_{m+1}\pm\cdots\pm a_{n-m}a_{n-m+1}\ldots a_n. \quad (6)$$

Among these emerge the pseudo-Euclidean spaces labeled $(1, m-1)$, which play an important role in the modern theoretical physics. Though the classical quadratic form seems to be more convenient for the spaces, the second degree of the intervals in some of the isotropic bases looks like:

$$|\mathbf{A}|^2 = (\mathbf{A},\mathbf{A}) = a_1a_2 + a_1a_3 + a_1a_4 + \cdots + a_{m-1}a_m = \sum_{k\neq l} a_ka_l.$$

For example, the square of interval of Minkowskian space $S^2 = (ct)^2 - x^2 - y^2 - z^2$ after the substitution

$$ct = \sqrt{3/8}(u+v+w+z), \quad x = \sqrt{1/8}(u-v+w-z), \\ y = \sqrt{1/8}(u+v-w-z), \quad z = \sqrt{1/8}(u-v-w+z)$$

(similar to (16)) gets an attractive symmetric form:

$$S^2 = uv + uw + uz + vw + vz + wz.$$

The expression (6) looks more concise in the cases with $n = m$, that is, when the dimension of the fundamental form coincides with the dimension of the space. In this case the nth degree of a vectors with respect to the corresponding basis takes on the form

$$|\mathbf{A}|^n = (\mathbf{A},\mathbf{A},\ldots,\mathbf{A}) = \pm a_1a_2\ldots a_n.$$

In these circumstances the specific role of the pseudo-Euclidean plane, where such correlations are held, is defined. It seems probable that there must exist a connection with associative-commutative algebras, that involves the appearance in the space of a large group of conformal reflections, only in spaces with $n = m$. At the same time the conformal reflections can be seen in a number of cases which follow from the works [10, 11] where the eight-dimensional biquaternions are examined, that, according to the above given axiom, have metric forms of the fourth degree which come outside the Liouville theorem. We can only hope that the property of some polylinear spaces has a vast group of conformal reflections which appears to be perspective in geometry as well as in physics.

On the other hand even superficial study of the properties of the polylinear spaces let us state that in some of them there are not only conformal, but also non-linear transformations that do not have analogies within common bilinear spaces. The presence of such transformations ensues merely from that the studied spaces require extension of the notion of orthogonality up to several respective members. As is well known, the nonlinear transformations that leave invariant ordinary orthogonality relates to conformal. In this connection it is natural to expect that the transformations retaining the transversality would occur preferable, too. This makes the existent polylinear spaces even more interesting.

4. Examples of polylinear spaces

There is a great number of polylinear spaces. The task to classify such spaces seems to be difficult even if we work with three-linear forms, not to mention the forms with a larger number of dimensions. But if we limit ourselves to the three-dimensional case, and if among symmetric three-dimensional spaces we examine those whose metric forms do not depend on permutation of vector components (it is suggested in the work [12], that examines a similar classification, to call them the *high-symmetric*) than we can single out 8 independent classes, where a fundamental canonical polyform can be related to each of them. The simplest look among all the forms has the following:

$$(\mathbf{A},\mathbf{A},\mathbf{A}) = a_1^3 + a_2^3 + a_3^3 = F_1;$$
$$(\mathbf{A},\mathbf{A},\mathbf{A}) = a_1^2a_2 + a_1^2a_3 + a_2^2a_1 + a_2^2a_3 + a_3^2a_1 + a_3^2a_2 = F_2;$$
$$(\mathbf{A},\mathbf{A},\mathbf{A}) = a_1a_2a_3 = F_3.$$

In the work [12] they are called *basic*. Any of the eight non-isomorphic high-symmetric tree-linear polyforms can be presented as a linear combination of the bases:

$$(\mathbf{A},\mathbf{A},\mathbf{A}) = \omega_1F_1 + \omega_2F_2 + \omega_3F_3.$$

But no matter how great the variety of spaces with three-linear symmetric form is, the space with the following form stands out with its concise symmetry:

$$(\mathbf{A},\mathbf{A},\mathbf{A}) = a_1a_2a_3.$$

As the result of its high involved symmetry we can confront the corresponding space with the algebra of commutative-associative numbers that is the sum of three real algebras. Let us call such hypercomplex system the *triple numbers* and label it as H_3. Mathematical, geometrical and may be physical structures related to the triple numbers are not trivial at all, that is proved in the works [13, 14] published in this issue. It will be noted that most three-linear polyforms cannot be juxtaposed by algebras in general [12].

In the 4-dimensional polylinear spaces with $n = m$ the basic forms are:

$$(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{A}) = a_1^4 + a_2^4 + a_3^4 + a_4^4; \tag{7}$$

$$\begin{aligned}(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{A}) = a_1^3(a_2+a_3+a_4) + a_2^3(a_1+a_3+a_4)\\ +a_3^3(a_1+a_2+a_4) + a_4^3(a_1+a_2+a_3);\end{aligned} \tag{8}$$

$$(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{A}) = a_1^2a_2^2 + a_1^2a_3^2 + a_1^2a_4^2 + a_2^2a_3^2 + a_2^2a_4^2 + a_3^2a_4^2; \tag{9}$$

$$\begin{aligned}(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{A}) = a_1^2(a_2a_3+a_2a_4+a_3a_4) + a_2^2(a_1a_3+a_1a_4+a_3a_4)+\\ a_3^2(a_1a_2+a_1a_4+a_2a_4) + a_4^2(a_1a_2+a_1a_3+a_2a_3);\end{aligned} \tag{10}$$

$$(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{A}) = a_1a_2a_3a_4, \tag{11}$$

and to each of them their particular, not isomorphic to others, geometries of the polylinear space.

As well as in the three-dimensional case the variety of four-dimensional polylinear spaces is not limited to these examples. It seems to be a very difficult task to present the full classification of corresponding geometries. Let us study at least one case before setting about its realization. For example, the geometry related to the most symmetric among the basic polyforms (7) – (11), and to be more specific (11). Its high symmetry again gives us an opportunity to confront the space defined by it to the algebra of commutative-associative hypercomplex numbers, that in order to be brief we will call the *Quadra-numbers* labeled as H_4. Some of the properties of the space, related to the Quadra-numbers are given in [15]. We can get the Quadra-number algebra by adding the axiom of real numbers to the axiom of composing and multiplication of the following objects: $A = a_1\cdot 1 + a_2\cdot I + a_3\cdot J + a_4\cdot K$ and $B = b_1\cdot 1 + b_2\cdot I + b_3\cdot J + b_4\cdot K$, where a_i and b_i – real numbers called the components, and $1, I, J, K$ the basic units. We accepting by definition that the sum of the numbers A and B is called the number

$$C = (a_1+b_1)\cdot 1 + (a_2+b_2)\cdot I + (a_3+b_3)\cdot J + (a_4+b_4)\cdot K,$$

and their product – another number of the same class:

$$\begin{aligned}D = (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4)\cdot 1 + (a_1b_2 + a_2b_1 + a_3b_4 + a_4b_3)\cdot I+\\ +(a_1b_3 + a_2b_4 + a_3b_1 + a_4b_2)\cdot J + (a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1)\cdot K,\end{aligned}$$

By the method given above we get the algebra of commutative-associative hypercomplex numbers, where the multiplication table of basic units have the form:

	1	I	J	K
1	1	I	J	K
I	I	1	K	J
J	J	K	1	I
K	K	J	I	1

.

It follows from the table that $I^2 = J^2 = K^2 = 1$, namely all its imaginary units are hyperbolic. We can get the same algebra another way: by applying for 2 times the algebra of the real number using two independent hyperbolical-imaginary units I and J the doubling operation. Let us denote the product of I and J as an independent object k, the number A from the corresponding multitude can be presented as a linear combination:

$$A = (a_1 + a_2 \cdot I) + (a_3 + a_4 \cdot I) \cdot J = a_1 + a_2 \cdot I + a_3 \cdot J + a_4 \cdot K,$$

where the symbol of the real unit 1, as it is accepted in the complex-numbers and quaternions, is omitted.

Let us call the numbers $\bar{A}, \hat{A}, \widetilde{A}$ *conjugate* to the number $A = a_1 + a_2 \cdot I + a_3 \cdot J + a_4 \cdot K$, if they look like:

$$\begin{aligned} \bar{A} &= a_1 - a_2 \cdot I + a_3 \cdot J - a_4 \cdot K, \\ \hat{A} &= a_1 + a_2 \cdot I - a_3 \cdot J - a_4 \cdot K, \\ \widetilde{A} &= a_1 - a_2 \cdot I - a_3 \cdot J + a_4 \cdot K. \end{aligned} \tag{12}$$

Notice that

$$\widehat{\bar{\widetilde{A}}} = A. \tag{13}$$

The product of such fours, as it is easy to check by the direct substitution, are always real numbers

$$\begin{aligned} A\bar{A}\hat{A}\widetilde{A} = a_1^4 + a_2^4 + a_3^4 + a_4^4 - 2a_1^2a_2^2 - 2a_1^2a_3^2 - 2a_1^2a_4^2 \\ - 2a_2^2a_3^2 - 2a_2^2a_4^2 - 2a_3^2a_4^2 + 8a_1a_2a_3a_4. \end{aligned} \tag{14}$$

By analogy with the algebra of complex numbers we will relate the value to the fourth degree of the corresponding number modulus and denote it as $|A|^4$. The introduced conception has the common properties of the modulus:

$$|\lambda A| = |\lambda| \cdot |A|, \qquad |AB| = |A| \cdot |B|,$$

where λ is a real, and A, B are complex numbers. In the product the property of mutually conjugated to result in the real number let us introduce into the examined algebra the operation of division, interpreted as an action inverse to multiplication. So, let us understand the number

$$A^{-1} = \frac{\bar{A}\hat{A}\widetilde{A}}{|A|^4} \tag{15}$$

under the number A^{-1} which is inverse to A. Only the numbers whose module is nonzero have their inverse analogues. Such numbers do not have such analogs. The examined algebra is associated with the form (11). It can be proved by examining a shift from the basis $1, I, J, K$ to the basis S_1, S_2, S_3, S_4,, whose objects are connected with the initial correlation:

$$S_1 = \frac{1}{4}(1 + I + J + K), \qquad S_2 = \frac{1}{4}(1 - I + J - K),$$
$$S_3 = \frac{1}{4}(1 + I - J - K), \qquad S_4 = \frac{1}{4}(1 - I - J + K). \tag{16}$$

These bases are the divisors of zero and are distinguished by the fact that their multiplication table is the most vivid one:

$$\begin{array}{c|cccc} & S_1 & S_2 & S_3 & S_4 \\ \hline S_1 & S_1 & 0 & 0 & 0 \\ S_2 & 0 & S_2 & 0 & 0 \\ S_3 & 0 & 0 & S_3 & 0 \\ S_4 & 0 & 0 & 0 & S_4 \end{array} \cdot$$

We will call the divisor of zero with such properties the *principle*, and the bases formed of them – the *absolute*. The feedback of the units $1, I, J, K$ with the principle zero divisor of the algebra H_4 is evaluated the following way:

$$1 = S_1 + S_2 + S_3 + S_4, \qquad I = S_1 - S_2 + S_3 - S_4,$$
$$J = S_1 + S_2 - S_3 - S_4, \qquad K = S_1 - S_2 - S_3 + S_4.$$

It is easy not only to sum but also multiply and divide the numbers from H_4 written in the absolute basis. So, the product of two numbers A and B looks is following:

$$(AB) = (a_1'b_1')S_1 + (a_2'b_2')S_2 + (a_3'b_3')S_3 + (a_4'b_4')S_4,$$

and their fraction reads

$$\frac{A}{B} = \frac{a_1'}{b_1'}S_1 + \frac{a_2'}{b_2'}S_2 + \frac{a_3'}{b_3'}S_3 + \frac{a_4'}{b_4'}S_4.$$

(Henceforth the components with primes will relate to the absolute basis). the absolute basis reveals the structure of the quadrahypeboloic number algebra, which is isomorphic to the algebra of real diagonal matrices. The group of mutually conjugated written in the absolute basis looks like:

$$A = a_1'S_1 + a_2'S_2 + a_3'S_3 + a_4'S_4,$$
$$\bar{A} = a_2'S_1 + a_1'S_2 + a_4'S_3 + a_3'S_4,$$
$$\hat{A} = a_3'S_1 + a_4'S_2 + a_1'S_3 + a_2'S_4,$$
$$\widetilde{A} = a_4'S_1 + a_3'S_2 + a_2'S_3 + a_1'S_4. \tag{17}$$

The modulus of the number A in such special basis looks like:

$$|A| = |a'_1 a'_2 a'_3 a'_4|^{1/4}, \tag{18}$$

that proves the correspondence of the algebra to geometry defined by the fundamental metric form (11). We can introduce the conception of function for the multitude of the Quadra-numbers. The exponential function is one of the most interesting. Under it we will understand the following series:

$$e^X = 1 + X + \frac{X}{2!} + \dots,$$

where X is an arbitrary Quadra-number. With the introduction of the exponential function we can examine along with the algebraic form of the number H_4 its exponential form. So, the number $A = a'_1 S_1 + a'_2 S_2 + a'_3 S_3 + a'_4 S_4$, where all the components of a'_i in the absolute basis are positive, corresponds to:

$$A = |A|\, e^{\alpha I + \beta J + \gamma K}, \tag{19}$$

where the positive value $|A|$ is its modulus. By analogy with the complex and double numbers we will call the real numbers α, β and γ, the *argument of the Quadra-number A*. The connection of the arguments with the components a'_i in the absolute basis looks like:

$$\alpha = \frac{1}{4}\ln\frac{a'_1 a'_3}{a'_2 a'_4} = \frac{1}{4}(\ln a'_1 - \ln a'_2 + \ln a'_3 - \ln a'_4),$$

$$\beta = \frac{1}{4}\ln\frac{a'_1 a'_2}{a'_3 a'_4} = \frac{1}{4}(\ln a'_1 + \ln a'_2 - \ln a'_3 - \ln a'_4),$$

$$\gamma = \frac{1}{4}\ln\frac{a'_1 a'_4}{a'_2 a'_3} = \frac{1}{4}(\ln a'_1 - \ln a'_2 - \ln a'_3 + \ln a'_4),$$

where $\ln x$ is a logarithmic function of the real x. As the hyperboloic analog to the Euler formula works for every imaginary unit:

$$e^{\alpha I} = \cosh\alpha + I\sinh\alpha,$$

then the following expression for the exponent from an arbitrary Quadra-number $X = \delta + \alpha I + \beta J + \gamma K$ is true:

$$e^X = (\cosh\delta + \sinh\delta)\cdot(\cosh\alpha + I\sinh\alpha)\cdot \\ \cdot(\cosh\beta + J\sinh\beta)\cdot(\cosh\gamma + K\sinh\gamma), \tag{20}$$

where $\cosh x$ and $\sinh x$ are hyperbolic sinus and cosine. We can introduce an analogous function for the quadranumerical variable X as the following rows:

$$\cosh X = 1 + \frac{X^2}{2!} + \dots, \qquad \sinh X = X + \frac{X^3}{3!} + \dots.$$

We can connect the notion of the derivative with the function of the quadranumerical variable by the direction and analyticity the same way as the corresponding ideas are introduced into the algebra of double numbers [2]. The analyticity of the function from H_4 denotes the independence of its derivative from directions, [5] $dF = F'da$, and appears in simultaneous execution of 12 equations, which are analogs to the Cauchy-Riemann terms for the complex and double variables:

$$\frac{\partial U}{\partial a_1} = \frac{\partial V}{\partial a_2} = \frac{\partial W}{\partial a_3} = \frac{\partial Q}{\partial a_4}, \qquad \frac{\partial U}{\partial a_2} = \frac{\partial V}{\partial a_1} = \frac{\partial W}{\partial a_4} = \frac{\partial Q}{\partial a_3},$$

$$\frac{\partial U}{\partial a_3} = \frac{\partial V}{\partial a_4} = \frac{\partial W}{\partial a_1} = \frac{\partial Q}{\partial a_2}, \qquad \frac{\partial U}{\partial a_4} = \frac{\partial V}{\partial a_3} = \frac{\partial W}{\partial a_2} = \frac{\partial Q}{\partial a_1}, \tag{21}$$

where

$$F(A) = U(a_1, a_2, a_3, a_4) + V(a_1, a_2, a_3, a_4)I + W(a_1, a_2, a_3, a_4)J + Q(a_1, a_2, a_3, a_4)K$$

is an analytical function of a quadranumerical variable, and U, V, W, Q are hypercomplex-conjugated functions of four real arguments. In the algebra of quadranumbers there are 16 typical unit objects $e_1 - e_{16}$ that have in their basis, where the form (11) is written, the following components:

$$\begin{array}{ll}
e_1 \leftrightarrow (1,1,1,1); & e_5 \leftrightarrow (-1,-1,-1,-1); \\
e_2 \leftrightarrow (1,-1,1,-1); & e_6 \leftrightarrow (-1,1,-1,1); \\
e_3 \leftrightarrow (1,1,-1,-1); & e_7 \leftrightarrow (-1,-1,1,1); \\
e_4 \leftrightarrow (1,-1,-1,1); & e_8 \leftrightarrow (-1,1,1,-1); \\
e_9 \leftrightarrow (1,-1,-1,-1); & e_{13} \leftrightarrow (-1,1,1,1); \\
e_{10} \leftrightarrow (1,1,-1,1); & e_{14} \leftrightarrow (-1,-1,1,-1); \\
e_{11} \leftrightarrow (1,-1,1,1); & e_{15} \leftrightarrow (-1,1,-1,-1); \\
e_{12} \leftrightarrow (1,1,1,-1); & e_{16} \leftrightarrow (-1,-1,-1,1).
\end{array}$$

The vectors e_i that correspond to the numbers can be used to illustrate the presence in the Quadra-space of two types of transversality, that generalize the idea of orthogonal directions for the Finslerian space. This is true that the 2 symmetrized forms (5) enter the Quadra-space. They look like:

$$S_1(\mathbf{a}, \mathbf{b}) = (\mathbf{a}, \mathbf{a}, \mathbf{a}, \mathbf{b}) + (\mathbf{a}, \mathbf{b}, \mathbf{b}, \mathbf{b}) \tag{22}$$

$$S_2(\mathbf{a}, \mathbf{b}) = (\mathbf{a}, \mathbf{a}, \mathbf{b}, \mathbf{b}). \tag{23}$$

The equality to zero of any of them means the transversality of the corresponding directions. By direct substitutions of the components of vectors e_i in (22) and (23) we can make ourselves absolutely sure of the fact that every vector of the multitude faces 1, form mutually transversal pairs of the first order, and of the second with 8 of them. We can construct the basis that is an analog to the orthogonal from the four the first order transversal vectors. One of the specific cases of the basis is the above examined four-set $1, I, J, K$. It is impossible to construct a basis from the second order transversal vectors as for each pair of the third and what is more fourth order do not have such correlation of directions.

Conclusion

The offered method of studying the examined class of Finslerian linear spaces, called polylinear, seems to be promising for it is based on the same principles as the scalar product. Let us note that the arising abilities allow us to move the focus of research from the common vivid base to the soil of mathematical constructions. Thus the pseudo-Euclidean spaces demonstrate advantages of the analogous substitution. Not all geometrical effects are vivid in these spaces but the extension of the scalar product in its time was very useful.

References

[1] I. M. Gelfant, *Lectures on Linear Algebra*, Nauka, M. 1966 (in Russian).
[2] I. L. Kantor, A. S. Solodovnikov, *Hypercomplex Numbers.* Springer, Berlin, 1989.
[3] B. A. Rosenfeld, *Multidimensional Spaces*, Nauka, M. 1966 (in Russian).
[4] A. I. Maltsev, *Foundations of Linear Algebra*, TTL, M. 1956 (in Russian).
[5] M. A. Lavrentiev, B. O. Shabat, *The Problems of Hydrodynamics and their Mathematical Models,* Science, M., 1977 (in Russian).
[6] B. A. Rosenfeld, *Non-Euclidian Spaces*, Nauka, M. 1969 (in Russian).
[7] G. S. Asanov, *Finslerian Extension of General Relativity*, Reidel, Dordrecht 1984.
[8] P. K. Rashevsky, *The Geometrical Theory of Partial Differential Equations*, 2-nd ed., Editorial URSS, M., 2003 (in Russian).
[9] H. Rund, *The Differential Geometry of Finsler Spaces,* Springer, Berlin, 1959.
[10] V. V. Kassandrov, Vestnik of Peoples Friendship University, Physics, **1**, 1993.
[11] V. V. Kassandrov, *Number, Time, Light*, http:www.chronos.msu.ru.
[12] G. I. Garas'ko, *Three-numbers, whose cube of norm is nondegenerate three-form*, Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004.
[13] S. V. Lebedev, *Properties of spaces connected with commutative-associative H_3 and H_4 algebras*, Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004.
[14] D. G. Pavlov, *Chronometry of the three-dimensional time*, Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004.
[15] D. G. Pavlov, *Four-dimensional time*, Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004.

Chronometry of the Three-Dimensional Time

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The concept of multi-dimensional time has often tried to take its deserved place in natural sciences, but each time, under the pressure of some paradox, it was rejected. Meanwhile, emerged a philosophical question: why does Space admit several dimensions while Time does not, still remained. In this work a new attempt has been made to resolve this matter, by switching from the traditional quadratic metrics to the Finslerian ones, which may admit several vector components as arguments for the metric function. Though this method enables us to build continuums of Time of any natural dimensionality, in order to point out the specificity of the topic, we shall focus on lower temporal dimensions.

MSC2000: 53B40, 53C60, 15A63, 15A69, 83D05.

1. Introduction

The idea of space is accepted much easier and vividly than the idea of time. This circumstance is conditioned by the fact that the space is looked over all at one time, and above all in the three-dimensional shape, meanwhile we see just a side of the time and only in one dimension. This situation forced some scientists "to get rid" of the time, either limiting to fixed problems or driving the time into the condition of an extra space dimension. The first approach is related to Archimedes, the latter approach for the first time appeared in the works of Galilei, reached perfection in Lagrange's and in fact reigns nowadays, – though special theory of relativity practically confronted the category of time to space, denoting them absolutely different in their essence, having differences already on the geometrical level.

There grows the belief formulated for the first time by Synge [1] that Euclid put the natural science on the wrong track, as he took the space but not the time as the fundamental idea of the science. The lack of any adopted term for time studying according to Synge is the proof of such disregard. He suggested that we should use the word "chronometry" to define the branch of science that deals with the idea of time in the same wide meaning as geometry does with the idea of space. Though Synge is unlikely to mean the multi-dimensional time, his statement is applicable to this aspect of the problem.

2. Two-dimensional time

The essence of the multy-dimensional time, that serves as an alternative to the multy-dimensional space, can be illustrated by a paradoxical-seeming statement:

practically all physicians know about the two-dimensional time, but by tradition go on looking at it in another way. We mean the pseudo-Euclidean plane. It is surprising that among all the Euclidean spaces only the two-dimensional is distinguished with its unique peculiarities, it is worth mentioning the following.

Firstly, the theorem of Liouville, that enumerates the types of possible conformal transformations, coming to translations, rotations, dilatations and inversions, is true for all the pseudo-Euclidean spaces with 3 or more dimensions. In the two-dimensional case the list of their conformal transformations is by far longer.

Secondly, there are several concepts of the total product of the plane vectors, and the majority of them have the inverse ones; meanwhile in other pseudo-Euclidean spaces only scalar product is introduced, as well as division is not defined at all.

Thirdly, isotropic vectors always divide the pseudo-Euclidean planes with the signature $(1, n-1)$ into 3 simply connected domains, with an exception of the plane, with 4 such domains.

Fourthly, it does not matter which of the two typical coordinates of the Euclidean space we will choose as the temporal and which as the spatial, as the result will change to permutation. Another case appears in planes with a bigger number of dimensions, where such symmetry collapses and to the time we can apply only change of the sign.

And finally, only the plane admits the accordance with the associative-commutative algebra, whose main objects are called the double numbers. Their algebra has all the characteristics of usual algebras of real and complex numbers, including the product commutativity, with an exception of presence of specific objects, called the divisors of zero. Each divisor of zero has a counterpart such that their product is a divisor. Though the double numbers are trivial in comparison with the complex, even such algebras cannot be related with pseudo-Euclidean spaces with more than 2 dimensions.

But, thinking that the uniform order starts with 3 and more dimensions, scientists, due to some reasons, don't notice or at best attribute it to the reducible nature of the two-dimensional space. It is interesting to note that we face practically the same in the Euclidean case: the two-dimensional representatives stand separately out and are juxtaposed with the algebra of complex numbers.

We can make a supposition basing on only these two examples that because of some reasons the connection of some metric spaces with the commutative-associative algebra make them in a way distinguished and that is why the very algebras and the corresponding spaces deserve a special attention.

When we stated above that we have a reason to treat the pseudo-Euclidean space as a special case of the multy-dimensional time, we based on the fact that in the space there is no objective reason for us to distinguish which of its directions can act as time and which not. Then we must admit that in such space all non-isotropic directions are equal in rights. Their differentiation by physical meaning takes place only after subjectively choosing one quadrant as the field of future.

Note. The subjective choice is related mostly to the world line, an element of whose length is interpreted as the proper time of an observer, and the future region is defined as the consequence of the line direction.

Only after the given procedure the points of the facing quadrant automatically acquire the meaning of the past actions, and the points of the two side – become absolutely distant. But few things will change on the pseudo-Euclidean plane if we choose to use any other quadrant as the field of the future, as only all the others will trade places. With an exception of this inessential-seeming moment, any further construction in the pseudo-Euclidean plane does not differ from the construction in its usual interpretation as the time-space.

But a move to 3 and more dimensions leads to the fact that the difference between the pseudo-Euclidean space-time and the dimension-corresponding pure time becomes principal, and moreover if we think of the conceptual multy-dimensional time as of a possible geometrical alternative to the space of the Special Theory of Relativity, it is important to revise not only mathematical, but also philosophical attitudes towards the structure of physical reality.

3. Three-dimensional time

To make a move from the two-dimensional time model to the three-dimensional let us use the observation that in the case of the pseudo-Euclidean plane the corresponding geometry becomes related with the idea of the commutative-associative hyper-complex number, which are related to the commutative-associative hyper-complex algebras. William Hamilton is the pioneer of hyper-complex numbers; while speaking at one of the sittings of the Royal Irish Academy he stated that if there existed geometry – the pure mathematical space science, there must be the same pure time science, and such a science should be algebra [2]. It is paradoxical but he on the example of the quaternions, discovered by himself, disproved the multitude of principally different algebras. But let us take his statement, as a presentiment of the great mathematician, and by analogy with the algebra of binary numbers we will try to make the algebra of triple number, and try to correspond with them geometry, or using Synge's suggestion, the chronometry of three-dimensional time.

The presence of the basis in binary numbers makes the expression for the second degree of the module to take an absolutely symmetrical form:

$$|\mathbf{X}|^2 = x'_1 x'_2, \tag{1}$$

It indirectly shows that there must be a basis for the numbers that admittedly can be an algebraic analog to the vectors of the three-dimensional time. In this basis the fourth degree of the module becomes connected with the next absolutely symmetrical form out of three components:

$$|\mathbf{X}|^3 = x'_1 x'_2 x'_3. \tag{2}$$

It is not difficult to make sure that the algebra of such numbers exists, it is commutative and associative, and is the direct sum of three real algebras that continues the tendency that started at the example of binary numbers, whose algebra becomes the direct sum of the two real. As is well known, the one-dimensional time can be compared with the real numbers themselves, that is another confirmation of the chosen algebraic way of searching for models of the multy-dimensional time.

The manifolds for which the differentials of the vector length are expressed by means of the types (1) – (2), are well known in geometry and are called the Finslerian spaces with the Berwald-Moor metric function [3]. Usually under the term Finslerian spaces we understand the manifold of the most common type with a null meaning of curvature and torsion. The concerned metric (2) is defines the linear space, that is why it is in near relation with Euclidean and pseudo-Euclidean spaces, though they do not look alike in everything.

Let us call the linear Finslerian spaces, whose metric function in one of the bases looks like:

$$F(x') = \Big|\prod_{i=1}^{n} x'_i\Big|^{1/n}, \tag{3}$$

the *n-dimensional time*. To have not only axiomatical but also physical right to use this name let us interpret every point of the spaces as an event, and every line as a world line of an inertial reference frame.

Note. The concept of an event is introduced in this way that though having something common with the classical analogue introduced by Minkowski, still differs from the latter. This is related to the fact that the concept of event in the multy-dimensional time stops having a single meaning and becomes dependent on the reference frame. In other words the same point of the space should be interpreted as different events if the world lines are separated by isotropic hypersurfaces. The concepts of time and space are as if substituting with one another. There are cases 2^n of such domains in n-dimensional time, and every point may have the same number of interpretations. But there does not emerge polysemy if we examine only the reference frames where the world lines lie only in the light cone, and the concept of event practically does not differ from its classical analogue.

In such reference frame the interval of proper time between an arbitrary pair of the equals the length of the vector related to the event. It follows from the symmetry of the examined spaces that all their non-isotropic directions are absolutely equal in rights if we decide to relate, according to the given above thesis, the length of the vector to the proper time in the distinguished reference frame then its justified to call the spaces, this time not by definition, rather than because of physical reasons, the multy-dimensional time.

But still preserves the question: whether such verities have any connection with the real world? To approach the answer let us try to examine the properties and peculiarities of the three-dimensional time. We will start from examining its structure and isotropic subspaces.

4. Light pyramids

The form (2) nullifies in the points that correspond to the three distinguished planes, defined by the equalization:

$$x_1' = 0, \quad x_2' = 0, \quad x_3' = 0. \tag{4}$$

The vectors lying on the plane have the zero meaning of the modulus and in this meaning are isotropic. At the same time, lines, that simultaneously belong to 2 planes (as well as the point of intersection of all the 3) automatically become marked out. As there are only three lines, it is quite natural to try to connect the vectors with the special basis. This basis is unique up to permutation and the form (2) given above defines the value of an arbitrary number module and also the length of the vector, – all being of the simplest shape. Concerning the originality of such basis, we will give it a proper name of the Absolute basis.

In this respect the concerned space turns out to be arranged in an absolutely another way, than the usual Euclidean and pseudo-Euclidean spaces, where there are no preferred bases (with an exception of the pseudo-Euclidean plane), and that is why we usually try to turn the studying of analogous geometries into a non-coordinate form. The existence of special bases in the multy-dimensional time means that if some day a connection between corresponding varieties and the physical reality will be found then some frame of reference will play a clearly distinguished role.

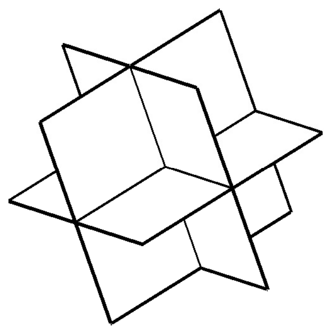

Figure 1: Isotropic planes of tree-dimensional time

The isotropic planes (4) can be thought about for example as they are presented on Fig. 1. As we can see on the picture the three-dimensional space is divided by isotropic planes into 8 equal camera-octants, that are domains of simple connectedness in fact. At the same time every camera is separated from the 3 side ones by the two-dimensional isotropic planes, it borders upon isotropic rays with another 3 cameras and with the opposite one it contacts through only one point. By analogy we can characterize, only taking into consideration the dimension, the mentioned above the two-dimensional time, where all the space is divided by isotropic lines into 4 camera-octants. Every quadrant is separated from 2 adjoining ones by isotropic rays, and with the opposite borders through a point. At the same time the one-dimension time also obeys the rule, as we can look upon the corresponding line as 2 opposite simply connected domains, divided by a special point, a zero that in a way can be considered to be an extreme singular case of the isotropic cone.

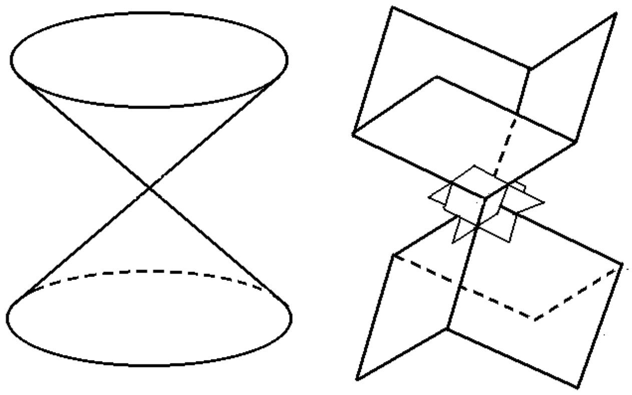

Figure 2: Light cones of tree-dimensional time (right) and tree-dimensional pseudo-Euclidian space (left)

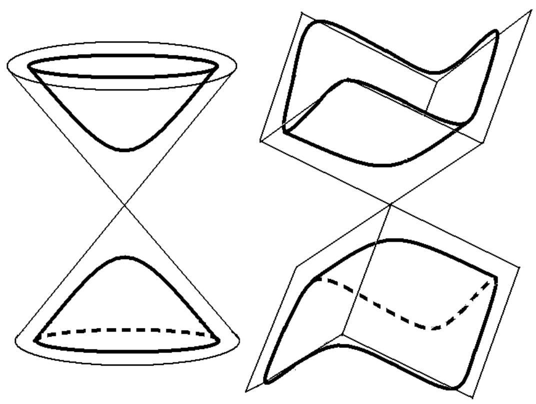

Figure 3: The fragments of unit hyperboloids

If we choose 2 facing camera-octants from the 8 of the three-dimensional time and examine their united border we will get a figure depicted on Fig. 2. Such the sub-space looks like a light cone of the Euclidean space (depicted on the same picture to the left side) but for the fact that the first does not have a continuous axis symmetry. There are non-zero vectors in the inside of both facing octants, and the ends of the unit length vectors form 2 planes of a specific hyperboloid, which is the Finslerian analogue of the double-band hyperboloid of the pseudo-Euclidean space. Both figures are depicted on Fig. 3, the left corresponds to the three-dimensional time and represents only a quarter of the hyperboloid of space, which has 8 cavities, each for every simply connected area. The points of the figure satisfy the equalization: $|x'_1 x'_2 x'_3| = 1$, and its general form is represented on Fig. 4.

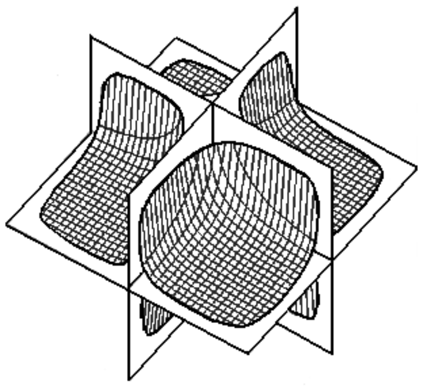

Figure 4: The eight-sheet hyperboloid of tree-dimensional time

Among the unit vectors that are set against one and the same plane of such hyperboloid continuous transfers, exercised by the Abelian two-parameter group of linear transformations, is possible. The transformations can be displayed as a diagonal matrix:

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \tag{5}$$

with $a_1 a_2 a_3 = 1$. Transformations of the group are invariant to the interval of the three-dimensional time (2) and that is why it is its motion. In their character the motions are similar to the boosts of the corresponding pseudo-Euclidean space with the only difference that the points of the line stay static in the one-parameter turnings in space-time, and in the analogous case of the concerned space – only one single point. We will call transformations of the group the *hyperbolic turning of the three-dimensional time.*

Among motions of the space, apart from turnings, we can single out a three-parameter group of parallel shifts, that are a common idea in linear planes. There is no other continued transformation that would be invariant to the interval in the three-dimensional time.

The isotropic edges and unit hyperboloids of the distinguished group of facing octants whose ends are to end at infinity are depicted on Fig. 2 and Fig. 3, but due to the limited plane of the draft, their ends are cut short, but not at a plane, common for pseudo-Euclidean space, but in a more sophisticated way according to the following considerations. If we intersect the border of one of the octants with the border of the facing octant dislocated along their mutual axis we will get

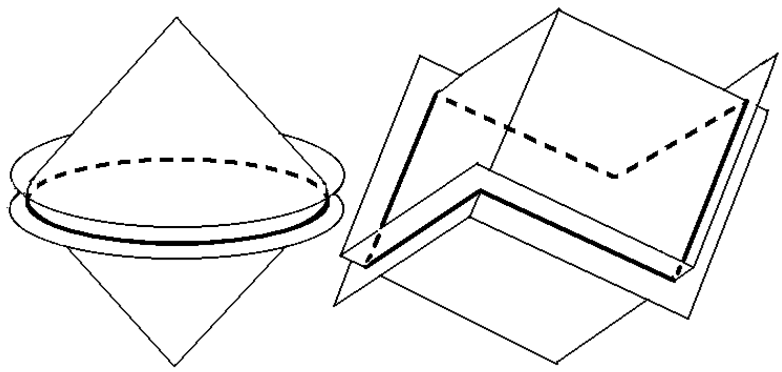

Figure 5: The two light cones couple intersection

a rectilinear hexagon, and not a plane but the broken as it is demonstrated on Fig. 5. The volume that belongs to the interior of both octants is a common cube, and the mentioned above hexagon is composed of its edges that do not intersect the main axis.

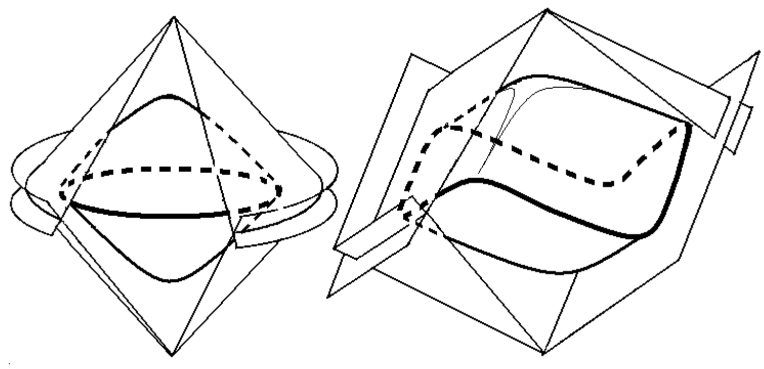

Figure 6: The two hyperboloids couple intersection with $0 < R < T$

Note. We can say that in case of the n-dimensional time the figure that is the interception of two deposed towards each other facing cameras, consists of a half of $(n-2)$ edges of the formed by it hypercube, on top of all only edges that do not have common points with the main axis of symmetry participate in the formation.

If we construct two sets of concentric hyperboloids (per se they are Finslerian generalizing of spheres) inside the octants that form the cube with their centers in the opposite tops, the intersection of pairs with equal radius will result into a set of continuous closed graphs, whose form depends on the ratio of the corresponding to the curve radius of the hyperboloid R to half of the main diagonal of the cube T. When the radius of hyperboloids equal 0 they coincide with the isotropic edges of the octants, and their interception is a broken in space hexagon already examined on Fig. 5. When $0 < R < T$ the hyperboloids are intercepted on curves that look

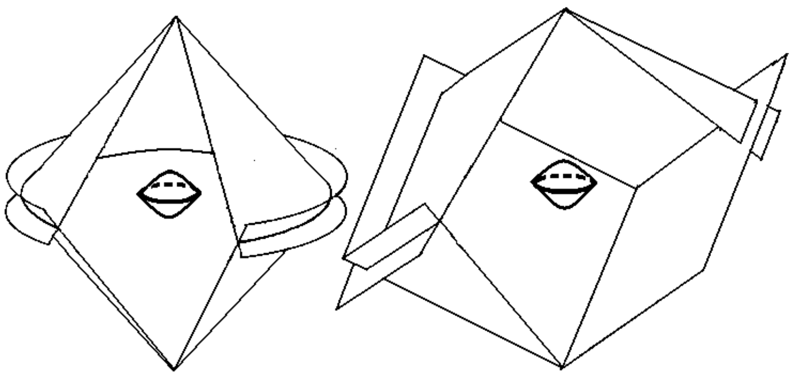

Figure 7: The two hyperboloids couple intersection with $R \approx T$

like the curve on Fig. 6. They are three-dimensional and have 6 round corners. While the value of the hyperboloid radius approaches to the value T the curves that are the result of their interception become more smooth and flattened out, and when $R \to T$ they turn into absolutely plane circumferences, though with infinitesimal radius Fig. 7.

In the three-dimensional pseudo-Euclidean space the analogous constructions lead to a group of concentric circumferences that lie in the same plane, you can see the circles on Fig. 5–7 to the right of them. The circumference that belongs to two light cones, that is corresponds to the interception of the pseudo-Euclidean sphere with $R = 0$ which in the Special Theory of Relativity is interpreted as a momentary position of the light front, that can be registered by the observer that is at the top of one of the cones, supposing that there is a flash at the top of the other. In general we should apply an analogous interpretation to the three-dimensional time case. So, the broken hexagon depicted on Fig. 5 can be interpreted as the multitude of points of the observer space, that is situated at the point T, with which it connects the momentary position of the light front, whose flash took place in $-T$. To make this situation true we must admit that the isotropic borders of the facing octants are analogues of the light cones of the past and future that corresponds in number of dimensions with the pseudo-Euclidean. This method looks rather natural and the only effort, in comparison with the common idea of the Special Theory of Relativity, we should make is to admit the borderness of the light cone. Taking into consideration that this borderness is executed in the space not available for the contemplation of the observer, the question whether is complies with the realities of our world turns out to be not so obvious.

Though we could save the name of light cones, usually used in the pseudo-Euclidean spaces in order not to emphasize peculiarities of geometry of the multy-dimensional time, for the isotropic borders of the simply-connected cameras, so let us call the corresponding figures the *light pyramids*, first of all singling out the pyramids of the past and future.

5. Planes of relative simultaneity

We should logically go further and accept an analogy not only between isotropic sub-spaces and the related to them light fronts but also we should put into correspondence with every common circle of two equal hyperboloids of the pseudo-Euclidean space an analogous curve, that is the interception of a pair of Finslerian spheres of the multy-dimensional time. There emerges quite a natural way to define the plane of the relative simultaneity of the three-dimensional time, as the same physical sense was played in the pseudo-Euclidean geometry by a plane represented with the above examined set of circles. Following the logic we should understand a multitude of points, equidistant in the meaning of the corresponding Finslerian metrics of two fixed points, under the simultaneous events of the multy-dimensional time. At the same time one of the fixed points coincides with the momentary position of the observer, and the second is the reflection of it with respect to the studies plenty of events.

The straight line that goes through the two points defines the inertial reference frame, but as it follows from the accepted definition of simultaneity now this property depends not only on the speed of the observer but also on his momentary position concerning the layer, to which he is going to give the equal time of performance. In the pseudo-Euclidean case (that has become practically classical) while defining the simultaneity meant only the relative speed of the relative speed of the reference frame, and the momentary position of the observer was not important. It is not so in the three-dimensional time and this circumstance seems to be one of the most important items, that differ the physical properties of the examined manifold from the common pseudo-Euclidean constructions.

It is convenient to describe the plane of simultaneity that corresponds to a fixed pair of points by an equalization that relates it coordinates to the coordinates of the initial affine space represented in the absolute basis. It is not difficult to get such equalization for an arbitrary pair of points, but it looks most vividly when momentary position of the observer is related to the point (T, T, T), and its reflection has coordinates $(-T, -T, -T)$. In this case the equality of intervals leads to the equalization:

$$|(x'_1 + T)(x'_2 + T)(x'_3 + T)| = |(T - x'_1)(T - x'_2)(T - x'_3)|, \tag{6}$$

then after opening the brackets it leads to:

$$x'_1 x'_2 x'_3 + (x'_1 + x'_2 + x'_3)T^2 = 0. \tag{7}$$

The plane corresponding to the equalization is depicted on Fig. 8.

The curves examined on Fig. 5 and Fig. 7 mark points on the plane in a certain sense equidistant from their geometrical center. Such curves in many ways are analogous to common concentric circles, though the related to it geometry does not coincides with the usual Euclidean.

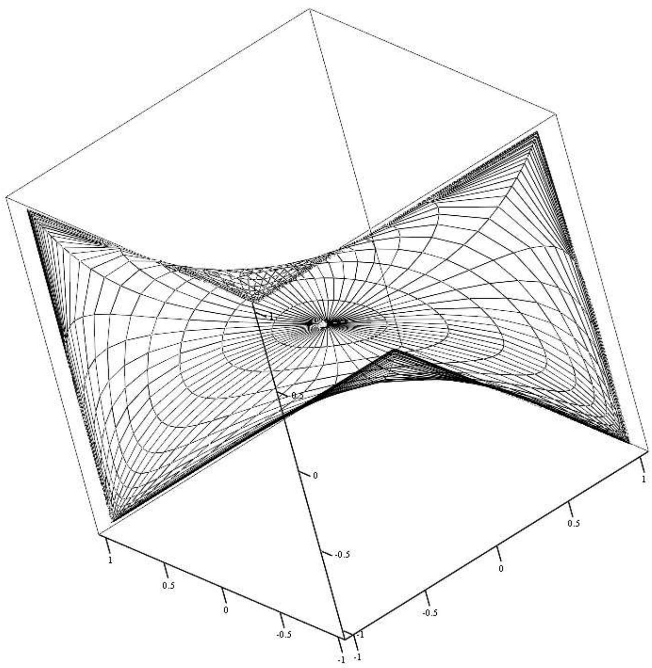

Figure 8: The simultaneous surface of three-dimensional time

On the other hand we can get a new group of curves, that corresponds to the multitude of radial lines of the Euclidean circle the canonic planes by intercepting the plane of simultaneity by canonic planes, called in the work [4] the *cones of rotation*, have tops in the point (T, T, T) and include the real axis. So, there is a net of curvilinear coordinates, that in the two-dimensional physical space play the same role as the polar scheme of coordinates does in the Euclidean plane.

Transformations that turn into themselves the plane of simultaneity so that the circles and radial curves at the same time map into the same curves and become in many ways analogous to spatial turns around the point of origin in the pseudo-Euclidean space, as the physical distance in either of the cases remain the same. But in the case of the three-dimensional time these transformations are not linear, and on top of all do not leave invariant the three-dimensional intervals.

6. Physical distance and speed

It could seem that we have approached to the possibility of introduction into the three-dimensional time of two-dimensional physical distance and speed, it is enough to bring on the simultaneity plane in correspondence the set of circumferential and radial curves with the lines of the polar reference frame. But it is not like this. The fact is that the examined multitude does not admit the introduction as one-digit such physical notions as the distance and speed at least if the construction is based on the starting measurement of time intervals. What seems to be practically an obvious property of the pseudo-Euclidean spaces turns out to be not-compatible with the idea if the multy-dimensional time. This circumstance not only decreases, but on the reverse increases the possibility of the multi-dimensional time to compete with the Minkowski space for being the geometrical basis of the real world. In fact, if we follow the idea of chronometry we should associate associate the time intervals, that are needed to send a desired signal and receive its reflection, with physical distance. But any attempt to unite this natural and vivid physical principle with the necessity of one-digitness comes upon obstacles. The idea of rejecting the one-digitness of the physical distance and speed seems to be a nice and far-reaching exit (cf. interpretations of quantum-mechanical uncertainty principle).

The above said does not mean that an entirely amorphous structure should replace the Euclidean geometry of the physical space. The analysis shows that our radical supposition touches upon not the quality, but only quantity aspect of the phenomenon. The distance and speed as independent physical categories are not completely excluded in the multy-dimensional time, but only change their status, getting the traits uncertainty on the initial geometric level. In particular the idea of equidistant in the physical meaning objects becomes dependent on which signals the observer, that defines this equidistance, uses as the reference. In its way the reference signals are defined by the principle of equality of proper times, where the hours pass in the corresponding inertial reference frames between sending, reflecting and receiving the signals. Taking into consideration that the time intervals are the only value that by definition are measurable in our Finslerian multitude, the task of distinguishing among the continuous specter of inclined world lines the ones would be characterized by the equality of intervals is quite possible. Let us note that we already used the method above, while defining the relatively simultaneous events. So, we can consider the signals to be etalons if their world lines start in one point, reach the plane of simultaneity and after refraction gather together and in another fixed point of the world line of the same observer. It is clear that all the intervals should be equal either before or after the refraction.

Such logic in constructing drives us to the fact that the physical space of the observer with its geometrical properties becomes in a way dependent on which set of reference signals define the geometry. So if the world lines of reference frames are practically parallel to the line of the observer, he starts to see a space, which in its characteristics practically coincides with the Euclidean. This is related to the fact that the ends of the vectors with the same value of the intervals in this cases lie (as it has been said above) on practically plane and ideal circle, and the latter while constructing the physical space plays the role of the Finslerian indicatrix. A common circle is the indicatrix of the two-dimensional Euclidean space. When tuning to the signals whose world lines are inclined more significantly, the ends of the corresponding vectors form this time not a circle, but a more sophisticated closed curve, which is not a plane one. At limit of the signals, whose speeds are interpreted as the light, this curve transforms into a broken hexagon, examined on

Fig. 5. The geometry of the two-dimensional physical space is the Finslerian, and it is this geometry that differs greatly from the Euclidean, but in connection with the fact that the indicatrix even in this limit case is still closed and flattened out. The differences between the two geometries are not significant, in connection with which it is probably possible to mix them up, especially if the experimental cases are limited to low speeds.

So, if we suppose that our real world has a direct connection with the examined Finslerian geometry, the appearance of Euclidean and pseudo-Euclidean ideas in observer outlook should be a natural process of consistent approaches to a more exact description. On the other hand in our everyday life we use signals whose speed is by far lower than the light when we try to find the zones that manage the world. As the matter of fact we use the light only to identify the objects, and the distance is defined by other slower means – for example by a ruler. This circumstance leads us to the fact that when in special experiments really high-speed signals become of great importance, the geometry is considered to be defined before hand, and that is why even abnormal results will be treated anyhow, but only not in the direction of revising the obvious geometrical properties.

Conclusion

Among all the above listed properties and peculiarities of the three-dimensional time, as a representative of a very specific class (the non-linear) of Finslerian spaces, we should treat as the most important the one, thanks to which it is related to the most fundamental notion of mathematics – the number – which is the object of algebra, that has the most common arithmetical properties. We should emphasize ones more the fact that neither Euclidean nor pseudo-Euclidean spaces with three or more dimensions do not possess the analogous qualities. The quaternions and biquaternions used in similar situations are not genuine numbers, as there algebra has commutative multiplication, as the result of which the construction of a valuable theory that would generalize the theory of functions of the complex variable is not possible (or is extremely difficult). At the same time the given above examples demonstrate how common Euclidean and pseudo-Euclidean conceptions can come out of the idea of substitution of the pseudo-Euclidean metric to multy-temporal case – rather interesting and actual.

References

[1] J. L. Synge, The New Scientist, 19th February 1959, p. 410.
[2] W. R. Hamilton, Trans. Roy. Irish. Acad., 1833–1835.
[3] G. S. Asanov, *Finslerian Extension of General Relativity*, Reidel, Dordrecht 1984.
[4] D. G. Pavlov, *Hypercomplex Numbers, Associated Metric Spaces, and Extension of Relativistic Hyperboloid*, arXiv:gr-qc/0206004.

Four-Dimensional Time

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The generalized metric space, which can be called *the flat four-dimensional time*, is based on Berwald-Moor's Finslerian concept of metric function. This structure allows us to introduce several physical notions, as: event, world lines, reference frames, set of relatively simultaneous events, proper time, three-dimensional distance, speed, etc. It is proved how from the location of the physical observer, associated with the world line, in absolutely symmetrical four-dimensional time, the contraposition of coordinates takes place (which define its proper time) with the ones that appear as the result of the measurements made with the help of sample signals. When the signals correspond to lines, which are practically parallel to the world line of the observer, he starts to see the three-dimensional space which at the limit is the Euclidean space.

MSC2000: 53B40, 53C60, 15A63, 15A69, 83D05.

1. Introduction

For the last 100 years the idea, that the Pseudo-Euclidean metric with an alternating-sign quadratic dependence on the length of the vectors from the magnitude of its components lays in the basis of geometry, has taken root in physics. But still numerous and various attempts to connect all the known natural forces nature with the metric and make true the idea of the total geometrization of physics have failed. This drives to the idea that the reason lies not in the lack of scientists' creativity, but in the metrics itself, even better to say in the classical quadratic form, in place of which it is admittedly to use other dependences. Unfortunately, this attitude, the possibility of which indicated Riemann [1], was for the first time studied by Finsler [2], and up to nowadays used by hundreds of investigator [3], did not give eventual pictures. Though nowadays the work in this direction is continued, it considerably differs from many of them, as it is based on the idea of scalar poly-products, which is new for the Finslerian geometry, and metric form that is connected with one of the most fundamental notions in mathematics – the real number.

2. Multidimensional time

The spaces that have unique correspondence with algebras, that are the sum of several real number algebras, stand out from Finslerian linear spaces. The metric functions do not depend on the point and in one of the bases look like:

$$F(x') = \left|\prod_{i=1}^{n} x'_i\right|^{1/n}, \tag{1}$$

where x_i' are the components of the vector and n is the number of dimensions. Such metric functions are well-known in the theory of Finslerian spaces and took the name of Berwald-Moor's function [3].

Geometries with such metrics in many ways are of the same type and the difference is related only to the dimension. The total equality of all non-isotropic directions is their main peculiarity. As any of such directions can be related to the proper time of the inertial reference frame, it is appropriate to call such spaces the *multi-dimensional time.*

Note. It seems that it is possible to relate a general line with an inertial reference frame in any linear space, where the element of the length is defined in every point. But in many spaces some reference frames do not admit the presence of isotropic connections with other lines that go in a parallel way with the given. For the viewer related to such reference frames, the existence of isotropic vectors, with which it is traditional to associate the light signals, becomes the origin to the idea of the physical distance and consequently the physical space.

The defined in this way spaces not always have the same shape as the one we got used to (in every day life and thanks to Euclid and Minkowski). At the same time we have to put a more general meaning than usually into the idea of physical space. On the other hand nothing prevent us from considering that in the sectors or dimensions, where isotropic connection is not set or have an extraordinary characteristics, that physical directions are undetectable, though representable from geometrical point of view. Consequently, it is quite logical to suppose the existence of some spaces, some parts of directions and even dimensions of which are not apparent from their physical side. From such point of view it would be interesting to analyze arbitrary linear spaces and in particular those, connected with quadratic forms and the Berwald-Moor's metrics treated over the field of complex numbers.

The chosen geometrical element of every n-dimensional time is its isotropic sub-space, that is a figure constructed from n-hyperplanes, that divide the multyformity into 2^n-equal simply connected cameras. Any of the cameras adjoins to the others, but for the facing, with which it borders in a point. The adjoining cameras can be classified according to the distinguished by the dimension of the frontier planes from 1 to $(n-1)$. All simply connected cameras are equal and have the shape of regular pyramids, n-hyperplanes of which start from the top and go to the infinity. We will call such pyramids, by analogy with isotropic cones of the Minkowski space, the light pyramids. Every *light pyramid* has n one-dimensional edges that can easily be connected with a special basis. In the basis the geometrical correlation of the multy-dimensional time appears in a vivid shape and, as such a basis is to permutation unique, it is quite natural to call it the *absolute*.

Any single vector that belongs to the inner area of a light pyramid can be continuously introduced into any other single vector that belongs to the same pyramid. The respective transformation form $(n-1)$-parametrical Abelian subgroup of movements, that leaves initial metric function (1) invariant. The metrics of such transformations in the absolute basis is reduced to the diagonal form:

$$\begin{pmatrix} a'_1 & 0 & \dots & 0 \\ 0 & a'_2 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & a'_n \end{pmatrix}, \tag{2}$$

where $\prod_{i=1}^{n} a'_i = 1$. The corresponding reflections can be classified as Hyperbolic turn (that in a way are analogous to the busts of the pseudo-Euclidean spaces) because such transformations leave on the place a point of convergence of the tops of all the pyramids and isotropic edges of the last at the same time turn into themselves. Among continuous movements of the multy-dimensional time along with hyperboloic turns there is also a n-parametrical subgroup of parallel transfers. The examined variety doesn't include any other continuous congruent transformations and that is why has less freedom than the spaces with quadratic types of metrics.

The very circumstance made Helmholtz, Lee, Weyl prove a number of theorems that stated that the oneness of the quadratic metrics [4 − 6]. The main emphasis was made to maximum mobility in quadratic spaces. This according to them gave grounds to reject all other metric forms in the meaning of the basis of the real space-time. Let us note without rejecting the theorem accuracy that its approval is based on the examination of only the distinguished linear transformations, which means that it gives a chance to other theorems, where non-linear symmetries play the same role. In contrast to continuous congruent transformations the discrete group of symmetry of the multy-dimensional time excels the corresponding Euclidean- and pseudo-Euclidean spaces, but this is not enough to compete with the latter one. What really makes the multy-dimensional time the multy-dimensional time interesting is the presence of distinguished groups of non-linear transformations which are practically as fundamental as the groups of movements.

Such transformations save invariant not the intervals, but specific scalar forms of several vectors, that do not have direct analogous quadratic spaces, and that is why are not well-studied.

It is better to come to the understanding of such polyforms through the generalizing of the idea of the scalar product. It turns out that in a number of Finslerian linear spaces the poly-linear symmetry form of n vectors [7] (its special case is the classical bilinear form) can play the role of the scalar product. Let us call the poly-linear form the *scalar poly-product.* Founding on this generalizing we can enlarge with some Finslerian spaces such fundamental ideas of geometry as the length, the angle, the orthogonality, etc., the introduction of which is difficult due to some problems [8].

In the absolute basis the *scalar poly-product* of the multy-dimensional time looks like:

$$(\mathbf{A},\mathbf{B},\ldots,\mathbf{Z})=\frac{1}{n!}\sum_{(i_1,i_2,\ldots,i_n)} a'_{i_1}b'_{i_2}\ldots z'_{i_n},\quad \text{at}\quad i_j\neq i_k,\quad \text{if}\quad j\neq k. \tag{3}$$

It is not difficult to believe that with $\mathbf{A}=\mathbf{B}=\ldots=\mathbf{Z}$ the form (3) turns into the metric function (1). We can build the geometry of the linear time in an arbitrary natural scale using the poly-linear symmetrical form (3). But let us focus on this case if we base on common ideas about physical measurements and vivid typological detailedness of the four-dimensional space [9].

3. Four-dimensional time

According to (3) the scalar poly-product, that defines the four-dimensional time, in the absolute basis looks like:

$$(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})=\frac{1}{4!}\sum_{(i_1,i_2,i_3,i_4)} a'_{i_1}b'_{i_2}c'_{i_3}d'_{i_4},\qquad \text{when}\quad i_j\neq i_k \text{ if } j\neq k, \tag{4}$$

it follows that the fourth degree of the vector length of such linear space is defined by the expression:

$$(\mathbf{X},\mathbf{X},\mathbf{X},\mathbf{X})=|\mathbf{X}|^4=x'_1x'_2x'_3x'_4. \tag{5}$$

While turning to the basis analogous to the orthonormalized [7] (it is more visual than in the absolute case) the expression transforms into a more complicated but still symmetrical form:

$$|\mathbf{X}|^4=x_1^4+x_2^4+x_3^4+x_4^4-2(x_1^2x_2^2+x_1^2x_3^2+x_1^2x_4^2+x_2^2x_3^2+x_2^2x_4^2+x_3^2x_4^2)+8x_1x_2x_3x_4. \tag{6}$$

In a number of cases it is more convenient to use the form picking out one of the coordinates, in particular x_1:

$$\begin{aligned}|\mathbf{X}|^4=x_1^4-2(x_2^2+x_3^2+x_4^2)x_1^2+8(x_2x_3x_4)x_1+&\\ +(x_2^4+x_3^4+x_4^4-2x_2^2x_3^2-2x_2^2x_4^2-2x_3^2x_4^2).&\end{aligned} \tag{7}$$

The main arguments in favor of the chance of confronting the four-dimensional time to the real physical world is the presence of a group of continuous symmetries [10], that can be examined as an alternative to the linear group of spatial turning of the Minkowsky space. Not a scalar poly-product of the four-dimensional time (4) is an invariant to the transformations, but a specific form, that is defined by 2 vectors:

$$S(\mathbf{A},\mathbf{B})=\frac{(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{B})}{(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{A})^{1/2}}+\frac{(\mathbf{A},\mathbf{B},\mathbf{B},\mathbf{B})}{(\mathbf{B},\mathbf{B},\mathbf{B},\mathbf{B})^{1/2}}. \tag{8}$$

Though the form $S(\mathbf{A},\mathbf{B})$ is not an additive quantity of the vectors that belong to the interior of domain of a light pyramid, it complies with other very important characteristics of the common scalar product, to be more specific: the symmetry, the rule of multiplication by the vector, the sign distinctness and the triangle rule [10]. According to this there exists a principal opportunity in the four-dimensional time to introduce the idea of the three-dimensional distance, that corresponds to most of common conceptions of the physical quantity, but for the additivity. From philosophical point of view the last characteristic is very important. No, really, why should the rule of composition differ from the one of three-dimensional distances, as both values are relative? Such linearity appears only when we work with big distances, as well as the non-linearity of the rule of speed composing is essential only in the relativist field. At the same time an additional fundamental constant – the maximum possible magnitude of the physical system, or, in other words, the radius of the Universe, acts as the light speed in the three dimensional distance. For everyday distances we can still use the linear approximation, but in the space scale, in case of logical appliance of the multy-dimensional time conception, certain corrections should be made.

4. Plenty of relatively simultaneous events

We should first of all clarify the situation about a number of simultaneous events to give the definition of the four-dimensional time, three-dimensional speed and distance. Let us understand under it the total of points equidistant (of course in the meaning of the accepted Finslerian metrics (5)) from a pair of fixed events. In contrast to the Minkowskian space, where a multitude of points constitute hyperplanes, in the four-dimensional time the corresponding planes are non-linear [10]. Their form depends not only on the direction of the world line, that connects the fixed points, but also on the magnitude of the interval that separates them. This is the most fundamental difference from the space of the Special Theory of Relativity, as the idea of simultaneity is defined now not only by the speed of the reference frame, but also by the interval of time that separates the instantaneous position of the observer and the examined spatial layer of events. So the relativism in the four-dimensional time touches upon not only the hyperbolic turns, with the help of which realizes the switch between one system to another, but also the transmission, that enables to change the reference point.

From philosophical point of view such generalization is quite logical, but in fact establishes a sort of relationship between the two subgroups of the total group of congruent symmetries. As an indirect affirmation of the made conclusion can serve the fact that in algebra transmissions lack the operation of composition, which are a part of the four-dimensional time, and hyperboloic turnings – multiplication, and mathematics do not question relationship between them. A natural way of introducing the idea of the physical distance in the four-dimensional time is offering a method that from conceptual point of view is analogous to the method of defining of the idea in the Minkowskian space. By definition under distance we can understand a value that equals (or is proportional) the tie intervals, that go along the world line of the observer, between sending some uniformly moving model signals to the world lines of the examined objects, and receiving the reflected signals. It leads to the fact that it is senseless to use the idea of distance towards single events in the four-dimensional time, and is productive concerning only chains of them, that are presented by certain lines. We can pay no attention to the fact in the Minkowskian space, as multitudes regarding simultaneous events are hyperplanes, as a result the distance defined for an arbitrary pair of parallel lines were still substantial and for a pair of points.

Not to overload the brief article with excessive community, but at the same time to be rather specific, we will give the result to which the described above algorithm drives only in one case – when the world line of the observer coincides with the real axis, it itself is situated at the point $(T, 0, 0, 0)$ and the necessary layer goes through the point $(0, 0, 0, 0)$ (Fig. 1) [Here and later on the appearing coordinates relate to the generalized orthogonal basis [7] that differs tremendously from the absolute].

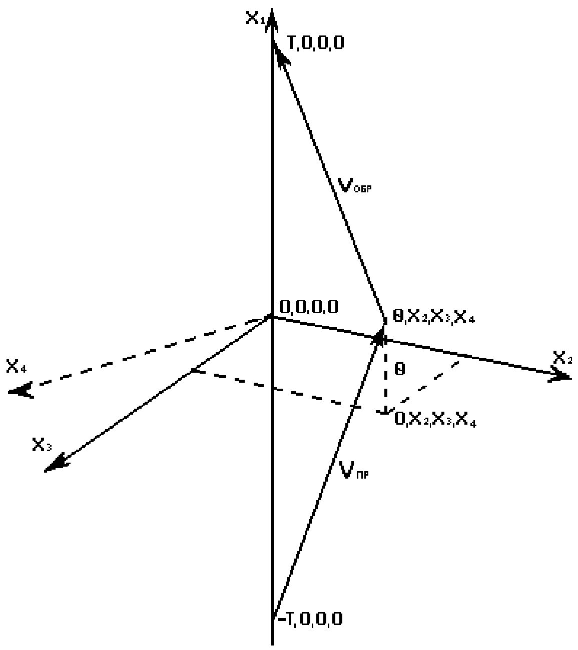

Figure 1: The world lines of direct and opposite signals with speed module

In this case the equalization, that relates the real coordinate θ of a point of the plane simultaneity to three other coordinates x_2, x_3 and x_4, follows from the rule of equality of the vector length that have the following components $(T + \theta, x_2, x_3, x_4)$ and $(T - \theta, -x_2, -x_3, -x_4)$. (Variable θ means deviation of concrete point from hyperplane $x_1 = 0$.) Using the expression for the magnitude of the interval (7) and at the same time concerning that for even degrees $(-x)^n = x^n$, we have:

$$(T+\theta)^4 - 2(x_2^2 - x_3^2 + x_4^2)(T+\theta)^2 + 8(x_2x_3x_4)(T+\theta) + (x_2^4 + x_3^4 + x_4^4 - 2x_2^2x_3^2 - 2x_2^2x_4^2 - 2x_3^2x_4^2) =$$
$$(T-\theta)^4 - 2(x_2^2 + x_3^2 + x_4^2)(T-\theta)^2 + 8(x_2x_3x_4)(T-\theta) + (x_2^4 + x_3^4 + x_4^4 - 2x_2^2x_3^2 - 2x_2^2x_4^2 - 2x_3^2x_4^2).$$

Opening the brackets and collecting terms we get:

$$T\theta^3 + (T^2 - x_2^2 - x_3^2 - x_4^2)T\theta + 2x_2x_3x_4T = 0. \tag{9}$$

introducing sizeless value $\eta = \theta/T, \chi_2 = x_2/T, \chi_3 = x_3/T, \chi_4 = x_4/T$ and taking into consideration that $T \neq 0$ we get a cubic equalization relatively to η:

$$\eta^3 + (1 - \chi_2^2 - \chi_3^2 - \chi_4^2)\eta + 2\chi_2\chi_3\chi_4 = 0. \tag{10}$$

Its real root characterizes the relative value of deflection of the simultaneity plane abscissa from the coming through its center according to the hyperplane $x_1 = 0$. We will call such parameter the *coefficient of non-platitude*. When $\chi_2 \approx \chi_3 \approx \chi_4 \to 0$, η also stems to 0, we mean around the point $(0, 0, 0, 0)$ the plane of the simultaneity turns into the hyperplane $x_1 = 0$.

The plane of simultaneity has physical meaning only inside the light pyramid, that has the world line of the observer, in other case it would be necessary to admit the physical meaning of the superlight speed. Following the method of the Special Theory of Relativity, with every vector that start at $(-T, 0, 0, 0)$ and ends at the plane of simultaneity, or in other words at $(\eta T, x_2, x_3, x_4)$ it would be quite natural to connect the world line of the signal, that has a definite uniform speed. We will transform the signals of the vectors, if they have equal interval values, according to the value of the speed module: $|\mathbf{V}_{\text{dir}}|$. Logically the signal, that is confronted to the vector, connecting the points $(\eta T, x_2, x_3, x_4)$ and $(T, 0, 0, 0)$, has the value that is inverse to the speed $|\mathbf{V}_{\text{rev}}|$. On contrast to the Minkowskian space such vectors have components that differ not only in sign but also in value (Fig. 1), to be more specific: $\mathbf{V}_{\text{dir}} \leftrightarrow (\eta T + T, x_2, x_3, x_4)$ and $\mathbf{V}_{\text{rev}} \leftrightarrow (T - \eta T, -x_2, -x_3, -x_4)$. In the Minkowskian space the coefficient of the non-platitude η for every point of the plane of the simultaneity equals 0, as the result the components of the vectors that correspond to direct and inverse signal look like: $\mathbf{V}_{\text{dir}} \leftrightarrow (T, x_2, x_3, x_4)$ and $\mathbf{V}_{\text{rev}} \leftrightarrow (T, -x_2, -x_3, -x_4)$.

To give a definition of distance between the real axis and an arbitrary line parallel to it, which is totally defined by 3 fixed coordinates x_2, x_3, x_4, we should have a model signal, or even better to say vectors related to it, with the help of which it is possible to make intervals that would equal the distance of different directions. As well as in the space of the Special Theory of Relativity, in the four-dimensional time it is more convenient to relate such symbol signals to isotropic vectors, that at one end have the same beginning and from the other – they set against the plane of simultaneity. In the Minkowskian geometry a number of ends of such vectors represent an intersection of two light cones: the future with the top at point $(-T, 0, 0, 0)$ and the past whose top is deposed to $(T, 0, 0, 0)$. As is well known the result of such interception is a common sphere, that lies completely in the hyperplane $x_1 = 0$. This is typical only for spaces with a quadratic metric type. In any case in the fur-dimensional time an analogous figure that is the result of interception of two facing light pyramids, is not plane though consists of linear elements.

Tit is better to make sure of it using the three- and four-dimensional time [12] as the example, in particular looking at Fig. 2 where it is demonstrated the interception of two light pyramids. For comparison, an interception of two light cones of the three-dimensional pseudo-Euclidean space is demonstrated on the same picture. In the three-dimensional time the interior of domain, that belongs to either of the pyramids, is a common cube, one diagonal of which is a segment of the real axis $[-T, T]$. At the same time the interception of two light pyramids results in a figure, built from $(n-2)$ edges of such cube, excluding the points $-T$ and T. In this case this is a hexagon $ABCDEF$ and it does not belong to the plane $x_1 = 0$, though compiles one of it rectilinear elements.

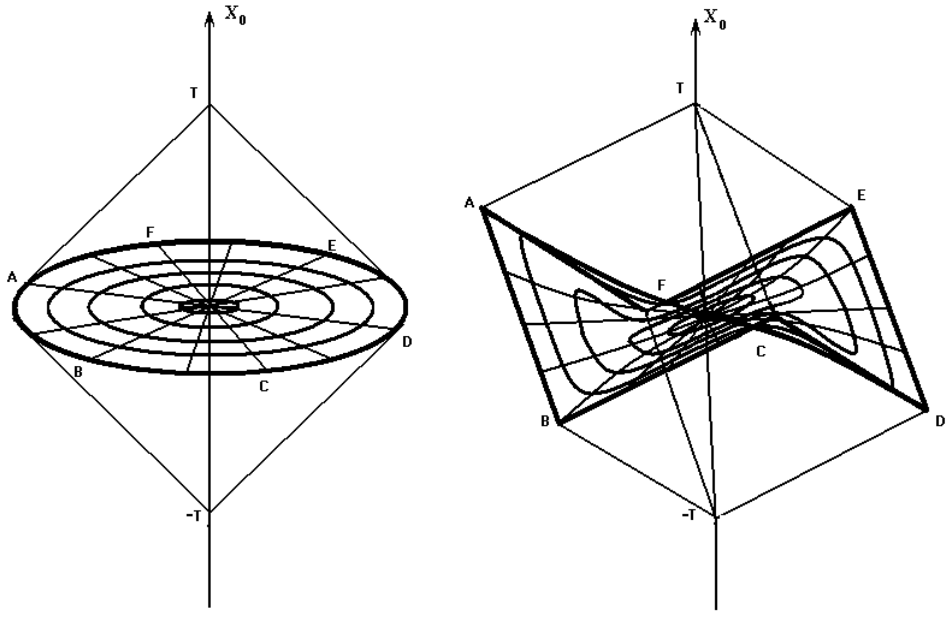

Figure 2: The simultaneous surface of three-dimensional time (right) and in three-dimensional pseudo-Euclidian space (left)

It is analogous in the four-dimensional time: the area that belongs to two facing light pyramids is a four-dimensional cube and the plane of the interception of their isotropic edges is built by 20 2-edges of the cube, that do not include the main diagonal $[-T, T]$. It is difficult to demonstrate this figure using a plane scheme that is why we will limit to the examined above a three-dimensional prototype. In the work [13] there was made an attempt to examine the corresponding dodecahedron (but it seems that the author has lost its principle four-dimensional character and depicted it as a common three-dimensional figure).

In the Minkowskian space the world lines that are parallel to the world line of the observer and touch the figure, which is the interception of two light cones, are accepted as equidistant points of the physical space of the observer, and the value proportional to the axis length of such double cone is referred as the distance. We can act in the analogous way in the four-dimensional time. In this case the parallel to the real axis lines, that come through the point of interception of the edges of two facing light pyramids, become equidistant from it, and in the role of the distant act the value that proportional to the main diagonal of the hypercube that is the result of such interception. In order to find the numerical value of it we should choose 2 real roots from the equalization:

$$x_1^4-2(x_2^2+x_3^2+x_4^2)x_1^2+8(x_2x_3x_4)x_1+(x_2^4+x_3^4+x_4^4-2x_2^2x_3^2-2x_2^2x_4^2-2x_3^2x_4^2)=0, \quad (11)$$

which are nothing but the abscises of the interception point of the line, which is related to the coordinates x_2, x_3, x_4, and 4 isotropic hyperplanes. One of the roots $x_{1,1}$ corresponds to the point that belong to the pyramid of the past, another $x_{1,2}$ – to the future, as the other 2 redundant roots $x_{1,3}$ and $x_{1,4}$ belong to the edges of the plane of the side pyramids. In this case we can consider the distance to be half of the sum of the first 2 roots: $R_c = 1/2(x_{1,1} + x_{1,2})$, while the index "c" emphasizes that the value is defined by light signals.

The three-dimensional space that appears as the result of such procedure is the Finslerian and is characterized by its indicatrix whose role plays the described above [13] dodecahedron. The space in its characteristics is quite close to the Euclidean, it comes from the convexity and two-dimensional restraint of its indicatrices, that does not differ greatly from the indicatrix of the Euclidean space, which is a common sphere. But the difference between the Euclidean sphere and the examined dodecahedron is rather principle to mix up their geometries. That is why there was made a conclusion in the work [13] that the idea that in the basis of the geometry of the real macro-world lies the four-dimensional time metrics. But still we think that while making the conclusion one very important circumstance, that when orientating in the real space the observer uses much slower signals rather than the light ones, was not taken into consideration. The light only helps, it is to identify the objects, as the comparison of their distances is realized by other slower means. The fact was not important in the Special Theory of Relativity as the indicatrix of the physical space did not depend on the speed of the signal. It is not like this in the multy-dimensional time. The more the relative speed of the probing signals differs from the light, the less the corresponding indicatrix distinguished from the hyper-plane, the more round become its angles and the more it looks like the three-dimensional sphere. At the limit when the relative speed of signals, with the help of which the physical space is examined, stems to 0, it stops being different from the Euclidean. So if we detect some static objects in the four-dimensional time with the help of the light, and define the distance with the help of other slower signals, so in this case we will come upon only the Euclidean geometry. Let us note that the very condition is complied in the vast majority of common for a man situations.

On the other hand it is not questioned that there is a principle opportunity to carry out an experiment in order to get to know which geometry better suits the real physical space – the Riemannian or the Finslerian. In this case it is important that the distance between fixed objects should be made by other light or slower signals. It is paradoxical but such experiments that do not accept double interpretation lack among the huge number of experimental materials. But the differences that should be traced are not large and that is why can be explained in different ways.

The above accepted conception of building the three-dimensional time explains why in absolutely equal in geometrical rights coordinates of the four-dimensional time the observer, associated with a world line, will register a significant difference between the coordinate that relate to his proper time and the other three. The answer lies in the topological difference between indicatrices of the geometrical and physical spaces. So if the first has the look of a specific 16-line hyperboloid, the second is a ring closed in two dimensions, its right form though depends on the used in measurements signals, is static from topological point of view.

5. Conclusions

Forms that save the scalar form (8), do not leave the intervals invariant, and tot ell the truth are not movements of the four-dimensional time. But as they turn the hyper-planes of the simultaneity (10) into themselves and do not change the three dimensional distances R_c they can act as common physical turns. There can emerge an explanation of the famous paradox – between the forward and rotatory movement. It is difficult to use the principle of relativity to the latter case, and the most famous attempt to examine it was made by Mach, who thought that the centrifugal forces owe their existence to the enormous mass of all the bodies in the Universe. According to Mach if we start turning the whole Universe a static small body will be affected by the centrifugal force that equals the force that emerge during the turning of the body itself. For many people it stays unclear the truth of the statement, and the question itself is still acute. In case we correspond to the real world in place of the Galileo or Pseudo-Euclidean metrics the geometry of the four-dimensional time the problem itself will not appear as the transformation that is responsible for the forward and rotatory movement, correspond to absolutely different continuous symmetries.

The analysis of the multiformity characteristics made in the work that claims to become an alternative to the Minkowski space is far from being finished. But the fact that we can give such condition for one of the most simple Finslerian metrics of the fourth degree that has nothing in common with the usual quadratic form, when it can stimulate not only classical but relative conceptions about the physical space, is worth paying attention to.

References

[1] B. Riemann, *About hypotheses in foundation of geometry*, – in the book.: About foundation of geometry.
[2] P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen*, Göttingen, 1918 (Dissertation).
[3] G. S. Asanov, *Finslerian Extension of General Relativity*, Reidel, Dordrecht, 1984.
[4] H. Helmholtz, *About facts in foundation of geometry*, – in the book.: About foundation of geometry.
[5] S. Lie, *Notes on Helmholtz paper "About facts in foundation of geometry"*, – in the book.: About foundation of geometry.
[6] H. Weyl, *Space, Time, Matter,* Dover, 1922.
[7] D. G. Pavlov, *Hypercomplex numbers, associated metric spaces, and extension of relativistic hyperboloid*, aArXiv:gr-qc/0206004.
[8] H. Rund, *The Differential Geometry of Finsler Spaces,* Springer, Berlin, 1959.
[9] R. V. Mikhailov, *On some questions of four dimensional topology: a survey of modern research*, Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004.
[10] D. G. Pavlov, *Nonlinear relativistic invariance for quadrahyperbolic numbers*, arXiv: gr-qc/0212090.
[11] D. G. Pavlov, *Four-dimensional time as alternative to Minkowski space-time*, Proceedings of International Conference "GEON-2003", Kazan, 2003.
[12] D. G. Pavlov, *Chronometry of the three-dimensional time*, Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004.
[13] G. Yu. Bogoslovskiy, *Status and Perspectives of Theory of Local Anisotropic Space-Time*, Physics of Nuclei and Particles. Moscow State University, M. 1997 (in Russian).

Properties of Spaces Connected with Commutative-Associative H_3 and H_4 Algebras

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In the first part of the paper, both a real axis of the space associated with the H_3 algebra and the lines parallel to this axis, are interpreted as the world lines of resting particles. A surface of simultaneity is used (in particular, in H_3) to define the distance between the real axis and a parallel to it, line of universe. It is shown that a coordinate system similar to the polar one can be introduced on this surface, which allows to reveal its simplest invariant transformations. In the second part of the paper, are described Lorentz transformations attached to the structure, which resemble the rotations within the space associated with the H_4 algebra.

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Introduction

The H_3 and H_4 algebras belong to the commutative–associative algebras of the H_n type which are of the simplest structure. These algebras are characterized by some preferred basis. The multiplication of numbers is realized in terms of this basis in a componentwise manner similarly to the addition in arbitrary algebras. On the other side, in H_n type algebras, which can be called hyperbolic, H_3 and H_4 algebras directly follow after the algebras of real (H_1) and double (H_2) numbers, which possessed important properties for their physical applications [6, 11]. We set forth an assumption of "inheriting" these properties by 3- and 4- dimensional algebras under consideration. As a motivation of this assumption we recall the relation between Berwald–Moor's metrics and H_4 algebra in Finsler generalization of the relativity theory [1]. From the point of view of possible applications, hyperbolic H_4 algebra is the most promising one because the $n = 4$ dimensional spaces have the topological preference [7]. However, H_3 algebra possesses one evident advantage. It is possible to use the computer visualization animation for figures, surfaces, and lines in the three dimensional metrical space associated with this algebra. Although it is not worth overestimating the analytical capacities of such applications, it gets a special visuality to geometric properties of this space. Therefore a sufficiently general approach to physical treatment of the hyperbolic space properties, offered in the first part of this paper, is represented for a space accounted with H_3 algebra. Its properties give the cube of norm as

$$|A|^3 = |a^1 a^2 a^3|,$$

where a^i are components of the vector in the preferred basis, combined from three numbers e_i, where $i = 1, 2, 3$, with properties $(e_i)^2 = e_i$, $e_i \cdot e_j = 0$ when $i \neq j$. Real numbers on a line can be shared in two classes: they are positive numbers, placed on the right side from zero, and the negative ones, placed on the left side from zero. Two isotropic lines in the double numbers algebra divide the pseudo-euclidian plane into 2^2 quadrants. Similarly to this the associated space is divided into 2^3 octants, and for all numbers appropriated to one octant points it is typical that the same sign combination of components is taken with respect to the preferred basis. The boundaries of the octants are three isotropic planes with equations $a^i = 0$, where $i = 1, 2, 3$. It will be noted also that since a hyperbolic algebras are algebras with a unity, defined by an expression

$$1 = e_1 + e_2 + \cdots + e_n,$$

two octants of the treated space can be preferably be selected. They are the octants, containing 1 and -1; they are characterized by numbers with all positive or all negative components, respectively.

Using considered algebras requires an availability of euclidian or pseudo-euclidian properties. In the order of algebras: the Dirac algebra [2], quaternions [3], biquaternions [5] – the existence of such properties provides a classical appearance of the norm of the number. However, there is a slight amount of such algebras, but amongst commutative-associative algebras only the double number algebra belongs to such class, in which a square of the norm of the numbers is given by

$$|A|^2 = |(a^1)^2 - (a^2)^2|$$

(see [4]). Chronogeometry method [8], [12] gives an other opportunity to establishing properties which are similar with the properties of euclidian or pseudo-euclidian spaces, in the spaces associated with the considered algebras; the first part of this paper is devoted to application of this method to H_3. Some more opportunity to establishing the sought properties appears on application of symmetric polyform associated with the algebra [9], which, for example, has the following form for H_3 algebra:

$$(A, B, C) = \frac{1}{3!}(a^1 b^2 c^3 + \cdots + a^3 b^2 c^1).$$

The second part of this paper is connected with such opportunity applied to H_4 algebra, where the form having appearance as pseudo-euclidian metric is determined by a polylinear form of four vectors.

1 A simultaneity surface in the commutative-associative algebras (as examplified by H_3)

1.1 *Axiomatics*

We shall treat the following statements, playing the role of axioms, as a principle to interpret physically the properties of the considered algebras class.

1. It is possible to connect an algebra number with some spatial-temporal event.

2. The real axis of the space, which direction is given by means of the unity of the algebras, is treated as a temporal axis, while the norm of the number is interpreted as an observer's time interval whose world line coincides with the vector corresponded to this number.

3. The increase of a relative velocity of particle or signal results in increasing an inclination of tangent line to the particles world line in the given point to the observer world line, and resting material points have world lines which are parallel to the observer line.

4. Light signals, which have a maximal velocity, are connected with isotropic hypersurfaces of the algebra; and it is supposed that the velocity of the light signals does not depend on their propagation direction. According to these statements two selected octants with 1 and -1, which are referred to above, are the analogs of the cone of the future and the past Minkowski space in the space associated with H_3 algebra, respectively. Contrary to the Minkowski space in the considered space a domain outside these cones also possesses isotropic directions, because consists of six side cones. In this paper we restrict our attention to the most common particular case, when the observer world line coincides with the real axis.

1.2 *Exponential form of the H_3 algebra number representation with respect to the basis* $(1, j, k)$

Any number in the selected basis is represented as:

$$A = a^1 \cdot e_1 + a^2 \cdot e_2 + a^3 \cdot e_3.$$

For an exponential function in terms of this basis the following formula takes place:

$$\exp(a^1 \cdot e_1 + a^2 \cdot e_2 + a^3 \cdot e_3) = \exp(a^1) \cdot e_1 + \exp(a^2) \cdot e_2 + \exp(a^3) \cdot e_3. \tag{1}$$

Since in the considered algebra we get $|A|^3 = |a^1 a^2 a^3|$, any number with $a^i > 0$ is represented as

$$A = |A| \cdot \exp(b^1 e_1 + b^2 e_2 + b^3 e_3)$$

with a restriction

$$b_1 + b_2 + b_3 = 0, \tag{2}$$

which implies the identity:

$$|\exp(b^1 e_1 + b^2 e_2 + b^3 e_3)| = 1.$$

The other basis of the algebra is composed from vectors:

$$\begin{cases} 1 = e_1 + e_2 + e_3 \\ j = \sin\varphi_0 \cdot e_1 + \sin(\varphi_0 + 2\pi/3) \cdot e_2 + \sin(\varphi_0 + 4\pi/3) \cdot e_3 \\ k = \cos\varphi_0 \cdot e_1 + \cos(\varphi_0 + 2\pi/3) \cdot e_2 + \cos(\varphi_0 + 4\pi/3) \cdot e_3 \end{cases} \tag{3}$$

The vectors appearing in this basis are mutually orthogonal (in the usual euclidian sense), while an arbitrary parameter φ_0 can be treated in a certain sense as the angle of a simultaneous rotation of a pair of vectors j, k around the real axis. If t, x, y – are coordinates of the number in a new basis, then according to the transformation rules of coordinates of the number we have a system in the other basis:

$$\begin{cases} a^1 = t + \sin\varphi_0 \cdot x + \cos\varphi_0 \cdot y \\ a^2 = t + \sin(\varphi_0 + 2\pi/3) \cdot x + \cos(\varphi_0 + 2\pi/3) \cdot y \\ a^3 = t + \sin(\varphi_0 + 4\pi/3) \cdot x + \cos(\varphi_0 + 4\pi/3) \cdot y \end{cases} \tag{4}$$

from which it follows that $t = (a^1 + a^2 + a^3)/3$. Therefore by (2) the number representable in a exponential form in the basis (1,j,k) is given by

$$A = |A| \cdot e^{\alpha \cdot j + \beta \cdot k}.$$

If we modify this exponential representation, introducing an definition $\rho = \sqrt{\alpha^2 + \beta^2}$, we obtain

$$A = |A| \cdot e^{\rho(\cos\varphi \cdot j + \sin\varphi \cdot k)}. \tag{5}$$

Thus, in agreement with (5), the number at this representation is given by three parameters: the norm of the number $|A|$, the "radial coordinate" ρ, and the "angle coordinate" φ. Making use of (1) and (3), formula (5) takes simple and elegant form in components:

$$\begin{cases} a^1 = |A| \cdot \exp(\rho \sin[\varphi_0 + \varphi]) \\ a^2 = |A| \cdot \exp(\rho \sin[\varphi_0 + 2\pi/3 + \varphi]) \\ a^3 = |A| \cdot \exp(\rho \sin[\varphi_0 + 4\pi/3 + \varphi]) \end{cases}$$

1.3 *Method of setting the distance between the real axis and the parallel line*

For determination of the distance between the world lines of resting particles, one of which lying on the real axis, we use the chronogeometry method. Consider the exchange of signals with the constant velocity $\nu \leq c$; for simplicity we shall arrange point-events of signal transmission and the reception of the reverse signal on the real axis symmetrically with respect to zero time moment. Because of an equality of lengths of straight and reverse signals velocity $|B - A_1| = |A_2 - B|$, so we have:

$$(a^1 + T)(a^2 + T)(a^3 + T) = (T - a^1)(T - a^2)(T - a^3),$$

where $a^i + T > 0$, $T - a^i > 0$, which after expanding takes form:

$$(a^1 + a^2 + a^3) \cdot T^2 + a^1 a^2 a^3 = 0. \tag{6}$$

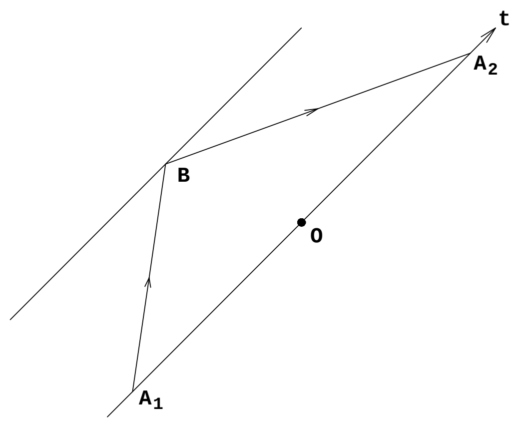

Figure 1: The measuring of a distance between world lines by prelight signals exchange.

The multitude of points-events satisfied to equation (6) form a surface of a simultaneity: it is for the observer on the real axis, being in the point with T coordinate, all these events are taking place in the same zero moment of time. Point $A = (0, 0, 0)$ belongs to the simultaneity surface, and the tangent plane to this surface in the origin has an equation:

$$a^1 + a^2 + a^3 = 0. \tag{7}$$

Substitution of (4) into (6) allows to obtain the equation of the simultaneity surface in form of the dependence of the time of the signal passing (on a clock of resting observer) T from introduced coordinates $\{t, x, y\}$ of point of the simultaneity surface:

$$T^2 = \frac{1}{12}(x^2 + y^2) - \frac{1}{3}\left\{t^2 + \frac{1}{t}\left[\frac{3}{4}xy(y \cdot \sin 3\varphi_0 - x \cdot \cos 3\varphi_0) + \right.\right.$$
$$\left.\left. + x^3 \sin\varphi_0 \sin(\varphi_0 + 2\pi/3)\sin(\varphi_0 + 4\pi/3) + y^3 \cos\varphi_0 \cos(\varphi_0 + 2\pi/3)\cos(\varphi_0 + 4\pi/3)\right]\right\}.$$

According to this equation (and similar equations for other algebras, in particularly, H_4 algebra) the first items on the right side have an euclidian form, and then they dominate on other remaining items, square of travel time of signal depends linearly on square of the euclidian distance in the world lines space, which can be useful for the next physical interpretations.

1.4 *The system of curvilinear coordinates of the simultaneity surface and the transformations mapping it to itself*

Keeping in mind an important of an invariant transformations in modern physics, we shall briefly consider the topic of finding the transformations of the simultaneity surface, mapping it to itself. We introduce two-dimension coordinate system $\{\rho, \varphi\}$ on this surface, somewhat analogous to polar coordinate system on two-dimension plane to get:

$$\begin{cases} a^1 = (T-\rho)\cdot e^{R(\rho,\varphi)\sin(\varphi_0+\varphi)} - T, \\ a^2 = (T-\rho)\cdot e^{R(\rho,\varphi)\sin(\varphi_0+2\pi/3+\varphi)} - T, \\ a^3 = (T-\rho)\cdot e^{R(\rho,\varphi)\sin(\varphi_0+4\pi/3+\varphi)} - T, \end{cases} \tag{8}$$

where the function $R = R(\rho,\varphi)$ taken from transcendent equation is obtaining by using the coordinates (8) into (6):

$$\bar{Z}^3 - \bar{Z}^2\left[e^{-R\sin(\varphi_0+\varphi)} + e^{-R\sin(\varphi_0+2\pi/3+\varphi)} + e^{-R\sin(\varphi_0+4\pi/3+\varphi)}\right]$$
$$+2\bar{Z}\left[e^{R\sin(\varphi_0+\varphi)} + e^{R\sin(\varphi_0+2\pi/3+\varphi)} + e^{R\sin(\varphi_0+4\pi/3+\varphi)}\right] - 4 = 0,$$

where $\bar{Z} = (T-\rho)/T$.

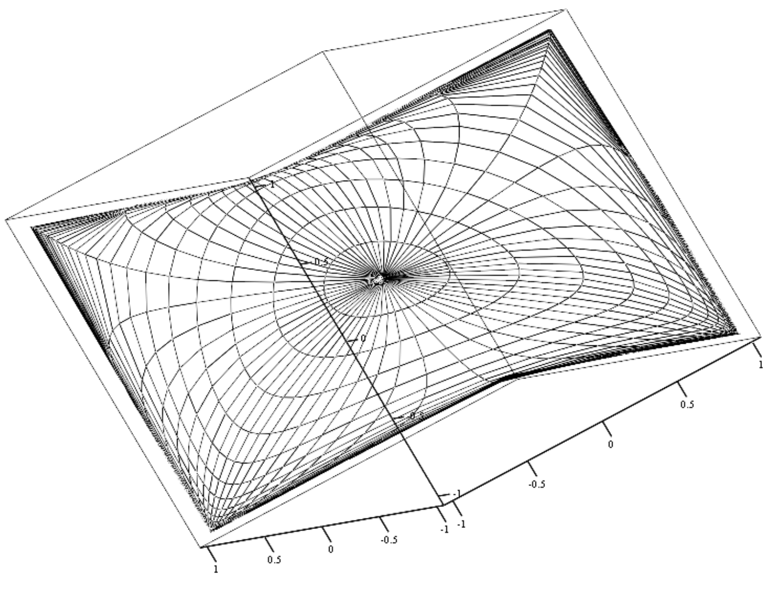

Figure 2: Curvilinear coordinates system ρ, ϕ on simultaneity surface.

In the vicinity of zero at $a^1, a^2, a^3 \ll 1,\ R \ll 1,\ \rho \ll 1$, the equations (8) are got simplified:

$$\begin{cases} a^1 \cong R\cdot T\cdot\sin(\varphi_0+\varphi), \\ a^2 \cong R\cdot T\cdot\sin(\varphi_0+2\pi/3+\varphi), \\ a^3 \cong R\cdot T\cdot\sin(\varphi_0+4\pi/3+\varphi), \end{cases}$$

so that

$$a^1 + a^2 + a^3 \cong 0 \text{ and } (a^1)^2 + (a^2)^2 + (a^3)^2 \cong (R\cdot T)^2. \tag{9}$$

Thus, according to (9), the coordinate system (8) is distinguished: *in the vicinity of zero the parameter R is proportional to euclidian distance from a point, located on the simultaneity surface, to the center of this surface, in which $R = 0$.*

Then independent transformations of the simultaneity surface we seek are "rotations" by angle $\Delta\varphi(\varphi \to \varphi + \Delta\varphi)$ and "a similarity transformations" with a coefficient $K(\rho \to K \cdot \rho)$.

2 The representation a Lorentz transformations by rotations in the space, associated with H_4 algebra.

Following [10], we define the inner product of two arbitrary (with positive values of components) vectors A and B in the space under consideration by a symmetric four-form of H_4 space as:

$$(A, B) := \frac{(A, A, B, B)}{|A| \cdot |B|}.$$

The inner product of two vectors satisfying to properties of positiveness, homogeneity, and normality:

1. $(A, B) > 0$;
2. $(kA, B) = (A, kB) = k(A, B)$;
3. $(A, A) = |A|^2$.

The inner product of units vectors $a = A/|A|$ and $b = B/|B|$ may be regarded as an angle characteristic, setting a relation between two directions defined by these vectors – it is expressed via quotient components of these vectors ($d = b/a$):

$$(a, b) = (d_1 d_2 + d_1 d_3 + \ldots d_3 d_4)/6. \tag{10}$$

Consider a basis in the space associated with H_4 algebra, consisting of these vectors:

$$\begin{cases} 1 = e_1 + e_2 + e_3 + e_4, \\ j' = 3e_1 - e_2 - e_3 - e_4, \\ k' = \sqrt{2}(2e_2 - e_3 - e_4), \\ l' = \sqrt{6}(e_3 - e_4). \end{cases}$$

We denote coordinates of relation of two considered vectors in a new basis via t_d, x_d, y_d, z_d and expressing (10) via these components, we obtain:

$$(a, b) = t_d^2 - x_d^2 - y_d^2 - z_d^2.$$

We shall denote the nonlinear transformation of 4-space, associated with H_4 algebra, which remains all vectors in the direction setting by vector A in rest, and retains the introduced inner product, as a *rotation* of vector B round a vector A. Thus, in addition to the other representations of Lorentz group [13] the representation by rotations round arbitrary time-like axis in the space, associated with H_4 algebra, can be used.

3 Results and conclusions

The method of determination of the distances between the world lines introduced for the space associated with a commutative-associative H_3 algebra (or H_4) allows to distinguish "a euclidian part".

A new geometric interpretation of the Lorentz transformations as rotations in the space connected with algebra H_4 is obtained. Arbitrary setting of a rotation axis is possible; all said above gives a hope on the application of such new interpretation in relativity physics.

References

[1] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theories*, Reidel, Dordrecht 1985.

[2] D. Hestenes, *Space-Time Algebra*, Gordon and Breach, N. Y. 1966.

[3] A. V. Berezin, Yu. A. Kurochkin, E. A. Tolkachev, *Quaternions in Relativity Physics,* Science and Technic, Minsk, 1989 (in Russian).

[4] I. L. Kantor, A. S. Solodovnikov, *Hypercomplex Numbers*. Springer, Berlin, 1989

[5] V. V. Kassandrov, *Algebraic structure of spatial-tempore and algebrodynamics*, Published by Russian UDN, M. 1992 (in Russian).

[6] M. A. Lavrentiev, B. O. Shabat, *The Problems of Hydrodynamics and their Mathematical Models,* Science, M., 1977 (in Russian).

[7] R. V. Mikhailov, *About some questions of 4-dimensional topology: Survey of modern investigations*, Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004.

[8] D. G. Pavlov, *Three dimensional time*, Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004.

[9] D. G. Pavlov, *The generalization of the inner product axioms*, Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004.

[10] D. G. Pavlov, *Non-linear Relativistic Invariance for Quadrahyperbolic Numbers*, arXiv:gr-qc/0212090.

[11] P. K. Rashevsky, *Riemann Geometry and Tensor Analysis,* Science, M., 1967 (in Russian).

[12] J. L. Synge, *Relativity: the General Theory,* North-Holland Publ., Amsterdam, 1960.

[13] F. I. Fedorov, *The Lorenz Group*, Science, M. 1979. (in Russian)

Generalized-Analytical Functions of Polynumber Variable

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We introduce the notion of the generalized-analytical function of the polynumber variable, which is a non-trivial generalization of the notion of analytical function of the complex variable and, therefore, may turn out to be fundamental in theoretical physical constructions. As an example we consider in detail the associative-commutative hypercomplex numbers H_4 and an interesting class of corresponding functions.

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1. Introduction

Let M_n be an n-dimensional elementary manifold and P_n denote the system of n-dimensional associative-commutative hypercomplex numbers (polynumbers, n-numbers), and a one-to-one correspondence between the sets be assigned. Under these conditions, we choose in P_n the basis

$$e_1, e_2, ..., e_n; \quad e_i e_j = p_{ij}^k e_k, \tag{1}$$

$$X = x^1 \cdot e_1 + x^2 \cdot e_2 + \cdots + x^n e_n \in P_n, \tag{2}$$

where $e_1, e_2, ..., e_n$ – symbolic elements, p_{ij}^k stand for characteristic real numbers, and $x^1, x^2, ..., x^n$ – real coordinates with respect to the basis $(e_1 \equiv 1, e_2, \ldots, e_n)$. Obviously, the numbers $x^1, x^2, ..., x^n$ can be used not only as the coordinates in P_n, but also as coordinates in the manifold M_n , so that $(x^1, x^2, ..., x^n) \in M_n$. Though in M_n we can go over to any other curvilinear reference frame, the reference frame $\{x^i\}$, as being built by the help the basis of polynumbers and a fixed one-to-one correspondence $M_n \leftrightarrow P_n$, ought to be considered preferable (as well as any other reference frame connected with this by non-degenerate linear transformation). The polynumber algebraic operations induce the same operations in the elementary manifold (formally) and in the tangent space at any point of manifold (informally). Accordingly, the tangent spaces to M_n are isomorphic to P_n.

The function

$$F(X) := f^1(x^1, \cdots, x^n)e_1 + \cdots + f^n(x^1, \cdots, x^n)e_n \tag{3}$$

of the polynumber variable, where f^i are sufficiently smooth functions of n real variables, will be considered to be a vector (contravariant) field in M_n. Hence, apart from addition and multiplication by number, any operation of multiplication of vector fields

$$f^k_{(3)} = f^i_{(1)} \cdot f^j_{(2)} \cdot p^k_{ij} \tag{4}$$

can also defined in M_n. It is useful but not obligatory to consider the space M_n to be the main ("the examined") object and the space P_n to be a sort of an instrument with the help of which the space M_n is "examined". In the general case the parallel transportation of a vector in the space P_n does not correspond to the "parallel transportation" of the same vector in the space M_n, so that for a due definition of absolute differential (or the covariant derivative) we are to have the connection objects or the quantities which may replace them. If we avoid introducing the pair $\{M_n, P_n\}$, restricting the treatment only to associative-commutative hypercomplex numbers, then it is natural to introduce the definitions

$$dX := dx^i \cdot e_i \tag{5}$$

and

$$dF(X) := F(X + dX) - F(X) = \frac{\partial f^i}{\partial x^k} \cdot e_i \cdot dx^k. \tag{6}$$

The function $F(X)$ of polynumber variable X is called *analytical*, if such a function $F'(X)$ exists that

$$dF(X) = F'(X) \cdot dX, \tag{7}$$

where the multiplication in the right-hand part means the polynumber operation. From (7) it follows that

$$\frac{\partial f^i}{\partial x^k} = p^i_{kj} \cdot f'^j. \tag{8}$$

Since with respect to the basis e_i with the components $e_1 = 1$ the equalities

$$p^i_{1j} = \delta^i_j \tag{9}$$

hold, we have

$$f'^i = \frac{\partial f^i}{\partial x^1}. \tag{10}$$

Inserting (10) in (8) yields the Cauchy-Riemann relations

$$\frac{\partial f^i}{\partial x^1} - p^i_{kj} \cdot \frac{\partial f^j}{\partial x^1} = 0 \tag{11}$$

for the functions under study. The number $n(n-1)$ of these relations is growing quicker that the number n of components of analytical function. This leads to the *functional restriction* of the set of such functions at $n > 2$. The present work is just attempting to elaborate a non-trivial extension of the notion of analytical function of polynumber variable subject to the condition that number of the Cauchy-Riemann-type conditions does nor exceed the number of unknown function-components. The first step in this direction has been made above when introducing the pair $\{M_n, P_n\}$. Therefore it seems natural to replace the differential 6) by means of the absolute differential

$$DF(X) := \bigtriangledown_k f^i \cdot e_i \cdot dx^k, \tag{12}$$

where

$$\bigtriangledown_k f^i := \frac{\partial f^j}{\partial x^k} + \Gamma^i_{kj} \cdot f^j \tag{13}$$

is the covariant derivative, and Γ^i_{kj} means "the connection coefficients". Instead of the formulas (8) and (10) we get

$$\bigtriangledown_k f^i = p^i_{kj} \cdot f'^j \quad \text{and} \tag{14}$$

$$f'^i = \bigtriangledown_1 \cdot f^i, \tag{15}$$

and the Cauchy-Riemann conditions take on the form

$$\bigtriangledown_k f^i - p^i_{kj} \cdot \bigtriangledown_1 f^j = 0. \tag{16}$$

Of course, "the connection objects" Γ^i_{kj} in the formula (13) are not obligatory to be uniform for all the set of functions obeying the conditions (16).

2. Definitions and basic implications

Let us call the function $F(X)$ *generalized-analytical*, if such a function $F'(X)$ exists that

$$\widetilde{D}F(X) = F'(X) \cdot dX, \tag{17}$$

where

$$\widetilde{D}F(X) \equiv \widetilde{\bigtriangledown}_k f^i \cdot e_i \cdot dx^k \tag{18}$$

and the definition

$$\widetilde{\bigtriangledown}_k f^i := \frac{\partial f^i}{\partial x^k} + \gamma^i_k \tag{19}$$

has been used. It is assumed that under the transition from one (curvilinear) coordinate system to another coordinate system the involved objects γ^i_k are transformed according to the law

$$\gamma^{i'}_{k'} = \frac{\partial x^k}{\partial x^{k'}} \cdot \frac{\partial x^{i'}}{\partial x^i} \cdot \gamma^i_k - \frac{\partial x^k}{\partial x^{k'}} \cdot \frac{\partial^2 x^{i'}}{\partial x^k \partial x^i} \cdot f^i. \tag{20}$$

It will be noted that such a definition entails that $\widetilde{\bigtriangledown}_k f^i$ behaves like a tensor. The quantities γ^i_k will be called the *gamma-objects*. In general we do not assume the relations

$$\gamma^i_k = \Gamma^i_{kj} \cdot f^j \tag{21}$$

with a single "connection object" Γ^i_{kj} for generalized-analytical functions. It would be more precise to say of the pair $\{f^i, \gamma^i_k\}$, such that the analytical function of polynumber variable is the pair $\{f^i, 0\}$, but this pair transform to the pair $\{f^i, \gamma^{i'}_{k'} \neq 0\}$ under going over from the special coordinate system to another curvilinear one.

From the definition of generalized-analytical functions it follows that

$$\widetilde{\nabla}_k f^i = p^i_{kj} \cdot f'^j \tag{22}$$

and

$$f'^j = \widetilde{\nabla}_1 f^i; \tag{23}$$

the respective generalized Cauchy-Riemann relations take on the form

$$\widetilde{\nabla}_k f^j - p^i_{kj} \widetilde{\nabla}_1 f^j = 0. \tag{24}$$

The number of unknown functions in the pair $\{f^i, \gamma^i_k\}$ equals $n+n^2 = n(n+1)$, – which is more than number $n(n-1)$ of the generalized Cauchy-Riemann relations (24). Thus, to use the notion of generalized-analytical function in theoretical-physical constructions it is necessary to additionally establish and formulate the set of requirements (possibly one requirement) which, when used in conjunction with the notion of generalized-analytical function, would lead unambiguously to equations of some field of physical meaning. Usually, they are n partial differential equations of second order for n independent function-component field.

If $\{f^i_{(1)}, \gamma^i_{(1)k}\}$ and $\{f^i_{(2)}, \gamma^i_{(2)k}\}$ – two generalized-analytical functions, then their arbitrary linear sum with real coefficients α, β is a generalized-analytical function. This ensues directly from the definition, on using also the formulae (22)–(24) and (20). Thus, we have

$$\alpha \cdot \{f^i_{(1)}, \gamma^i_{(1)k}\} + \beta \cdot \{f^i_{(2)}, \gamma^i_{(2)k}\} = \{\alpha \cdot f^i_{(1)} + \beta \cdot f^i_{(2)}, \alpha \cdot \gamma^i_{(1)k} + \beta \cdot \gamma^i_{(2)k}\}. \tag{25}$$

Now, let us consider the polynumber product of two generalized-analytical functions $f^i_{(1)}$ and $f^j_{(2)}$:

$$f^k_{(3)} = f^i_{(1)} \cdot f^j_{(2)} \cdot p^k_{ij} \tag{26}$$

and try to find the object $\gamma^i_{(3)k}$ such that the pair $\{f^i_{(3)}, \gamma^i_{(3)k}\}$ be generalized-analytical function. To this end we formally differentiate the left and right parts of (26) with respect to x^k and use the formula (22), obtaining

$$\frac{\partial f^i_{(3)}}{\partial x^k} + \gamma^i_{(3)k} = p^{i_1}_{kj} p^i_{i_1 i_2} f'^j_{(1)} f'^{j_2}_{(2)} + p^{i_2}_{kj} p^i_{i_1 i_2} f^{i_1}_{(1)} f'^j_{(2)}. \tag{27}$$

Owing to the formula

$$p^r_{im} \cdot p^m_{kj} = p^r_{km} \cdot p^m_{ij} \tag{28}$$

(which is an implication of the properties of associativity and commutativity of polynumbers), we can write

$$\frac{\partial f^i_{(3)}}{\partial x^k} + \gamma^i_{(3)k} = p^i_{kj} p^j_{i_1 i_2} (f'^{i_1}_{(1)} f^{i_2}_{(2)} + f^{i_1}_{(1)} f'^{i_2}_{(2)}), \tag{29}$$

where

$$\gamma^i_{(3)k} = p^i_{i_1 i_2} \cdot (\gamma^{i_1}_{(1)k} f^{i_2}_{(2)} + f^{i_1}_{(1)} \gamma^{i_2}_{(2)k}). \tag{30}$$

The result (29) can conveniently be represented in terms of the absolute differential as follows:

$$D[F_{(1)}(X) \cdot F_{(2)}(X)] = [DF_{(1)}(X)] \cdot F_{(2)}(X) + F_{(1)}(X) \cdot [DF_{(2)}(X)] \quad \text{or} \tag{31}$$

$$D[F_{(1)}(X) \cdot F_{(2)}(X)] = [F'_{(1)}(X) \cdot F_{(2)}(X) + F_{(1)}(X) \cdot F'_{(2)}(X)] \cdot dX. \tag{32}$$

From the last formula we obtain the relation

$$[F_{(1)}(X) \cdot F_{(2)}(X)]' = F'_{(1)}(X) \cdot F_{(2)}(X) + F_{(1)}(X) \cdot F'_{(2)}(X). \tag{33}$$

It remains to clarify whether the transformation law of the objects $\gamma^i_{(3)k}$ under the transitions to arbitrary coordinate system is correct. With this aim the formula (30) should be written in a varied form:

$$\gamma^i_{(3)k} = p^i_{i_1 i_2} \cdot (\gamma^{i_1}_{(1)k} f^{i_2}_{(2)} + f^{i_1}_{(1)} \gamma^{i_2}_{(2)k}) + (\Gamma^i_{km} p^m_{i_1 i_2} - \Gamma^m_{ki_1} p^i_{mi_2} - \Gamma^m_{ki_2} p^i_{i_1 m}) \cdot f^{i_1}_{(1)} f^{i_2}_{(2)}, \tag{34}$$

where $\Gamma^j_{im} \equiv 0$ with the respect to our special coordinate system; however, under the transition to an arbitrary coordinate system the objects Γ^j_{ik} transform like ordinary connection objects and in general $\Gamma^{j'}_{i'k'} \neq 0$. The condition $\Gamma^j_{ik} \equiv 0$ can also be replaced to apply the more general condition

$$\Gamma^i_{km} p^m_{i_1 i_2} - \Gamma^m_{ki_1} p^i_{mi_2} - \Gamma^m_{ki_2} p^i_{i_1 m} \equiv 0 \tag{35}$$

and, moreover, the three coefficients Γ in (35) can be regarded as different. It is possible to restrict ourselves to but the class of generalized-analytical function obeying the property

$$({}^{(1)}\Gamma^i_{km} p^m_{i_1 i_2} - {}^{(2)}\Gamma^m_{ki_1} p^i_{mi_2} - {}^{(3)}\Gamma^m_{ki_2} p^i_{i_1 m}) \cdot f^{i_1}_{(1)} f^{i_2}_{(2)} \equiv 0. \tag{36}$$

Given the special coordinate system. If one has $\Gamma^i_{jk} \equiv {}^{(1)}\Gamma^i_{jk} \equiv {}^{(2)}\Gamma^i_{jk} \equiv {}^{(3)}\Gamma^i_{jk} \equiv 0$, then the tensor p^k_{ij} is transported "parallel" without any changes in components.

Thus, the poly-product of two generalized-analytical functions of polynumber variable is again a generalized-analytical function, and the formula (33) takes place for derivatives if one adopts that the "connection coefficients" associated to the tensor p^k_{ij} with respect to the special coordinate system vanishes identically over all three indices. In terms of the pairs $\{f^i, \gamma^i_k\}$ the poly-product of two generalized-analytical function can be written as follows:

$$\{f^{i_1}_{(1)}, \gamma^{i_1}_{(1)}\} \cdot \{f^{i_2}_{(2)}, \gamma^{i_2}_{(2)}\} = \{p^i_{i_1 i_2} f^{i_1}_{(1)} f^{i_2}_{(2)}, p^i_{i_1 i_2} \cdot (\gamma^{i_1}_{(1)k} f^{i_2}_{(2)} + f^{i_1}_{(1)}) \gamma^{i_2}_{(2)k})\}. \tag{37}$$

So, the polynomial or the converged series with real or polynumber coefficients of one or several generalized-analytical functions is a generalized-analytical function. The ordinary differentiation rules are operative for the respective derivative (which was denoted my means of the prime ($'$)) of such polynomials and series, whenever the tensor p^k_{ij} with respect to the special coordinate system vanishes identically over all three indices.

Since in such a theory of generalized-analytical functions of polynumber variable (in which the "connection objects" as well as the gamma-objects are different for each tensor and, generally speaking, for each index), the concept of "parallel transportation" is deprived of the geometrical simplicity that is characteristic of the spaces of affine connection, the Riemannian and pseudo-Riemannian spaces included. This notwithstanding, the concepts of absolute differential and covariant derivative can readily be extended on the basis of invariance of their form with respect to any curvilinear coordinate system. The covariant derivative $\widetilde{\bigtriangledown}_k$ for arbitrary tensor is defined quite similarly to the way which is followed to define the covariant derivative $\bigtriangledown_k$ in the spaces of affine connection; at the same time, for each tensor and probably for each index there exist, in general, their own "connection objects" or gamma-objects. The respective differential is constructed in accordance with the definition

$$\widetilde{D} := dx^k \cdot \widetilde{\bigtriangledown}_k. \tag{38}$$

Here, the converted indices can not be ignored, for "connection coefficients" correspond to them.

The Cauchy-Riemann relations (24) are necessary and sufficient conditions in order that f^i be a generalized-analytical function. Let us show that these relations can be written in an explicitly invariant form if the matrix composed of the numbers

$$q_{ij} = p^r_{im} p^m_{rj}, \tag{39}$$

is non-singular, that is if

$$q = \det(q_{ij}) \neq 0. \tag{40}$$

In this case the inverse matrix (q_{ij}) forms the tensor (q^{ij}) showing the properties

$$q_{jk} q^{ki} = q^{ik} q_{kj} = \delta^i_j. \tag{41}$$

Whence, when the formula (22) is applied instead of the formulae (23) and (24), we get the invariant expression for the derivative

$$f'^i = q^{is} p^r_{sm} \widetilde{\bigtriangledown}_r \cdot f^m \tag{42}$$

and for the Cauchy-Riemann relations

$$\widetilde{\bigtriangledown}_k f^i - p^i_{kj} \cdot q^{js} p^r_{sm} \widetilde{\bigtriangledown}_r f^m = 0. \tag{43}$$

Let us turn to the generalized-analytical functions $F_{(1)}(X)$ and $F_{(2)}(X)$, which are constrained by the relation

$$F_{(2)}(X) = F(X) \cdot F_{(1)}(X), \tag{44}$$

where $F(X)$ – some function of polynumber variable. The function is generalized-analytical in the field where the function $F_{(1)}(X)$ is not a divisor of zero. In this case

$$F(X) = \frac{F_{(2)}(X)}{F_{(1)}(X)}, \tag{45}$$

$$\widetilde{D}F(X) = \frac{F_{(1)}(X)\widetilde{D}[F_{(2)}(X)] - \widetilde{D}[F_{(1)}(X)]F_{(2)}(X)}{F^2_{(1)}(X)} \tag{46}$$

or

$$F'(X) = \frac{F_{(1)}(X)F'_{(2)}(X) - F'_{(1)}(X)F_{(2)}(X)}{F^2_{(1)}(X)}. \tag{47}$$

If

$$F(X) = F_{(2)}[F_{(1)}(X)], \tag{48}$$

then the function $F(X)$ is generalized-analytical with

$$F'(X) = F'_{(2)}(F_{(1)}) \cdot F'_{(1)}(X). \tag{49}$$

3. Similar geometries and conformal transformations

Actually, we are interested in not only the pair $\{M_n, P_n\}$ and generalized-analytical functions $\{f^i, \gamma^i_k\}$ but (eventually) possible ways of application of these notions to constructing physical models and solving new physical problems. Two spaces in which congruences of extremals (geodesics) coincide are similar in many respects. The extremals are meant to be solutions to set of equations for definition of curves over which the length of the curve acquires its extremum; alternatively, they are meant to be the curves which in a given geometry are defined to be geodesics (for example, geodesics in geometries of affine connection). However, for some physical as well as mathematical problems it is not of great importance which length element is used in applied space, – a real use is made to only the set of equations that define extremals (or to extremals proper). We shall say that two n-dimensional geometries are *similar*, if there exist such coordinate systems and parameters along curves that with respect to them the equations for extremals are equivalent and the initial and/or final date set forth in one space may also be given in another space.

All the set of generalized-analytic functions can be broken into the subsets $\{f^i, \Gamma^k_{ij}\}$ that involve the same connection coefficients Γ^k_{ij}, so that for all generalized-analytic functions from the subset the relation

$$\Gamma^i_{kj} f^j = \gamma^i_k \tag{50}$$

is fulfilled. It should be stressed (once more) that the coefficients Γ^k_{ij} are independent of any choice of functions in the subset $\{f^i, \Gamma^k_{ij}\}$. Generally speaking, the subset may be formed by only one generalized-analytic function. If f^i and γ^i_k are prescribed, then the relations (50) can be treated to be a set of equations for definition of the coefficients Γ^k_{ij}. Having find and fixed them, they can be applied for all tensors and indices, thereafter we get a due possibility to work with the space of affine connection $L_n(\Gamma^k_{ij})$ in which the set of equations for geodesics is of the form

$$\frac{d^2x^i}{d\sigma^2} = -\Gamma^i_{kj}\frac{dx^k}{d\sigma}\frac{dx^j}{d\sigma}. \tag{51}$$

Generally speaking, in this way we loose the possibility to use the polynumber product for construction of new generalized-analytical functions and should give up the simple differentiation rules (33). In the last case the covariant derivative $\widetilde{\nabla}_k$ in the special coordinate system must act on the tensor p^i_{kj}. In order to have simultaneously on the subset $\{f^i, \Gamma^k_{ij}\}$ the polynumber product of generalized-analytical functions and the rules (33), which application yields again a generalized-analytical function, we are to restrict ourselves to the functions subjected to the condition (36) with $\Gamma^i_{jk} \equiv {}^{(1)}\Gamma^i_{jk} \equiv {}^{(2)}\Gamma^i_{jk} \equiv {}^{(3)}\Gamma^i_{jk}$.

Let us require that the space $L_n(\Gamma^i_{jk})$ be similar to a Riemannian or pseudo-Riemannian one $V_n(g_{ij})$, where g_{ij} is a (fundamental) metric tensor. Then instead of (50) we get the system of equations

$$\left[\frac{1}{2}g^{im}\left(\frac{\partial g_{km}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^m}\right) + \frac{1}{2}(p_k\delta^i_j + p_j\delta^i_k) + S^i_{kj}\right]\cdot f^j = \gamma^i_k, \tag{52}$$

where S^i_{kj} stands for an arbitrary tensor (torsion tensor) obeying the property of skew-symmetry with respect to two indices, and p_i may be arbitrary one-covariant tensor [1]. This system may be used to define the fundamental tensor g_{ij}.

There exist such Finslerian spaces which are not of Riemannian or pseudo-Riemannian type, but in which, however, one has the system of equations

$$\frac{d^2x^i}{d\sigma^2} = -\Gamma^i_{kj}[L(dx;x)]\cdot\frac{dx^k}{d\sigma}\frac{dx^j}{d\sigma}, \tag{53}$$

where the coefficients $\Gamma^i_{kj}[L(dx;x)]$ are defined by means of a relevant metric function $L(dx^1,\dots,dx^n;x^1\dots,x^n)$ of Finsler type. The corresponding Finsler spaces are similar to spaces of affine connection endowed with the connection coefficients Γ^i_{kj} deviated possibly from the coefficients $\Gamma^i_{kj}[L(dx;x)]$ by occurrence of an additive torsion tensor and/or an additive tensor $\frac{1}{2}(p_k\delta^i_j + p_j\delta^i_k)$ [1].

Let a generalized-analytical functions define spaces of the affine connection $L_n({}^{(1)}\Gamma^k_{ij})$ and $L_n({}^{(2)}\Gamma^k_{ij})$ similar to corresponding Riemannian or pseudo-Riemannian spaces $V_n(g_{ij})$ and $V_n(K_V^2 g_{ij})$ and/or the Finslerian spaces $F_n[L(dx;x)]$ and $F_n[K_F L(dx,x)]$, where $K_V(x^1,..,.x^n) > 0$, $K_F(x^1,...,x^n) > 0$ – scalar functions (invariants). Then the transformation (coordinate and/or in the space of generalized-analytical functions) going over the set $f^i_{(1)}$ in the set $f^i_{(2)}$, can be called conformal, for under this procedure one has

$$g_{ij}(x) \to K_V^2(x) \cdot g_{ij} \qquad \text{and} \tag{54}$$

$$(dx; x) \to K_F(x) \cdot L(dx; x). \tag{55}$$

4. Possible additional requirements

From the definition of a generalized-analytical function it follows that it is possible to present the function by choosing two arbitrary one-covariant fields $f^i(x^1, \dots, x^n)$ and $f'^i(x^1, \dots, x^n)$. Then the formula (23) entails the following representation for the gamma-objects:

$$\gamma^i_k = -\frac{\partial f^i}{\partial x^k} + p^i_{kj} f'^j \tag{56}$$

The Cauchy-Riemann conditions are fulfilled automatically. So, to get the field equations for the unknown function-components $f^i(x^1, \dots, x^n)$ and $f'^i(x^1, \dots, x^n)$, it is necessary to set forth at least $2n$ additional relations, for example, some partial differential equations of the first-order with respect to $f^i(x^1, \dots, x^n)$ and $f'^i(x^1, \dots, x^n)$.

(1): Let us consider the subset of generalized-analytical functions f^i such that

$$\tilde{D}F(x) \equiv 0, \quad \leftrightarrow \quad \tilde{\nabla}_k f^i \equiv 0, \quad \leftrightarrow \quad f'^i \equiv 0 \tag{57}$$

In this case the Cauchy-Riemann conditions are fulfilled automatically and arbitrary vector-function coupled with $\gamma^i_k = -\frac{\partial f^i}{\partial x^k}$, that is the pair $\{f^i, -\frac{\partial f^i}{\partial x^k}\}$, is a generalized-analytical function. It is important to note that the properties of polynumbers do not influence this procedure. In other words, this subset (treated on the level of the Cauchy-Riemann conditions) are independent of any choice of the system of polynumbers.

(2): If instead of the conditions (57) we assume the relations

$$\tilde{D}F(X) = \lambda \cdot F(X) \cdot dX, \quad \leftrightarrow \quad \tilde{\nabla}_k f^i = \lambda \cdot p^i_{kj} \cdot f^j, \quad \leftrightarrow \quad f'^i = \lambda \cdot f^i, \tag{58}$$

where λ is a real number, then the pairs $\{f^i, -\dfrac{\partial f^i}{\partial x^k} + \lambda p^i_{kj} f^j\}$ with arbitrary vector-functions f^i will form the subset of the generalized-analytical functions which to some extent account for properties of polynumbers.

(3): Farther generalizing the requirements (57) and (58) can be formulated in the form

$$F'(X) = \Lambda \cdot F(X), \tag{59}$$

where

$$\Lambda = \lambda^1 e_1 + \lambda^2 e_2 + \dots + \lambda^n e_n \tag{60}$$

an arbitrary polynumber. In this case the pair

$$\left\{ f^i, -\frac{\partial f^i}{\partial x^k} + p^i_{kj} p^j_{mr} \lambda^m f^j \right\} \tag{61}$$

will be the generalized-analytical functions.

(4): Using the formulas (23) and (24), we can prove the following statement. If the relations

$$1)\ \Gamma^i_{kj} f^j = \gamma^i_k, \tag{62}$$

$$2)\ \Gamma^i_{1j} p^j_{kr} - p^i_{kj} \Gamma^j_{1r} = 0, \tag{63}$$

$$3)\ \frac{\partial \Gamma^i_{1r}}{\partial x^k} - \frac{\partial \Gamma^i_{kr}}{\partial x^1} + \left[(\Gamma^i_{kj} - p^i_{km} \Gamma^m_{1j}) \Gamma^j_{1r} - \Gamma^i_{1j} (\Gamma^j_{kr} - p^j_{km} \Gamma^m_{1r}) \right] = 0 \tag{64}$$

hold, then together with the generalized-analytical pair $\{f^i, \gamma^i_k\}$, the pair

$$\{f'^i, \Gamma^i_{kj} f'^j\}, \{f''^i, \Gamma^i_{kj} f''^j\}, \ldots, \{f^{(m)i}, \Gamma^i_{kj} f^{(m)j}\}, \ldots \tag{65}$$

are also generalized-analytical. In the last formulas the notation

$$f^{(m)i} \equiv \frac{\partial f^{(m-1)j}}{\partial x^1} + \Gamma^i_{1j} f^{(m-1)j} \tag{66}$$

has been used.

(5): One additional requirements can sound: for the subset $\{f^i, \Gamma^i_{kj}\}$ of generalized-analytical functions a Riemannian or pseudo-Riemannian geometry $V_n(g_{ij})$ similar to the affine connection geometry $L_n(\Gamma^i_{jk})$ can be found.

(6): If a Finsler space $F_n[L(dx; x)]$ is similar to a space of affine connection, then one among possible requirements can claim that the subset $\{f^i, \Gamma^i_{jk}\}$ give rise to an affine connection geometry similar to the Finsler geometry $F_n[L(dx; x)]$.

(7): Let

$$x^i = x^i(\tau) \tag{67}$$

be a parametric presentation of some curve joining two points $x^i_{(0)} = x^i(0)$, $x^i_{(0)} = x^i(0)$, that is, the parameter along curves varies in the limits $\tau \in [0; 1]$. Let us consider the functional with integration along indicated curve

$$I[x^i(\tau)] = \int_0^1 F(X)\, dX = \left[\int_0^1 p^i_{kj} f^k(x^1(\tau), \ldots, x^n(\tau)) dx^j \right] \cdot e_i = \left[\int_0^1 p^i_{kj} f^k \frac{dx^j}{d\tau} \right] \cdot e_i, \tag{68}$$

where $F(X)$ – some generalized-analytical function, and require that value of the integral (68) be independent of integration way, in which case the variation of this functional at fixed ends of curves should vanish, that is the Euler conditions

$$\frac{d}{d\tau} \left(p^i_{kj} f^j \right) - p^i_{mj} \frac{\partial f^j}{\partial x^k} \frac{dx^m}{d\tau} = 0 \tag{69}$$

or

$$\left(p^i_{kj} \frac{\partial f^j}{\partial x^m} - p^i_{mj} \frac{\partial f^j}{\partial x^k} \right) \cdot \frac{dx^m}{d\tau} = 0 \tag{70}$$

must be valid. Assuming that $x^i(\tau)$ are arbitrary smooth functions, from these equations we get

$$p^i_{kj} \frac{\partial f^j}{\partial x^m} - p^i_{mj} \frac{\partial f^j}{\partial x^k} = 0, \tag{71}$$

or, recollecting that $\{f^i, \gamma^i_k\}$ is a generalized-analytic pair,

$$p^i_{kj} \gamma^i_m - p^i_{mj} \gamma^j_k = 0. \tag{72}$$

From these relations it ensues that for the functions f^i the Cauchy-Riemann conditions (11) hold fine.

Thus, the assumption of independence of the integral (68) of the path leads to the conclusion that the function $F(X)$ is analytical, that is such an assumption is superfluous for non-trivial generalization of the concept of analyticity.

5. Case H_4

It is convenient to work with the associative-commutative hypercomplex numbers in term of the ψ-basis which relates to the basis

$$e_1 = 1, e_2 = j, e_3 = k, e_4 = jk, \qquad j^2 = k^2 = (jk)^2 = 1 \tag{73}$$

by means of the linear dependence

$$e_i = s^j_i \cdot \psi_j, \tag{74}$$

where

$$s^j_i = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad s^k_i \cdot s^j_k = 4 \cdot \delta^j_i. \tag{75}$$

For the basis elements $\psi_1, \psi_2, \psi_3, \psi_4$ the multiplication law

$$\psi_i \cdot \psi_j = p^{(\psi)k}_{ij} \cdot \psi_k \tag{76}$$

involves the characteristic numbers

$$p^{(\psi)k}_{ij} = \begin{cases} 1, & \text{if } i = j = k, \\ 0, & \text{in other cases} \end{cases} \tag{77}$$

We shall use the following notation:

$$X = x^1 e_1 + ... + x^4 e_4 = \xi^1 \psi_1 + ... + \xi^4 \psi_4 \tag{78}$$

and

$$F(X) = \varphi^1(\xi^1, ..., \xi^4) \cdot \psi_1 + \varphi^4(\xi^1, ..., \xi^4) \cdot \psi_4. \tag{79}$$

Thus, if $\varphi^i(\xi^1, ..., \xi^4)$ – a generalized-analytical function of the H_4-variable used, then such a vector-function $\varphi'^i(\xi^1, ..., \xi^4)$ can be found that

$$\frac{\partial \varphi^i}{\partial \xi^k} + \gamma_k^{(\psi)i} = p_{kj}^{(\psi)i} \cdot \varphi'^j. \tag{80}$$

Taking into account (77), we get

$$\left.\begin{array}{llll}
\dfrac{\partial \varphi^1}{\partial \xi^1} + \gamma_1^{(\psi)1} = \varphi'^1, & \dfrac{\partial \varphi^1}{\partial \xi^2} + \gamma_2^{(\psi)1} = 0, & \dfrac{\partial \varphi^1}{\partial \xi^3} + \gamma_3^{(\psi)1} = 0, & \dfrac{\partial \varphi^1}{\partial \xi^4} + \gamma_4^{(\psi)1} = 0, \\
\dfrac{\partial \varphi^2}{\partial \xi^1} + \gamma_1^{(\psi)2} = 0, & \dfrac{\partial \varphi^2}{\partial \xi^2} + \gamma_2^{(\psi)2} = \varphi'^2, & \dfrac{\partial \varphi^2}{\partial \xi^3} + \gamma_3^{(\psi)2} = 0, & \dfrac{\partial \varphi^2}{\partial \xi^4} + \gamma_4^{(\psi)2} = 0, \\
\dfrac{\partial \varphi^3}{\partial \xi^1} + \gamma_1^{(\psi)3} = 0, & \dfrac{\partial \varphi^3}{\partial \xi^2} + \gamma_2^{(\psi)3} = 0, & \dfrac{\partial \varphi^3}{\partial \xi^3} + \gamma_3^{(\psi)3} = \varphi'^3, & \dfrac{\partial \varphi^3}{\partial \xi^4} + \gamma_4^{(\psi)3} = 0, \\
\dfrac{\partial \varphi^4}{\partial \xi^1} + \gamma_1^{(\psi)4} = 0, & \dfrac{\partial \varphi^4}{\partial \xi^2} + \gamma_2^{(\psi)4} = 0, & \dfrac{\partial \varphi^4}{\partial \xi^3} + \gamma_3^{(\psi)4} = 0, & \dfrac{\partial \varphi^4}{\partial \xi^4} + \gamma_4^{(\psi)4} = \varphi'^4.
\end{array}\right\} \tag{81}$$

These relations involve the expression for the derivative

$$\varphi'^i = \frac{\partial \varphi^i}{\partial \xi^{i_-}} + \gamma_{i_-}^{(\psi)i} \tag{82}$$

($i = i_-$, for which no summation is assumed), and also the Cauchy-Riemann relations

$$\frac{\partial \varphi^i}{\partial \xi^k} + \gamma_k^{(\psi)i} = 0, \ i \neq k. \tag{83}$$

The space H_4 is the metric (Finslerian) space in which the length element ds is expressible through the form $d\xi^1 d\xi^2 d\xi^3 d\xi^4$ in a conic region defined possibly in various ways. Let us stipulate that

$$ds = \sqrt[4]{d\xi^1 d\xi^2 d\xi^3 d\xi^4}, \tag{84}$$

assuming that the region is prescribed by the inequalities

$$d\xi^1 \geq 0, \ d\xi^2 \geq 0, \ d\xi^3 \geq 0, \ d\xi^4 \geq 0 \ . \tag{85}$$

Let us consider the four-dimensional Finslerian geometry with the length element of the form

$$ds = \sqrt[4]{\kappa^4 \cdot d\xi^1 d\xi^2 d\xi^3 d\xi^4}, \tag{86}$$

where $\kappa \equiv \kappa(d\xi^1 d\xi^2 d\xi^3 d\xi^4) > 0$. Such a geometry is not Riemannian or pseudo-Riemannian. Let us show that such a geometry is similar (according to terminology adopted above) to some affine geometry with a connection $L_4(\Gamma_{kj}^i)$. Let us write equations for extremals of this Finslerian space by using the tangential equation of indicatrix [2]:

$$\Phi(p_1,...,p_4;\xi^1,...,\xi^4)=0\ , \tag{87}$$

where

$$\Phi(p;\xi)=p_1p_2p_3p_4-\left(\frac{\kappa}{4}\right)^4\ , \tag{88}$$

and

$$p_i=\frac{\partial(ds)}{\partial(d\xi^i)}=\frac{1}{4}\cdot\frac{\sqrt[4]{\kappa^4\cdot d\xi_1 d\xi_2 d\xi_3 d\xi_4}}{d\xi^i}. \tag{89}$$

Then the set of equations for definition of extremals reads

$$\left.\begin{array}{c}\dfrac{d\xi^1}{\dfrac{\partial\Phi}{\partial p_1}}=...=\dfrac{d\xi^4}{\dfrac{\partial\Phi}{\partial p_4}}=\dfrac{dp_1}{-\dfrac{\partial\Phi}{\partial\xi^1}}=...=\dfrac{dp_4}{-\dfrac{\partial\Phi}{\partial\xi^4}},\\ \Phi(p,\xi)=0;\end{array}\right\} \tag{90}$$

or

$$d\xi^i=\frac{\partial\Phi}{\partial p_i}\cdot\lambda\cdot d\tau,\ dp_i=-\frac{\partial\Phi}{\partial\xi^i}\cdot\lambda\cdot d\tau,\ \ \Phi(p;\xi)=0, \tag{91}$$

where τ – a parameter along extremals, and $\lambda\equiv\lambda(p;\xi)\neq 0$ – a function. For the tangential equation of the indicatrix (87), (88) the set of equations (91) takes on the form

$$\dot{\xi^i}=\frac{p_1p_2p_3p_4}{p_i}\cdot\lambda,\quad \dot{p^i}=(\frac{1}{4})^4\frac{\partial k^4}{\xi^i}\cdot\lambda,\quad p_1p_2p_3p_4=\left(\frac{k}{4}\right)^4, \tag{92}$$

with

$$\dot{\xi^i}=\frac{d\xi^i}{d\tau},\ \ \dot{p_i}=\frac{dp_i}{d\tau}. \tag{93}$$

Let us consider $\lambda=\lambda(\xi)>0$ to be a function of only coordinates. Then, by explicating p_i, we get the set of equations for definition of extremals in the Finslerian space (86) in the form

$$\ddot{\xi^i}=-\Gamma^i_{kj}\ \dot{\xi^k}\dot{\xi^j}, \tag{94}$$

where

$$\Gamma^i_{kj}=-\begin{cases}\dfrac{\partial ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial\xi^j}, & \text{if }\ i=j=k,\\ \delta^i_k\dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^j}, & \text{in other cases;}\end{cases} \tag{95}$$

$$\sigma = \left(\frac{\kappa}{4}\right)^4 \cdot \lambda, \tag{96}$$

λ_0 and σ_0 are constants of relevant dimensions. Let us write down explicitly the coefficients Γ^i_{kj}:

$$(\Gamma^1_{kj}) = -\begin{pmatrix} \dfrac{\partial ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial\xi^1} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^2} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^3} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{97}$$

$$(\Gamma^2_{kj}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^1} & \dfrac{\partial ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial\xi^2} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^3} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{98}$$

$$(\Gamma^3_{kj}) = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^1} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^2} & \dfrac{\partial ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial\xi^3} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^4} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{99}$$

$$(\Gamma^4_{kj}) = -\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^1} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^2} & \dfrac{\partial ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial\xi^3} & \dfrac{\partial ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial\xi^4} \end{pmatrix}. \tag{100}$$

It will be noted that instead of the matrices (97)−(100) one can take their transforms. Thus, the Finslerian geometry with the length element (86) is similar to the geometry of the affine connection $L_4[\Gamma^i_{kj} + S^i_{kj} + \frac{1}{2}(p_k\delta^i_j + p_j\delta^i_k)]$, where S^i_{kj} – a tensor which is assumed to be skew-symmetric with respect to the subscripts, and p_k stands for an arbitrary one-covariant tensor.

Let us consider the generalized-analytical functions φ^i of H_4-variable that obey the additional condition 3), that is the pair

$$\left\{\varphi^i, -\frac{\partial\varphi^i}{\partial\xi^k} + p^{(\psi)i}_{kj}\mu^j\varphi^j\right\}, \tag{101}$$

where

$$\Lambda = \lambda^i \cdot e_i = \mu^j \cdot \psi_j. \tag{102}$$

Let us select from such pairs a subset $\{\varphi^i, \Gamma^i_{kj}\}$, where Γ^i_{kj} are defined by the matrices transposed to the matrices (97) – (100). In this way, the requirement 6) is retained. Then the pair (101) should fulfill the 16 relations (50) the first four of which are

$$\frac{\partial \varphi^1}{\partial \xi^1} = \mu^1 \varphi^1 + \frac{\partial\, ln\left(\dfrac{\lambda}{\lambda_0}\right)}{\partial \xi^1}\varphi^1, \qquad \frac{\partial \varphi^1}{\partial \xi^2} = \frac{\partial\, ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial \xi^2}\varphi^1, \tag{103}$$

$$\frac{\partial \varphi^1}{\partial \xi^3} = \frac{\partial\, ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial \xi^3}\varphi^1, \qquad \frac{\partial \varphi^1}{\partial \xi^4} = \frac{\partial\, ln\left(\dfrac{\sigma}{\sigma_0}\right)}{\partial \xi^4}\varphi^1.$$

For the compatibility it is necessary and sufficient that the mixed derivatives obtained with the help of the formulae (103) be equal. A part of these equations, except for three ones, is automatically satisfied. If we consider all the 16 equations, not confining ourselves to the first four equations, we get the following 12 conditions:

$$\frac{\partial^2 ln\left(\dfrac{\kappa}{\kappa_0}\right)^4}{\partial \xi^i \xi^j} = 0,\ i \neq j; \tag{104}$$

from which it ensues that

$$ln\left(\frac{\kappa}{\kappa_0}\right)^4 = a_1(\xi^1) + a_2(\xi^2) + a_3(\xi^3) + a_4(\xi^4) \tag{105}$$

or

$$\kappa = \kappa_0 \cdot \exp\{[a_1(\xi^1) + a_2(\xi^2) + a_3(\xi^3) + a_4(\xi^4)]/4\}, \tag{106}$$

where a_i are four arbitrary functions of one real argument. Then from equations (103) and relevant equations for other components of the generalized-analytical function, we get

$$\varphi^i = \varphi^i_{(0)}\left(\frac{\kappa}{\kappa_0}\right)^4\left(\frac{\lambda}{\lambda_0}\right) b_i(\xi^{i-}) \cdot exp(\mu^{i-}\xi^i), \tag{107}$$

where

$$a_i(\xi^{i-}) = ln\left|b_i(\xi^{i-})\right|. \tag{108}$$

Thus, despite of two additional requirement, the generalized-analytical function (107) in general case is not reducible to an analytical function of H_4-variable, and besides we obtain the expression (106) for the coefficients κ in the metric function of the Finslerian space with the length element (86). If $\frac{\lambda}{\lambda_0} = \left(\frac{\kappa_0}{\kappa}\right)^4$, then φ^i is an analytical function.

If

$$\kappa = \kappa_0 \cdot \exp\{[(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + (\xi^4)^2]/4\}, \tag{109}$$

then with respect to the coordinates x^i

$$\kappa = \kappa_0 \cdot exp\{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\}. \tag{110}$$

Conclusion

Having introduced the concept of the generalized-analytical function of polynumber variable in the present work, we have made the first step in the direction of constructing a relevant theory aiming to develop theoretical-physical models. An important ingredient of such investigations must be search for additional requirements to be obeyed by the generalized-analytical functions and for the consequences implied by the requirements. The conditions that lead to trivial results – to analytical functions – should especially be analyzed. This may admit formulating the properties that are forbidden to attribute proper generalized-analytical functions of polynumber variable (in contrast to analytical functions proper). As it has been shown above, the independence of integral of integration path relates to such properties. Of course, it is necessary to carry out a particular attentive study to compare the properties of analytical functions of complex variable and generalized-analytical functions of polynumber variable in case of the dimension exceeding 2. It can be hoped, therefore, that the concepts and results of the present work may face future novel theoretical-physical applications.

References

[1] P. K. Rashevsky, *Riemann Geometry and Tensor Analysis,* Science, M., 1967 (in Russian).

[2] P. K. Rashevsky, *The Geometrical Theory of Partial Differential Equations*, 2-nd ed., Editorial URSS, M., 2003 (in Russian).

On Some Questions of Four-Dimensional Topology: a Survey of Modern Research

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It is emphasized that 4-dimensionality plays special role in almost all modern physical theories, within the palette of higher-dimensional physical models which, having additional dimensions often give a new freedom. The goal of the paper is to offer a brief survey of some problems in 4-dimensional topology. The addressed topics are: the S-cobordism problem, false and exotic 4-dimensional manifolds and the Schoenflies Conjecture.

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Introduction

Our physical intuition distinguishes four dimensions in a natural correspondence with material reality. Four dimensionality plays special role in almost all modern physical theories. High dimensional quantum fields theory and string theory are considered together with their compactifications, i. e. the main space, describing the reality is a product of a four-dimensional manifold with some compact high-dimensional space. In this way we come to the well-known Kaluza-Klein model and ten-dimension superstring theory.

It is an interesting fact that the dimension four is a more complicated dimension from pure mathematical point of view. It seems that there is a contradiction with our intuition in understanding of the dimension concept, really, new dimensions give us new complexity. But it is not true in general. Additional dimensions often give a new freedom. It is natural that we must have some golden mean in this approach, in which we don't have a necessary freedom, but low-dimensional methods weakly work. In topology this mean is dimension four.

The goal of this note is to give a small survey of some problems in four-dimensional topology.

S-cobordism problem

One of the main questions of geometric topology is the problem to classify manifolds lying in a given category with respectively chosen equivalence relation. Working in the topological category, the question about classification of topological manifolds up to homeomorphism rises, for example, assuming compactness, connectness and closedness. In dimension one we have only circles, in dimension two we come to the complete classification: every connected closed compact manifold is homeomorphic to the two-dimensional sphere with handles and Mobius bands. In this case, the fundamental group is a complete topological invariant. In the dimension three the question about classification becomes a hard problem, the existence of the connected 1-connected three-dimensional manifold, which is not homeomorphic to the three-dimensional sphere, is a well-known Poincare Problem. It is interesting that in dimension ≥ 5 many difficulties, occurring place in low dimensions, are disappear. First of all, this fact is connected with the concept of general position in high-dimensional spaces. Roughly speaking, in many important cases small deformations give possibility to cancel self-intersections of complexes. But in low-dimensional case we can not do the same.

Let's introduce one of the central equivalence relation in the topology of manifolds, so called s-cobordism relation. Let M_1 and M_2 be n-dimensional manifolds. We say that they are *cobordant* if there exists a $(n+1)$-dimensional manifold W, such that $\partial W = M_1 \cup M_2$. Further, if the embeddings $M_i \to W,\ i = 1, 2$ are homotopical equivalences, then this cobordism is called *h-cobordism* (and manifolds are $h - cobordant$). Every homotopical equivalence defines an element from the Whitehead group, which depends only on the fundamental group of a given manifold (or in general, fundamental group of cell complex). The Whitehead group can be defined as a quotient of the K_1-functor of the integral group ring of the fundamental group by the natural action of group. In this way, the homotopical equivalence represents a trivial element of the Whitehead group if and only if it is homotopic to the composition of elementary cell extensions and collapsings, i.e. so-called simple homotopy equivalence. H-cobordism with simple homotopy equivalence is called *s-cobordism*. In particular, every homotopy equivalence between 1-connected manifolds is homotopic to the simple one.

The main result of the high-dimensional topology is the following Theorem (see [1], [2]).

The S-cobordism Theorem. Let $n \geq 5$. The connected h-cobordism W between n-dimensional manifolds M_1 and M_2 is homeomorphic to the direct product $W \equiv M_1 \times I$, if and only if this cobordism is an s-cobordism.

In particular, if we consider only 1-connected manifolds then arbitrary h-cobordism between them is a direct product. The higher-dimensional Poincare Conjecture then follows from this, i.e. every homotopical sphere is homeomorphic to the standard one in dimension ≥ 5. The proof of the s-cobordism Theorem fails in the case of dimension 4 and analogical statement presents an open problem:

Problem. Does the s-cobordism Theorem hold in dimension 4?

The proof of the high-dimensional s-cobordism Theorem is based on handlebody decomposition of the manifold W and reduction of a given manifold to the structure of the direct product of M_i with interval. The crucial point in this method is so-called Whitney trick. It gives a possibility to cancel the intersection points of the immersed submanifolds due to the embedding of a 2-dimensional disk (Whitney's disk), (see [1]). The main obstruction to extend the proof on the case of dimension four is the fact that Whitney trick does not work in dimension four. Actually, it is well-known that every 2-dimensional complex can be isotopically reduced to the embedded one in the 5-dimensional manifold. But in dimension four it is not true in general and we can consider the Whitney's disk only as immersed one. This easy fact destroys all prove of the s-cobordism theorem in the case of dimension 4.

To get over the difficulties related to the immersed Whitney disc, some new methods have been developed. The method given by A. Casson is most effective. The meaning of this method is to paste a self-intersection step by step by new immersed discs. This process can be extended infinitely long but the neighborhood of the final 2-complex is a handle, which is homotopically equivalent to the standard one. This idea was used by M. Freedman in the proof of the topological Poincare Conjecture in dimension four.

In general, as it was mentioned above, the s-cobordism problem in dimension 4 is still open. The analog of the s-cobordism Theorem was proved by M. Freedman and P. Teichner in 1996 in the class of 4-dimensional manifolds with fundamental groups of the subexponential growth (more precisely, of the growth $\leq 2^n$) [4].

False and exotic 4-dimensional manifolds

There is a natural question of comparison of given equivalence relations, i.e. homotopical equivalence, homeomorphisms, diffeomorphisms, in the class of manifolds of a fixed dimension. So, any two continuously homeomorphic smooth manifolds are diffeomorphic in the dimension less than four. The situation in dimension four is much more complicated.

A manifold N is called a *false copy* of the manifold M if N is homotopically equivalent to M but not homeomorphic to M. N is called an *exotic copy* of M if N and M are homeomorphic, but not diffeomorphic as manifolds.

The existence of the false and exotic spheres is connected with the topological and smooth versions of the Poincare Conjecture respectively. The smooth Poincare Conjecture is true in the dimensions less than four: there are no exotic three (and less) dimensional spheres. The analysis of the high-dimensional question leads to the beautiful theory of exotic spheres: there exist 28 7-dimensional manifolds, which are homeomorphic to the standard 7-dimensional sphere, but not diffeomorphic, due to wonderful result of Milnor. The most intriguing case is again dimension four. This is the only dimension, in which the existence of the exotic spheres is still open.

The situation with exotic copies of $\mathbb{R}^4$ is also very surprising. It is known that there does not exist any exotic $\mathbb{R}^n$ in dimension $n \neq 4$ and the analogical question was open for a long time in dimension four. In eighties due to the results of Freedman and Donaldson it was proved that there exist infinitely many smooth pair-wise nondiffeomorphic four-dimensional manifolds, such that each of them is homeomorphic to $\mathbb{R}^4$. The proof of this fact essentially used the methods of mathematical physics: instantons, Yang-Mills connections etc (see [5]). One of the main invariants of 1-connected four-dimensional manifolds is so-called intersection form, i.e. symmetric bilinear form, define on the second cohomologies of a given manifold. Classical Whitehead's theorem says that two given 1-connected oriented closed smooth four-dimensional manifolds are homotopically equivalent if and only if they have isomorphic intersection forms. In this connection, there is an actual question to classify all symmetric bilinear forms which can be realized as intersection form for some four-dimensional manifold. M. Freedman has shown that every symmetric bilinear form can be realized as an intersection form of some compact 1-connected four-dimensional manifolds and that there exist no more than two manifolds with given form. Donaldson classified all intersection forms of smooth manifolds and concluded from this the existence of the exotic structures on $\mathbb{R}^4$. The structure of exotic $\mathbb{R}^4$ is very complicated and takes important place in modern research. There are still many open questions related to such manifolds. In particular, does there exist any exotic $\mathbb{R}^4$ such that it can not be divided by properly embedded $\mathbb{R}^3$ onto two exotic pieces (Problem 4.43 (D), [6]).

The false four-dimensional manifolds construction requires an application of other techniques. As it was mentioned above, there are no false four-dimensional spheres (four-dimensional topological Poincare Conjecture). Very often the question about homeomorphicity of a given homotopic four-dimensional manifolds is very difficult. One of the first such type examples of four-dimensional manifolds is Cappell-Shaneson construction (see [2]): there exists a false projective $\mathbb{R}P^4$, which is homotopically equivalent but not diffeomorphic to $\mathbb{R}P^4$. This space is not PL-homeomorphic to $\mathbb{R}P^4$.

Finishing this section let's present more open problems in dimension four, related to exotic structures. The reader can find many classical and modern problems of this type the Kirby Problem List [6] (see also [7]).

Problem (4.77 [6]): An exotic smooth structure on $\mathbb{R}^4$ with $\mathbb{R}^1$ is diffeomorphic to $\mathbb{R}^5$. How can we usefully see the exotic $\mathbb{R}^4$ in $\mathbb{R}^5$?

Problem (4.86 [6]): Do all closed, smooth 4-manifolds have more than one smooth structure? (The generalization of the smooth 4-dimensional Poincare Conjecture).

Problem (4.87 [6]): Does every non-compact, smooth 4-manifold have an uncountable number of smoothings?

Schoenflies Conjecture

Consider one more problem, which has the solution in all dimensions besides four. This problem is about knotting in codimension equal to one. Recall that the embedding $f\colon M^m \to N^{n+m}$ is called *locally-flat* if the image of each point in N^{m+n}

has neighborhood U such that the pair $(Im(f) \cap U, U)$ is homeomorphic (piecewise-linearly, in the case we work in this category) to the pare $(D^m \times D^n, D^m \times \{0\})$.

Conjecture Let $f\colon S^n \to S^{n+1}$ be a piece-linear locally-flat embedding. Then $S^{n+1} \setminus im(f)$ is 2-component and the closure of each of the components is a piece-linear n-dimensional ball.

Roughly speaking, this conjecture states that n-dimensional sphere can not knot in $(n+1)$-dimensional one. This conjecture turn out to be true in dimensions $n+1 \neq 4$. But in the case of dimension four, again we can not apply the methods which we use in other dimensions.

Finishing this note, we want to emphasize that there exist not so much fields in mathematics which use so different methods as four-dimensional topology. The problems of four-dimensional topology lead to the difficult questions of group theory. This is a theory of growth in groups, Andrews-Curtis-type problems, lower central series in groups etc. Also we can see many applications of high-dimensional methods in dimension four, for example, surgery exact sequences, methods of the link and knot theory. The dimension four is the unique dimension from the topological point of view, where we can find so many application of different techniques and which has so many open problems, the development of new techniques of algebra and topology will be needed for their solution.

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References

[1] Rourke, C. P., Sanderson, B. J, *Introduction to piecewise-linear topology,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 69. Springer-Verlag, New York-Heidelberg, 1972.

[2] Mandelbaum, R., *Four-dimensional topology: an introduction,* Bull. Amer. Math. Soc., **2**, (1980), 1–159.

[3] Freedman M. H. and Quinn F., *Topology of 4-manifolds,* Princeton Mathematical Series, 39., Princeton University Press, Princeton, NJ, 1990.

[4] Freedman, M. H., Teichner, P., 4-*manifold topology. I. Subexponential groups,* Invent. Math., **122**, (1995), 509–529.

[5] Freed, D. S., Uhlenbeck, K. K., *Instantons and Four-Manifolds,* Mathematical Sciences Research Institute Publications, Springer-Verlag, New York, 1991.

[6] Kirby R., *Problems in low-dimensional topology,* Geometric topology (Athens, GA, 1993), 35–473, Amer. Math. Soc., Providence, RI, 1997.

[7] Quinn, F., *Problems in low-dimensional topology. Surgery and geometric topology* (Sakado, 1996), Sci. Bull. Josai Univ., (1997), **2**, 97–104.

Normal Conjugation on the Polynumber Manifold

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The polynumber space is a linear space with several poly-linear forms. We introduce the notion of the normal conjugation on the non-degenerated n-numbers manyfold. The normal conjugation is an $(n-1)$-nary operation which is commutative for each argument but, in general, is not associative. Such an operation is equivalent to the usual conjugation for complex and hyperbolic numbers. The normal conjugation may be applied to scrutinize the algebraic and geometric structure of the n-numbers coordinate space. It is also useful to introduce the notions of the scalar product and angular characteristics of two and more numbers (vectors).

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Introduction

The polynumber spaces are the examples of vector spaces, where the poly-forms of several arguments play the role of the fundamental metric forms. [1]. Such spaces are principally different from the habitual Euclidian and pseudo-Euclidian spaces. Therefore they demand the development of the notions of the angle, orthogonality, scalar product etc. The necessity of the proper investigations is caused by the frequent attempts to consider the Finslerian spaces (the polynumber spaces as a rule are the ones) as the geometrical fundament of physics. [2, 3]. The physics progress strongly depends on the adequacy of it's mathematical apparatus and geometrical ideas.

Surprisingly, the first known mention about such spaces belongs to Riemann. In 1854th while entering the professor post of the Goettingen university he read a famous lecture, in which he noticed that beside the usual quadratic metric forms the linear element may be represented as a fourth degree root from the differential expression of the same degree [4]. Per se he described the particular case of the spaces which later were named the Finslerian spaces.

Finslerian metric functions are very multifarious even for the linear spaces. Therefore they require the individual approach for every single case. However, if the hypercomplex numbers stand behind the Finslerian spaces it is possible to suggest the unified algorithm, some elements of it are represented bellow.

The exclusive role of the polynumber spaces is beyond any doubt. Despite of it they are very rarely mentioned in the modern geometrical literature. Obviously, it is explained by the seaming simplicity of the polynumber algebraic structure. It does not encourage neither the scrutiny of the polynumbers them selves, nor the scrutiny of the spaces related to them. However, even the thoroughly examined complex numbers recently brought the surprise to the mathematicians. It turned out that the fractals may be built on the ground of the complex numbers. This fact makes us think that we may expect something similar from others of the hypercomplex numbers. The simplicity of the fractal construction algorithm underlines the potential variety hiding behind the most trivial number structures.

Such notions as the scalar product, orthogonality, angle between two vectors are the essential parts of the Euclidean space theory apparatus. These notions are naturally generalized for the pseudo-Euclidean spaces. The approach given bellow allows the similar generalization of the concerned notions for the polynumber spaces.

The polynumber spaces P_n with $n > 2$ are not Euclidean or pseudo-Euclidean. Thus, if $e_1,\ e_2,\ \ldots,\ e_n \in P_n$ – the basis and

$$e_i e_j = p^k_{ij} e_k, \tag{1}$$

$$P_n \ni X = x^1 e_1 + x^2 e_2 + \ldots + x^n e_n, \tag{2}$$

then n-th degree of the number X norm may be expressed with the n-linear symmetric form

$$(X, Y, \ldots, Z) = \omega_{i_1 i_2 \ldots i_n} x^{i_1} y^{i_2} \ldots z^{i_n} \tag{3}$$

of one argument X. When $n > 2$ with two arguments X and Y we obtain $(n-1)$ different forms, therefore we can introduce the scalar product and the angle between two vectors (numbers) in several ways.

Besides the metric form (3) we may take other invariant forms in the P_n-space, the bilinear for example.

$$((X, Y)) = q_{ij} x^i y^j, \tag{4}$$

where

$$q_{ij} = C p^k_{im} p^m_{kj}, \tag{5}$$

$C \neq 0$ – some real number. For every concrete polynumber system this number may be chosen according to the simplicity and symmetry of the obtained formulas. As it follows from the definition, the given form is symmetric, i.e. $((X, Y)) = ((Y, X))$.

Thus, the P_n-space is n-dimensional space with several poly-linear forms. Two of the forms are dedicated: the metric form of the n-th order and the bilinear form.

The notion of the conjugated number is related (complex numbers, quaternions) with the changing of the sign of imaginary (symbolic) units. This makes us introduce $(n-1)$ conjugations in general and use the number itself and it's $(n-1)$ conjugations to construct of them the polynumber $(|\,X|^n \cdot 1 + 0e)$.

Normal conjugation

We shall call the n-numbers *nondegenerated*, if the matrix (q_{ij}) (5) is nondegenerated, i.e.

$$det(q_{ij}) \neq 0. \tag{6}$$

In this case, besides the two-times covariant tensor q_{ij}, the two times contravariant tensor q^{ij} is defined in the P_n-space.

Let us define the $(n-1)$-nary operation of the *normal conjugation of a complex* $\{X_{(1)}, X_{(2)}, ..., X_{(n-1)}\}$ with the following way:

$$[X_{(1)}, X_{(2)}, ..., X_{(n-1)}] = \omega_{i_1 i_2 ... i_{n-1} i_n} q^{i_n k} x^{i_1}_{(1)} ... x^{i_{n-1}}_{(n-1)} e_k. \tag{7}$$

It is obvious from this formula that the normal conjugation operation is commutative for every argument, but, generally, is not associative. The constant C in the formula (5) may be chosen with the following condition: $[1, 1, ..., 1] = 1$.

We shall say that the number $Z = [X_{(1)}, X_{(2)}, ..., X_{(n-1)}]$ *is normally conjugated* to the complex of numbers $\{X_{(1)}, X_{(2)}, ..., X_{(n-1)}\}$.

Let's define the *scalar product of the number X and the complex* $\{X_{(1)}, X_{(2)}, ..., X_{(n-1)}\}$ with the bilinear form

$$((X, Z)) = (X, X_{(1)}, X_{(2)}, ..., X_{(n-1)}). \tag{8}$$

Let us introduce the designation

$$\tilde{X} = [X, X, ..., X], \tag{9}$$

then

$$((X, \tilde{X})) = |\, X|^n, \tag{10}$$

If in the given polynumber system the n-th degree of the number X norm may be expressed as

$$|\, X|^n = (X, X, ..., X). \tag{11}$$

According to the definition, the number $\tilde{X}$ is normally conjuncted to the number X.

Now shall we illustrate the introduced notions with some examples.

Complex Numbers

$$X = x^1 + ix^2, \qquad i^2 = -1, \tag{12}$$

$$(X, Y) = x^1 y^1 + x^2 y^2, \tag{13}$$

$$(\omega_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{14}$$

$$(q_{ij}) = 2C \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

Let's take $C = \frac{1}{2}$, then

$$(\omega_{ik} q^{kj}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{16}$$

$$\tilde{X} = x^1 - ix^2, \tag{17}$$

i.e. the normal conjugation for complex numbers is the usual conjugation. The scalar product of the numbers X and Y is

$$((X, \widetilde{Y})) = x^1 y^1 + x^2 y^2 = (X, Y). \tag{18}$$

Thus

$$((X, \tilde{X})) = |\,X|^2, \tag{19}$$

$$X \cdot \tilde{X} = |\,X|^2 \cdot 1 + 0 \cdot i. \tag{20}$$

Hyperbolic numbers, H_2

$$X = x^1 + jx^2, \qquad j^2 = 1, \tag{21}$$

$$(X, Y) = x^1 y^1 - x^2 y^2, \tag{22}$$

$$(\omega_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{23}$$

$$(q_{ij}) = 2C \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{24}$$

Let us take $C = \frac{1}{2}$, then

$$(\omega_{ik} q^{kj}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{25}$$

$$\tilde{X} = x^1 - jx^2, \tag{26}$$

i.e. the normal conjugation for hyperbolic numbers is the usual conjugation. The scalar product of the numbers X and Y is

$$((X, \tilde{Y})) = x^1 y^1 - x^2 y^2 = (X, Y). \tag{27}$$

Thus

$$((X, \tilde{X})) = |\,X|^2, \tag{28}$$

$$X \cdot \tilde{X} = |\,X|^2 \cdot 1 + 0 \cdot j. \tag{29}$$

Hypercomplex Numbers H_3

The most easy way is to work in ψ-basis:

$$X = x^1\psi_1 + x^2\psi_2 + x^3\psi_3, \tag{30}$$

$$p^k_{ij} = \begin{cases} 1, & \text{if} \quad i = j = k, \\ 0, & \text{in all other cases}, \end{cases} \tag{31}$$

$$(q_{ij}) = C \cdot diag(1, 1, 1), \tag{32}$$

$$(X, Y, Z) = \frac{1}{6}(x^1y^2z^3 + x^1y^3z^2 + x^2y^1z^3 + x^2y^3z^1 + x^3y^1z^2 + x^3y^2z^1). \tag{33}$$

Let us take $C = \frac{1}{3}$, then

$$[X, Y] = \frac{1}{2}[(x^2y^3 + x^3y^2)\psi_1 + (x^1y^3 + x^3y^1)\psi_2 + (x^1y^2 + x^2y^1)\psi_3], \tag{34}$$

$$[1, 1] = 1, \tag{35}$$

$$\tilde{X} = x^2x^3\psi_1 + x^1x^3\psi_2 + x^1x^2\psi_3, \tag{36}$$

$$X \cdot \tilde{X} = |\, X|^3 \cdot 1 + 0 \cdot e, \tag{37}$$

if the norm $X \in H_3$ is defined with

$$|\, X|^3 = x^1x^2x^3. \tag{38}$$

The scalar product of the complex $\{X, Y\}$ and the number Z is the scalar

$$((Z, [X, Y])) = (X, Y, Z). \tag{39}$$

The bilinear form (4) from two numbers X and Y looks like

$$((X, Y)) = \frac{1}{3}(x^1y^1 + x^2y^2 + x^3y^3). \tag{40}$$

Let's find all numbers of H_3, that satisfy the equation

$$\tilde{X} = X. \tag{41}$$

Solving the system of three quadratic equations with three unknowns we have five roots: $(0, 0, 0)$, $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$. The four latter numbers (if we consider them radius-vectors) constitute the regular tetrahedron while the first number is it's center.

If $X, Y \in H_3$ are the divisors of zero for the normal conjugation (i. e. $[X, Y] = 0$, with $X \neq 0,\ Y \neq 0$), they have to be the divisors of zero for the polynumber multiplication.

Any number $Y \in H_3$ may be represented as

$$Y = [1, Z], \quad \text{where} \quad Z = (-y^1 + y^2 + y^3, y^1 - y^2 + y^3, y^1 + y^2 - y^3). \tag{42}$$

Let's scrutinize the eigenvectors and eigenvalues problem that is

$$[1, Y] = \lambda Y, \tag{43}$$

where λ – some real or complex number. All eigenvalues are real: $\lambda_1 = 1$, $\lambda_{2,3} = -\frac{1}{2}$, because the matrix of the linear transformation in the right side of the formula (43) is symmetric. Eigenvectors appropriate to the first eigenvalue constitute a straight line $1t$, where t – parameter along the straight line. The eigenvectors appropriate to the eigenvalue $(-\frac{1}{2})$, constitute a plain, which is Euclid-perpendicular to the straight line along the unity and contains the coordinate zero. I.e. this plain is strained on two radius-vectors. For example: $(2, -1, -1)$, $(0, 1, 1)$.

Formulas (30) – (40) may be automatically generalized for polynumbers[1] H_n with replacement $3 \to n$, $C = \frac{1}{n}$.

The examples given above makes us suppose (while comparing the formulas (20), (29) and (37))that for the complex and H_n numbers the following formula is true.

$$X \cdot \tilde{X} = |\, X|^n \cdot 1 + 0 \cdot e. \tag{44}$$

It is also possible, that it is true for any non-degenerated polynumbers, but this requires further prove.

We may say that X is *"orthogonal"* for Y, if

$$((X, \tilde{Y})) = 0. \tag{45}$$

Notice that this notion in general is not symmetric for $n > 2$, i. e. the fact that X is orthogonal for Y, does not mean that Y is orthogonal for X. Is these two are orthogonal to each other then X and Y are *mutually orthogonal.*

If we have $(n-1)$-number complex (some of numbers may coincide), and Z is a normally conjugated for this complex, then X is *"orthogonal" for the given complex,* if

$$((X, Z)) = 0. \tag{46}$$

Angular parameters of several numbers

In the polynumber spaces $n > 2$ we can introduce the angle between two numbers (vectors) with several ways. In this paper we use the algebraic approach based on the triangle-formula analog form the Euclid space.

Let us illustrate this on the H_3 example. If X and Y are such that

$$x^i > 0, \quad y^i > 0, \quad i = 1, 2, 3. \tag{47}$$

[1] H_n – the hypercomplex numbers isomorphous to the real square diagonal matrixes algebra $n \times n$.

In this case they are not the divisors of zero. Shall we find the expression for the norm of the cube of their summa $Z = X + Y$

$$|\,Z|^3 = (X+Y, X+Y, X+Y) = |\,X|^3 + 3(X, X, Y) + 3(X, Y, Y) + |\,Y|^3. \quad (48)$$

Let's introduce two hyperbolic angles β_X, β_Y according to the formulas:

$$\cosh\beta_X = \frac{(X, X, Y)}{|\,X|^2|\,Y|}, \quad \cosh\beta_Y = \frac{(X, Y, Y)}{|\,X|\,|\,Y|^2}, \quad (49)$$

then

$$|\,Z|^3 = |\,X|^3 + |\,Y|^3 + 3|\,X|^2|\,Y|\,\cosh\beta_X + 3|\,X|\,|\,Y|^2\,\cosh\beta_Y. \quad (50)$$

These two hyperbolic angles β_X, β_Y we shall call the *angular characteristics* of the pair of numbers X, Y.

Let us elucidate the meaning of the forms that appear in formulas (48), (49). For this let us consider the complex $\{X, Y\}$ and the normal conjugated number for this complex $W = [X, Y]$. The form (X, X, Y) is a scalar product of X and complex $\{X, Y\}$, and the form (X, Y, Y) is a scaler product of Y and the same complex.

If X, Y are not divisors of zero, but also they do not satisfy the (47) conditions, then the right sides of (49) may take negative values. If we want to preserve the formula (50), then the angular characteristics β_X, β_Y become, in general, complex numbers β_X, β_Y. Opposite, if we want to have real angular characteristics, we have to change the formulas (49) and (50). For example, if the right side in the first formula (49) is lesser than zero, then we can replace $\cosh\beta_X$ by $\sinh\beta_X$ in this formula and in (50).

Why do we need two angular characteristics for two numbers(vectors) in three-dimensional H_3 instead of one angle in the three-dimensional Euclid space? It is related with the fact that H_3-space and all the polynumber spaces of the dimension ≥ 2 has marked out directions and planes., i.e. they are anisotropic.

Fractals

Over the last thirty years there was an impetuous progress of the direction of the dynamic systems theory related with complex fractals. [5]. The most brilliant representatives of latter are the Julia and Mandelbrot sets. Lots of beautiful and useful results have shaded the important fact that they all were obtained from the complex numbers and the Euclid plane basis. Opposite, the construction of multi-dimensional fractals based on quaternions was not impressive after all.

The deepest cause of the problems, appearing on this way, is the principal impossibility to generalize the theory of analytical functions of the complex variable for the quaternions. This impossibility is caused by the non-commutativity of quaternionic multiplication.

The polynumbers structure does not contain the difficulties that appear in non-commutative or un-associative number algebras. Therefore we may expect that there is a possibility to construct the fractals based on polynumbers, and such fractals could be much more interesting than the quaternionic ones. Turning to the H_n-numbers for example it is easy to see that it is impossible to construct interesting fractals using the usual for Julia sets dependencies. For example:

$$X_{(i+1)} = X_{(i)}^2 + C. \tag{51}$$

It is related with the very simple structure of H_n-numbers, H_3 in particular. In the special basis the analytical functions of H_n-variable break up to n functions of one variable. Therefore the iterative process may be turned to n independent one dimensional iterative process, which is not very interesting. But there is a great possibility to introduce some additional operations for the polynumbers (one of them is the normal conjugation). These new operations may be used to build more complicated non-breaking iterative processes.

Thus, we can propose several simple non-trivial iterative processes for H_3: $X_{i+1} = F(X_i)$:

1. $F(X) = \tilde{X} + C$,
2. $F(X) = [X, \tilde{X}] + C$,
3. $F(X) = [X, [X, 1]] + C$,
4. $F(X) = [X, [X, [X, 1]]] + C$,
5. $F(X) = X \cdot [X, 1] + C$,
6. $F(X) = [X, [X, C]] - 1$,
7. $F(X) = [X \cdot X, X] + C = [X, X] \cdot [X, 1] + C$,

where $C \in H_3$. The initial numbers for these iterative processes were taken on the planes which are perpendicular (in Euclid meaning) to the straight $1 \cdot t$. The parameter t indicates the point where the straight and the plane intersect. With $t = 0$ the plain contains the coordinate zero. It is interesting that with $C = 0$, $t = 0$ processes 2,3,4 gives the convergence area that looks like a round hexagon.

The scrutiny for the convergence of the process 1 gives some interesting in geometric aspect three-dimensional convergence areas. Appropriate results for process 7 are even more interesting.

Conclusion

The constructions proposed above, of course, may be generalized farther. So, we can examine the n-dimensional linear space with poly-linear symmetric form (3), divide the arguments manifold on two complexes and say that this form is a scalar product of these two complexes. This shall cause a further generalization of the notions introduced above.

It is undoubted that the normal conjugation has it's own algebraic meaning. We have proposed useful generalization algorithm for well-known from the Euclid and pseudo-Euclid spaces geometrical objects and values, such as scalar product, orthogonality, angles an so on.

The introduction of additional operations on the hypercomplex numbers turns them into something more than linear algebras. These operations allow us to obtain the geometries which have much more inner symmetries than the polynumbers themselves contain. It would be appropriate to introduce the term "linear geometry", besides the usual "linear algebra". The new term contains the old one plus all possible independent poly-linear linear operations which natural follow from some constructions of linear algebra itself.

The construction of many-dimensional fractal sets is one of the perspective directions of applying the potential of such linear geometries.

Probably, we should underline once again, that the fractal sets, constructed by the mean of the introduced by authors specific $(n-1)$-nary operation, are the objects of polynumber, instead of arbitrary, space. This fact makes them perspective, unlike the quaternion-based fractals. It is well known that the quaternionic multiplication is not commutative. Therefore quaternions have poor mathematical perspectives. Thus, it is impossible to create a complete analytical functions theory. Since there no such problem with polynumbers and taking into account the hypothetic possibility of the replacement of the Minkowsky space with one of the poly-spaces [6], the proposed approach seems to be very perspective.

References

[1] D. G. Pavlov, *Generalization of scalar product axioms,* Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004.

[2] G. Yu. Bogoslovskiy, *The Theory of Locally Anisotropic Space-Time,* MSU, M., 1992.

[3] G. S. Asanov, *Finslerian Extension of General Relativity,* Dordrecht, 1984.

[4] B. Riemann, *About hypotheses in foundation of geometry,* – in the book: *On Foundation of Geometry.*

[5] H. O. Peitgen, D. H. Richter, *The Beauty of Fractals: Images of Complex Dynamical Systems,* Springer-Verlag, NY, 1986.

[6] D. G. Pavlov, *Four-dimensional time,* Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004.

Generalized Analytical Functions and the Congruence of Geodesics

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The author defines the notion of generalized-analytic function of polynumber variable, a primary step towards constructing a relevant theory able to develop theoretical-physical models. This notion is a non-trivial fundamental generalization of the corresponding one of *analytical function* of complex variable, which is essential for theoretical physical constructions. An important obstruction in building this novel notion consists in locating appropriate additional requirements to be obeyed by the generalized-analytical functions and in dealing with the consequences implied by these requirements. The conditions that lead to trivial results (i.e., to analytical functions) are first to be analyzed. Several main properties of generalized-analytical functions of polynumber variable are shown to extend the known ones for the proper classical case (e.g., the path-independence of integral).

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Introduction

Very impressive success of the theory of complex variable and it's applications to physics makes us to search for a generalization of this theory for spaces of more than two dimensions. It is possible that the construction of polynumber variable [1] is one of such generalizations. We have to put some additional conditions to allow automatically apply such functions for theoretic physics models and some concrete physical questions.

Generalized analytical function (for further details see [1]) – is the pair $\{f^i;\gamma^i_k\}$:

$$\frac{\partial f^i}{\partial x^k}+\gamma^i_k=p^i_{kj}\dot f^j,\quad \text{or}\quad \tilde\nabla_k f^i=p^i_{kj}\dot f^j, \tag{1}$$

where f^i, $\dot f^i$ – single-covariant vector fields in the space $\{\mathbf{M_n};\mathbf{P_n}\}$, $\mathbf{M_n}$ – n-dimensional elementary manifold admissive of inter-single-valued correspondence $\mathbf{M_n}\leftrightarrow\mathbf{P_n}$ on n-dimensional space of polynumbers $\mathbf{P_n}$, and the objects γ^i_k while switching to another frame of reference transform as the objects $(\Gamma^i_{kj}f^i)$, where Γ^i_{kj} – objects of the affine connectivity. We postulate that one of the necessary properties of the space $\{\mathbf{M_n};\mathbf{P_n}\}$ is that it's tangent space is isomorphous to the space of associative-commutative hypercomplex numbers (polynumbers) $\mathbf{P_n}$. in any point $X\in\{\mathbf{M_n};\mathbf{P_n}\}$. Due to the presence of the inter-single-valued transformation $\mathbf{M_n}\leftrightarrow\mathbf{P_n}$ we may introduce the special frames of references in the space $\{\mathbf{M_n};\mathbf{P_n}\}$.

In such frames of reference we define the rules of polynumber multiplication, which does not depend on the concerned point. If $\mathbf{P_n} \ni e_1, e_2, ..., e_n$ – a basis, then

$$e_i e_j = p^k_{ij} e_k. \tag{2}$$

Let ε^i – the coordinates of a unity breakdown then

$$\varepsilon^i p^k_{ij} = \delta^k_j. \tag{3}$$

Using this formulae and the formulae (1), we obtain an explicit stating for the generalized derivative

$$\dot{f}^i = \varepsilon^k \tilde{\nabla}_k f^i \tag{4}$$

and the Cauchy-Riemann correlations:

$$\tilde{\nabla}_k f^i - p^i_{kj} \varepsilon^m \tilde{\nabla}_m f^j = 0. \tag{5}$$

We may juxtapose a manifold of the affine connectivity $\mathbf{L_n}(\Gamma^i_{kj})$ to any generalized-analytical function $\{f^i; \gamma^i_k\}$. The objects of the affine connectivity Γ^i_{kj}, are a solutions of the equation set

$$\Gamma^i_{kj} f^j = \gamma^i_j. \tag{6}$$

Thus defined manifold of functions with a same object of connectivity forms a manifold (a function class), noted $\{f^i; \Gamma^i_{kj}\}$.

In the space of the affine connectivity always exist a parameter τ, such that the equation set of geodetic $x^i = x^i(\tau)$ takes [2] form

$$\frac{d^2 x^i}{d\tau^2} + \Gamma^i_{kj} \frac{dx^k}{d\tau} \frac{dx^j}{d\tau} = 0. \tag{7}$$

If we replace the connectivity object Γ^i_{kj} with a different one:

$$\tilde{\Gamma}^i_{kj} = \Gamma^i_{kj} + \frac{1}{2}(p_k \delta^i_j + p_j \delta^i_k) + S^i_{kj}, \tag{8}$$

where p_i – an arbitrary single-covariant field, and S^i_{kj} – an arbitrary tensor, which is antisymmetric by the down indexes, i.e. torsion tensor, then the geodetic remain the same. (see, for example, [2]).

Geodetic congruence, appropriate for generalized-analytical function

Let $\{f^i; \gamma^i_k\}$ – generalized-analytical function, and vector field f^i defines the congruence of geodetic with connection object (8), where the object Γ^i_{kj} is related with the concerned generalized-analytical function by the relation (6), moreover the tangent vector along the geodetic $x^i = x^i(\tau)$ is

$$\frac{dx^i}{d\tau} = f^i. \tag{9}$$

Then the differential equations (7) with Γ^i_{kj} replaced by $\tilde{\Gamma}^i_{kj}$ become relations that define generalized-analytical function

$$f^k \tilde{\nabla}_k f^i + (p_m f^m) f^i = 0, \tag{10}$$

or

$$f^k p^i_{kj} \dot{f}^i + (p_m f^m) f^i = 0. \tag{11}$$

Thus, to define geodetics congruence by the way given above (or, as we speak farther, to have X-property), the generalized-analytical function has to satisfy the relations (10), (11).

We call generalized-analytical functions with X-property the X-*functions.*

The equation set (11) is a set of linear equations for n unknowns $\dot{f}^i$ is consistent, since there certainly is one solution.

$$\dot{f}^i = -(p_m f^m)\varepsilon^i. \tag{12}$$

If the matrix

$$(a_{ij}) = f^k p^i_{kj} \tag{13}$$

is non-degenerate in some area, then (12) is the only solution of system (11) in this area of the space $\{\mathbf{M_n}; \mathbf{P_n}\}$.

The n-th degree of "norm" in the polynumber $X \in P_n$ space may be expressed in terms of the form

$$\Omega(X) = det(x^k p^i_{kj}). \tag{14}$$

This form's value does not depend on basis:

$$\Omega(YX) = \Omega(Y)\Omega(X) \tag{15}$$

with any X, $Y \in P_n$; at last $\Omega(1) = 1$. Thus, we may define the n-th degree of "norm" by

$$|\,X|^n = \Omega(X) \tag{16}$$

or by

$$|\,X|^n = |\Omega(X)|. \tag{17}$$

On account of the said above we may expect that the solutions of the equation (10) will strongly depend on X-function equal zero or not.

Let us demonstrate that for arbitrary polynumbers the analytic function

$$F(X) = \omega X + V_0 \tag{18}$$

(ω – an arbitrary real number, and V_0 – an arbitrary polynumber) is namely a function that define the congruence of the geodetics, i.e. an X-function.

If $\omega \neq 0$, then it may be written as

$$F(X) = \omega(X - X_0), \tag{19}$$

where X_0 – an arbitrary polynumber. Let us substitute (18) into (10) and, taking into account that for analytic functions $\gamma^i_k = 0$, we obtain

$$f^i[\omega + (p_m f^m)] = 0. \tag{20}$$

Since p_m – m for arbitrary functions-components, we may always construct such m components, that $(p_m f^m) = -\omega$. Which was to be proved.

Let us find out a kind of curves, defined by the function (18). To do it, we have to find a general solution of the system of ordinary differential equations

$$\frac{dx^i}{d\tau} = \omega x^i + v^i_0. \tag{21}$$

It has the appearance of

$$x^i = v^i_0 \tau + a^i e^{\omega\tau}. \tag{22}$$

We imply by the congruence of curves in some area of n-dimensional space the $(n-1)$-parametric family of curves. At that one and only one curve passes through every point of this n-dimensional space.

There is $(2n+1)$ independent real parameters and the parameter along the curve in the general solution (22). Therefore parameters v^i_0, a^i, ω have to be expressible as $(n-1)$ independent parameter for equations (22) to define the congruence. And the region of variation of the parameter τ may be limited according to the values of these $(n-1)$ independent parameters. If we fix the direction of the parameter τ changing (for example – from lesser to bigger values), every curve gets a direction, i.e. it has a view of a current line or a "field line".

Despite of simplicity of appearance of the general solution (22), these formulas define a great variety of congruences of curves. And not all of them are straight, i.e. geodetic. Thus, the manifold of solutions of (10 includes the X-functions as a subset. This means that the fulfilment of (10) is necessary, but not enough for the generalized-analytical function to have the X-property.

In physics we often meet the condition $\nabla_i f^i = 0$. The law of conservation of charge and the 4-vector calibration of electro-magnetic field are expressed like that for example.

Let us calculate the same convolution product for the generalized-analytic function. We obtain

$$\tilde{\nabla}_i f^i = p^i_{ij} \dot{f}^j. \tag{23}$$

For X-function in case the condition (12) is satisfied we have

$$\tilde{\nabla}_i f^i = -(p_m f^m), \tag{24}$$

and for X-function (18), (19)

$$\tilde{\nabla}_i f^i = n\omega. \tag{25}$$

Examples of analytic X-function

Complex numbers

Let us take up the analytic function

$$F(z) = u(x, y) + iv(x, y) \tag{26}$$

of complex variable

$$z = x + iy, \quad i^2 = -1. \tag{27}$$

For first, let us write out the matrix(13)

$$(a_{ij}) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \tag{28}$$

and calculate it's determinant

$$det(a_{ij}) = u^2 + v^2. \tag{29}$$

Thus, for complex numbers the following formulae(16) is true

$$det(a_{ij}) = |F(z)|^2. \tag{30}$$

Let us solve the equation set (10). In this case it takes form

$$\begin{cases} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + (p_1 u + p_2 v)u = 0, \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + (p_1 u + p_2 v)v = 0. \end{cases} \tag{31}$$

Using the Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{32}$$

From this equation set we have two sets:

$$\begin{cases} (u^2 + v^2)\left[\frac{\partial u}{\partial x} + (p_1 u + p_2 v)\right] = 0, \\ (u^2 + v^2)\frac{\partial u}{\partial y} = 0, \end{cases} \tag{33}$$

$$\begin{cases} (u^2 + v^2)\frac{\partial v}{\partial x} = 0, \\ (u^2 + v^2)\left[\frac{\partial u}{\partial y} + (p_1 u + p_2 v)\right] = 0. \end{cases} \tag{34}$$

Let us examine these equation set in the area $u^2 + v^2 \neq 0$. In that case, reducing by this non-zero factor and writing the integrability conditions of the obtained equation sets, we have

$$\frac{\partial}{\partial x}(p_1 u + p_2 v) = \frac{\partial}{\partial y}(p_1 u + p_2 v) = 0 \quad \Rightarrow \quad (p_1 u + p_2 v) = const, \tag{35}$$

and the only solution in this case

$$F(z) = \omega z + w_0, \tag{36}$$

where ω – an arbitrary real number, $w_0 = u_0 + iv_0$ – an arbitrary complex number.

Let us calculate a convolution $\nabla_i f^i$ of two X-functions (36), we get

$$\nabla_i f^i = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2\omega, \tag{37}$$

which matches the formula (25).

So we have proven that all analytic X-functions of complex variable have the appearance of (36). There is no analytic X-function of complex variable (excluding a constant), for which $\nabla_i f^i \equiv 0$.

Hyperbolic numbers, H_2

Let us consider an analytic function

$$F(z) = u(x, y) + jv(x, y) \tag{38}$$

of hyperbolic variable

$$z = x + jy, \quad j^2 = 1. \tag{39}$$

Let's calculate the matrix (13)

$$(a_{ij}) = \begin{pmatrix} u & v \\ v & u \end{pmatrix} \tag{40}$$

and it's determinant

$$det(a_{ij}) = u^2 - v^2. \tag{41}$$

Thus, if $v = \pm u$ the matrix (a_{ij}) is degenerate, and for hyperbolic numbers formulae (16) is true (16)

$$det(a_{ij}) = |F(z)|^2, \tag{42}$$

if we take the square of norm in H_2 space as

$$|z|^2 = x^2 - y^2. \tag{43}$$

The relations (10) for hyperbolic numbers have the same appearance as for complex numbers, and the Cauchy-Riemann equations change a bit :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{44}$$

– therefore we only have to change the common factor in the equations (33), (34) to $(u^2 - v^2)$. Doing that, we obtain not a single, but three qualitative different solutions:

$$F_{(0)}(z) = \omega z + w_0, \tag{45}$$

where ω – an arbitrary real number and w_0 – an arbitrary hyperbolic number;

$$F_{(1)}(z) = f_{(1)}(x+y)(1+j), \tag{46}$$

where $f_{(1)}(\xi)$ – an arbitrary one real number function;

$$F_{(2)}(z) = f_{(2)}(x-y)(1-j), \tag{47}$$

where $f_{(2)}(\xi)$ – an arbitrary one real number function. In the ψ-basis:

$$\begin{gathered} \psi_{1,2} = \tfrac{1}{2}(1 \pm j), \quad \psi_1\psi_1 = \psi_1, \quad \psi_2\psi_2 = \psi_2, \psi_1\psi_2 = 0, \\ x + jy = (x+y)\psi_1 + (x-y)\psi_2 = \xi^1\psi_1 + \xi^2\psi_2 \end{gathered} \tag{48}$$

the two latter X-functions take the appearance of

$$F_{(1)}(z) = 2f_{(1)}(\xi^1)\psi_1, \quad F_{(2)}(z) = 2f_{(2)}(\xi^2)\psi_2, \tag{49}$$

at that $|F_{(1)}(z)| = 0, \quad |F_{(2)}(z)| = 0.$

So, the analytic X-functions of H_2 variable are more multifarious than appropriate functions of complex variable. It is related with the presence of the divisors of zero in H_2 algebra.

Let us calculate the scalar $\nabla_i f^i$ of three obtained X-functions:

$$\nabla_i f^i_{(0)} = 2\omega, \quad \nabla_i f^i_{(1)} = 2\dot{f}_{(1)}(x+y), \quad \nabla_i f^i_{(2)} = 2\dot{f}_{(2)}(x-y). \tag{50}$$

Note that there are no analytic X-functions of H_2 variable (excluding a constant) for which $\nabla_i f^i \equiv 0$.

Hypercomplex numbers H_4

These polynumbers algebra is isomorphous to the algebra of real diagonal square matrices 4×4. It is the most easy to work with such numbers in ψ-basis: ψ_1, ψ_2, ψ_3, ψ_4;

$$\psi_i\psi_j = p^k_{ij}\psi_k, \quad p^k_{ij} = \begin{cases} 1, & \text{if} \quad i = j = k, \\ 0, & \text{in all other cases.} \end{cases} \tag{51}$$

An arbitrary analytic function of H_4-variable has an appearance of:

$$F(x) = \varphi^1(\xi^1)\psi_1 + \varphi^2(\xi^2)\psi_2 + \varphi^3(\xi^3)\psi_3 + \varphi^4(\xi^4)\psi_4, \tag{52}$$

where φ^i – arbitrary even functions of a real variable, and ξ^i – the coordinates of $X \in P_n$ in ψ-basis. The matrix (13) has the appearance of

$$(a_{ij}) = \begin{pmatrix} \varphi^1 & 0 & 0 & 0 \\ 0 & \varphi^2 & 0 & 0 \\ 0 & 0 & \varphi^3 & 0 \\ 0 & 0 & 0 & \varphi^4 \end{pmatrix} \tag{53}$$

and it's determinant equal

$$det(a_{ij}) = \varphi_1\varphi_2\varphi_3\varphi_4. \tag{54}$$

Thus, for hypercomplex numbers H_4 the formulae (16) is true:

$$|F|^4 = det(a_{ij}), \tag{55}$$

if the fourth degree of norm in H_4 space is

$$|\,X|^4 = \xi^1\xi^2\xi^3\xi^4. \tag{56}$$

The equation set (10) after substituting (52) into it is written like: (52)

$$\varphi^i\left[\frac{\partial\varphi^{i_-}}{\partial\xi^{i_-}} + p_m\varphi^m\right] = 0, \tag{57}$$

where $i \equiv i_-$ (no summation). As we noted above, the qualitative difference of the equation set (57) solutions is related with the presence of the divisors of zero in the polynumbers system. Let us classify polynumbers $X \neq 0$ in H_4 space in the following way:

A) X is not a divisor of zero;

B) three coordinates $\xi^i, \xi^j, \xi^k,\ \ i \neq j,\ i \neq j,\ j \neq k$ not equal zero, and the fourth coordinate equal zero;

C) only two coordinates ξ^i and ξ^j, $i \neq j$ differ from zero, and another two coordinate equal zero;

D) only one coordinate ξ^i is not zero.

According to this classification we classify the solutions of the equation set (57):

A)
$$F_{(0)}(X) = \omega X + W_0, \tag{58}$$

where ω – an arbitrary real number, a W_0 – an arbitrary polynumber;

B)
$$F_{(i,j,k)}(X) = \omega(\xi^i\psi_{i_-} + \xi^j\psi_{j_-} + \xi^k\psi_{k_-}) + \zeta_0^i\psi_{i_-} + \zeta_0^j\psi_{j_-} + \zeta_0^k\psi_{k_-}, \tag{59}$$

where ω, ζ_0^m – four arbitrary real numbers for each X-function of this kind;

C)
$$F_{(i,j)}(X) = \omega(\xi^i\psi_{i_-} + \xi^j\psi_{j_-}) + \zeta_0^i\psi_{i_-} + \zeta_0^j\psi_{j_-}, \tag{60}$$

where ω, ζ_0^m – three arbitrary real numbers for each X-function of this kind;

D)
$$F_{(i)}(X) = \varphi^i(\xi^{i_-})\psi_{i_-}, \tag{61}$$

where $\varphi^i(\xi^{i_-})$ – an arbitrary flat function of a real variable for each X-function of this kind;

Let us calculate the scalar $\nabla_m\varphi^m$ of each obtained X-function.

$$\begin{array}{ll} \text{A)}\quad \nabla_m\varphi^m = 4\omega, & \text{B)}\quad \nabla_m\varphi^m = 3\omega, \\ \text{C)}\quad \nabla_m\varphi^m = 2\omega, & \text{D)}\quad \nabla_m\varphi^m = \dot{\varphi}^i(\xi^{i_-}). \end{array} \tag{62}$$

Thus, there are no analytic X-functions of H_4 variable (excluding a constant) for which $\nabla_m\varphi^m \equiv 0$.

Non-degenerate X-functions

Let us call the X-function *non-degenerate* if it is not a divisor of zero, i.e. $|\,F(X)| \neq 0$.

Then it follows from the above-stated that such generalized-analytic function has an appearance of

$$\{f^i; \gamma^i_k\} = \{f^i; -\frac{\partial f^i}{\partial x^k} + \delta^i_k a(x)\}, \tag{63}$$

where f^i – an arbitrary flat vector field, and $a(x)$ an arbitrary scalar field. Thus, there are non-degenerate X-functions for any polynumbers, all of them have an appearance of (63), at that

$$\dot{f}^i = \varepsilon^i a(x), \qquad \tilde{\nabla}_i f^i = na(x). \tag{64}$$

Formally the non-constant non-degenerate X-functions with $\tilde{\nabla}_i f^i = 0$ do exist, but they are trivial, since the scalar field $a(x)$ at that identically equal zero. Mark that the derivative of the non-degenerate X-function in the basis generally has an appearance of $e_1 = 1, e_2, ..., e_n$

$$\dot{F}(X) = a(x) + 0e_2 + 0e_3 + ... + e_n. \tag{65}$$

Let us find out the conditions for the product of two non-degenerate X-functions $F_{(1)}(X), F_{(2)}(X)$ to be a non-degenerate X-function $F_{(3)}(X)$ too. Since $|\,F_{(1)}(X)F_{(2)}(X)| = |\,F_{(1)}(X)|\,|\,F_{(2)}(X)|$,, the function $F_{(3)}(X)$ is non-degenerate.

All we have to do is to check the fulfillment of the formulae (63) for it. From the article [1] we take the formulae for the polynumber product of two generalized-analytical functions:

$$\{f^i_{(1)}; \gamma^i_{(1)k}\}\{f^i_{(2)}; \gamma^i_{(2)k}\} = \{f^i_{(3)}; \gamma^i_{(3)k}\}, \tag{66}$$

where

$$\gamma^i_{(3)k} = p^i_{i_1 i_2}(f^{i_2}_{(2)}\gamma^{i_1}_{(1)k} + f^{i_1}_{(1)}\gamma^{i_2}_{(2)k}). \tag{67}$$

Let us demand all γ-objects in the formulae (67) to have the appearance, defined by the formulae (63). Then, after some transforms we obtain:

$$a_{(3)}\delta^i_k = a_{(1)}p^i_{kj}f^j_{(2)} + a_{(2)}p^i_{kj}f^j_{(1)}. \tag{68}$$

These are the wanted conditions: two nondegenerate X-functions must satisfy them in order to their product be nondegenerate X-function too.

Conclusion

In this article have introduced the generalized-analytical functions of an arbitrary polynumber variable, which have been called the non-degenerate X-functions and they are the equivalent of function $F(z) = z$ of complex variable z. While these functions are not divisors of zero, they may define a congruence of geodetics in space $\{\mathbf{M_n}; \mathbf{P_n}\}$. At that the derivative of such function is a poly-numeral unity multiplied by a scalar field. Formally the non-constant non-degenerate X-functions with $\tilde{\nabla}_i f^i = 0$ do exist, but they are trivial, since the scalar field $a(x)$ at that identically equal zero. Possible, namely the non-degenerate X-functions shall play the very same fundamental role as the complex variable z does in theory of analytical functions of complex variable, i.e. non-degenerate X-function $F(z) = z$.

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References

[1] G. I. Garas'ko, *Generalized-analytical functions of polynumber variable,* Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004, 77 – 90, 2004.

[2] P. K. Rashevsky, *The Geometrical Theory of Partial Differential Equations*, 2-nd ed., Editorial URSS, M., 2003 (in Russian).

The Notions of Distance and Velocity Modulus in the Linear Finsler Spaces

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In the framework of 4-dimensional linear spaces with Berwald-Moor metrics, are determined formulas for the 3-dimensional distance and for the the velocity modulus. The used algorithm is applicable both for the Minkowski space and for an arbitrary multi-linear Finsler space with fixed time-like component. The constructed modulus coincides with the corresponding expression of the Galilean space for small (non-relativistic) velocities, while at maximal velocities (i.e., for world lines lying on the surface of the cone), this modulus equals unity. Further, the notion of the surface of relative simultaneity (employed in special relativity too) is used to construct the 3-dimensional distance. The formulas for the velocity transformation which describe the change between inertial frames are obtained as well. In the case when both velocities are directed along one of the three selected future straight lines, it is shown that the obtained relations coincide with the analogous relations of Special Relativity - unlike the general case. Moreover, for the Berwald-Moor space are obtained the expressions for the transformations which play the same role as the Lorentz transformations in the Minkowski space. It is proved that, if the 3-space coordinate axes are straight lines along which the velocities are added as in special relativity, then if considering the velocity of the new inertial frame collinear to the one of these coordinate axis, one can see that both the transformation of this coordinate and of the time coordinate coincide with the Lorentz transformations, while the transformations of the two transversal coordinates differ from the corresponding Lorentz transformations. The main addressed issues can be summarized as follows: physical interpretation of the main geometrical objects, definitions of distance and velocity modulus in the Minkowski space and in the *H4* space, addition of velocities, and transition from the motionless inertial frame to the moving one.

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Introduction

The geometry of the classical (non-relativistic) space is usually connected with the names of Galileo and Newton. It can be considered the second order approximation with regard to the small parameter (the ratio of the velocity modulus to the speed of light) of the Minkowski space geometry. But there are other geometries whose metrics is not quadratic, for which the corresponding limit transition leads to the Galilean space, that is, to the classical non-relativistic mechanics.

Starting with four dimensions that are definitely present in the physical world and wishing first of all to regard the simplest metrics of the fourth order, it seems necessary to begin with the linear space with the Berwald-Moor metrics. In one of the basis its interval can be represented as the product of four coordinates

$$S = \sqrt[4]{\xi_1 \xi_2 \xi_3 \xi_4}\,. \tag{1}$$

This space we designate as H_4 [1]. The metrics function (1) is a particular case of the more general metrics function ([2], [3])

$$S = \xi_1^{(1+r_1+r_2+r_3)/4} \xi_2^{(1+r_1-r_2-r_3)/4} \xi_3^{(1-r_1+r_2-r_3)/4} \xi_4^{(1-r_1-r_2+r_3)/4}, \tag{2}$$

for which all the parameters r_1, r_2, r_3 are set equal to zero. The important property of H_4 is that it is connected with the commutative associative algebra and has an analogue of the scalar product that can be introduced as a symmetric poly-linear form of several vectors [4].

Notice, that despite of its exotic view, the eq. (1) metrics can be regarded as a 4-dimensional generalization of the usual quadratic form characteristic for the pseudo-Euclidean plane.

$$S^2 = x_0^2 - x_1^2 \tag{3}$$

in the special basis constructed out of the isotropic vectors can be presented as

$$S^2 = \xi_1 \xi_2. \tag{4}$$

This is already enough to expect the space with the eq. (1) metrics to have properties close to the properties of the pseudo-Euclidean space (especially, 2-dimensional one), one of such properties being certain relativistic features.

Remark: To simplify the formulas we will usually write the tensor indices as subscripts and sometimes will not write the vector coordinates as the differences between their ends. This should not lead to errors or misunderstandings since we use only affine spaces and the lifting and lowering of the indices is not used.

Physical interpretation of the main geometrical objects

Regarding a 4-dimensional multiple set as a model of the space-time, one should first of all look for the effects taking place in its 3-dimensional subspace. The last one should be preferably able to be interpreted as the regular 3-dimensional classical space of the observer. In Minkowski space (and its Riemann generalizations) the Euclidean properties of its 3-dimensional subspace are present in the fundamental metrics form containing the positively defined quadratic components. As a result of this, the methodological problems of comparing the properties of such multiple sets with the properties of the real 3-dimensional space (undoubtedly close to the Euclidean geometry), arise only as the corollaries of the rejection of the absolute simultaneity.

Leaving the Minkowski space with its quadratic form for the Finsler space, particularly to H_4, where the intervals are expressed by the fourth-order form, the observer "living" in such a space could not be sure what kind of geometry he finds around him. To answer this question let us find out which objects of this multiple set are related to the common physical notions and values. But before that, let us first give the interpretations of the analogues geometric objects connected with the special relativity (SR). These interpretations are:

1. point in the 4-dimensional space – event;
2. straight line – world line of the inertial frame;
3. distance between the points on the straight line – interval between the events;
4. set of isotropic (with the zero interval) straight lines crossing at one point – light cone;
5. hyper-surface with the points that are equidistant from the fixed point – space-time hyper-sphere or set of events equidistant in the observer's proper time from the fixed event;
6. hyper-surface with the points equidistant from the two fixed points – set of the relatively simultaneous events in the selected inertial frame whose world line passes through the fixed points;
7. straight lines parallel to the fixed line – set of points that are motionless in the 3-dimensional space of the observer located in the fixed inertial frame.

For geometrical objects in H_4 (as well as for many other Finsler spaces) practically the same physical interpretations can be used. The differences reveal themselves only in particular cases, and for H_4 they constitute the following three facts: instead of a circular light cone there is a cone with flat sides; the set of relatively simultaneous events (i.e. the set of events equidistant from the two fixed points of space-time) is now not flat but it presents rather complicated hyper-surface; and instead of the pseudo-Euclidean sphere consisting of three hyperboloids (second order surfaces) there is now a hyper-surface consisting of 16 hyperboloids (fourth order surfaces). All these circumstances follow from the fact that now the interval is not the square root of quadratic form, but the fourth order root of the fourth order form (1).

The special basis in which the H_4 interval has the laconic form (1) is connected with the special isotropic vectors. In the analogous basis in SR the square of the interval looks rather unusually too:

$$S^2 = \xi_1\xi_2 + \xi_1\xi_3 + \xi_1\xi_4 + \xi_2\xi_3 + \xi_2\xi_4 + \xi_3\xi_4. \tag{5}$$

Such representation of the Minkowski space interval is rarely used, therefore, not to step away from the usual SR constructions, let us transform the metrics of

H_4 to the basis that is a Berwald-Moor analogue of the ortho-normal basis [5]. To do this we use the linear substitution:

$$\xi_i = A_{ij} x_j, \ (A_{ij}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \ A_{ik} A_{kj} = 4\delta_{ij} \tag{6}$$

– and obtain the following expression for the fourth power of the interval :

$$\begin{aligned} S^4 = x_0^4 - 2x_0^2(x_1^2 + x_2^2 + x_3^2) + 8x_0 x_1 x_2 x_3 + \\ + x_1^4 + x_2^4 + x_3^4 - 2x_1^2 x_2^2 - 2x_1^2 x_3^2 - 2x_2^2 x_3^2. \end{aligned} \tag{7}$$

Raising the square of the Minkowski interval

$$S^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{8}$$

to the second power to have the powers of four in both expressions, we get the polynomial similar to that in the r.h.s of eq. (7):

$$S^4 = x_0^4 - 2x_0^2(x_1^2 + x_2^2 + x_3^2) + x_1^4 + x_2^4 + x_3^4 + 2x_1^2 x_2^2 + 2x_1^2 x_3^2 + 2x_2^2 x_3^2. \tag{9}$$

In the regions characteristic for the non-relativistic physics where $|v_\alpha| \ll 1$, $v_\alpha = x_\alpha / x_0$, $\alpha = 1, 2, 3$, the expressions (7) and (9) coincide within the accuracy of the second power of small parameter $|v_\alpha|$. This justifies the mentioned limit transition from the H_4 geometry to the Galilean geometry, that is to the geometry of classical Newtonian physics.

Definitions of distance and velocity modulus in Minkowski space

The observer in Minkowski space who attributes the equal distances (in the 3-dimensional sense) to a certain set of events can follow a simple geometric rule – intercross two spheres with the same radii (hyperboloids in Minkowski space) located in two different centers (Fig.1). The straight line passing through the centers of these hyperbolic spheres can be associated with the inertial frame in which the events on the cross-section of the corresponding hyperboloids appear to be equidistant.

With no loss of generality, the centers of hyperboloids could be located on the time axis x_0 symmetrically from the coordinates origin, i.e., in the points $(-T, 0, 0, 0)$ and $(T, 0, 0, 0)$. To obtain the equation of the intersection of the two pseudo Euclidean spheres with radii S and with centers in these points, one has to solve the system of equations

$$\left. \begin{aligned} S^2 = (T + x_0)^2 - x_1^2 - x_2^2 - x_3^2, \\ S^2 = (T - x_0)^2 - x_1^2 - x_2^2 - x_3^2. \end{aligned} \right\} \tag{10}$$

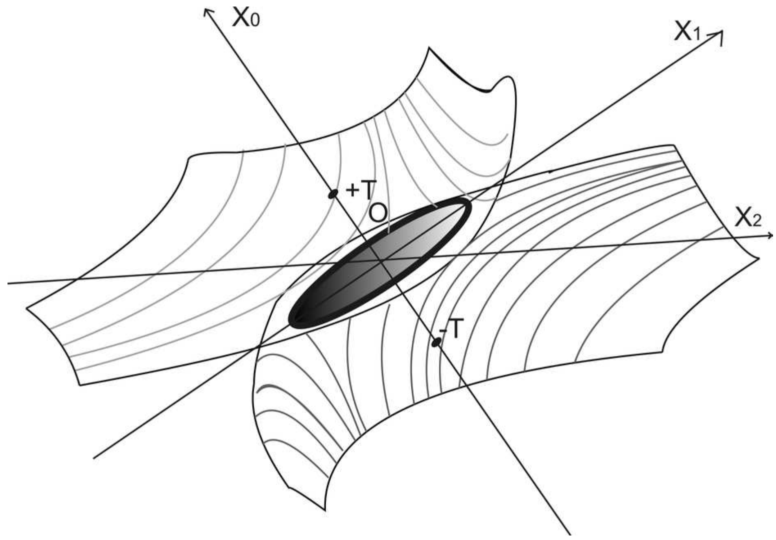

Figure 1: The cross-section of the hyperboloids in Minkowski space

After addition and subtraction of these equations, one gets

$$\left.\begin{aligned} S^2 = T^2 + x_0^2 - x_1^2 - x_2^2 - x_3^2, \\ 0 = 2Tx_0, \end{aligned}\right\} \tag{11}$$

The first of these equations describes the single-cavity spherical hyperboloid (time axis is the axis of its symmetry), the second equation describes the hyper-plane $x_0 = 0$ orthogonal to the x_0 axis (Fig. 2).

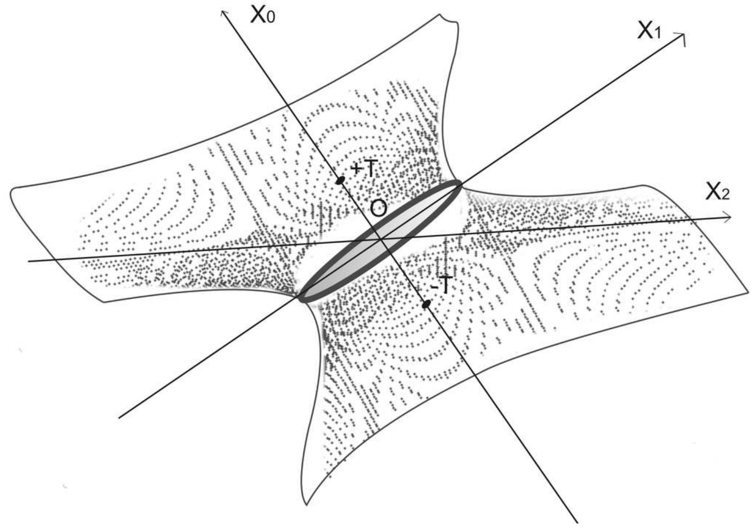

Figure 2: The cross-section of the plane and hyperboloid in Minkowski space

Varying the value of the interval S from 0 to T, we get a set of 2-dimensional surfaces put into one another. Each of these surfaces should be attributed a real positive value, which the observer in the fixed frame could call 'distance'. To do this, he should attribute these values to the arbitrary points of the surfaces and expand these values on all the other points of the surfaces. The simplest realization of this procedure is the drawing of a line across all the surfaces, then the linear parameter along this line could be used as 'distance'. In particular, one can use the x_1 axis as this line. Taking the linear rule for the correlation between the distance l and the coordinate x_1, i.e. substituting $x_0 = 0$, $x_1 = l$, $x_2 = 0$, $x_3 = 0$ into the first equation in (11), we obtain the expression

$$l = \sqrt{T^2 - S^2}, \tag{12}$$

Eq. (12) gives the relation between the distance l and the radius (interval) S of the hyperboloids that were used to find the surfaces with the same value of distances. This relation can be used to rewrite the first of eqs. (11) in the form of the well known in SR expression for the 3-dimensional distance between the x_0 axis and the world lines parallel to it:

$$l = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{13}$$

The described procedure of obtaining the expression for the distance is never used in SR, but it is equivalent to one that is used. Here we need such a complicated procedure to perform the analogous construction in H_4 space in which the SR algorithms do not lead to the result.

To obtain the 3-dimensional velocity in Minkowski space one can use similar speculations. Two points $(x_{(1)0}, x_{(1)1}, x_{(1)2}, x_{(1)3})$ and $(x_{(2)0}, x_{(2)1}, x_{(2)2}, x_{(2)3})$, (the second one is in the cone of future of the first one) define a vector with coordinates $(x_{(2)i} - x_{(1)i})$ that can be rewritten as

$$(x_{(2)i} - x_{(1)i}) \equiv (x_{(2)0} - x_{(1)0})v_i, \tag{14}$$

where $v_0 \equiv 1$, and the components v_1, v_2, v_3 generate the 3-dimensional velocity vector. Then the interval for these three points can be expressed with the help of the velocity components

$$S_{21} = (x_{(2)0} - x_{(1)0})\sqrt{1 - v^2}, \tag{15}$$

where

$$v = \sqrt{v_1^2 + v_2^2 + v_3^2}. \tag{16}$$

Notice, that the modulus of the 3-dimensional velocity in SR has the property

$$S_{21} = (x_{(2)0} - x_{(1)0})f(v), \tag{17}$$

where $f(v)$ is a function of one real variable. If vector $(1, v_1, v_2, v_3)$ and, consequently, vector $(x_{(2)0} - x_{(1)0},\ x_{(2)1} - x_{(1)1},\ x_{(2)2} - x_{(1)2},\ x_{(2)3} - x_{(1)3})$ approach the isotropic direction, then $v \to 1$.

Definition of the distance and the velocity modulus in the H_4 space

Let us take as a definition that in the H_4 space in the same way as in Minkowski space, the cross-section of the two spheres (hyperboloids) with equal radii but different centers is a set whose points are spatially equidistant from the observer whose world line passes through these centers. Apart from the analogy with the pseudo-Euclidean case, this statement is supported by the equality of the proper times for the signals emitted from the $(-T, 0, 0, 0)$ point and coming to the points of the cross-section of two hyperboloids and the proper times of the back signals emitted in these points and coming to the point $(T, 0, 0, 0)$. From the point of view of the observer whose world line passes through these points, i.e. coincides with the x_0 axis, and who can use only the information concerning himself and these signals, the latter reflect from the points of the 3-dimensional space equidistant from the observer. The total travel time (on the "signal's watch") appears to be equal to $2S$ for all pairs of signals and does not depend on the direction of travel. The watch of the observer, who considers himself motionless, will read the interval $2T$. Therefore, neither the readings of the signal's watch, nor the readings of the observer's watch do not contradict the suggestion that the distances from the observer to the world lines passing through the points of the cross-section of two hyperboloids, are the same. Consequently, they are completely characterized by the two values S and T.

To get the equation for the surface of the cross-section of the two hyperboloids with the centers in the points $(-T, 0, 0, 0)$ and $(T, 0, 0, 0)$ in the H_4 space, substitute first $(T + x_0, x_1, x_2, x_3)$ and then $(T - x_0, -x_1, -x_2, -x_3)$ instead of (x_0, x_1, x_2, x_3) into eq. (7):

$$\left.\begin{aligned} S^4 &= (T + x_0)^4 - 2(T + x_0)^2(x_1^2 + x_2^2 + x_3^2) + 8(T + x_0)x_1x_2x_3 + \\ &\qquad + x_1^4 + x_2^4 + x_3^4 - 2x_1^2x_2^2 - 2x_1^2x_3^2 - 2x_2^2x_3^2\,, \\ S^4 &= (T - x_0)^4 - 2(T - x_0)^2(x_1^2 + x_2^2 + x_3^2) - 8(T - x_0)x_1x_2x_3 + \\ &\qquad + x_1^4 + x_2^4 + x_3^4 - 2x_1^2x_2^2 - 2x_1^2x_3^2 - 2x_2^2x_3^2\,. \end{aligned}\right\} \tag{18}$$

Taking as in the case of Minkowski space the sum and the difference of these equations, one gets

$$\left.\begin{aligned} S^4 &= x_0^4 + 2x_0^2(3T^2 - x_1^2 - x_2^2 - x_3^3) + 8x_0x_1x_2x_3 + T^4 - \\ &- 2T^2(x_1^2 + x_2^2 + x_3^2) + x_1^4 + x_2^4 + x_3^4 - 2(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2)\,, \\ 0 &= x_0^3 + (T^2 - x_1^2 - x_2^2 - x_3^2)x_0 + 2x_1x_2x_3\,. \end{aligned}\right\} \tag{19}$$

It is not so easy to draw even schematically the 2-dimensional surfaces corresponding to eq. (18) in the 4-dimensional space. That is why to illustrate the result we will use the similar surface in the 3-dimensional case, Fig. 3, corresponding to

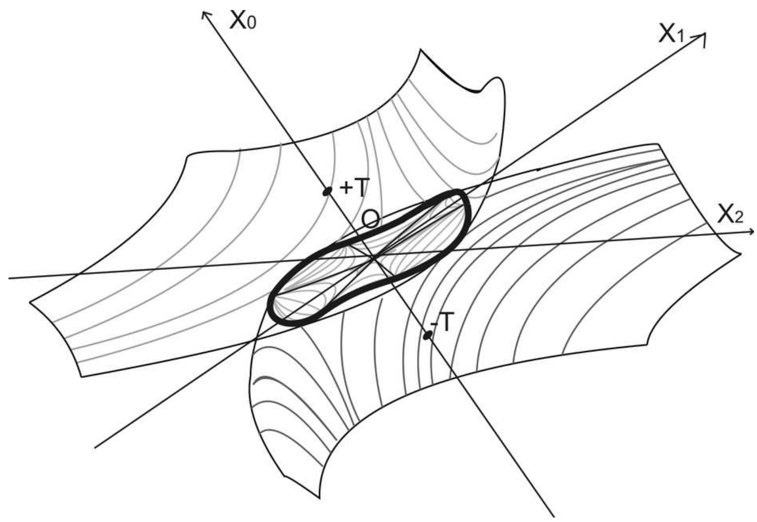

Figure 3: The cross-section of the hyperboloids in H_3 space

H_3 space, which is constructed similarly to H_4 and has the following metrics in the isotropic basis

$$S^3 = \xi_1\xi_2\xi_3. \tag{20}$$

Having passed from eqs. (18) to eqs. (19), we pass from the cross-section of two hyperboloids to the cross-section of two new hyper-surfaces. The first of them is in a sense equivalent to the single-cavern hyperboloid of the Minkowski space, and the second is analogous to the hyper-plane $x_0 = 0$ of the pseudo-Euclidean space, because there are equal intervals from every point of it to the points $(-T, 0, 0, 0)$ and $(T, 0, 0, 0)$, Fig.4. But now the second equation of eqs. (19) defines the essentially nonlinear surface, this being the result of using the Finsler metrics that has higher order than the quadratic one. From the physical point of view, such hyper-surface could be related to the notion of relative simultaneity. This is reasonable only in case when both the inertial frame is fixed, and the characteristic scale T (that gives the time between the instantaneous location of the observer and the event with regard to which the simultaneity is defined) are fixed. In the pseudo-Euclidean case this scale is unnecessary, since the hyper-surface related to the notion of the relative simultaneity remained the same for every interval separating the observer and the layer of the relatively simultaneous events. In the linear Finsler spaces this is not so, and this leads to the reconsideration of the properties of time, at least, for the spaces with the non-quadratic metrics.

The cross-section of the hyperboloids (18) with the centers at points $(-T,\ 0,\ 0,\ 0)$ and $(T,\ 0,\ 0,\ 0)$ is such a set of events that the observer whose world line passes through these points would consider equidistant from himself (from his world line). Varying the interval S from 0 to T, we obtain the set of 2-dimensional surfaces enclosed in each other, each of which corresponds to a certain spatial distance. To characterize each of these 2-dimensional sets with one and

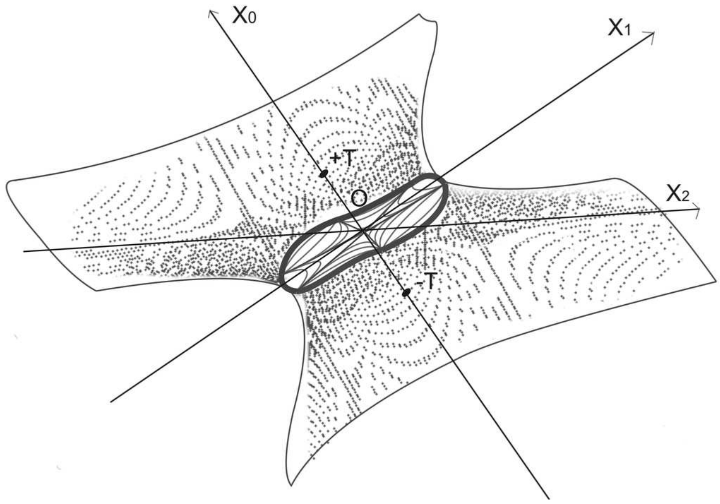

Figure 4: The cross-section of the special surface and hyperboloid in H_3 space

the same value of distance automatically, it is sufficient to attribute certain values of distances to at least one of the points on each surface, and then extend these values over all the points of the corresponding surface. As in the pseudo-Euclidean case mentioned above, one can take a straight line crossing all these surfaces, and call the linear parameter l along this line the 'distance' already not in the regular pseudo-Euclidean space, but in the linear Finsler space-time.

The analysis of eq. (19) shows that all the straight lines passing through the coordinates origin and lying on one of the three planes (x_1, x_2), (x_1, x_3) or (x_2, x_3) belong to the surfaces of relative simultaneity of the H_4 space, corresponding to this equation. Particularly, one of these lines is the x_1-axis, therefore, relating the distance l and the coordinate x_1, one gets the distance l from the observer to the motionless (with regard to him) observers for whom the initial hyperboloids have the radii equal to S and the half of the interval between their centers is equal to T. Substituting $x_1 = l,\ x_2 = 0,\ x_3 = 0$ into eq. (19), one gets

$$\left.\begin{aligned} S^4 &= x_0^4 + 2x_0^2(3T^2 - l^2) + T^4 - 2T^2l^2 + l^4, \\ 0 &= x_0^3 + (T^2 - x_1^2 - x_2^2 - x_3^2)x_0. \end{aligned}\right\} \tag{21}$$

The second equation gives $x_0 = 0$, therefore, the first equation gives

$$S^4 = T^4 - 2T^2l^2 + l^4. \tag{22}$$

Solving this equation for l, one gets

$$l = \sqrt{T^2 - S^2}. \tag{23}$$

Thus, the 3-dimensional distance from the world line $(0, 0, 0)$ to the parallel world line (x_1, x_2, x_3) is expressed by the formula

$$l(T, x_1, x_2, x_3) = \sqrt{T^2 - S^2(T, x_1, x_2, x_3)}, \tag{24}$$

where $S^2(T, x_1, x_2, x_3)$ is the square root of the r.h.s of the first of eqs. (19) in which x_0 is the real cubic root of the second of eqs. (19))).

The expression for the 3-dimensional distance, being essentially different from the regular spherically symmetric form (13), contains the parameter T lacking in SR. These differences lead to rather unusual properties of the 3-dimensional distances in H_4. Particularly, the distance from world line AA to the world line B is usually not equal to the distance from world line B to the world line A. But such effects reveal themselves only when any of the values $|x_\alpha|$ can not be neglected with regard to T. If we can neglect the third and higher powers of the ratio $|x_\alpha|/T$ with regard to unity, then the expression for the distance (24) takes the form

$$l(T, x_1, x_2, x_3) \simeq \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{25}$$

New qualitative feature that appears when constructing the surface of relatively simultaneous events in H_4 and that distinguishes it from Minkowski space case is the need for the concrete parameter T measured in the units of length. It seems logical to connect this characteristic scale, which is absent in SR, to the observer, that is to the reference frame, and interpret it as an additional parameter characterizing the reference frame and providing the possibility to construct the fixed surface of relative simultaneity.

Let us now pass to the 3-dimensional velocity.

Two points $(x_{(1)0}, x_{(1)1},\ x_{(1)2},\ x_{(1)3})$ and $(x_{(2)0},\ x_{(2)1},\ x_{(2)2},\ x_{(2)3})$ in the H_4 space (the last point is in the cone of future of the first one) define the vector with coordinates $(x_{(2)i} - x_{(1)i})$ that can be rewritten with the help of velocity as

$$(x_{(2)i} - x_{(1)i}) \equiv (x_{(2)0} - x_{(1)0})v_i, \tag{26}$$

where $v_0 \equiv 1$, while the components v_1, v_2 and v_3 form the 3-dimensional velocity vector. Then the interval between these two points can be expressed by the components of velocity as follows

$$S_{21} = (x_{(2)0} - x_{(1)0})\sqrt[4]{W}, \qquad \text{where} \tag{27}$$

$$W = (1 + v_1 + v_2 + v_3)(1 + v_1 - v_2 - v_3)(1 - v_1 + v_2 - v_3)(1 - v_1 - v_2 + v_3). \tag{28}$$

The modulus v of the 3-dimensional velocity in H_4 must have the property

$$S_{21} = (x_{(2)0} - x_{(1)0})f(v), \tag{29}$$

where $f(v)$ is a function of one real variable. If only one of the components of the 3-dimensional velocity differs from zero, for example, v_1, then, naturally, $v = |v_1|$, and expression (27) gives

$$S_{21} = (x_{(2)0} - x_{(1)0})\sqrt{1 - v^2}. \tag{30}$$

In general case, the speculations similar to those for 3-dimensional distance give

$$\sqrt{1 - v^2} = \sqrt[4]{W(v_1, v_2, v_3)}, \qquad \text{or} \tag{31}$$

$$v = \sqrt{1 - \sqrt{W(v_1, v_2, v_3)}}. \tag{32}$$

In the non-relativistic approximation

$$v \simeq \sqrt{v_1^2 + v_2^2 + v_3^2}. \tag{33}$$

If vector $(1, v_1, v_2, v_3)$, and, consequently, vector $(x_{(2)0} - x_{(1)0},\ x_{(2)1} - x_{(1)1},\ x_{(2)2} - x_{(1)2},\ x_{(2)3} - x_{(1)3})$ approach the isotropic direction, then $v \to 1$. Notice also, that in general case, $W(-v_1, -v_2, -v_3) \neq W(v_1, v_2, v_3)$.

Addition of velocities

The symmetry group $G_1(H_4)$ preserves invariant the interval (1) and consists of linear continuous transformations

$$x'_i = \frac{1}{4} A_{ik} D_{km} A_{mj} x_j, \tag{34}$$

where

$$(D_{km}) = diag(\exp\varepsilon_0, \exp\varepsilon_1, \exp\varepsilon_2, \exp\varepsilon_3), \tag{35}$$

The real parameters ε_i vary in $(-\infty, \infty)$ and suffice the condition

$$\varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0, \tag{36}$$

This group can be parameterized with the three real values, V_1, V_2, V_3 that can have the meaning of the components of velocity obtained by the motionless object after the transformation ((35)

$$\exp\varepsilon_i = \frac{A_{ij} V_j}{\sqrt{1 - V^2}}, \tag{37}$$

where $i, j = 0, 1, 2, 3$; $V_0 = 1$. If an object had the velocity components (v_1, v_2, v_3) in the initial reference frame, then in the new reference frame it will have

$$\left.\begin{aligned} v'_1 &= \frac{v_1 + V_1 + v_2 V_3 + v_3 V_2}{1 + v_1 V_1 + v_2 V_2 + v_3 V_3}, \\ v'_2 &= \frac{v_2 + V_2 + v_1 V_3 + v_3 V_1}{1 + v_1 V_1 + v_2 V_2 + v_3 V_3}, \\ v'_3 &= \frac{v_3 + V_3 + v_1 V_2 + v_2 V_1}{1 + v_1 V_1 + v_2 V_2 + v_3 V_3}. \end{aligned}\right\} \tag{38}$$

The definition of the $G_1(H_4)$ group gives

$$(x'_{(2)0} - x'_{(1)0})\sqrt{1 - (v')^2} = (x_{(2)0} - x_{(1)0})\sqrt{1 - v^2}, \tag{39}$$

Thus, the formula for the 3-dimensional velocity in the new reference frame is

$$v' = \sqrt{1 - \frac{(1-v^2)(1-V^2)}{(1+v_1V_1+v_2V_2+v_3V_3)^2}}, \tag{40}$$

because the transformations (34) – (37) give

$$x'_{(2)0} - x'_{(1)0} = \frac{1+v_1V_1+v_2V_2+v_3V_3}{\sqrt{1-V^2}}(x_{(2)0} - x_{(1)0}). \tag{41}$$

If the components of v_α and V_α contain only one component different from zero each, and these correspond to the same specially chosen direction, for example, $(v_1, 0, 0)$ and $(V_1, 0, 0)$, then the formulas (38) coincide with the corresponding formulas for addition of velocities in SR.

Transition from the motionless inertial frame to the moving one

In this Section we will regard the transition from the old (no strokes) reference frame to the new (stroked) inertial frame moving with the velocity (V_1, V_2, V_3) relatively to the old one. That is, the point that has the velocity (V_1, V_2, V_3) in the old frame will have the velocity $(0, 0, 0)$ in the new one. The formulas (34) – (36) will remain the same, while the formula (37) will be

$$\exp(-\varepsilon_i) = \frac{A_{ij}V_j}{\sqrt{1-V^2}}, \tag{42}$$

That is, the transitions from one frame to another considered here and in the previous Section are reverse to each other. Notice, that the change of (V_1, V_2, V_3) to $(-V_1, -V_2, -V_3)$ in (34) – (37) does not give the transition reverse to (34) – (37).

So, the transition to the frame moving with velocity (V_1, V_2, V_3) in the old coordinates can be expressed by the new ones as

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{4\sqrt{1-V^2}} \cdot \hat{A} \cdot \begin{pmatrix} (1+V_1+V_2+V_3)(x'_0+x'_1+x'_2+x'_3) \\ (1+V_1-V_2-V_3)(x'_0+x'_1-x'_2-x'_3) \\ (1-V_1+V_2-V_3)(x'_0-x'_1+x'_2-x'_3) \\ (1-V_1-V_2+V_3)(x'_0-x'_1-x'_2+x'_3) \end{pmatrix}, \tag{43}$$

where matrix $\hat{A}$ has the components A_{ij} (6).

Let us regard this transition for the case when all the components but one of the velocity of the new frame in the old frame coordinates along the three special directions are equal to zero, for example, $V_1 \neq 0$, but $V_2 = 0$ and $V_3 = 0$. Then

$$V = |V_1|, \tag{44}$$

and formulas (43) take the form

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-V_1^2}} & \frac{V_1}{\sqrt{1-V_1^2}} & 0 & 0 \\ \frac{V_1}{\sqrt{1-V_1^2}} & \frac{1}{\sqrt{1-V_1^2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-V_1^2}} & \frac{V_1}{\sqrt{1-V_1^2}} \\ 0 & 0 & \frac{V_1}{\sqrt{1-V_1^2}} & \frac{1}{\sqrt{1-V_1^2}} \end{pmatrix} \cdot \begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}, \tag{45}$$

or

$$\left. \begin{aligned} x_0 &= \frac{x'_0 + V_1 x'_1}{\sqrt{1-V_1^2}} \quad x_1 = \frac{V_1 x'_0 + x'_1}{\sqrt{1-V_1^2}} \\ x_2 &= \frac{x'_2 + V_1 x'_3}{\sqrt{1-V_1^2}} \quad x_3 = \frac{V_1 x'_2 + x'_3}{\sqrt{1-V_1^2}} \end{aligned} \right\}. \tag{46}$$

Such transformation of the coordinates $(x'_0, x'_1) \leftrightarrow (x_0, x_1)$ coincide with the corresponding transformation in SR, and the transformation $(x'_2, x'_3) \leftrightarrow (x_2, x_3)$ differs from the corresponding transformation in SR where $x_2 = x'_2$, $x_3 = x'_3$.

Conclusion

The H_4 space which is the space of associative commutative hyper-complex numbers (poly-numbers) is rather simple from the algebraic point of view – it is isomorphic to the algebra of the square diagonal real matrices 4×4. This space is an anisotropic metric Finsler space with the three parametric Abel symmetry group and it can not be reduced to a space with the quadratic metrics function. It is the simultaneous consideration of the algebraic and geometric properties of H_4 that leads to the appearance of a non-trivial mathematical object. As it was shown in this paper, the consideration of the physical contents of H_4 together with its algebraic and geometrical structures makes it even more complicated and interesting, despite its initial algebraic simplicity: in the non-relativistic limit (neglecting second and higher orders of the ratio of the velocity of the physical object to the velocity of light), it is indistinguishable both from the Galilean space (the classical mechanics space) and from the Minkowski space (SR). Moreover, even in the general case there are some special directions and 2-dimensional planes for which the properties of H_4 coincide with the corresponding properties of the Minkowski space for the same directions and planes.

The difference between the H_4 space and the Minkowski space is due to the anisotropy of the first one and to the physical effects proportional to the third and higher powers of the ratio of the velocities of the physical objects to the velocity of light. That is why, to our view, the question, which of the spaces is most adequate for the description of the real World is open. In any case, the need for the thorough investigation of the H_4 space and other similar spaces is even more obvious, since they happened to be left aside from the mainstream of modern geometric and physical research. This means that they could contain essences close to the properties of the real World.

It should be underlined that the approach developed in this paper to get the modulus of the 3-dimensional velocity and the 3-dimensional spatial distance is applicable for any linear Finsler space for which there is a special coordinate system where one time and three space coordinates can be separated.

References

[1] D. G. Pavlov, *Four-dimensional Time,* Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004, 33.

[2] G. Yu. Bogoslovsky, H. F. Goenner, *On the possibility of phase transitions in the geometric structure of space-time,* Phys. Lett. A 244, N 4, (1998) 222–228.

[3] G. Yu. Bogoslovsky, H. F. Goenner, *Finslerian spaces possessing local relativistic symmetry,* Gen. Relativ. Gravit. 31, N 10, (1999) 1565–1603.

[4] D. G. Pavlov, *Hypercomplex Numbers, Associated Metric Spaces, and Extension of Relativistic Hyperboloid,* ArXiv: gr-qc/0206004.

[5] D. G. Pavlov, *Generalization of scalar product axioms,* Hypercomplex Numbers in Geometry and Physics, **1**, Vol 1, 2004, 5.

Generalization of Conformal Transformations

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Conformal transformations of the Euclidean complex plane are deeply involved in many mathematical and physical-mathematical problems formulated on the plane, connected to completeness and in providing solutions for these problems. This is not the case for the Euclidean, pseudo-Euclidean or polynumber spaces of dimension greater than two. Using the concepts of analogical geometries, the author generalizes conformal transformations not only to the case of Euclidean or pseudo-Euclidean spaces, but also to the case of Finsler spaces, like this is done in spaces with affine connection. Examples of such transformations for the case of complex and hypercomplex numbers $H4$ are included, and it is shown that they form a group of transitions between projective Euclidean structures of a distinguished class which is fixed by a choice of metric structure in affine coordinates. It is pointed out that the relation between the generalized conformal transformations and generalized analytical functions might provide advances in solving fundamental problems in theoretical and mathematical physics.

MSC2000: 53B40, 83D05, 22F05, 30C35.

Introduction

Conformal transformations play a distinguished role in mathematics and physics. Riemannian and pseudo-Riemannian spaces of constant curvature are not less important (among such spaces are Lobachevsky space and spherical space), their homogeneity is as complete as in the case of a Euclidean space, since their motion groups have the same number of parameters as in the Euclidean case [1]. This work studies only Finslerian spaces admitting an affine coordinate system, so in the case of metric spaces, that is why we consider the length element to be the basic concept, and the concept of angle will be considered secondary. The proposed approach (of course, changed slightly) can be also applied for spaces (geometries) having the length element not defined, but with angles between vectors defined in each point.

If V_n is a Riemannian or a pseudo-Riemannian space with coordinates x^i and a metric tensor $g_{ij}(x)$, then the connection coefficients Γ^i_{kl} in this space are well-known to be defined by the following formula:

$$\Gamma^i_{kl}(g) = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right). \tag{1}$$

If

$$G_{ij}(x) = \Lambda(x) \cdot g_{ij}(x), \tag{2}$$

where $\Lambda(x) > 0$ is a scalar function defined on coordinates, then

$$\Gamma^i_{kl}(G) = \Gamma^i_{kl}(g) + \frac{1}{2\Lambda}\left(\frac{\partial\Lambda}{\partial x^l}\delta^i_k + \frac{\partial\Lambda}{\partial x^k}\delta^i_l - g^{im}\frac{\partial\Lambda}{\partial x^m}g_{kl}\right). \tag{3}$$

Spaces with metric tensors g_{ij} and G_{ij} are called conformally connected [1].

Since connectivity coefficients are transformed by the following formulas when changing the coordinate system:

$$\frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{kl} = \Gamma^{i'}_{n'p'}\frac{\partial x^{n'}}{\partial x^k}\frac{\partial x^{p'}}{\partial x^l} + \frac{\partial^2 x^{i'}}{\partial x^k \partial x^l}. \tag{4}$$

These are conformal transformations of coordinates, realized by functions f^i in some area $W_n \subset V_n$, where the metric tensor g_{ij} does not depend on the point of the space, and they satisfy the following system of equations:

$$\frac{\partial^2 f^i}{\partial x^k \partial x^l} = \frac{1}{2\Lambda}\left(\frac{\partial\Lambda}{\partial x^l}\delta^m_k + \frac{\partial\Lambda}{\partial x^k}\delta^m_l - g^{mp}\frac{\partial\Lambda}{\partial x^p}g_{kl}\right)\frac{\partial f^i}{\partial x^m}. \tag{5}$$

The convolution of both sides of the equations (5) and the tensor g^{kl} over both indexes gives us the following:

$$g^{kl}\frac{\partial^2 f^i}{\partial x^k \partial x^l} = \frac{2-n}{2\Lambda}g^{kl}\frac{\partial\Lambda}{\partial x^k}\frac{\partial f^i}{\partial x^l}. \tag{6}$$

Thus the functions realizing conformal transformation in Euclidean and pseudo-Euclidean spaces are the solutions of the differential equation (6).

For analytical functions of a complex variable (the first type conformal transformations of the Euclidean plane) and for complex conjugate analytical function of a complex variable (the second type conformal transformations of the Euclidean plane)

$$\Lambda = \left(\frac{\partial f^1}{\partial x^1}\right)^2 + \left(\frac{\partial f^1}{\partial x^2}\right)^2, \tag{7}$$

and equations (5) are valid in the area of analyticity and simple-connectedness.

Generalization of conformal transformations in Euclidean and pseudo-Euclidean spaces

The concept of analogical geometries was introduced in [2]. It is proposed to call geometries analogical in some areas, if these geometries have same dimensions and if there exists a mapping of one area onto another, under which some set of geodesics (extremals) of one geometry is mapped exactly on some set of geodesics (extremals) of the second geometry. Under certain assumptions the similarity of geometries means that there exist coordinate systems in which the differential equations of geodesics (extremals) coincide.

If in some geometry of affine connectedness we add to the connectivity coefficients the tensor

$$T^i_{kl} = \frac{1}{2}(p_k\delta^i_l + p_l\delta^i_k) + S^i_{kl}, \tag{8}$$

where p_i is an arbitrary covariant field, and S^i_{kl} is an arbitrary tensor field, antisymmetric with respect to the lower two indexes, then the geodesic curves will remain the same [1].

Let functions f^i map an area in a Euclidean or a pseudo-Euclidean space with a metric tensor g_{ij} bijectively onto another area in the same space, and suppose also that these functions satisfy the following system of equations:

$$\frac{\partial^2 f^i}{\partial x^k \partial x^l} = \left[\frac{1}{2}(p_l\delta^m_k + p_k\delta^m_l) - g^{mp}\frac{\partial L}{\partial x^p}g_{kl}\right]\frac{\partial f^i}{\partial x^m}\,, \tag{9}$$

where p_i is a covariant vector field and L a scalar field. Then this map (a coordinate transformation) will be called elementary generalized conformal.

Notice that in the case of using the additional term (8) with non-zero torsion tensor S^i_{kl} to obtain the formulas (9) instead of a generalization additional conditions appear:

$$S^m_{kl}\frac{\partial f^i}{\partial x^m} = 0\,, \tag{10}$$

as far as all the other additive terms in both sides of the system (9) are symmetric under the permutations of indexes k and l.

It follows from the definition of elementary generalized conformal transformations of Euclidean and pseudo-Euclidean spaces that these transformations and functions f^i realizing them are closely connected with the concept of projective Euclidean geometries [1].

Thus each function (a component) of an elementary generalized conformal transformation satisfies the following scalar equation:

$$g^{kl}\frac{\partial^2 f^i}{\partial x^k \partial x^l} = g^{kl}\left(p_k - \frac{n}{2}\frac{\partial L}{\partial x^k}\right)\frac{\partial f^i}{\partial x^l}. \tag{11}$$

Though for proper generalized conformal transformations the formula (2) is not valid, we will suppose by definition that

$$\Lambda = \Lambda_0 \cdot \exp(L). \tag{12}$$

In certain sense the scalar field Λ defined this way will be a characteristic for the squared coefficient of the space "stress-strain" under an elementary generalized conformal transformation.

To show the non-triviality of such a generalization let us perform a solution of the system (9):

$$f^i = \frac{x^i}{a + b\cdot g_{kl}x^k x^l}, \tag{13}$$

where a and b – are real numbers and

$$\Lambda = \frac{d}{(a - b \cdot g_{kl} x^k x^l)^2}, \tag{14}$$

where d is a real number.

In the case of the Euclidean (complex) plane (x, y)

$$z = x + iy, \qquad F(z) = f^1 + i f^2, \tag{15}$$

the function (13)

$$F(z) = \frac{z}{a + bz\bar{z}} \tag{16}$$

is neither analytical nor complex conjugate analytical when $a \neq 0$ and $b \neq 0$, but it realizes an elementary generalized conformal transformation of the plane. When $a = 0$ this function becomes complex conjugate analytical

$$F(z) = \frac{1}{b\bar{z}}, \tag{17}$$

which corresponds to a conformal map of the second type. When $b = 0$ the function $F(z)$ is analytical,

$$F(z) = \frac{1}{a} z, \tag{18}$$

which corresponds to a conformal map of the first type.

Polynumbers H_4

In the space H_4 the fourth power of the length element written in the basis ψ looks like

$$(ds)^4 = d\xi^1 d\xi^2 d\xi^3 d\xi^4, \tag{19}$$

and a conformally connected geometry will have the length element

$$(ds)^4 = \Xi d\xi^1 d\xi^2 d\xi^3 d\xi^4, \tag{20}$$

where $\Xi > 0$ is a scalar field. This geometry is similar to the geometry of affine connectedness with the connectivity coefficients [2]

$$\Gamma^i_{kj} = \frac{1}{2}(p_k \delta^i_j + p_j \delta^i_k) - p^i_{kj} \frac{1}{\Xi} \frac{\partial \Xi}{\partial \xi^{j_-}} + S^i_{kj}, \tag{21}$$

where

$$\psi_k \psi_j = p^i_{kj} \psi_i, \qquad p^i_{kj} = \begin{cases} 1, & \text{if } i = j = k, \\ 0, & \text{otherwise,} \end{cases} \tag{22}$$

p_k, $S^i_{kj} = -S^i_{jk}$ are arbitrary tensor fields.

Thus we obtain the system of equations for functions f^i, which realize an elementary generalized conformal transformation in the coordinate space of polynumbers H_4:

$$\frac{\partial^2 f^i}{\partial x^k \partial x^l} = \left[\frac{1}{2}(p_l \delta_k^m + p_k \delta_l^m) - p_{kl}^m \frac{\partial L}{\partial x^{l-}}\right] \frac{\partial f^i}{\partial x^m}, \tag{23}$$

where

$$\Xi = \Xi_0 \cdot \exp(L). \tag{24}$$

Any function analytical with respect to the variable H_4 realizing a one-to-one correspondence between two ares contained in the coordinate space of polynumbers H_4 satisfies the system (23), and at the same time

$$p_i = 0, \qquad \Xi = \dot{f}^1 \dot{f}^2 \dot{f}^3 \dot{f}^4, \qquad L = \ln|\Xi/\Xi_0|. \tag{25}$$

Functions analytical with respect to the variable H_4 are not the only solutions of the system (23). Another solution of this system is the function

$$f^i = \frac{f_0^i \ln\left|\dfrac{\xi^{i-}}{\xi_0^{i-}}\right|}{a + b\ln\left|\dfrac{\xi^1\xi^2\xi^3\xi^4}{\xi_0^1\xi_0^2\xi_0^3\xi_0^4}\right|}, \tag{26}$$

which becomes analytical with respect to the variable H_4 only when $b = 0$. In the formula (26) a, b, ξ_0^i, f_0^i are constants but, of course, they are not all independent. For the function (26)

$$\Xi = \frac{const}{\xi^1\xi^2\xi^3\xi^4}. \tag{27}$$

As far as in the space H_4 the following tensor can be defined

$$q_{ij} = p_{ik}^m p_{mj}^k, \qquad (q_{ij}) = diag(1,1,1,1), \tag{28}$$

there also exists a twice contravariant tensor q^{ij},

$$(q^{ij}) = diag(1,1,1,1). \tag{29}$$

This is why each component of an elementary generalized conformal transformation of H_4 should satisfy the following scalar equation:

$$q^{kl}\frac{\partial^2 f^i}{\partial x^k \partial x^l} = q^{kl}\left(p_k - \frac{\partial L}{\partial x^k}\right)\frac{\partial f^i}{\partial x^l}. \tag{30}$$

Comparing the equations (11), (30) and taking into account the formulas (12) (24), we see that the scalar equation (11), solutions of which are the functions realizing generalized conformal transformations in the four-dimensional Euclidean space, and the scalar equation (30) describing the functions realizing generalized conformal transformation in the space H_4 have the same structure:

$$\delta^{kl}\frac{\partial^2 f^i}{\partial x^k \partial x^l} = \delta^{kl}\left(p_k - 4\frac{\partial l}{\partial x^k}\right)\frac{\partial f^i}{\partial x^l}, \tag{31}$$

where the coefficient λ of linear "stress-strain" can be expressed in the terms of a scalar field l for the both four-dimensional Euclidean space and space H_4 with the same formula

$$\lambda = \lambda_0 \exp(l). \tag{32}$$

Notice, however, that we cannot claim that p_k and l are the same in the four-dimensional Euclidean space and in the space H_4. At the same time it would be very interesting to find such a class of elementary generalized conformal transformations, that for all its elements the covariant field $\left(p_k - 4\dfrac{\partial l}{\partial x^k}\right)$ would be the same in the four-dimensional Euclidean space and in the space H_4, i.e. that in both cases the functions f^i would satisfy the same scalar equation not only formally. Linear transforms automatically form a subset of such a class of transformations.

Generalized conformal transformations

The preceding constructions allow us to suppose that the system of equations defining elementary generalized conformal transformations of a metric geometry (at this moment Finsler geometry is developed more than enough for the needs of theoretical and mathematical physics) admitting affine coordinates and for which all its conformally connected spaces are always similar to some geometry of affine connectedness has the following most general view in the affine coordinates:

$$\frac{\partial^2 f^i}{\partial x^k \partial x^l} = \left[\frac{1}{2}(p_l\delta^m_k + p_k\delta^m_l) - \Delta^{pm}_{kl}\frac{\partial L}{\partial x^p}\right]\frac{\partial f^i}{\partial x^m}, \tag{33}$$

where Δ^{pm}_{kl} is a symmetric with respect to the lower indexes number tensor in an affine coordinate system of the initial metric geometry, L and p_k are a scalar and a covariant fields; and for conformal transforms the coefficient λ of linear "stress-strain" is expressed in the terms of the scalar field L with the formula

$$\lambda = \lambda_0 \exp(L/m) \equiv \lambda_0 \exp(l). \tag{34}$$

Here λ_0 is a real number and m is a natural number, equal to the order of the Finsler geometry form, by which the length element is expressed, for instance, for Euclidean and pseudo-Euclidean geometries $m = 2$ and for H_4-numbers $m = 4$.

It follows from the formulas (33) that any linear non-degenerate transformation is elementary generalized conformal with

$$p_i = 0, \qquad L = const. \tag{35}$$

Though we do hope that for all possible tensors Δ_{kl}^{pm} the concept of Finsler geometry is enough (it is possible that the concept of polynomial geometry [3] might be enough), this conjecture (same as the stronger one) needs a rigorous proof.

For non-degenerate polynumber spaces P_n there always exists a tensor q^{ij} (see (28), (29)), that is why in such spaces elementary generalized conformal transformations satisfy the following scalar equation:

$$q^{kl}\frac{\partial^2 f^i}{\partial x^k \partial x^l} = \left(p_k q^{km} - q^{kl}\Delta_{kl}^{pm}\frac{\partial L}{\partial x^p}\right)\frac{\partial f^i}{\partial x^m}. \tag{36}$$

Elementary generalized conformal transformations (33) do not form a group. But all their products (i.e. consequent executions) together with the inverse ones do form a group, which will be denoted as $G_n(\Delta_{kl}^{pm})$ and called a group of generalized conformal transformations. The elements of this group are the solutions of the system

$$\frac{\partial^2 f^i}{\partial x^k \partial x^l} = \left[\frac{1}{2}(p_l\delta_k^m + p_k\delta_l^m) - \Delta_{kl}^{pm}\frac{\partial L}{\partial x^p}\right]\frac{\partial f^i}{\partial x^m} - \left[\frac{1}{2}(p'_r\delta_s^i + p'_s\delta_r^i) - \Delta_{sr}^{pi}\frac{\partial L'}{\partial f^p}\right]\frac{\partial f^s}{\partial x^k}\frac{\partial f^r}{\partial x^l}, \tag{37}$$

where p_l, p'_k, L, L' are some fields, Δ_{sr}^{pi} is the same scalar tensor as in the system of equations (33); and the derivatives $\dfrac{\partial L}{\partial f^p}$ are meant to be explicitly expressed it terms of partial derivatives by x^i.

Generalized conformal transformations can be viewed as transitions in the uniquely characterized by the tensor Δ_{sr}^{pi} subset (class) of projective Euclidean spaces. Let us emphasize once again that it is enough to investigate elementary generalized conformal transformations, as far as an arbitrary generalized conformal transformation can be constructed as a product of an elementary and an inverse to an elementary transformation (of course another one if we do not wish to obtain an identity transform). Riemannian and pseudo-Riemannian conformally Euclidean spaces are always spaces of constant curvature [1], hence the proposed constructions define a class of Finsler spaces, which can be called Finsler spaces of constant curvature. Thus generalized conformal transformations form a group of transitions between the elements of such a class of Finsler spaces.

Generalized analytical functions

If the initial metric space with a number tensor Δ_{kl}^{pm} corresponding to it is polynumber $P_n \ni X$, then analytical functions realize conformal transformations in the area where the Jacobean of their coordinates is different from zero, and a concept of generalized analytical functions can be introduced in this space [3]. Of course, in this case functions realizing generalized conformal transformations are generalized analytical functions of the given polynumber variable. The following problem seems to be more interesting: find a class $\Upsilon(\Delta_{kl}^{pm}) \ni F(X)$ of generalized analytical functions, each element of which is a solution of the system (37).

Notice that if $F_{(1)}(X)$, $F_{(2)} \in \Upsilon(\Delta_{kl}^{pm})$, then $F_{(1)}\left(F_{(2)}\right) \in \Upsilon(\Delta_{kl}^{pm})$. It follows from the group properties of generalized conformal transformations.

A generalized analytical function of a polynumber variable $X \in P_n$,

$$F(X) = f^1(x^1, x^2, ..., x^n)e_1 + f^2(x^1, x^2, ..., x^n)e_2 + ... + f^n(x^1, x^2, ..., x^n)e_n, \tag{38}$$

$X = x^i e_i$, e_i is a basis, satisfies the correlations

$$\frac{\partial f^i}{\partial x^k} + \gamma^i_k = p^i_{kj}\dot{f}^j, \tag{39}$$

where $\dot{f}^j$ is a generalized derivative, tensor p^i_{kj} is defined by the correlations

$$e_k e_j = p^i_{kj} e_i, \tag{40}$$

and the object γ^i_k should change under transition to another coordinate system according to the following law

$$\gamma^{i'}_{k'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^i}\gamma^i_k - \frac{\partial x^k}{\partial x^{k'}}\frac{\partial^2 x^{i'}}{\partial x^k \partial x^i} f^i. \tag{41}$$

If ε^i are the coefficients of the unit's decomposition in the basis e_i then taking into account the following formula:

$$\varepsilon^k p^i_{kj} = \delta^i_j, \tag{42}$$

from the formula (39) we get

$$\dot{f}^i = \varepsilon^m \frac{\partial f^i}{\partial x^m} + \varepsilon^m \gamma^i_m \tag{43}$$

and an analogue of Cauchy-Riemann correlations:

$$\frac{\partial f^i}{\partial x^k} + \gamma^i_k - p^i_{kj}\left(\varepsilon^m \frac{\partial f^j}{\partial x^m} + \varepsilon^m \gamma^j_m\right) = 0. \tag{44}$$

The conditions of correlations (39) integrability (with respect to the functions f^i) are as follows:

$$\frac{\partial}{\partial x^m}\left(-\gamma^i_k + p^i_{kj}\dot{f}^j\right) = \frac{\partial}{\partial x^k}\left(-\gamma^i_m + p^i_{mj}\dot{f}^j\right). \tag{45}$$

If the polynumbers system P_n is non-degenerate and the generalized derivative is also a generalized analytical function $\{\dot{f}^i, \dot{\gamma}^i_k\}$ then each component f^i formally satisfies the following scalar equation:

$$q^{mk}\dot{\tilde{\nabla}}_m \tilde{\nabla}_k f^i = Q^i_r \ddot{f}^r, \tag{46}$$

where

$$Q^i_r = q^{mk} p^i_{kj} p^j_{mr}. \tag{47}$$

For analytical functions of a complex variable this equation becomes

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) f^i = 2\ddot{f}^i \tag{48}$$

and is identical. Thus the field $(2\ddot{f}^i)$ can be considered as the field of a field source f^i for the operator

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}. \tag{49}$$

Consider a two-dimensional non-homogeneous (with the right-hand side) hyperbolic equation in partial derivatives.

$$\left(\frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x_s^2}\right) u_s = f_s(t, x_s), \tag{50}$$

where t is time, x_s is the coordinate along the string, $u_s(t, x_s)$ is the amplitude of small lateral oscillations of the string, $\rho f_s dx_s$ is the lateral force acting on an element $(x_s, x_s + dx_s)$ of the string, ρ is the mass density. When changing the variables

$$f^i = u_s, \quad at = x, \quad y = x_s\,, \quad \frac{1}{a^2} f_s(t, x_s) = 2 f^i(x, y) \tag{51}$$

the equations (48) and (50) switch places but the right-hand side of the equation (48) is an analytical function of a complex variable (x, y), which restricts sufficiently the variety of sources.

Thus if the source function (the right-hand side) of a two-dimensional non-homogeneous hyperbolic equation (a wave equation) written in a special form (48) is an analytical function of a complex variable, then one of the solutions of this equation will be the second antiderivative of the source function divided by two.

Except for the equation (48) each analytical function of a complex variable satisfies the Laplace equation, which can be obtained analogically to how the equation (46) was obtained, having changed the tensor q^{mk} into the tensor g^{mk}, which is inverse to g_{ij}, the metric tensor of the Euclidean plane:

$$g^{mk} \dot{\tilde{\nabla}}_m \tilde{\nabla}_k f^i = 0 \quad \Rightarrow \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f^i = 0. \tag{52}$$

Similar equations are valid also for analytical functions of an H_2-variable,

$$X = x + jy, \qquad j^2 = 1, \tag{53}$$

but the elliptic and hyperbolic types of equations switch places:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f^i = 2\ddot{f}^i, \quad \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) f^i = 0. \tag{54}$$

So, if the source function (the right-hand side) of a two-dimensional non-homogeneous Laplace equation is an analytical function of an H_2-variable, then one of the solutions of this equation will be the second antiderivative of the source function divided by two.

Thus when changing $C \leftrightarrow H_2$ not only the wave equation and Laplace equation "switch places", but one of them loses the source (the non-homogeneous right side) and another one gains it. It is quite reasonable now to suppose that such symmetry might take place for polynumbers of dimension greater than two and not only for analytical but also for generalized analytical functions.

The scalar equation (46) for analytical functions of an H_4-variable in the coordinate system of the ψ-basis (22) becomes

$$\left(\frac{\partial^2}{\partial (\xi^1)^2} + \frac{\partial^2}{\partial (\xi^2)^2} + \frac{\partial^2}{\partial (\xi^3)^2} + \frac{\partial^2}{\partial (\xi^4)^2}\right) f^i = \ddot{f}^i, \tag{55}$$

or in the coordinate system (x^0, x^1, x^2, x^3) of the basis $\{1,\, j,\, k,\, jk\}$ consisting of the unit and three symbol units $j^2 = k^2 = (jk)^2 = 1$:

$$\left.\begin{aligned} &\xi^1 = x^0 + x^1 + x^2 + x^3,\ \ \xi^2 = x^0 + x^1 - x^2 - x^3,\\ &\xi^3 = x^0 - x^1 + x^2 - x^3,\ \ \xi^4 = x^0 - x^1 - x^2 + x^3 \end{aligned}\right\} \tag{56}$$

that same equation becomes

$$\left(\frac{\partial^2}{\partial(x^0)^2} + \frac{\partial^2}{\partial(x^1)^2} + \frac{\partial^2}{\partial(x^2)^2} + \frac{\partial^2}{\partial(x^3)^2}\right) f^i = 4\ddot{f}^i. \tag{57}$$

Thus if the source function (the right-hand side) of a four-dimensional non-homogeneous Laplace equation is an analytical function of an H_4-variable, then one of the solutions of this equation will be the second antiderivative of the source function divided by four.

Notice also that in the equations (48), the first one in (54) and (57) one can take an arbitrary linear combination of the source function's components and not change the coordinates, because the index i is free in both sides, and also use the symmetry (which the corresponding polynumbers do not have) of the scalar operators from the right-hand side to change the coordinates not "shuffling" the components of analytical functions. These circumstances extend in a way the corresponding set of source functions.

Conclusion

In the present paper a generalization of conformal transformations of a metric space is proposed. If we restrict ourselves to considering the spaces admitting affine coordinates then generalized conformal transformations of a given metric space can be considered as the group of transitions between the elements of some class of spaces of constant curvature.

If the problem of finding a one-to-one correspondence (modulo a discrete group of transformations) between generalized conformal transformations of the space P_n and generalized analytical functions of the polynumber variable P_n is solved then it is reasonable to hope to build a powerful mathematical instrument for solving mathematical problems and problems of theoretical physics appearing in the spaces P_n.

References

[1] P. K. Rashevsky, *Riemann Geometry and Tensor Analysis,* Science, M., 1967 (In Russian).

[2] G. I. Garas'ko, *Generalized analytical functions of a polynumber variable,* Hypercomplex Numbers in Geometry and Physics, **1** (2004), 75–88.

[3] D. G. Pavlov, *Generalization of axioms of scalar product,* Hypercomplex Numbers in Geometry and Physics, **1** (2004), 5–19.

Some Properties of Bicomplex Numbers

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In this article the commutative algebra of bicomplex numbers is considered with the metric $(+--+)$. This algebra of the 4-th rank has the properties of division, conjugation, taking the root and factorization together with the direct analog of the Euler's formula like usual complex numbers. It is shown, that rotations are representable in this algebra without obstruction of commutativity. The presence of divisors of zero is inherently related to a relativistic interval.

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I. Introduction

Objects like quaternions were proposed already in the 18-th century by Euler and Gauss, while the quaternions had got their classical realization in 1843 due to W. R. Hamilton as vector extension over the field of complex numbers [1, 2, 3, 4]. Their vector part present from itself a generalized imaginary part and form the three dimensional quaternion vector space. D. C. Maxwell has formulated the electrodynamics just in the language of quaternions, but they had not entered into the standard mathematical tool of the XX-th century. Only now they are included into the basic mathematical packages MathCAD^{TM} and Mathematica^{TM} discovering interesting applications in Computational Mathematics (for example, processing of images) and many physical domains including Mechanics and Special Theory of Relativity [4], Theory of Elementary Particles and Astrophysics, Field Theory and Optics. In Physics of beams of charged particles quaternions are effective, for example, in the solution of the problem of transportation of spin [5, 6]. Evidently, that despite elegance of differential geometry of quaternions the absence of commutativity does not permit their generalization in the region of theory of functions of hypercomplex variables. For the considered here numbers an attempt of such enlargement was done for the first time in the theory of functions of space complex variable (TFSCV, see. [7, 22]).

In analytical investigations and models we come non rarely across with expressions containing not only complex numbers, but 2×2 matrix also. Nevertheless, a matrix representation may have more convenient alternatives. Quantum Mechanics, for example, may be elegantly formulated with the help of Geometric Algebra. In others situations it is non rarely more suitable tackle with transformations of entirely commutative scalar expressions rather than traditional quaternions which are carriers of vector properties. It is especially actual in combination with functions of complex variable. The corresponding practical cases include, for example, an analysis of proper modes of some boundary problems [8, 9], a transportation of a beam of charged particles and its dynamics in accelerators [10] and electronic devices [11, 12]. It is possible to suppose, that the space of scalar or pseudoscalar numbers may implicitly include (encapsulate) elementary matrix transformations through enlarged properties of "scalar quaternions". This space is the union of two independent fields of usual complex numbers (associated as a rule with time and space coordinates).

L. Levin was one of the first [9] who practically implemented scalar bicomplex objects for analysis of electromagnetic waves spreading in different wave transferring structures: with dielectrics and ferromagnetic, surface anisotropy and crimps. He has introduced a phenomenologically additional imaginary unit (see (1.1)), that to distinguish complex numbers corresponding for different properties of time variable (and/or tangent phase coordinate) on the one side and space (either transversal/angle) variables on the other side. The corresponding imaginary units form the commutative group:

$$i^2 = -1, \ j^2 = -1, \ \ ij = ji \neq -1 \text{ or } \sqrt{-1}. \tag{1.1}$$

Using this approach Levin has gotten compact scalar dispersion equation for normal modes with four-component complex numbers. Further development of this method [13, 14, 15] allowed to characterize rigorously a self-consistent system, in which a beam interacts with a slow-wave structure in a solenoidal field. It was shown in [13], that the usual matrix approach gives an equivalent solution of a system of dispersion equations and finally leads to precisely the same increment of a threshold current of regenerative transversal instability of a "beam break-up" (BBU). Nevertheless, a usage of scalar quaternions simplifies significantly calculations and gives much more transparent physical solution. For example, a collective frequency $\widetilde{\nu}$ found algebraically from a unique hypercomplex dispersion equation has clear meaning of its components: $Re_i Re_j \widetilde{\nu}$ is the detuning of collective frequency relative to the eigenmode frequency; $Im_i Im_j \widetilde{\nu}$ is the angular velocity of rotation of a degenerate collective dipole mode; while $Im_i Re_j \widetilde{\nu} \pm Im_j Re_i \widetilde{\nu}$ give increments of right-hand and left-hand polarized collective modes of gyromagnetic instability. Note, that in work [10] a deficiency of additional imaginary unit has led to an incorrect mixing between degrees of freedom and an incorrect result for the threshold current of transversal instability.

The commutative algebra for the corresponding hypercomplex numbers was introduced in [13] for partial applications of physics of beams in accelerators in [7, 22] for more general region of physical problems. It was defined as a closed generalization over different i- and j- fields of complex numbers, which form a commutative algebra of the 4-th rank with division and basic attributes of usual complex numbers. In this article we give basic properties and the simplest analytical extension. We consider here as equivalent such terms as: ≪a four-component≫, ≪hypercomplex≫, ≪bicomplex≫ number and ≪a scalar quaternion≫. In the algebra of manifolds the considered numbers can be related to the bicomplex variety of poly-numbers.

II. Elementary properties of the commutative algebra of four-component numbers

We write down a four-component number, which looks like usual quaternion (but it is not such):

$$\widetilde{a} = \alpha_0 + i\alpha_1 + j\alpha_2 + ij\alpha_3, \tag{2.1}$$

where components $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are real; i, j are independent imaginary units and ij is the hypercomplex (compound) unit from (1.1).

We consider in this work hypercomplex numbers (2.1) as having commutativity and associativity, distributivity and closeness relative to multiplication and division.

In particular, the product of two simple complex numbers from different i- and j- spaces form "a scalar quaternion", representing the three dimensional space as the particular case of the four dimensional hyperspace (when $\left\| \begin{matrix} \alpha_0 & \alpha_1 \\ \alpha_2 & \alpha_3 \end{matrix} \right\| = 0$):

$$(a + ib) \cdot (c + jd) = \alpha_0 + i\alpha_1 + j\alpha_2 + ij\alpha_3, \quad \text{where } \alpha_0 = ac,\ \alpha_1 = bc,\ \alpha_2 = ad,\ \alpha_3 = bd. \tag{2.2}$$

It is possible to consider spaces of usual complex numbers as two dimensional projections of hypercomplex numbers. Therefore, it is natural to redefine operators of real and imaginary parts in the following way:

$$Re_i\,\widetilde{a} = \alpha_0 + j\alpha_2, \quad Im_i\,\widetilde{a} = \alpha_1 + j\alpha_3, \tag{2.3}$$

where the imaginary units i and j denote the corresponding projection space as the domain of action of the corresponding operation.

Consider now the Pauli matrices $\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{\sigma}_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$, $\hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as operators acting, for example, only in the j-space. Then putting $\widetilde{a}$ being the row matrix $\begin{pmatrix} Re_j\,\widetilde{a} \\ Im_j\,\widetilde{a} \end{pmatrix}$, we can transfer to the algebraic form, using the corresponding rules of substitution:

$$\hat{\sigma}_1\widetilde{a} \to j\widetilde{a}^{*j}, \quad \hat{\sigma}_2\widetilde{a} \to -\widetilde{a} \qquad \text{and} \quad \hat{\sigma}_3\widetilde{a} \to \widetilde{a}^{*j}, \tag{2.4}$$

that is, matrix operators may be formally presented as $\hat{\sigma}_1 \to j()^{*j}$, $\hat{\sigma}_2 \to -1$ and $\hat{\sigma}_3 \to ()^{*j}$.

Like the algebra of spin matrices from (2.4) we have an analogous relation:

$$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3\hat{\sigma}_3\widetilde{a} = \hat{\sigma}_1\hat{\sigma}_2\widetilde{a} = -j\hat{\sigma}_3\widetilde{a}; \quad \hat{\sigma}_2\hat{\sigma}_3\widetilde{a} = -j\hat{\sigma}_1\widetilde{a}; \quad \hat{\sigma}_3\hat{\sigma}_1\widetilde{a} = -j\hat{\sigma}_2\widetilde{a}.$$

The convenience is in the commutativity of operators in such writing. Thus, an arbitrary matrix 2×2 operator $\hat{U}$ in the complex (j-) space can be presented, for example, in such form:

$$\hat{U} \equiv \rho\hat{E} - j(\lambda\hat{\sigma}_1 + \mu\hat{\sigma}_2 + \nu\hat{\sigma}_3) \to \rho + j\mu + (\lambda - j\nu)()^{*j},$$

where $\hat{E}$ is the unit 2×2 matrix, $\rho^2+\lambda^2+\mu^2+\nu^2=\det\hat{U}$, and ρ,λ,μ,ν are real numbers describing related with the j-space operator $\hat{U}$.

It remains to generalize an action of a matrix 2×2 operator together with the corresponding representation of rotations on the entire $i\ j$- hyperspace. We can formally substitute the complex unit j on i in $\hat{U}$ and $\hat{\sigma}_2$ (that is, $\hat{\sigma}_2\to ij$):

$$\hat{U}=\rho\hat{E}-i\left(\lambda\hat{\sigma}_1+\mu\hat{\sigma}_2+\nu\hat{\sigma}_3\right)\to\rho+j\mu-\left(ij\lambda+i\nu\right)()^{*J}. \tag{2.5}$$

If $\hat{U}$ is a unimodular matrix and $\rho^2+\lambda^2+\mu^2+\nu^2=1$, then (2.5) represents rotations in the four dimensional i,j- space.

Before giving a definition of an entire length in this hyperspace we define a partial determinant in each projection space:

$$\det_i\tilde{a}=\left(\mathrm{Re}_i\tilde{a}\right)^2+\left(\mathrm{Im}_i\tilde{a}\right)^2=\tilde{a}\cdot\tilde{a}^{i*}\equiv|\tilde{a}|_i^2=\alpha_0^2+\alpha_1^2-\alpha_2^2-\alpha_3^2+2j\left(\alpha_0\alpha_2+\alpha_1\alpha_3\right). \tag{2.6}$$

From the rules of commutativity (1.1) and definitions (2.1, 2.3, 2.6) it follows the following evident identities:

$$\begin{aligned}
&\tilde{a}\cdot\tilde{b}=\tilde{b}\cdot\tilde{a},\\
&\mathrm{Re}_i\mathrm{Re}_j\tilde{a}=\mathrm{Re}_j\mathrm{Re}_i\tilde{a}=\alpha_0\equiv\mathrm{Re}_{ij}\tilde{a}=\mathrm{Re}_{ji}\tilde{a}\equiv\mathrm{Re}\,\tilde{a},\\
&\mathrm{Im}_i\mathrm{Re}_j\tilde{a}=\mathrm{Re}_j\mathrm{Im}_i\tilde{a}=\alpha_1,\\
&\mathrm{Im}_i\mathrm{Im}_j\tilde{a}=\mathrm{Im}_j\mathrm{Im}_i\tilde{a}=\alpha_3\equiv\mathrm{Im}_{ij}\tilde{a}=\mathrm{Im}_{ji}\tilde{a}\equiv\mathrm{Im}\,\tilde{a},\\
&(\tilde{a}^{*i})^{*j}\equiv\tilde{a}^{*i*j}=\tilde{a}^{*j*i}\equiv(\tilde{a}^{*j})^{*i}=\alpha_0-i\alpha_1-j\alpha_2+ij\alpha_3,\\
&\tilde{a}+\tilde{a}^{*i}=2\mathrm{Re}_i\tilde{a},\tilde{a}-\tilde{a}^{*i}=2i\mathrm{Im}_i\mathrm{Re}_j\tilde{a}+2j\mathrm{Re}_i\mathrm{Im}_j\mathrm{Re}_i\tilde{a},\\
&(\tilde{a}+\tilde{a}^{*i*j})+C.C._{\cdot i}=(\tilde{a}+\tilde{a}^{*i*j})+C.C._{\cdot j}=4\mathrm{Re}_i\mathrm{Re}_j\tilde{a}\equiv4\mathrm{Re}\,\tilde{a},\\
&\det{}_i\det{}_j\,\tilde{a}\equiv\left||\tilde{a}|_j^2\right|_i^2=\left||\tilde{a}|_i^2\right|_j^2\equiv\det{}_j\det{}_i\,\tilde{a}\equiv\|\tilde{a}\|^4.
\end{aligned} \tag{2.7}$$

The given previously rules and relations describe a simple scalar unification-superposition of two fields of complex numbers. These relations may be useful in some applications, where there is necessary a useful algebraic form of wave conductors [9], wake fields [8], polarimetry and analytical representation of magneto-static fields [14]). Nevertheless, for working with such forms it is necessary to construct a complete algebra of hypercomplex space, which is closed with respect to the operations of multiplication and division, raising to a power, taking a root.

Therefore we postulate additional to (1.1) rules, which give in general the following table of multiplication of base units:

1	i	j	k
i	-1	k	$-j$
j	k	-1	$-i$
k	$-j$	$-i$	1

Here $k \equiv ij$ is the hypercomplex unit and as it can be lightly seen the metric is $(+--+)$.

The remaining properties of «scalar quaternions » and the corresponding functional analytical extensions may be derived like the theory of complex numbers. For example, it is not difficult to see, that:

$$1/ij = ij; \quad \sqrt{1} = \pm 1, \ \pm ij; \quad ij = \exp\bigl(\pm(i+j)\pi/2\bigr) \tag{2.9}$$

Thus in this algebra of the fourth rang the quadratic root has four values. Another example is the rule of multiplication of hypercomplex numbers $\widetilde{a} = \alpha_0 + i\alpha_1 + j\alpha_2 + ij\alpha_3$
and $\widetilde{b} = \beta_0 + i\beta_1 + j\beta_2 + ij\beta_3$:

$$\begin{gathered}\widetilde{a} \cdot \widetilde{b} = \alpha_0\beta_0 + \alpha_3\beta_3 - \alpha_1\beta_1 - \alpha_2\beta_2 + i(\alpha_1\beta_0 + \alpha_0\beta_1 - \alpha_3\beta_2 - \alpha_2\beta_3) \\ +j(\alpha_2\beta_0 + \alpha_0\beta_2 - \alpha_3\beta_1 - \alpha_1\beta_3) + ij(\alpha_3\beta_0 + \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_0\beta_3).\end{gathered}$$

III. Conjugation and absolute value, divisors of zero

The complete conjugation can be defined through partial conjugations:

$$\tilde{a}^* = \tilde{a}^{*i}\tilde{a}^{*j}\tilde{a}^{*i*j} \tag{3.1}$$

We give several useful properties of conjugation which follow from (2.8):

$$\begin{gathered}\tilde{a} + \tilde{a}^{*i}\tilde{a}^{*j} + \tilde{a}^{*i*j} = 4\mathrm{Re}_i\mathrm{Re}_j\tilde{a} \\ \tilde{a}^{*i*j}\tilde{a} = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + 2ij(\alpha_3\alpha_3 - \alpha_1\alpha_2)\end{gathered} \tag{3.2}$$

and in the general case

$$\tilde{a} + \tilde{a}^{*i*j} \neq 2\mathrm{Re}\,\tilde{a}, \quad \tilde{a} + \tilde{a}^* \neq 2\mathrm{Re}\,\tilde{a}.$$

The natural means to define a complete determinant through partial determinants (2.6) is:

$$\begin{gathered}\det\tilde{a} = \det_i\det_j\tilde{a} \equiv \Bigl|\,|\tilde{a}|_j^2\Bigr|_i^2 \equiv \tilde{a}\tilde{a}^{*i}\tilde{a}^{*j}\tilde{a}^{*i*j} = \\ = \tilde{a}\cdot\tilde{a}^* = \left(\alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2\right)^2 + 4\left(\alpha_0\alpha_2 + \alpha_1\alpha_3\right)^2.\end{gathered} \tag{3.3}$$

It can be mentioned, that the determinant (3.3) may vanish for some nonzero components α_n. The corresponding numbers are divisors of zero. We have the matter of such numbers, for example, when either $|\alpha_0| = |\alpha_3| \neq 0$ for $\alpha_1 = \alpha_2 = 0$, or when $|\alpha_1| = |\alpha_2| \neq 0$ for $\alpha_0 = \alpha_3 = 0$.

Complete determinants in contradistinction to partial determinants are real and nonnegative. Therefore, we define an absolute value (or through an arithmetical root of the fourth order):

$$\|\tilde{a}\| \equiv N(\tilde{a}) = \sqrt[4]{\det \tilde{a}} \equiv \sqrt[4]{\tilde{a}\tilde{a}^{*i}\tilde{a}^{*j}\tilde{a}^{*i*j}} = \sqrt[4]{\left|\left|\tilde{a}\right|_j^2\right|_i^2}. \tag{3.4}$$

Note, that the numbers $i \pm j$, $1 \pm ij$ have the zero norm (or the hyper-length). As we shall see below the numbers $2\pi(i \pm j)$ and $\pi(i \pm j)$ are the hyper-periods for the hyperbolic functions $\cosh(\widetilde{x}), \sinh(\widetilde{x})$ and $\tanh(\widetilde{x})$, $\operatorname{cotanh}(\widetilde{x})$, as well as $2\pi(1 \pm ij)$ and $\pi(1 \pm ij)$ are the hyper-periods for the trigonometric functions $\cos(\widetilde{x}), \sin(\widetilde{x})$ and $\tan(\widetilde{x})$, $\operatorname{cotan}(\widetilde{x})$ respectively.

It can be easily seen, that (3.4, 3.3) coincide with expressions for the norm obtained in [7, 22] for polynomial and bi-exponential as well representation of a bicomplex (in polar system of coordinates for each of projection space). It is interesting, that for «three dimensional» bicomplex (2.2) we have the length: $\|\widetilde{a}\| = \sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2}$, which reduces to the Euclidean form due to the relation $\alpha_0\alpha_3 = \alpha_1\alpha_2$. Indeed, for a conventional vector $\{x, \ y, \ z\}$ to get $x^2 + y^2 + z^2 = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2$ the corresponding components of a bicomplex can be found, for example, as follows:

$$\alpha_0 = x, \ \alpha_1 = y, \ \alpha_2 = xz/\sqrt{x^2 + y^2}, \ \alpha_3 = yz/\sqrt{x^2 + y^2}.$$

The complete determinant introduced above may be used directly for a search of inverse value of a bicomplex number with the zero norm:

$$\tilde{a}^{-1} \equiv \frac{1}{\tilde{a}} = \frac{\tilde{a}^*}{\det \tilde{a}}. \tag{3.5}$$

One can obtain (3.5) also through subsequent transformations in projection spaces applying the rules given above we get:

$$\frac{1}{\tilde{a}} = \frac{\tilde{a}^{*j}}{\left|\tilde{a}\right|_j^2} \equiv \frac{\tilde{a}^{*i}}{\left|\tilde{a}\right|_i^2} \equiv \frac{\tilde{a}^{*i}}{\tilde{a} \cdot \tilde{a}^{*i}} = \frac{\tilde{a}^{*i} \cdot \left(\tilde{a}\tilde{a}^{*i}\right)^{*j}}{\tilde{a}\tilde{a}^{*i} \cdot \left(\tilde{a}\tilde{a}^{*i}\right)^{*j}} = \frac{\tilde{a}^{*i}\tilde{a}^{*j}\tilde{a}^{*i*j}}{\tilde{a}\tilde{a}^{*i}\tilde{a}^{*j}\tilde{a}^{*i*j}} = \frac{\tilde{a}^*}{\|\tilde{a}\|^4}.$$

Inverted divisors of zero (hyper-zeros) may be interpreted as hyper-infinities of the ij-algebra.

It is not difficult to verify, that the norm (a determinant) of a product is equal to the product of norms (determinants):

$$\left\|\tilde{a} \cdot \tilde{b}\right\| = \|\tilde{a}\| \cdot \left\|\tilde{b}\right\|, \quad \det_{i,j} \tilde{a}\tilde{b} = \det_{i,j} \tilde{a} \det_{i,j} \tilde{b}.$$

Apart from complex numbers and conventional modules the modules of hypercomplex numbers may be, nevertheless, smaller than one of its components. Along with presence of divisors of zero it makes the Frobenius' theorem inapplicable ([17], [22]) for the given metric.

We show that an interval between two events is similar to a norm of a bicomplex. For a bi-square of an interval we have: $dS^4 = c^4dt^4 + dl^4 - 2c^2dt^2dl^2$. On the other hand, the determinant (3.3) has the form:

$$\det \tilde{a} = \left(\alpha_0^2 + \alpha_1^2\right)^2 + \left(\alpha_2^2 + \alpha_3^2\right)^2 + 4\left(\alpha_0\alpha_2 + \alpha_1\alpha_3\right)^2 - 2\left(\alpha_0^2 + \alpha_1^2\right)\left(\alpha_2^2 + \alpha_3^2\right).$$

We complement the definition $\det \widetilde{a} = dS^4$ with the following:

$$c^2dt^2dl^2 = \left(\alpha_0^2 + \alpha_1^2\right)\left(\alpha_2^2 + \alpha_3^2\right). \tag{3.6}$$

The system of these two equations can be reduced to a conventional square equation in the domain of real numbers:

$$T^2 - \left(\left(\alpha_0^2 + \alpha_1^2\right)^2 + \left(\alpha_2^2 + \alpha_3^2\right)^2 + 4\left(\alpha_0\alpha_2 + \alpha_1\alpha_3\right)^2\right)T + \left(\alpha_0^2 + \alpha_1^2\right)^2\left(\alpha_2^2 + \alpha_3^2\right)^2 = 0,$$

solutions of which always exist and are equal to $T_1 = c^4dt^4$, $T_2 = dl^4$, since the determinant of the equation

$$D = \left(\left(\alpha_0^2 + \alpha_1^2\right)^2 - \left(\alpha_2^2 + \alpha_3^2\right)^2\right)^2 + 4\left(\alpha_0\alpha_2 + \alpha_1\alpha_3\right)^2\left(\left(\alpha_0^2 + \alpha_1^2\right)^2 + \left(\alpha_2^2 + \alpha_3^2\right)^2 + 4\left(\alpha_0\alpha_2 + \alpha_1\alpha_3\right)^2\right)$$

is always positive.

Thus, the norm of a bicomplex number may be presented in a form of a relativistic interval. In an important particular case, when $\alpha_2 = 0 = \alpha_1$, we have

$$|cdt| = |\alpha_0|, \quad |dl| = |\alpha_3| \qquad \text{(see below (4.3))}.$$

IV. The Euler's formula, factorization and taking a root

Prior defining the rooting for an arbitrary bicomplex we consider two particular cases.

The first case is related to a product of two complex numbers $a + ib$ and $c + jd$ (see (2.2)). This simple case corresponds to the matrix 2×2 operator (or rotation) applied to «plane» vector (that is, usual complex number), belonging to the i-space ($Im_j = 0$). Indeed, from (3.3) and (2.2) we have $\|\widetilde{a}\| = \sqrt{a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2}$ and from (2.5): $\hat{U} \to \rho - i\nu + j\mu - ij\lambda$ putting the «scalar quaternion» $\{\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3\}$ proportional $\{\rho,\ -\nu,\ \mu,\ -\lambda\}$.

Evidently, in this case a root of the n-th order is taken trivially:

$$\sqrt[n]{\widetilde{a}} = \sqrt[n]{(a + ib)\cdot(c + jd)} = \sqrt[n]{\|\tilde{a}\|}\exp\left[(i\arctan b/a + j\arctan d/c + 2\pi(ki + lj))/n\right], \tag{4.1}$$

where $k, l = \{0, 1..., n-1\}$ are natural numbers.

Thus, the period of the exponential function in our hyper-space is $2\pi(ki + lj)$. In the general case this gives n^2 values for $\sqrt[n]{\widetilde{a}}$.

Another interesting case is a hypercomplex represented by only two components: $\widetilde{A} = a + ijd$. From (2.8) and the Taylor's expansion it is possible to get the basic formula for such number:

$$\exp(ij\varphi) = \cosh\varphi + ij\sinh\varphi. \tag{4.2}$$

For $|d/a| \neq 1$ there is the following representation:

$$\tilde{A} = a + ijd = \sqrt{|a^2 - d^2|}\exp\left(ij\,\operatorname{arctanh}\frac{d}{a}\right) \tag{4.3}$$

Note that $\operatorname{arctanh} d/a$ is real for $|d/a| < 1$ otherwise it is complex in either i-, or in the j- space. Analogously, for $|d/a| > 1$ there exists an additional «symmetric» representation in the i, j-space

$$\begin{aligned}\tilde{A} = ij\,(d + ija) = ij\sqrt{|a^2 - d^2|}\exp\left(ij\,\operatorname{arctanh} a/d\right) = \\ = \sqrt{|a^2 - d^2|}\exp\left((i + j)\,\pi/2 + ij\,\operatorname{arctanh} a/d\right)\end{aligned} \tag{4.4}$$

From the comparison of (4.4) and (4.3) we get the hyper-extension of the known formula $\arctan x = (\pi/2)\,sgnx - \arctan(1/x)$:

$$\operatorname{arctanh} x = -\,(i + j)\,\frac{\pi}{2} sgn\,(|x| - 1) + \operatorname{arctanh}\frac{1}{x}. \tag{4.5}$$

Thus the range of definition of inverse hyperbolic tangent is expended over entire real axis resulting in extension of the function values into the space of scalar quaternions.

Using (4.3), we can extract square root from a simple two-component number $\tilde{B} = \tilde{A}^2 = b + ijc$:

$$\sqrt[n]{\tilde{B}} = \sqrt[n]{\left\|\tilde{B}\right\|}\exp\left(\frac{2\pi(ki + lj)}{n} + \frac{1}{n} ij\,\operatorname{arctanh}\left(\frac{c}{b}\right)\right) \tag{4.6}$$

where $|c/b| \neq 1$ and $k, l = \{0, 1..., n - 1\}$.

Supposing that $\widetilde{B} = \widetilde{A}^2$ it is possible to accomplish a verifying comparison for $\tilde{A} \equiv a + ijd$ and $\sqrt{\widetilde{B}}$. Substituting in (4.6) $b = a^2 + d^2$ and $c = 2ad$ we have:

$$\sqrt{\tilde{B}} = \pm\sqrt{|a^2 - d^2|}\cdot\left(\cosh\frac{\varphi}{2} + ij\sinh\frac{\varphi}{2}\right), \quad \text{where} \quad \varphi = \operatorname{arctanh}\left(\frac{2ad}{a^2 + d^2}\right). \tag{4.7}$$

Simple transformations of hyperbolic functions in (4.7) give:

$$\sqrt{(a + ijd)^2} = \begin{pmatrix} \pm a \pm ijd \\ \pm d \pm ija \end{pmatrix}, \tag{4.8}$$

where different combinations of signs give eight values of the radical $\sqrt{\widetilde{B}}$. Nevertheless, only four among them are linearly independent in the sense of (2.9), whereas others are obtained by the way of the multiplication on ij.

In general, when $\left\|\widetilde{A}\right\| \neq 0$, we can generalize the Euler's formula in the following way:

$$\tilde{A} \equiv a + ib + jc + ijd = \exp\left(\alpha_0 + i\alpha_1 + j\alpha_2 + ij\alpha_3\right) \equiv \exp(\tilde{a}) \qquad (4.9)$$

where a relation between $\widetilde{A}$ and $\widetilde{\alpha}$ may be found from the system:

$$\alpha_0 = \ln\left\|\tilde{A}\right\|, \qquad \text{and}$$
$$\begin{cases} b_N = \sin\alpha_1\cos\alpha_2\cosh\alpha_3 - \cos\alpha_1\sin\alpha_2\sinh\alpha_3 \\ c_N = \cos\alpha_1\sin\alpha_2\cosh\alpha_3 - \sin\alpha_1\cos\alpha_2\sinh\alpha_3 \\ d_N = \sin\alpha_1\sin\alpha_2\cosh\alpha_3 \end{cases}, \qquad (4.10)$$

where $b_N = b/\left\|\widetilde{A}\right\|$, $c_N = c/\left\|\widetilde{A}\right\|$ and $d_N = d/\left\|\widetilde{A}\right\|$ are normalized components.

Like a three dimensional rotation presented by a usual quaternion [1], (4.9–4.10) present the rotation $\alpha_1,\ \alpha_2,\ \alpha_3$ in the considered hyper-space. The degenerate case (2.2, 4.1) may be interpreted by analogy with the Cardan suspension (when $\alpha_3 = 0$ in (4.9)).

Note that apart from usual complex numbers and cases (2.2, 2.5, 4.1) the normalized components $b_N,\ c_N,\ d_N$ in the general case may vary in the entire real domain from $-\infty$ up to $+\infty$.

There (4.10) can be reduced to the algebraic system of two variables $\tan\alpha_1$ and $\tan\alpha_2$:

$$\begin{cases} \tan^2\alpha_1 - \tan^2\alpha_2 = \left(b_N^2 - c_N^2\right)\left(1 + \tan^2\alpha_1\right)\left(1 + \tan^2\alpha_2\right) \\ \left(b_N\tan^2\alpha_1\tan\alpha_2 + c_N\tan\alpha_1\tan^2\alpha_2 = d_N\right)\left(\tan^2\alpha_1 - \tan^2\alpha_2\right) \end{cases} \qquad (4.11)$$

and

$$\alpha_3 = \ln\left(\sin\left(\alpha_1 + \alpha_2\right) / \left(c_N + b_N\right)\right). \qquad (4.12)$$

System (4.11) may be resolved in an explicit form, but the expressions we obtained with the help of symbol methods appear to be too complicated, that to give them here.

To provide in (4.12) $\sin\left(\alpha_1 + \alpha_2\right) / \left(c_N + b_N\right) > 0$ one can always find suitable solutions (4.11) in the form $\alpha_{1,2} + \pi m$ due to the periodicity of the tangent. For $b = -c$ we formally have a singularity in (4.12). Nevertheless, this singularity is removable by the way of complex conjugation (4.9–4.12) (in the i- or j- space) and applying the conjugation once again (in the same space) to a result obtained on the right side.

For taking a root it can be proposed also another way of factorization of «the scalar quaternion» (2.2) in multipliers:

$$\alpha_0 + i\alpha_1 + j\alpha_2 + ij\alpha_3 = (a + ib)\cdot(c + jd)\cdot(e + ijf), \qquad (4.13)$$

where a, b, c, d, e, f are real. Put in (4.13) for simplicity, that $\alpha_0 = 1 = a = c = e$. Then (4.13) leads to the following algebraic system:

$$\begin{cases} \alpha_3 = bd(1 - bdf) + f \\ \alpha_2 = d(1 - bdf) - bf \\ \alpha_1 = b(1 - bdf) - df \end{cases} \tag{4.14}$$

The solutions b, d, f of System (4.14) are expressible in an explicit form much more compactly, than the solution of the System (4.11). It can be shown, that solutions of (4.14) always exist and they are real. For one of the solutions there is a singularity (for example, for $\alpha_2 + \alpha_1\alpha_3 = 0$), which is removable. Thus, a nonzero ($\left\|\widetilde{A}\right\| \neq 0$) «scalar quaternion» can be represented as a product of elementary multipliers (two complexes and one hypercomplex), and the root can be taken in accordance with (4.13), (4.9), either (4.1), or (4.6).

V. Differentiability

Let us take a hypercomplex function of the argument $\tilde{z} = x + iy + js + ijt$ in the general form: $\tilde{f}(\tilde{z}) = u(x, y, s, t) + iv(x, y, s, t) + jw(x, y, s, t) + ijq(x, y, s, t)$. The Cauchy-Riemann conditions are expressible, as it is known by $n(n-1)$ number of equations, where n is the dimension of a space. A function $\tilde{f}(\tilde{z})$ is differentiable, if the following twelve relations are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial s} = \frac{\partial q}{\partial t}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{\partial q}{\partial s} = -\frac{\partial w}{\partial t},$$

$$\frac{\partial w}{\partial x} = \frac{\partial q}{\partial y} = -\frac{\partial u}{\partial s} = -\frac{\partial v}{\partial t}, \qquad \frac{\partial q}{\partial x} = -\frac{\partial w}{\partial y} = -\frac{\partial v}{\partial s} = \frac{\partial u}{\partial t} \tag{5.1}$$

In the work [16] an attempt of a nontrivial generalization of a notion of an analytical function was done, for which a number of conditions analogous to that of Cauchy-Riemann would not exceed a number of component functions. In the work [22] it was done with elegant compactization of Cauchy-Riemann conditions in the more familiar form, where the variables are generalized, that is, not single dimensional but two dimensional (conventional complex) arguments from projection space were taken. In the case when these arguments are representable in the three dimensional form (2.2), the Cauchy-Riemann conditions take a visual and simple form, and in the four-dimensional hyper-space of arguments Conditions (5.1) can be reduced to six complex equations using complex exponents and complex projections of bicomplex functions [22].

VI. Discussion

Among a variety of associative algebras (see, for example, [17, 18, 19]) the given associative-commutative algebra has the complete heritage of properties of complex numbers. A key possibility is in construction of the corresponding theory of a function of a hypercomplex variable [20]. Divisors of zero, usually understandable as world's lines [20], coincide with generalized functions in this algebra, which points out on a fundamental importance of these objects. Divisors of zero are inherently related to a relativistic interval, which in its turn is absolutely similar to a norm of a hypercomplex number in the complete agreement with the classical statements of STR.

Thus, the serious steps in the development of bicomplex numbers are made, new foundations of their differential and integral calculus are constructed, mappings by conformal type, «hyper-surfaces» and some analytical extensions [22], generalized-analytic functions of hypercomplex variables [18].

The application of this hyper-space has shown an effectiveness in problems with a outlined direction of a space-time interaction (especially of that of quasi-periodic), propagation and energy exchange.

One can expect a further development of a further development and new applications of this ij-algebra and the corresponding TFSCV in the Fundamental, Mathematical and Applied Physics, Computational Mathematics, Biophysics and Molecular Chemistry.

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References

[1] W. R. Hamilton, *Lectures on quaternions,* Dublin, Hodges & Smith, 1853; *Elements of Quaternions,* Chesley Publ. Co. N.Y., 1969.

[2] G. Korn, T. Korn, *Mathematical Handbook,* McGraw-Hill Company. 1968;

[3] E. A. Carataev, *Quaternions and three dimensional rotations,* http://karataev.nm.ru/quat3rot.pdf

[4] Hamaker J. P., *Understanding radio polarimetry,* Astron. Astrophys. Suppl. Ser. 143, (2000) 515–534; http://aanda.u-strasbg.fr:2002/articles/aas/ps/2000/09/h1201.ps.gz

[5] A. P. Efremov, *Quaternions: algebra, geometry and physical theories,* Hypercomplex Numbers in Geometry and Physics, 1, 111–127, http://polynumbers.ru

[6] G. H. Hoffstaetter, *Successive approximations for charged particle motion,* arXiv: physics/0006008 v 1, 5 Jun 2000

[7] K. Heinemann and G. H. Hoffstatter, Phys. Rev. E, 54 (1996) 4240; Official DESY Report 96-078.

[8] V. I. Eliseev, A. S. Foht, *Methods of functions of space complex varaible,* Kiev, 1984, 57 p. (Preprint/AS UkSSR, Institute of Mathematics: 84.61).

[9] A. V. Smirnov, Nucl. Instrum. & Meth. in Phys. Res., NIM A, 469 (1) (2001) 21

[10] L. Lewin, *Theory of Waveguides,* London, Newnes-Butterworths, 1979.

[11] E. L. Burshtein, G. V. Voskresensky, *"Linear Electron Accelerators with Intense Beams",* Atomizdat, Moscow, 1970 (in Russian).

[12] L. A. Vainshtein, V. A. Solntsev, *Lectures on RF Electronics,* Sov. Radio, Moscow, 1973 (in Russian).

[13] L. A. Vainshtein, D. E. Vakman, *Frequency separation in the theory of oscillations and waves,* Nauka, Moscow, 1983 (in Russian).

[14] A. V. Smirnov, *Beam Interaction with Fundamental and Asymmetric Modes in Tapered Linac Sections,* Ph. D. Dissertation, Moscow Engineering Physics Institute, MEPhI, 1985 (in Russian).

[15] A. V. Smirnov et al, in Proc. of Particle Accelerator Conf. (PAC2001), IL., Chicago, 18–22 June(2001) 2293; http://accelconf.web.cern.ch/ AccelConf/p01/papers/wpah087.pdf

[16] A. V. Smirnov, *Commutative algebra of scalar quaternions,* Vladicaucasus Mathematical Journal, V. 6, N 2 (2004), http://www.vmj.ru/articles/2004_2_6.pdf (attn.: misprints!)

[17] A. V. Smirnov, in Proc. of Particle Accelerator Conference (PAC'97), Vancouver, B.C., Canada, 12–16 May (1997) 894; Nucl. Instrum. and Meth. NIM A349 (1994) 295

[18] L. S. Pontrjagin, *Generalization of Numbers,* Nauka, Moscow, 1986.

[19] G. I. Garas'ko, *Generalized-analytical functions of a polynumber varaible.* Hypercomplex Numbers in Geometry and Physics, 1, 75–88, http://polynumbers.ru

[20] A. N. Averkin, *On the Analytical Electrodynamics,* http://ephir.narod.ru/11.htm

[21] D. G. Pavlov, *All about commutative-associative numbers,* http://polynumbers.ru

[22] I. L. Kantor, A. S. Solodovnikov, *Hypercomplex Numbers.* Springer, Berlin, 1989

[23] V. I. Eliseev. *Introduction in methods of theory of functions of a space complex varaible.* Center of Scientific-Technical Creative Work of Youth "Algorithm", Moscow, NIAT. 1990, http://www.maths.ru/

Philosophical and Mathematical Reasons for Finsler Extensions of Relativity Theory

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The paper provides arguments to show that Finsler Geometry is by far the best candidate to provide relevant models for the further development of Physics (in general) and Relativity Theory (in particular). In the 4-dimensional case, the quadratic Berwald-Moor Finsler model is associated to a metric *algebra* (called *quad-numbers* algebra), which is commutative, associative, and it is isomorphic to the direct sum of four real algebras. The *geometry* beyond this simple algebra is non-trivial, because of diverse non-linear operations related with the generalized conformal transformations. The complexity brought by the Berwald-Moor Finsler spaces allows, using a new extended analysis apparatus, to construct three-dimensional time-varying fractals in the multi-dimensional space, by means of quad-numbers. The perspective brought by these spaces might shed a new view on Klein's Erlangen Program, where geometry focused on studying symmetry groups. Now it is suggested to generalize the notion of "symmetry" and to widen the classic case, based on isometric and conformal transformations, by introducing generalized conformal transformations.

MSC2000: 15A69, 53B40, 83D05.

Historically, first mention on possibility of existence of geometries with linear element not related to the square root from quadratic form of component's differentials, belongs to Riemann. As a result, calling such geometries Riemannian should be quite appropriate. However, nowadays such geometries are known as Finsler geometries. Partially, Riemann himself is guilty of this curious incident, as, since the possibility of non-quadratic metrics stated, Riemann decided that such geometries are too complex, interpreted poorly and hardly possess any significant matter. Strange enough, but the most part of modern physicists, in fact, realizes the same. One of the aims of this work is to shake this unfair confidence, at least partially, and show that Finsler Geometry could be that very arena where further development of Physics generally and Relativity Theory specifically will continue.

Of course, inattention to Finsler Geometry is not for no reason at all. Pessimistic conclusion was made by Riemann and many of his followers for some significant considerations. The first reason, carrying subjective nature, – related to pretended complexity and poor clearness -– is mentioned above. Proof of some theorems [2, 3] in the end of XIX century have became an another deterrent. According to these theorems, symmetry groups, which leave linear component as invariant, have maximum quantity of independent parameters just in case of non-quadratic geometries. Probably, one shouldn't doubt that using manifolds with maximum symmetries in physics is the most convenient today. The third reason is related to absence of widely used in Riemannian constructions natural alternative to dot product until recently. Thus, so-called Finsler Metric Tensor [4] have got ascendancy in Finsler Geometry. Since in fact being a slight modification of Riemannian Metric Tensor, it's usage only aggravated the problem. The next argument against Finsler Geometry is observed high grade of isotropy of our three-dimensional space. Since Finsler metrics lead to anisotropy, and we have to use non-quadratic geometries, those should be close to Riemannian. And the last, seemingly not related to geometry at all, reason is Frobenius Theorem which stated, that hypercomplex algebras with usual arithmetic laws of multiplication and addition are related with real and complex numbers only. Some mathematicians, mistakenly based upon this theorem, concluded that there is no number classification beyond complex numbers, thus eliminating a significant stimulus to research geometries related to hypercomplex algebras.

As it will be shown further, all mentioned above arguments against applying Finsler Geometry to the real space-time are at least shaken, if not overcame; therefore there is no reason to treat Finsler Geometry as a secondary add-on to Riemannian.

* * *

Let us begin from the thesis on complexity and poor clearness of Finsler Geometry. In fact, the only difference of some Finsler Spaces from the usual (and in many respects, clear just because of this) Euclidean geometry is that a set of dots equidistant from some fixed dot (nothing but the definition of a sphere) in Finsler Geometry turns out to be a bit more complex hypersurface. However, even in already almost classical Minkowsky Geometry such set is different from Euclidean sphere and appears as a union of three hyperboloids: two real and one imaginary. Usually they are represented in three-dimensional affine space as in fig. 1, a. Today the one's hardly surprised to see this surface really appears to be a sphere in geometrical sense. In primary linear Finsler spaces it is almost the same as in the pseudoeuclidean case, with the only difference, that in affine space the sphere (indicatrix) may appear even more exotic, for example, like on fig. 1, b.

Following spheres, the one can define other elementary surfaces, according the euclidean term «hypercomplex» for example. Thereto is enough to notice, that hyperplane in quadratic spaces is a set of dots with any dot equidistant from two fixed dots. (fig. 2, a). In spaces with another metric the hypersurface defined alike does not coincide with affine plane and could appear, say, like on fig. 2, b. When operating these and some other primary geometrical objects, geometry, significantly different from Euclidean, appears quite simple and reasonable if not evident.

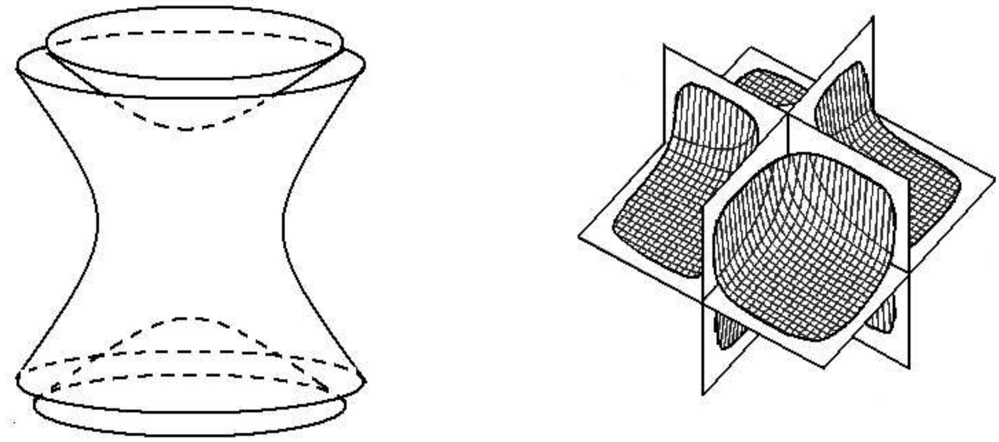

Figure 1: Indicatrix examples in three-dimensional a) pseudoeuclidean and b) Finsler spaces.

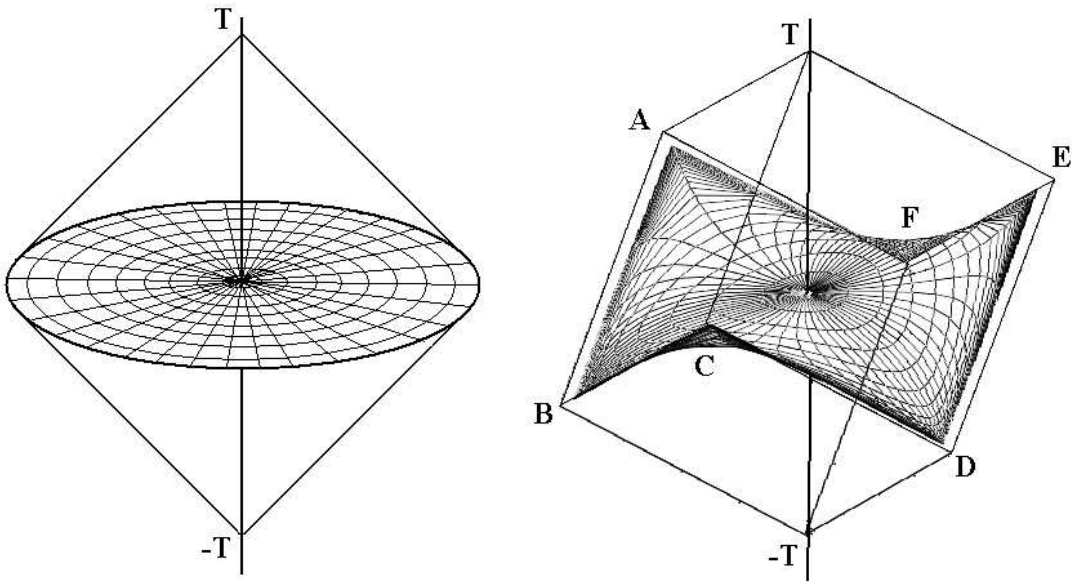

Figure 2: Example of surfaces with dots, equidistant from two fixed dots (T and $-T$) in three-dimensional a) pseudoeuclidean and b) Finsler spaces.

Thus, in non-quadratic spaces, surfaces, which are hyperplanes in metrical sense, are not such in affine sense. It turns out to be a significant condition, as it leads to denying automatic use of many Riemannian methods based upon the plane properties, particularly, the tangent bundle method. At the same time, such a denial should not be absolute. The one just should be very cautious with formal adoptions and, if needed, develop natural alternatives.

* * *

Passing on to the problem of the unique richness of symmetries in quadratic (i.e., Riemannian) geometries, the one should notice, that this fact only relates to the simple movement symmetries, i.e., to linear transformations, which save distances between dots invariant. However, even in the case of quadratic spaces, except metrically marked out isometric transformations, there are metrically marked out transformations, which save not distances, but angles invariant. Sometimes such transformations are called conformal movements. But these are the same symmetries, but non-linear. So, quite unexpectedly, some Finsler spaces, while yielding in variety of isometric transformations to quadratic spaces, are significantly richer in variety of non-linear symmetries [6].

But, speaking of symmetries, there is an even more important circumstance. Symmetries of spaces, including Finsler spaces, are not limited to isometries and conformal transformations. There are also metrically marked up transformations, saving invariant not distances and angles only, but some specific Finsler Geometry values, related not only to one or two vectors, but three and more. Such values can not be expressed in quadratic spaces, therefore there is no alike transformations in Riemannian Geometry in principle. Such transformations exist only in Finsler spaces and probably are even more interesting than conformal. Consequently, in some Finsler spaces, as for original compensation for relative poverty in isometries, not only greater groups of conformal transformations arise, but also conditions for appearance of the whole classes of new metrically marked up non-linear representations. We shall temporary call such representations as generalized conformal [6].

Today, only separate representations of generalized conformal transformations are known, and only for specific Finsler spaces. At that, in our opinion, their full enumeration is a priority task for modern Finsler Geometry. It should be noticed, that stated task is quite complex, because it supposes work not only with groups of symmetries, but also their generalizations, for example, such with some sets of transformations related not only by bi-, but also n-ary operations, and that, in turn, might not have direct or reverse conversion. In fact, it's a question of a new view on famous Klein's Erlangen Program, where the aim of geometry was marked as researching symmetry groups. Now it is suggested to concentrate on generalizing of the term 'symmetry' and proceed from its quite a special case, based on isometric and conformal transformations to generalized conformal transformations.

* * *

Let us turn to dot product problem and closely related to it metric tensor term. In twenties, in previous century Sing, Taylor and Berwald proposed a definition of the Finsler metric tensor, which on the one hand derived one of the main properties of Riemannian metric tensor (having two indices) and on the other hand – reflected specific property of non-quadratic geometries to base upon the 'generalized sphere' term (so called indicatrix). While being quite comfortable, it is sufficiently limited because of this metric tensor is based not on a private for current Finsler geometry dot product generalization, but adopts tensor structure from opposite kind (i.e. quadratic) nature. Figuratively, Finsler geometry was built not on specific foundation, but on the basement for another building. The one shouldn't expect any elegance from the obtained structure. And at that time the most part of mathematicians and physicists are based in their conclusions about poor perspective of Finsler geometry on regarding that very structure.

Meanwhile, there is a whole class of Finsler spaces, capable of natural generalization of a dot product. In such spaces polylinear symmetrical form from m vectors [5] plays a role of bilinear symmetrical form; and the square of distance is replaced by it's n-th power. Principal consequence of such generalization is in necessity to revise existing practice of using two-index Finsler metric tensor without any alternative and to use a tensor related to corresponding polylinear form instead. It is obvious that such tensor will have more than two indices. Advantage of such exchange, except it is absolutely natural, is that a number of independent variables increases greatly with number of dimensions staying the same. Thus, in 4-dimensional spaces with cubic form metric tensor has not nine independent components as in General Relativity Theory, but twelve. In spaces with quadranar form metric tensor already has thirty five independent components. In fact, it means that equation systems, defining fields of a new tensor in corresponding Finsler manifolds will be sufficiently richer than Einstein's equation system, at that staying in four-dimensional space!

It should be noted, that such way of researching Finsler geometry and its physical interpretations ideologically quite close to GRT, related however with necessity in a new mathematical apparatus.

* * *

The problem of isotropy of directions is not less interesting. Traditionally, it is considered, that if based on condition that the real space-time is 4-dimensional, then almost the only metrics, consistent with experimentally observed high grade of isotropy of surrounding three-dimensional space, are Galileo and Minkowsky metrics, holding Euclidean space as a subspace. At that it's usually missed, that these 4-dimensional spaces themselves are typically anisotropic, as any time-like direction sufficiently differs from any space-like direction. Besides, there are examples of 4-dimensional Finsler spaces, not including Euclidean subspace, but from the viewer's point percepted 3-dimensional space will *appear* as almost isotropic, at least at some range of parameters. Moreover, even today the one can call at least two Finsler metric functions leading to such quasi-isotropical effect.

One of such Finsler metrics is related not to quadratic, but to cubic form, and takes shape of symmetrical cubic polynomial:

$$S^3 = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4\,. \tag{1}$$

The first one to detect this metric in a context of its possible relation to real space-time geometry was Professor Vladimir Chernov from Samara Aerospace University, so I'll permit myself to call this metric and a corresponding space by his name. Unusual property of Chernov's space is that it doesn't contain any continuous rotations, in other words, the transformation group similar to Lorentz group is null-parametric. However in spite of this, statement about this space to compete successfully with Minkowsky space as the arena for physical phenomenon is not absurd. Point is that, while absent as an isometric symmetries, Lorentz group may be present in it as a subgroup of generalized conformal transformations. Unfortunately, Chernov space is not researched well yet to unambiguously state this. I am sure that further research of this space and groups of its non-linear symmetries will help us to exceed the limits of usual quadratic representations and to make a step to better understanding of sufficiently more complex Finsler spaces with metrics of even greater power.

Particularly, 4-dimensional space with Berwald-Moor metric refers to such spaces. It's metric in one of its' basis is given by symmetrical quadratic polynomial (in fact, monomial):

$$S^4 = x_1 x_2 x_3 x_4. \tag{2}$$

By the way, it would not be out of place to notice, that quadratic form of Minkowsky space in similar basis, which consists of four vectors in the light cone, is also given by a symmetrical, but square polynomial:

$$S^2 = x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4. \tag{3}$$

This really wonderful condition, which strangely almost wasn't discussed by physicists, in our opinion is a key to understanding close relationships between three geometries, one universally recognized and two with metrics (1) and (2) (which, maybe will be as popular as the first one in future). Whatever the case, in any of these geometries from the viewer's point, surrounding world appears as a pair of one-dimensional time and three-dimensional space, where bundles in some range of parameters correspond to Euclidean geometry. In other words, in defined approximation we have a time-space geometry with classical Galileo–Newton physics.

* * *

Strangely enough, the most substantial and complex from these three spaces is related to Berwald-Moor metric function, which, though appearing as the most simple, leads to geometries, including as sequential approximations two others, and at one geometry of classical physics, as it is was marked up earlier. But there is a very important condition related to this space. In contrast to Galileo, Minkowsky and Chernov spaces, it is directly connected to the most fundamental mathematical term -– *the number*. At that this connectivity realized almost the same as between Euclidean plane geometry and complex numbers algebra; also as between 4-dimensional euclidean space geometry and quaternion algebra. Unlike the last, corresponding to the space with Berwald-Moor metric algebra (let us call it quad-numbers algebra) is commutatively-associative, and it is isomorphous to direct sum of four real algebras. It's hard to imagine a simpler four-component algebra.

From the other side, geometry beyond this simple algebra is far from trivial. It turns out to be so because of variety of non-linear operations related with generalized conformal transformations, almost alike as existence of infinite-parametric group of conformal transformations of Euclidean plane leads to conversion of, in fact, elementary algebra of complex numbers to the powerful complex analysis apparatus. It is known, that one of the consequences of such symbiosis is an ability to construct on the complex plane quite interesting geometric objects: algebraic fractals, carrying names of G. Julia and B. Mandelbrot among of them. Beauty and harmony of complex fractals is so deep, that, while staring at them, a will to construct something similar in multidimensional spaces arises. It is conceived, that using quad-numbers we are offered such ability. In this case fractal objects may appear three-dimensional and time-varying.

The idea, that physics can be based on hypercomplex number structures belongs to William Hamilton, who is a well-known inventor of the first hypercomplex algebra – quaternion algebra. I should notice, that he is also known for many other inventions, but he treated quaternions as his main discovery. He selflessly devoted the most part of his life to quaternions. And, however, neither he nor the others could put into practice the finest idea on connecting physics with quaternions, the last still remain the most well-known hypernumbers, and their applications are broad and interesting. Who knows, maybe the matter, failed to be done with quaternions by Hamilton could be achieved with simpler quad-numbers, and someday, we will exclaim like Pythagorean: ≪All existing is expressed in numbers≫. Of course, instead of a narrow sense of numbers as rational, real or complex we will take up significantly broader circle of their hypercomplex extensions and instead of usual Euclidean or pseudoeuclidean geometry – their Finsler generalizations.

Literature:

[1] B. Riemann, 1868. *On the hypotheses which lie at the foundation of geometry.* In Ewald, William B., ed., 1996. *From Kant to Hilbert: A Source Book in the Foundations of Mathematics,* 2 vols. Oxford Univ. Press: 652–661.

[2] H. Helmholtz, 1876. *The origin and meaning of geometrical axioms.* Ibidem, 663–688.

[3] S. Lee, *Comments on Hilbert's work "The origin and meaning of geometrical axioms".* Ibidem, 690–696.

[4] H. Rund, *The Differential Geometry of Finsler Spaces,* Springer, Berlin, 1959.

[5] D. Pavlov, *Generalization of scalar product axioms,* Hypercomplex Numbers in Geometry and Physics, 1 (1), 2004.

[6] G. I. Garas'ko, *Generalization of conformal transformations,* Hypercomplex Numbers in Geometry and Physics, 1 (3), 2005.

The Relation of Elementary Generalized Conformal Transformations with Generalized Analytic Functions in the Polynumber Space

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In this paper the author continues the study of connections between functions, providing elementary generalized conformal transformations in the space of nondegenerate poly-numbers and generalized analytical functions of the same polynumber variable. Concrete examples are considered for complex and hypercomplex numbers H_4. By these examples it is shown that the unique and sufficient requirement for an extraction of physically valuable subset of generalized analytic functions of a polynumber variable is the following: each physically valuable generalized analytic function of the polynumber variable Pn may be obtained by a certain method from a generalized conformal transformation of the space Pn, such that, when generalized conformal transformations are accomplished by components of analytic functions, are obtained analytic functions.

MSC2000: 30G35, 30G30, 30C35.

1 Introduction

In the complex plane conformal transformations are accomplished by analytic functions of the complex variable (conformal transformations of kind I) or complex conjugate analytic functions (transformations of kind II). For polynumbers of dimension greater than two it is necessary to make some generalization of the notion of analyticity of functions [1] and conformal transformations [2], therefore relations between these notions it is necessary to study anew.

The polynumber space $P_n \ni X$,

$$X = x^i e_i \,, \quad e_k e_j = p^i_{kj} e_i \,, \tag{1}$$

where x^i are coordinates in the basis e_i is called non-degenerate, if components of the tensor

$$q_{ij} = p^m_{ik} p^k_{mj} \tag{2}$$

form the non-degenerate matrix and hence it is possible to construct twice contravariant q^{ij}:

$$q^{im} q_{mj} = q_{jm} q^{mi} = \delta^i_j \,. \tag{3}$$

The function

$$\Phi(X) = \varphi^1(x^1, x^2, ..., x^n) e_1 + \varphi^2(x^1, x^2, ..., x^n) e_2 + ... + \varphi^n(x^1, x^2, ..., x^n) e_n \tag{4}$$

of the polynumber variable $X \in P_n$ is called *generalized analytic* [1], if there exists such object of connectedness $\tilde{\Gamma}^i_{kj}$ and such function

$$\dot{\Phi}(X) = \dot{\varphi}^1(x^1, x^2, ..., x^n)e_1 + \dot{\varphi}^2(x^1, x^2, ..., x^n)e_2 + ... + \dot{\varphi}^n(x^1, x^2, ..., x^n)e_n\,, \quad (5)$$

that

$$\tilde{D}\Phi(X) = \dot{\Phi}(X)dX\,, \quad (6)$$

where

$$\tilde{D}\Phi(X) = (\tilde{\nabla}_k\varphi^i)dx^k e_i\,, \quad \tilde{\nabla}_k\varphi^i = \frac{\partial\varphi^i}{\partial x^k} + \tilde{\Gamma}^i_{kj}\varphi^j\,. \quad (7)$$

From formulas (6), (7) we get

$$\frac{\partial\varphi^i}{\partial x^k} + \tilde{\Gamma}^i_{kj}\varphi^j = p^i_{kj}\dot{\varphi}^j\,. \quad (8)$$

If ε^i – are coordinates of the unit in the basis e_i, then

$$\varepsilon^k p^i_{kj} = \delta^i_j, \quad (9)$$

therefore

$$\dot{\varphi}^i = \varepsilon^m\left(\frac{\partial\varphi^i}{\partial x^m} + \tilde{\Gamma}^i_{mj}\varphi^j\right)\,. \quad (10)$$

Thus, generalized analytic functions must satisfy the Cauchy-Riemann relations:

$$\frac{\partial\varphi^i}{\partial x^k} + \tilde{\Gamma}^i_{kj}\varphi^j = p^i_{kj}\varepsilon^s\left(\frac{\partial\varphi^j}{\partial x^s} + \tilde{\Gamma}^j_{sm}\varphi^m\right)\,, \quad (11)$$

or briefly

$$\tilde{\nabla}_k\varphi^i = p^i_{kj}\varepsilon^s\tilde{\nabla}_s\varphi^j\,. \quad (12)$$

Functions φ^i are components of a contra-variant tensor. If $\tilde{\Gamma}^j_{sm} = 0$, then $\Phi(X)$ – is an analytic function of the polynumber variable P_n.

A bijective mapping of one domain $O_X \ni X$ on the same or on another domain $O_Y \ni Y$

$$y^i = f^{(i)}(x^1, x^2, ..., x^n) \quad (13)$$

of the polynumber space $P_n \supset O_X, O_Y$ is called *elementary generalized conformal* [2], if functions $f^{(i)}$ are solutions of the system of differential equations

$$\frac{\partial^2 f^{(i)}}{\partial x^k \partial x^l} = \Gamma^m_{kl}\frac{\partial f^{(i)}}{\partial x^m}\,, \quad (14)$$

where

$$\Gamma^m_{kl} = \frac{1}{2}(p_l\delta^m_k + p_k\delta^m_l) - \Delta^{pm}_{kl}\frac{\partial L}{\partial x^p}\,, \quad (15)$$

p_l, L – are some tensor fields, while Δ^{pm}_{kl} – is a number tensor, which is defined by the metric of the polynumber space, that is, by the tensor p^i_{kj}. Functions $f^{(i)}(x^1, x^2, ..., x^n)$ – are scalar functions of a point of this space.

Groups of generalized conformal transformations include in itself elementary generalized conformal transformations, inverse to them and all products of the aforementioned of them.

2 Construction of generalized analytic functions

Suppose that it is known to us an elementary generalized conformal transformation (13)

$$O_X \xrightarrow{f} O_Y \tag{16}$$

in the space P_n, when n scalar functions are known to us $f^{(i)}(x^1, ..., x^n)$, of which it is possible to generate n covariant tensors

$$\varphi_i^{(k)} = \frac{\partial f^{(k)}}{\partial x^i} \,. \tag{17}$$

There exist many different means of a construction of generalized analytic functions on the base of functions $\varphi_i^{(k)}$. Among them especially important are those means of construction, which give analytic functions, if functions $f^{(i)}(x^1, ..., x^n)$ are components of analytic functions. We give two the most interesting from our point of view and simple means.

2.1 I-st mean

Since the polynumber space is non-degenerate, then there exists a tensor q^{ij} (2), (3); hence it is possible to construct n contra-variant vectors

$$\varphi^{(s)i} = q^{ij}\varphi_j^{(s)} \,. \tag{18}$$

For functions $\varphi^{(s)i}$ be components of generalized analytical function $\Phi^{(s)}(X)$ there is necessary and sufficient the satisfaction of the Cauchy-Riemann relations (11). Substituting (18) into (11) and using the formula

$$\frac{\partial f^{(s)}}{\partial x^m}\frac{\partial x^k}{\partial f^{(s)}} = \delta_m^k \tag{19}$$

and (3) we get

$$\tilde{\Gamma}^i_{kj} + q^{im}\tilde{\Gamma}^r_{km}q_{rj} = p^i_{kt}\varepsilon^s\left(\tilde{\Gamma}^t_{sj} + q^{tm}\tilde{\Gamma}^r_{sm}q_{rj}\right) \,. \tag{20}$$

This system of linear equations relative to $\tilde{\Gamma}^i_{kj}$ contains n^3 indeterminates. Not all of these equations are linearly independent, at least n^2 linear dependences of linear correlations between these linear equations, that to establish this, it is necessary to convolute the left and the right parts of this system (20) with the tensor ε^k. The general solution of this system of linear equations can be always presented as the sum of a partial solution, which is in this case

$$\tilde{\Gamma}^i_{(p)kj} = -q^{im}\Gamma^r_{km}q_{rj} \,, \tag{21}$$

and a general solution $\tilde{\Gamma}^i_{(0)kj}$ of the corresponding homogeneous equation. The object

$$\tilde{\Gamma}^i_{(0)kj} = p^i_{kt}D^t_j(x^1, ..., x^n) + \delta^i_k d_j(x^1, ..., x^n) \,, \tag{22}$$

where D^t_j, d_j – are arbitrary fields, is the solution of the homogeneous system of equations (20) and formally contains (n^2+n) arbitrary functions, but we were not successful in proving rigorously for arbitrary polynumbers P_n, that (22) is the general solution of the homogeneous system of equations. Nevertheless, an arbitrariness, which is contained in (22), is sufficient for a construction of generalized analytic functions of complex and H_4 variables.

Thus, if $f^{(i)}(x^1, x^2, ..., x^n)$ – are functions accomplishing an elementary generalized conformal transformation, then the functions $\varphi^{(s)i}$ are components of n generalized analytic functions

$$\Phi^{(s)}(X) = \varphi^{(s)i} e_i \tag{23}$$

with objects of connectedness

$$\tilde{\Gamma}^i_{kj} = \tilde{\Gamma}^i_{(p)kj} + \tilde{\Gamma}^i_{(0)kj}\,, \tag{24}$$

at the same time due to the available arbitrariness there must be satisfied the necessary condition: if $\Phi^{(s)}(X)$ – is an analytic function, then

$$\tilde{\Gamma}^i_{kj} \equiv 0\,. \tag{25}$$

Mention that this condition can always be satisfied, if $\tilde{\Gamma}^i_{(0)kj}$ is a general solution of the homogeneous system of equations (20), since the Cauchy-Riemann relations are satisfied on the condition (25) for analytic functions of the variable P_n.

2.2 II-nd mean

Construct a tensor of the form

$$\omega_{ij} = a_{(st)}(x^1, ..., x^n) \frac{\partial f^{(s)}}{\partial x^i} \frac{\partial f^{(t)}}{\partial x^j}\,, \tag{26}$$

where $a_{(st)}$ are scalar functions of a point, where a matrix $(a_{(st)})$ is non-degenerate. Then the matrix (ω_{ij}) is also non-degenerate, therefore, it is possible to construct the tensor ω^{ij}:

$$\omega^{ik}\omega_{kj} = \omega_{jk}\omega^{ki} = \delta^i_k\,. \tag{27}$$

If a matrix $(a_{(st)})$ is non-symmetric, then the matrix (ω_{ij}) – is also non-symmetric. Partial derivatives of elements of the matrix (ω_{ij}) are defined by the formula

$$\frac{\partial \omega_{ij}}{\partial x^k} = \frac{\partial a_{(st)}}{\partial x^k} \frac{\partial f^{(s)}}{\partial x^i} \frac{\partial f^{(t)}}{\partial x^j} + \Gamma^m_{ki}\omega_{mj} + \omega_{im}\Gamma^m_{kj}\,, \tag{28}$$

then partial derivatives of the contra-variant tensor ω^{ij} can be calculated by the formula

$$\frac{\partial \omega^{ir}}{\partial x^k} = -\omega^{ip} \frac{\partial \omega_{pj}}{\partial x^k} \omega^{jr}\,. \tag{29}$$

Construct components of the generalized analytic function by the following procedure:

$$\varphi^{(s)i} = \omega^{ir} \frac{\partial f^{(s)}}{\partial x^r}\,. \tag{30}$$

Substitute $\varphi^{(s)i}$ in the Cauchy-Riemann relations for generalized analytical functions, we get the system of linear equations for the definition of $\tilde{\Gamma}^i_{kq}$, moreover, the general solution of the corresponding homogeneous system will be the same, as in the I-st mean, while a partial solution will be of the form

$$\tilde{\Gamma}^i_{(p)kq} = \Gamma^i_{kq} + \omega^{ir} \frac{\partial a_{(st)}}{\partial x^k} \frac{\partial f^{(s)}}{\partial x^r} \frac{\partial f^{(t)}}{\partial x^q}\,. \tag{31}$$

If a matrix (a_{st}) consists of numbers, then we get

$$\tilde{\Gamma}^i_{(p)kq} = \Gamma^i_{kq}\,. \tag{32}$$

A general solution of the corresponding homogeneous system should be chosen, as above by the way that a resulting solution would have the property: functions (30) are components of an analytic function, $\tilde{\Gamma}^i_{kq} \equiv 0$.

3 Complex numbers

Let

$$F(z) = f^{(1)} + i f^{(2)} = u + iv \tag{33}$$

be the analytic function of the complex variable $z = x + iy$.

In the paper [2] the objects Γ^m_{kl} were obtained for components of an analytic function $F(z)$ of the complex variable,

$$\Gamma^m_{kl} = \frac{1}{2\Lambda}\left(\frac{\partial \Lambda}{\partial x^l}\delta^m_k + \frac{\partial \Lambda}{\partial x^k}\delta^m_l - g^{mp}\frac{\partial \Lambda}{\partial x^p} g_{kl} \right), \tag{34}$$

where

$$(g_{ij}) = (g^{ij}) = diag(1,\, 1)\,, \qquad \Lambda = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 . \tag{35}$$

3.1 I-st mean

Taking into account that

$$(q_{ij}) = (q^{ij}) = diag(1,\, -1)\,, \tag{36}$$

we get

$$(\varphi^{(1)i}) = \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right), \quad (\varphi^{(2)i}) = \left(\frac{\partial v}{\partial x}, -\frac{\partial u}{\partial x} \right). \tag{37}$$

The components $\varphi^{(1)i}$ and $\varphi^{(2)i}$ are components of two analytic functions. The first is the derivative of the initial analytic function and since the components of the second function satisfy the Cauchy-Riemann relations, then it is analytic.

Choose $\tilde{\Gamma}^i_{(0)kj}$ in such a way that for a conformal transformation of the first kind $\tilde{\Gamma}^m_{kl} = 0$. Substitute (34) into (21), add $\tilde{\Gamma}^m_{(0)kl}$ (22) and this sum equate to zero. As the result we get the system of equations

$$-\frac{1}{2}\left(q^{im}\frac{\partial \ln|\Lambda|}{\partial x^m}q_{kj} + \frac{\partial \ln|\Lambda|}{\partial x^k}\delta^i_j - q^i_k\frac{\partial \ln|\Lambda|}{\partial x^p}q^p_j\right) + p^i_{kt}D^t_j + \delta^i_k d_j = 0\,, \tag{38}$$

where

$$(q^i_k) = diag(1\,, -1)\,. \tag{39}$$

From this system we find:

$$\left.\begin{aligned} D^1_1 &= \frac{1}{2}\frac{\partial \ln|\Lambda|}{\partial x} - d_1\,,\ D^1_2 = \frac{1}{2}\frac{\partial \ln|\Lambda|}{\partial y} - d_2\,, \\ D^2_1 &= -\frac{1}{2}\frac{\partial \ln|\Lambda|}{\partial y}\,, \qquad D^2_2 = \frac{1}{2}\frac{\partial \ln|\Lambda|}{\partial x}\,, \end{aligned}\right\} \tag{40}$$

where d_1 and d_2 – are arbitrary functions.

3.2 II-nd mean

Let

$$(a_{(st)}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{41}$$

that is,

$$\omega_{ij} = \frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} + \frac{\partial v}{\partial x^i}\frac{\partial v}{\partial x^j}\,. \tag{42}$$

Using the Cauchy-Riemann relations for the analytic function $F(z) = u + iv$ we get

$$(\omega_{ij}) = \Delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{43}$$

where

$$\Delta = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\,, \tag{44}$$

and hence

$$(\omega^{ij}) = \frac{1}{\Delta}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{45}$$

Construct functions $\varphi^{(s)i}$:

$$(\varphi^{(1)i}) = \left(\omega^{ij}\frac{\partial u}{\partial x^j}\right) = \frac{1}{\Delta}\begin{pmatrix} \dfrac{\partial u}{\partial x} \\ -\dfrac{\partial v}{\partial x}\end{pmatrix}, \tag{46}$$

$$(\varphi^{(2)i}) = \left(\omega^{ij}\frac{\partial v}{\partial x^j}\right) = \frac{1}{\Delta}\begin{pmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial x}\end{pmatrix}. \tag{47}$$

Since

$$\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)^{-1} = \frac{1}{\Delta}\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right), \quad \left(\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial x}\right)^{-1} = \frac{1}{\Delta}\left(\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right), \tag{48}$$

$\varphi^{(1)i}$ and $\varphi^{(2)i}$ – are components of analytic functions of a complex variable. Elucidate whether it is possible to choose in this case functions D^t_l and d_l in the objects of connectedness $\tilde{\Gamma}^m_{kl}$ such that $\tilde{\Gamma}^m_{kl} = 0$. For this it is necessary to solve the system of linear equations

$$\frac{\partial L}{\partial x^l}\delta^m_k + \frac{\partial L}{\partial x^k}\delta^m_l - g^{mp}\frac{\partial L}{\partial x^p}g_{kl} + p^m_{kt}D^t_l + \delta^m_k d_l = 0 \tag{49}$$

relative to unknown functions $D^t_l,\ d_l$. This system of equations is compatible and has the following solution:

$$\left.\begin{aligned} D^1_1 &= -\frac{1}{2}\frac{\partial L}{\partial x} - d_1\,, \quad D^1_2 = -\frac{1}{2}\frac{\partial L}{\partial y} - d_2\,, \\ D^2_1 &= \frac{1}{2}\frac{\partial L}{\partial y}\,, \qquad\qquad D^2_2 = -\frac{1}{2}\frac{\partial L}{\partial x}\,. \end{aligned}\right\} \tag{50}$$

4 Polynumbers H_4

In the space H_4 an arbitrary analytic function in the ψ-basis has the form

$$F(X) = f^{(1)}(\xi^1)\psi_1 + f^{(2)}(\xi^2)\psi_2 + f^{(3)}(\xi^3)\psi_3 + f^{(4)}(\xi^4)\psi_4\,, \tag{51}$$

where

$$X = \xi^i\psi_i\,, \qquad \psi_i\psi_j = p^k_{ij}\psi_k\,, \qquad p^k_{ij} = \delta_{ij}\delta^k_{i_-}\,, \tag{52}$$

$i = i_-$, but by this pair of indices the summation is not accomplished. The system of equations for functions $f^{(i)}$, which implement elementary generalized conformal transformations in the coordinate space of polynumbers H_4, has the following form:

$$\frac{\partial^2 f^{(i)}}{\partial\xi^k\partial\xi^l} = \left[\frac{1}{2}(p_l\delta^m_k + p_k\delta^m_l) - p^m_{kl}\frac{\partial L}{\partial\xi^{l_-}}\right]\frac{\partial f^{(i)}}{\partial\xi^m}\,. \tag{53}$$

An arbitrary analytic function of the variable H_4, which accomplish a bijective mapping of some domain of the coordinate space H_4 on the same or some other domain of the same space, defines the conformal transformation and satisfies the system of equations (53) with

$$p_i = 0\,, \qquad L = -\ln\left|\frac{\dot f^{(1)}\dot f^{(2)}\dot f^{(3)}\dot f^{(4)}}{const}\right|. \tag{54}$$

4.1 I-st mean

Since in the space H_4 it can be generated the tensor

$$q_{ij} = p^m_{ik}p^k_{mj}, \qquad (q_{ij}) = diag(1,1,1,1), \tag{55}$$

then there exists twice contra-variant tensor q^{ij}, moreover,

$$(q^{ij}) = diag(1,1,1,1). \tag{56}$$

Evidently: if $f^{(s)}$ – are components of an analytic function, then also $\varphi^{(s)i}$ (18) – are components of an analytic function. Try to choose $\tilde\Gamma^i_{(0)kj}$ (22) such that in this case $\tilde\Gamma^i_{kj} = 0$. Solving the system of linear equations

$$q^{im}p^r_{km}\frac{\partial L}{\partial\xi^{m_-}}q_{rj} + p^i_{kt}D^t_j + \delta^i_k d_j = 0 \tag{57}$$

relative to variables D^i_j and d_j we get:

$$D^i_j = -\delta^i_j\frac{\partial L}{\partial\xi^{j_-}} - d_j\,. \tag{58}$$

Thus, the condition of being zero of the object $\tilde\Gamma^i_{kj}$, when functions $f^{(s)}$ implement a conformal transformation, can always be accomplished by putting

$$d_i = 0\,, \qquad D^i_k = -\delta^i_k\frac{\partial L}{\partial\xi k_-}\,. \tag{59}$$

4.2 II-nd mean

Define the tensor ω_{ij} by the following way:

$$\omega_{ij} = \frac{\partial f^{(1)}}{\partial\xi^i}\frac{\partial f^{(1)}}{\partial\xi^j} + \frac{\partial f^{(2)}}{\partial\xi^i}\frac{\partial f^{(2)}}{\partial\xi^j} + \frac{\partial f^{(3)}}{\partial\xi^i}\frac{\partial f^{(3)}}{\partial\xi^j} + \frac{\partial f^{(4)}}{\partial\xi^i}\frac{\partial f^{(4)}}{\partial\xi^j}\,, \tag{60}$$

then if $f^{(s)}$ are components of an analytic function of the variable H_4, we get

$$(\omega_{ij}) = \begin{pmatrix} (\dot f^{(1)})^2 & 0 & 0 & 0 \\ 0 & (\dot f^{(2)})^2 & 0 & 0 \\ 0 & 0 & (\dot f^{(3)})^2 & 0 \\ 0 & 0 & 0 & (\dot f^{(4)})^2 \end{pmatrix}, \tag{61}$$

$$\left.\begin{array}{ll}(\varphi^{(1)i}) = \left(\dfrac{1}{\dot f^{(1)}(\xi^1)}, 0, 0, 0\right), & (\varphi^{(2)i}) = \left(0, \dfrac{1}{\dot f^{(2)}(\xi^2)}, 0, 0\right), \\ (\varphi^{(3)i}) = \left(0, 0, \dfrac{1}{\dot f^{(3)}(\xi^3)}, 0\right), & (\varphi^{(4)i}) = \left(0, 0, 0, \dfrac{1}{\dot f^{(4)}(\xi^4)}\right).\end{array}\right\} \tag{62}$$

Therefore, $\varphi^{(s)i}$ are components of an analytic function of the variable H_4, hence it is necessary to demand, that in this case objects $\tilde\Gamma^m_{kl}$ vanish, that is, the following system of equations be satisfied:

$$-p^m_{kl}\frac{\partial L}{\partial \xi^{l-}} + p^m_{kt}D^t_l + \delta^m_k d_l = 0\,. \tag{63}$$

This system of linear equations relative to D^t_l and d_l is compatible and has the following solution:

$$D^i_j = \delta^i_j \frac{\partial L}{\partial \xi^{j-}} - d_j\,. \tag{64}$$

Conclusion

In the work [1] it was mentioned that a formulated in it notion of generalized analytic function is too common and there are necessary some additional conditions (or a condition) for extracting from this set functions of physically valuable subset. At the same time the notion of conformal transformations in the publication [2] is generalized, by our opinion, in the minimal possible way. Therefore, we are convinced that the unique and sufficient demand for an extraction of physically valuable subset of generalized analytic functions of a polynumber variable is the following: each physically valuable generalized analytic function of the polynumber variable P_n may be obtained by one or another method from generalized conformal transformation of the space P_n such that when generalized conformal transformations are accomplished by components of analytic functions, there would be obtained analytic functions. In the present work it is shown that to establish such correspondence between generalized conformal transformations and physically valuable classes of generalized conformal transformations is quite possible.

References

[1] G. I. Garasko, *Generalized analytic functions of a polynumber variable,* Hypercomplex Numbers in Geometry and Physics, **1 (1)** (2004), 75 – 88.

[2] G. I. Garasko, *Generalization of conformal transformations,* Hypercomplex Numbers in Geometry and Physics, **1 (3)** (2005), 16 – 25.

4-Momentum of a Particle and the Mass Shell Equation in the Entirely Anisotropic Space-Time

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This work treats the model for the entirely anisotropic flat space-time, whose metric is a generalization of the Finslerian Berwald-Moor metric. The action for a massive particle in such an anisotropic space has been determined proceeding from the concepts of relativistic invariance and minimality along a straight world line. The variational principle is used to obtain the formulas that relate the 4-momentum of a particle to the 3-velocity of the latter. The relevant mass shell has been shown to be invariant with respect to the relativistic symmetry group of the entirely anisotropic space-time.

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Introduction

In terms of general relativity, the space-time is known to be Riemannian. According to the Einstein equations, the distribution and motion of matter determine only the space-time curvature and have never affect the geometry of tangent spaces. In other words, irrespective of the distribution and properties of the material medium that fills the Riemannian space-time, any flat tangent space-time remains the space-time of special relativity, i.e. Minkowski space.

Generally speaking, a flat space-time does not exhibit the Minkowski geometry in all cases. Such geometry arises only if a given flat space-time admits the 6-parameter Lorentz group to be a homogeneous isometry group. The 6-parameter Lorentz group is known to include the 3-parameter Lorentz boosts and the 3D rotation subgroup. If, however, the isotropy of 3D space is broken in a way, the flat space-time metric can no longer be described by the quadratic form of the coordinate differentials, but is a certain (and, generally speaking, arbitrary enough) homogeneous function of the differentials of degree two. In this case, the flat space-time is said to have the Finslerian geometry [1].

Attempts have long been made (see [2]–[7] for instance) to generalize the field theory and the Einstein equations for the Finslerian space-time. That task is difficult because, first of all, the Finslerian metric tensor depends on not only the base manifold points, but also the geometric objects of, generally, arbitrary nature. Therefore, any noticeable relevant advance in the field involves additional physical concepts. In particular, the extremely bright concept of Lorentz symmetry violation at relative velocities of inertial reference frames in immediate proximity to light velocity should be noted. The concept was suggested in refs. [8]–[9] to be the most probable cause of the absence of so-called GZK effect [10]–[11] and has led finally to construction of the viable model for flat space-time with partially broken 3D isotropy [12]. The model is described by the following Finslerian metric:

$$ds^2 = \left[\frac{(dx_0 - \boldsymbol{\nu} d\boldsymbol{x})^2}{dx_0^2 - d\boldsymbol{x}^{\,2}}\right]^r (dx_0^2 - d\boldsymbol{x}^{\,2})\,, \tag{1}$$

where unit vector $\boldsymbol{\nu}$ indicates a preferred direction in 3D space; the dimensionless parameter r determines the anisotropy magnitude, i.e. the degree of deviation of Finslerian metric (1) from the metric of isotropic Minkowski space. In this case, the Minkowski metric is clearly ultimate case of the metric (1) at $r = 0$.

Another ultimate case ($r = 1$) is also of importance when constructing a consistent dynamics pattern for space-time manifold. In this case, according to Eq. (1), the metric ds degenerates into a total differential of absolute time. This transformation of the metric leads us to conclude that some phase transitions can occur in the space-time geometric structure and may be related to the phase transitions arising in the system of the interacting fundamental fields in case the gauge symmetry breaks spontaneously. We shall discuss this aspect in more detail below; but would pay attention now to another important circumstance that concerns the specific form of the Finslerian metric (1).

Any of the Finslerian metrics $ds^2 = f((dx_0 - \boldsymbol{\nu} d\boldsymbol{x})^2/(dx_0^2 - d\boldsymbol{x}^{\,2}))(dx_0^2 - d\boldsymbol{x}^{\,2})$ (where $f((dx_0 - \boldsymbol{\nu} d\boldsymbol{x})^2/(dx_0^2 - d\boldsymbol{x}^{\,2}))$ is, in many respects, an arbitrary function of its own argument) also describes a certain flat Finslerian space-time with partially broken 3D isotropy, i.e. an axially symmetric Finslerian space. At the same time, if and only if f is of the form $f = ((dx_0 - \boldsymbol{\nu} d\boldsymbol{x})^2/(dx_0^2 - d\boldsymbol{x}^{\,2}))^r$, the respective metric (1) will describe the flat anisotropic space-time, which permits not only the 1-parameter group of rotations about vector $\boldsymbol{\nu}$, but also the homogeneous 3-parameter group of isometries that consists of only noncompact transformations. Such transformations link the physically equivalent inertial reference frames in the anisotropic space-time (1) and are called the generalized Lorentz transformations, or the generalized Lorentz boosts. As a result, we may assert that, when going over from the Minkowski space to the Finslerian space (1) with partially broken 3D isotropy, the Lorentzian space-time symmetry proves to be also broken, but the relativistic symmetry represented by the group of generalized Lorentz boosts remains valid [13]–[21].

In terms of the above described Finslerian model, the anisotropy of the flat space-time is produced by the relativistically-invariant axially-symmetric fermion-antifermion condensate formed under spontaneous breaking of the initial gauge symmetry and when the fundamental matter fields acquire masses. Contrary to the standard Higgs mechanism and to its alternative pattern [22]-[23], which treats the scalar fermion-antifermion condensate instead of the Higgs condensate, the vacuum rearrangement accompanied by formation the relativistically-invariant axially-symmetric fermion-antifermion condensate leads to changing the flat space-time geometry, namely, the Finslerian geometry with the metric (1) replaces the Minkowski geometry. In this case, as noted above, such a geometric phase transition preserves the relativistic symmetry, but violates the Lorentz symmetry of the theory.

Lately, another (string-motivated) approach to the problem of breaking the Lorentz symmetry is developed along with the Finslerian approach. The fact is that, even if a base unified theory exhibits the Lorentz symmetry at the most fundamental level, that symmetry can be broken spontaneously due to formation of the condensate of the vector or (for instance) tensor field. The assumed occurrence of such a condensate, or a constant classical field against the Minkowski space background implies that it may affect the dynamics of the fundamental fields and, thereby, modify the Standard Model of strong, weak, and electromagnetic interactions. Since the constant classical field is transformed under passive transformations as a Lorentzian vector or tensor, this effect will properly be allowed for by extension of the Standard Model Lagrangian using the additional terms, which are every possible Lorentz-covariant convolutions of the condensate with the standard fundamental fields. The phenomenological theory based on the given Lorentz-covariant modification of the Standard Model was called the Standard Model Extension (SME) [24]–[27]. By its construction, that model is not Lorentz-invariant because its Lagrangian fails to remain invariant under active Lorentzian transformations of the fundamental fields against the background of fixed condensate. Additionally, in the SME context, the Lorentz symmetry violation with respect to the active Lorentzian transformations implies also the relativistic symmetry violation because the presence of non-invariant condensate breaks the physical equivalence of the various inertial reference frames.

Of course, it cannot be excluded that the Nature is so organized that, on the Planck energy scales, not only Lorentzian symmetry, but also the above-described generalized Lorentzian symmetry, will prove to be broken either entirely or partially [28]. Even in this case, however, the Finslerian geometric space-time model may prove to be more adequate compared with the Riemannian model. Although the like viewpoint has been expressed in [29], it is apt to note that the absence of some local isometry group in the Finslerian space-time necessitates additional physical criteria that make it possible to select only those Finslerian metrics from their set, which permit description of the geometric properties of space-time manifold. For example, authors of [30]–[32] used the occurrences of the conformal and projective structures of the Finslerian space as such criteria to show that the Finslerian space-time that satisfies the criteria must be Berwald's special Finslerian space.

Returning to the Finslerian spaces that permit the homogeneous noncompact 3-parameter isometry groups and, hence, have the relativistic symmetry, the present work will pay main attention below to further investigating the flat Finslerian space-time with entirely broken 3D isotropy.

Relativistically invariant flat Finslerian space-time with entirely broken 3D isotropy

In terms of the relativity theory, the basic property of axially-symmetric Finslerian space-time (1) is that the latter is also relativistically symmetric. In other words, the transformation relations that hold among the various inertial reference frames belong to the group of its isometries and, in their turn, form a separate 3-parameter group. As to the axial asymmetry, it means that under transition from the Minkowski space to the Finslerian space-time (1), the 3D space isotropy is broken but partially. If the anisotropy of the fermion-antifermion condensate formed under one or another spontaneous violation of the initial gauge symmetry of the system of interacting fundamental fields is taken to be the source of the 3D space anisotropy, the following conclusion will get evident. If the axially-symmetric condensate produces the axially-symmetric relativistically-invariant Finslerian space-time (1) and if, apart from the axially symmetric condensate, the entirely anisotropic condensate can be formed, the latter must generate the entirely anisotropic relativistically invariant Finslerian space-time. The most general form of the respective entirely anisotropic Finslerian metric has been found in [33] and proved to depend on three dimensionless parameters r_1, r_2 and r_3 and to be presented as

$$\begin{aligned} ds = \; & (dx_0 - dx_1 - dx_2 - dx_3)^{(1+r_1+r_2+r_3)/4}(dx_0 - dx_1 + dx_2 + dx_3)^{(1+r_1-r_2-r_3)/4} \\ \times \; & (dx_0 + dx_1 - dx_2 + dx_3)^{(1-r_1+r_2-r_3)/4}(dx_0 + dx_1 + dx_2 - dx_3)^{(1-r_1-r_2+r_3)/4}. \end{aligned} \tag{2}$$

The range of admissible values of r_1, r_2 and r_3 is restricted by the conditions

$$\begin{gathered} 1 + r_1 + r_2 + r_3 \ge 0\,,\; 1 + r_1 - r_2 - r_3 \ge 0\,, \\ 1 - r_1 + r_2 - r_3 \ge 0\,,\; 1 - r_1 - r_2 + r_3 \ge 0 \end{gathered}$$

and takes the form of the regular tetrahedron $A\,B\,C\,D$, shown in Fig. 1.

At $r_1 = r_2 = r_3 = 0$, the metric (2) reduces to the fourth root of the product of four 1-forms:

$$\begin{aligned} ds_{B-M} = \; & [\,(dx_0 - dx_1 - dx_2 - dx_3)(dx_0 - dx_1 + dx_2 + dx_3) \\ \times \; & (dx_0 + dx_1 - dx_2 + dx_3)(dx_0 + dx_1 + dx_2 - dx_3)\,]^{1/4}\,. \end{aligned} \tag{3}$$

If, following ref. [34], new coordinates ξ_i are introduced so that

$$\xi_i = A_{ij}x_j\,, \qquad A_{ij} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix},$$

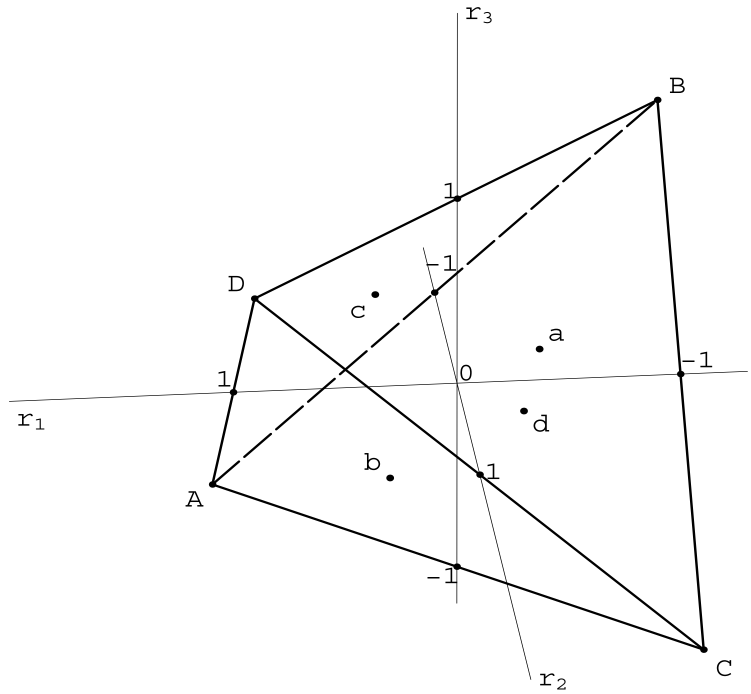

Figure 1: The range of admissible values of r_1, r_2 and r_3.

expression (3) in these coordinates will take the standard form of the Berwald-Moor metric [35]–[36], i.e. $ds_{B-M} = \sqrt[4]{\xi_1\xi_2\xi_3\xi_4}$. So, we see that the Berwald-Moor metric presented by expression (3) is the particular case of metric (2) at $r_1 = r_2 = r_3 = 0$ (this is the central point of tetrahedron $ABCD$ in Fig. 1).

The tetrahedron vertex A is in correspondence with the r_α values $(r_1 = 1, r_2 = -1, r_3 = -1)$, vertex B with $(r_1 = -1, r_2 = -1, r_3 = 1)$, vertex C with $(r_1 = -1, r_2 = 1, r_3 = -1)$, and vertex D with $(r_1 = 1, r_2 = 1, r_3 = 1)$. At each of the vertices, the metric (2) that describes the space-time with entirely broken 3D isotropy degenerates into the respective 1-form, i.e. into the total differential of absolute time:

$$ds_A = dx_0 - dx_1 + dx_2 + dx_3\,; \quad ds_B = dx_0 + dx_1 + dx_2 - dx_3\,;$$
$$ds_C = dx_0 + dx_1 - dx_2 + dx_3\,; \quad ds_D = dx_0 - dx_1 - dx_2 - dx_3\,.$$

If this observation is now compared with the above mentioned fact that the metric (1) describing the space-time with partially broken 3D isotropy gets also degenerated at $r = 1$ into the total differential of absolute time, the idea arises that the absolute time is not a stable degenerate state of space-time and may turn into either partially anisotropic space-time (1) or entirely anisotropic space-time (2). In any case, the respective geometric phase transition from the absolute time to 4D space-time may be treated to be an Act of Creation of 3D space. This phenomenon is accompanied by rearrangement of the vacuum state of the interacting fundamental field system, resulting in that elementary particles acquire masses. Only after the above described process is complete, the concepts of spatial extension and of reference frame get physically meaningful; (in a massless world, a spatial extension of anything, as well as one or another reference frame, is meaningless to speak of [1]). Finally, attention should be paid also to the fact that, formally, the absolute time serves as the connecting link, via which the correspondence principle is satisfied for the Finslerian spaces with partially and entirely broken 3D isotropy.

With the view to studying the fine structure of the geometric phase transitions, it is expedient to examine some Finslerian metrics obtainable using metric (2) as generatrix and selecting the appropriate characteristic subsets from the total set of the admissible values of the parameters r_1, r_2 and r_3.

According to Fig. 1, the relation $r_3 = -1-r_1-r_2$ holds among the parameters r_α on the ABC face. Therefore, we use (2) to obtain

$$\begin{aligned} ds_{_{ABC}} = {} & (dx_0 - dx_1 + dx_2 + dx_3)^{(1+r_1)/2}(dx_0 + dx_1 - dx_2 + dx_3)^{(1+r_2)/2} \\ \times {} & (dx_0 + dx_1 + dx_2 - dx_3)^{-(r_1+r_2)/2}\,. \end{aligned} \tag{4}$$

At the central point d of face ABC, the parameters r_α are $r_1 = r_2 = r_3 = -1/3$, while (4) reduces to the cubic root

$$ds_{_d} = \sqrt[3]{(dx_0 - dx_1 + dx_2 + dx_3)(dx_0 + dx_1 - dx_2 + dx_3)(dx_0 + dx_1 + dx_2 - dx_3)}\,. \tag{5}$$

On the BCD face, the relation $r_3 = 1 + r_1 - r_2$ holds among the parameters r_α. Accordingly, formula (2) gives

$$\begin{aligned} ds_{_{BCD}} = {} & (dx_0 - dx_1 - dx_2 - dx_3)^{(1+r_1)/2}(dx_0 + dx_1 + dx_2 - dx_3)^{(1-r_2)/2} \\ \times {} & (dx_0 + dx_1 - dx_2 + dx_3)^{-(r_1-r_2)/2}\,. \end{aligned} \tag{6}$$

At the central point a of face BCD, $r_1 = -1/3$, $r_2 = r_3 = 1/3$ and, according to (6), we again obtain the metric in the form of cubic root:

$$ds_{_a} = \sqrt[3]{(dx_0 - dx_1 - dx_2 - dx_3)(dx_0 + dx_1 - dx_2 + dx_3)(dx_0 + dx_1 + dx_2 - dx_3)}\,. \tag{7}$$

On the ABD face, $r_3 = 1 - r_1 + r_2$, resulting in

$$\begin{aligned} ds_{_{ABD}} = {} & (dx_0 + dx_1 + dx_2 - dx_3)^{(1-r_1)/2}(dx_0 - dx_1 - dx_2 - dx_3)^{(1+r_2)/2} \\ \times {} & (dx_0 - dx_1 + dx_2 + dx_3)^{(r_1-r_2)/2}\,. \end{aligned} \tag{8}$$

At the central point c of face ABD, $r_2 = -1/3$, $r_1 = r_3 = 1/3$ and the metric takes the form

$$ds_{_c} = \sqrt[3]{(dx_0 - dx_1 - dx_2 - dx_3)(dx_0 - dx_1 + dx_2 + dx_3)(dx_0 + dx_1 + dx_2 - dx_3)}\,. \tag{9}$$

[1] It should be noted that as early as in one of the first unified gauge theories (namely, the conformal Weyl theory [37], [38]) the very notion of space-time interval gets physically meaningful only after violation of local conformal symmetry and after the initially massless Abelian vector gauge field acquires mass.

On the last (fourth) face ACD, $r_3 = r_1 + r_2 - 1$ and we get

$$\begin{aligned} ds_{ACD} = &\ (dx_0 + dx_1 - dx_2 + dx_3)^{(1-r_1)/2}(dx_0 - dx_1 + dx_2 + dx_3)^{(1-r_2)/2} \\ \times &\ (dx_0 - dx_1 - dx_2 - dx_3)^{(r_1+r_2)/2} . \end{aligned} \tag{10}$$

At the central point b of face ACD, $r_1 = r_2 = 1/3$, $r_3 = -1/3$. Therefore,

$$ds_b = \sqrt[3]{(dx_0 - dx_1 - dx_2 - dx_3)(dx_0 - dx_1 + dx_2 + dx_3)(dx_0 + dx_1 - dx_2 + dx_3)} . \tag{11}$$

Let us find out at last what is the form that metric (2) takes on six edges of the tetrahedron $ABCD$, starting from edge BD. According to Fig. 1, this edge is the intersection of faces ABD and BCD. Therefore, the relations

$$\begin{aligned} 1 - r_1 + r_2 - r_3 = 0 , \\ 1 + r_1 - r_2 - r_3 = 0 \end{aligned}$$

hold among the parameters r_α at that intersection, whence $r_3 = 1$, $r_1 = r_2 = \tilde{r}$. As a result, we get

$$ds_{BD} = \left[\frac{(dx_0 - dx_3) - (dx_1 + dx_2)}{(dx_0 - dx_3) + (dx_1 + dx_2)}\right]^{\tilde{r}/2} \sqrt{(dx_0 - dx_3)^2 - (dx_1 + dx_2)^2} . \tag{12}$$

In the middle of edge BD, $r_3 = 1$, $r_1 = r_2 = \tilde{r} = 0$, and expression (12) reduces to the two-dimensional Minkowski metric $ds^2 = (dx_0 - dx_3)^2 - (dx_1 + dx_2)^2$.

Consider edge AD, which is the intersection of faces ABD and ACD. The relations

$$\begin{aligned} 1 - r_1 + r_2 - r_3 = 0 , \\ 1 - r_1 - r_2 + r_3 = 0 \end{aligned}$$

hold among the parameters r_α at that edge, resulting in $r_1 = 1$, $r_2 = r_3 = \tilde{r}$ and

$$ds_{AD} = \left[\frac{(dx_0 - dx_1) - (dx_2 + dx_3)}{(dx_0 - dx_1) + (dx_2 + dx_3)}\right]^{\tilde{r}/2} \sqrt{(dx_0 - dx_1)^2 - (dx_2 + dx_3)^2} . \tag{13}$$

In the middle of edge AD, $r_1 = 1$, $r_2 = r_3 = \tilde{r} = 0$, and expression (13) reduces again to the two-dimensional Minkowski metric $ds^2 = (dx_0 - dx_1)^2 - (dx_2 + dx_3)^2$.

Edge CD is the intersection of faces ACD and BCD. At that edge, the relations

$$\begin{aligned} 1 - r_1 - r_2 + r_3 = 0 , \\ 1 + r_1 - r_2 - r_3 = 0 \end{aligned}$$

hold among the parameters r_α, resulting in $r_2 = 1$, $r_1 = r_3 = \tilde{r}$. Accordingly,

$$ds_{CD} = \left[\frac{(dx_0 - dx_2) - (dx_1 + dx_3)}{(dx_0 - dx_2) + (dx_1 + dx_3)}\right]^{\tilde{r}/2} \sqrt{(dx_0 - dx_2)^2 - (dx_1 + dx_3)^2} . \tag{14}$$

In the middle of edge CD, $r_2 = 1$, $r_1 = r_3 = \tilde{r} = 0$, so that $ds^2 = (dx_0 - dx_2)^2 - (dx_1 + dx_3)^2$.

Edge CB is the intersection of faces ABC and BCD. At that edge, the relations

$$1 + r_1 + r_2 + r_3 = 0\,,$$
$$1 + r_1 - r_2 - r_3 = 0$$

hold among the parameters r_α, whence $r_1 = -1$, $r_2 = -r_3 = \tilde{r}$, resulting in

$$ds_{CB} = \left[\frac{(dx_0 + dx_1) - (dx_2 - dx_3)}{(dx_0 + dx_1) + (dx_2 - dx_3)}\right]^{\tilde{r}/2} \sqrt{(dx_0 + dx_1)^2 - (dx_2 - dx_3)^2}\,. \qquad (15)$$

In the middle of edge CB, $r_1 = -1$, $r_2 = -r_3 = \tilde{r} = 0$ and $ds^2 = (dx_0 + dx_1)^2 - (dx_2 - dx_3)^2$.

Edge AB is the intersection of faces ABC and ABD. At that edge, the relations

$$1 + r_1 + r_2 + r_3 = 0\,,$$
$$1 - r_1 + r_2 - r_3 = 0$$

hold among the parameters r_α, i.e. $r_2 = -1$, $r_1 = -r_3 = \tilde{r}$, resulting in

$$ds_{AB} = \left[\frac{(dx_0 + dx_2) - (dx_1 - dx_3)}{(dx_0 + dx_2) + (dx_1 - dx_3)}\right]^{\tilde{r}/2} \sqrt{(dx_0 + dx_2)^2 - (dx_1 - dx_3)^2}\,. \qquad (16)$$

In the middle of edge AB, $r_2 = -1$, $r_1 = -r_3 = \tilde{r} = 0$ and $ds^2 = (dx_0 + dx_2)^2 - (dx_1 - dx_3)^2$.

The last edge AC is the intersection of faces ABC and ACD. At that edge, the relation

$$1 + r_1 + r_2 + r_3 = 0\,,$$
$$1 - r_1 - r_2 + r_3 = 0$$

hold among the parameters r_α, whence $r_3 = -1$, $r_1 = -r_2 = \tilde{r}$ and

$$ds_{AC} = \left[\frac{(dx_0 + dx_3) - (dx_1 - dx_2)}{(dx_0 + dx_3) + (dx_1 - dx_2)}\right]^{\tilde{r}/2} \sqrt{(dx_0 + dx_3)^2 - (dx_1 - dx_2)^2}\,. \qquad (17)$$

In the middle of edge AC, $r_3 = -1$, $r_1 = -r_2 = \tilde{r} = 0$ and $ds^2 = (dx_0 + dx_3)^2 - (dx_1 - dx_2)^2$.

The next section will treat the relativistic point mechanics of a particle in the entirely anisotropic space-time (2) using essentially the transformations that constitute the homogeneous 3-parameter noncompact group of the isometries of that space-time. By its meaning, the group is the relativistic symmetry group for space-time (2) and was found [39] to be Abelian, while its determinant linear transformations are of the form

$$x'_i = D\, L_{ik}\, x_k\,, \qquad \text{where} \tag{18}$$

$$D = e^{-(\,r_1\,\alpha_1 + r_2\,\alpha_2 + r_3\,\alpha_3\,)}\,, \tag{19}$$

L_{ik} designates the unimodular matrices, with

$$L_{ik} = \begin{pmatrix} \mathcal{A} & -\mathcal{B} & -\mathcal{C} & -\mathcal{D} \\ -\mathcal{B} & \mathcal{A} & \mathcal{D} & \mathcal{C} \\ -\mathcal{C} & \mathcal{D} & \mathcal{A} & \mathcal{B} \\ -\mathcal{D} & \mathcal{C} & \mathcal{B} & \mathcal{A} \end{pmatrix}\,, \tag{20}$$

$$\mathcal{A} = \cosh\alpha_1\cosh\alpha_2\cosh\alpha_3 + \sinh\alpha_1\sinh\alpha_2\sinh\alpha_3\,,$$

$$\mathcal{B} = \cosh\alpha_1\sinh\alpha_2\sinh\alpha_3 + \sinh\alpha_1\cosh\alpha_2\cosh\alpha_3\,,$$

$$\mathcal{C} = \cosh\alpha_1\sinh\alpha_2\cosh\alpha_3 + \sinh\alpha_1\cosh\alpha_2\sinh\alpha_3\,,$$

$$\mathcal{D} = \cosh\alpha_1\cosh\alpha_2\sinh\alpha_3 + \sinh\alpha_1\sinh\alpha_2\cosh\alpha_3$$

and $\alpha_1\,,\alpha_2\,,\alpha_3$ being the group parameters. The transformations inverse to (18) are of the form

$$x_i = D^{-1}\,L^{-1}_{ik}\,x'_k\,, \tag{21}$$

where

$$L^{-1}_{ik} = \begin{pmatrix} \tilde{\mathcal{A}} & -\tilde{\mathcal{B}} & -\tilde{\mathcal{C}} & -\tilde{\mathcal{D}} \\ -\tilde{\mathcal{B}} & \tilde{\mathcal{A}} & \tilde{\mathcal{D}} & \tilde{\mathcal{C}} \\ -\tilde{\mathcal{C}} & \tilde{\mathcal{D}} & \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ -\tilde{\mathcal{D}} & \tilde{\mathcal{C}} & \tilde{\mathcal{B}} & \tilde{\mathcal{A}} \end{pmatrix}\,, \tag{22}$$

$$\tilde{\mathcal{A}} = \cosh\alpha_1\cosh\alpha_2\cosh\alpha_3 - \sinh\alpha_1\sinh\alpha_2\sinh\alpha_3\,, \tag{23}$$

$$\tilde{\mathcal{B}} = \cosh\alpha_1\sinh\alpha_2\sinh\alpha_3 - \sinh\alpha_1\cosh\alpha_2\cosh\alpha_3\,, \tag{24}$$

$$\tilde{\mathcal{C}} = \sinh\alpha_1\cosh\alpha_2\sinh\alpha_3 - \cosh\alpha_1\sinh\alpha_2\cosh\alpha_3\,, \tag{25}$$

$$\tilde{\mathcal{D}} = \sinh\alpha_1\sinh\alpha_2\cosh\alpha_3 - \cosh\alpha_1\cosh\alpha_2\sinh\alpha_3\,. \tag{26}$$

Considering that, similar to the Lorentz transformations in the Minkowski space, the transformations (18) link the various inertial reference frames in the Finslerian space (2), it is expedient to replace the group parameters $\alpha_1\,,\ \alpha_2\,,\ \alpha_3$ with the components $v_1 = dx_1/dx_0\,,\ v_2 = dx_2/dx_0\,,\ v_3 = dx_3/dx_0$ of the velocity of primed reference frame. Although the respective relations can be found in [39], we shall still present them here:

$$v_1 = (\tanh\alpha_1 - \tanh\alpha_2\tanh\alpha_3)/(1 - \tanh\alpha_1\tanh\alpha_2\tanh\alpha_3)\,,$$

$$v_2 = (\tanh\alpha_2 - \tanh\alpha_1\tanh\alpha_3)/(1 - \tanh\alpha_1\tanh\alpha_2\tanh\alpha_3)\,,$$

$$v_3 = (\tanh\alpha_3 - \tanh\alpha_1\tanh\alpha_2)/(1 - \tanh\alpha_1\tanh\alpha_2\tanh\alpha_3)\,.$$

The inverse relations are

$$\begin{aligned}
\alpha_1 &= \frac{1}{4}\ln\frac{(1+v_1-v_2+v_3)(1+v_1+v_2-v_3)}{(1-v_1-v_2-v_3)(1-v_1+v_2+v_3)}\ ,\\
\alpha_2 &= \frac{1}{4}\ln\frac{(1-v_1+v_2+v_3)(1+v_1+v_2-v_3)}{(1-v_1-v_2-v_3)(1+v_1-v_2+v_3)}\ ,\\
\alpha_3 &= \frac{1}{4}\ln\frac{(1-v_1+v_2+v_3)(1+v_1-v_2+v_3)}{(1-v_1-v_2-v_3)(1+v_1+v_2-v_3)}\ .
\end{aligned}$$

Relativistic point mechanics in the entirely anisotropic flat Finslerian space-time

We shall proceed from the considerations of relativistic invariance and minimality along a straight world line to write an action S for a free particle in the entirely anisotropic flat Finslerian space (2) :

$$S = -mc\int\limits_a^b ds\ , \tag{27}$$

where ds is an interval in the Finslerian space (2). Variation of this action is

$$\begin{aligned}
\delta S = & -\int\limits_a^b (p_0 d\delta x_0 - p_1 d\delta x_1 - p_2 d\delta x_2 - p_3 d\delta x_3) = \\
= & \ (-p_0\delta x_0 + p_1\delta x_1 + p_2\delta x_2 + p_3\delta x_3)|_a^b + \\
& + \int\limits_a^b [(dp_0/ds)\delta x_0 - (dp_1/ds)\delta x_1 - (dp_2/ds)\delta x_2 - (dp_3/ds)\delta x_3]ds\,.
\end{aligned} \tag{28}$$

If the world line is varied under condition $(\delta x_i)|_a = (\delta x_i)|_b = 0$, the principle of least action gives $p_i = const$, i.e. the rectilinear inertial motion. In turn, variation of the coordinates of point b under condition $p_i = const$ gives

$$p_0 = -\frac{\partial S}{\partial x_0}\,, \quad p_\alpha = \frac{\partial S}{\partial x_\alpha}\,; \qquad \alpha = 1\,,2\,,3\,, \tag{29}$$

whence it becomes clear the p_i is a canonical 4-momentum of particle in the Finslerian space (2). Having been expressed via 3-velocity $v_\alpha = dx_\alpha/dx_0$, the components of the 4-momentum take the form

$$\begin{aligned}
p_0 = \frac{ds}{dx_0}\left(\frac{dx_0}{ds_{_{B-M}}}\right)^4 \{ \ & 1 - v_1^2 - v_2^2 - v_3^2 - 2v_1v_2v_3 \\
& + \ r_1[(1 - v_1^2 + v_2^2 + v_3^2)v_1 + 2v_2v_3] \\
& + \ r_2[(1 + v_1^2 - v_2^2 + v_3^2)v_2 + 2v_1v_3] \\
& + \ r_3[(1 + v_1^2 + v_2^2 - v_3^2)v_3 + 2v_1v_2]\}\,,
\end{aligned} \tag{30}$$

$$p_1 = \frac{ds}{dx_0}\left(\frac{dx_0}{ds_{B-M}}\right)^4 \{\ (1 - v_1^2 + v_2^2 + v_3^2)v_1 + 2v_2v_3 \\ +\ r_1[1 - v_1^2 - v_2^2 - v_3^2 - 2v_1v_2v_3] \\ +\ r_2[(1 + v_1^2 + v_2^2 - v_3^2)v_3 + 2v_1v_2] \\ +\ r_3[(1 + v_1^2 - v_2^2 + v_3^2)v_2 + 2v_1v_3]\}\,, \tag{31}$$

$$p_2 = \frac{ds}{dx_0}\left(\frac{dx_0}{ds_{B-M}}\right)^4 \{\ (1 + v_1^2 - v_2^2 + v_3^2)v_2 + 2v_1v_3 \\ +\ r_1[(1 + v_1^2 + v_2^2 - v_3^2)v_3 + 2v_1v_2] \\ +\ r_2[1 - v_1^2 - v_2^2 - v_3^2 - 2v_1v_2v_3] \\ +\ r_3[(1 - v_1^2 + v_2^2 + v_3^2)v_1 + 2v_2v_3]\}\,, \tag{32}$$

$$p_3 = \frac{ds}{dx_0}\left(\frac{dx_0}{ds_{B-M}}\right)^4 \{\ (1 + v_1^2 + v_2^2 - v_3^2)v_3 + 2v_1v_2 \\ +\ r_1[(1 + v_1^2 - v_2^2 + v_3^2)v_2 + 2v_1v_3] \\ +\ r_2[(1 - v_1^2 + v_2^2 + v_3^2)v_1 + 2v_2v_3] \\ +\ r_3[1 - v_1^2 - v_2^2 - v_3^2 - 2v_1v_2v_3]\ \}\,, \tag{33}$$

where

$$(dx_0/ds)\left(ds_{B-M}/dx_0\right)^4 = (1 - v_1 - v_2 - v_3)^{(3-r_1-r_2-r_3)/4} \\ \times\ (1 - v_1 + v_2 + v_3)^{(3-r_1+r_2+r_3)/4} \\ \times\ (1 + v_1 - v_2 + v_3)^{(3+r_1-r_2+r_3)/4} \\ \times\ (1 + v_1 + v_2 - v_3)^{(3+r_1+r_2-r_3)/4}\,, \tag{34}$$

Here, ds is metric (2); ds_{B-M} is the Berwald-Moor metric (3). It should be noted that, starting from formula (30), we put $m = c = 1$ in all the relations.

According to expressions (30)–(33), four quantities (energy p_0 and 3-momentum p_α) are functions of three components, v_α, of particle velocity. The relations (30)–(33), therefore, may be treated to be the equations that determine the parametric form of mass shell, while v_α is taken to be the internal coordinates on that mass shell. We shall demonstrate below that the mass shell equation can be obtained to be an algebraic relation for p_i. As to the associate physical aspect, we see that, just as it should be, the energy p_0 determined by (30) reaches its absolute minimum $p_0 = 1$ at $v_\alpha = 0$, i.e. for a particle at rest. However, it is of importance to note that, apart from the rest energy $p_0 = 1$, a particle that resides in the entirely anisotropic space (2) still has a non-zero rest momentum. In virtue of (31)–(33), we get $p_1 = r_1$, $p_2 = r_2$, $p_3 = r_3$ at $v_\alpha = 0$. Moreover, according also to (31)–(33), the 3-momentum direction of the particle differs from its 3-velocity direction, thereby demonstrating that the free particle motion in the entirely anisotropic space is analogous to the motion of a quasiparticle in an entirely anisotropic crystalline medium.

Similarly to the case of the Minkowski space, the 3-velocity of a particle can be found if its 4-momentum in the entirely anisotropic space is known. To obtain the appropriate respective formula, we shall start from certain useful intermediate relations that are valid in virtue of (30)–(33), namely,

$$\frac{p_0+p_3}{p_1+p_2}=\frac{(1-v_3)(1+r_3)+(v_1+v_2)(r_1+r_2)}{(1-v_3)(r_2+r_3)+(v_1+v_2)(1+r_3)}\,, \tag{35}$$

$$\frac{p_0-p_1}{p_2-p_3}=\frac{(1+v_1)(1-r_1)+(v_2-v_3)(r_2-r_3)}{(1+v_1)(r_2-r_3)+(v_2-v_3)(1-r_1)}\,, \tag{36}$$

$$\frac{p_0+p_1}{p_2+p_3}=\frac{(1-v_1)(1+r_1)+(v_2+v_3)(r_2+r_3)}{(1-v_1)(r_2+r_3)+(v_2+v_3)(1+r_1)}\,. \tag{37}$$

These relations lead to the following set of three linear equations with respect to v_α:

$$a_{\gamma\alpha}v_\alpha=b_\gamma\,, \tag{38}$$

where

$$\begin{aligned}
a_{11}={}&\;a_{12}=(p_0+p_3)(1+r_3)-(p_1+p_2)(r_1+r_2)\,,\\
a_{13}={}&\;b_1=(p_1+p_2)(1+r_3)-(p_0+p_3)(r_2+r_3)\,,\\
a_{21}={}&-\;b_2=(p_0-p_1)(r_2-r_3)-(p_2-p_3)(1-r_1)\,,\\
a_{22}={}&-\;a_{23}=(p_0-p_1)(1-r_1)-(p_2-p_3)(r_2-r_3)\,,\\
a_{31}={}&\;b_3=(p_2+p_3)(1+r_1)-(p_0+p_1)(r_2+r_3)\,,\\
a_{32}={}&\;a_{33}=(p_0+p_1)(1+r_1)-(p_2+p_3)(r_2+r_3)\,.
\end{aligned}$$

At $r_1=r_2=r_3=0$, i.e. in the case of the Berwald-Moor space with metric (3), the set (38) takes the form

$$\begin{aligned}
(p_0+p_3)v_1+(p_0+p_3)v_2+(p_1+p_2)v_3&=(p_1+p_2)\,,\\
(p_3-p_2)v_1+(p_0-p_1)v_2+(p_1-p_0)v_3&=(p_2-p_3)\,,\\
(p_2+p_3)v_1+(p_0+p_1)v_2+(p_0+p_1)v_3&=(p_2+p_3)\,.
\end{aligned} \tag{39}$$

These relations solve the set (39).

$$v_1=\frac{p_1({p_0}^2-{p_1}^2+{p_2}^2+{p_3}^2)-2p_0p_2p_3}{p_0({p_0}^2-{p_1}^2-{p_2}^2-{p_3}^2)+2p_1p_2p_3}\,,$$

$$v_2=\frac{p_2({p_0}^2+{p_1}^2-{p_2}^2+{p_3}^2)-2p_0p_1p_3}{p_0({p_0}^2-{p_1}^2-{p_2}^2-{p_3}^2)+2p_1p_2p_3}\,,$$

$$v_3=\frac{p_3({p_0}^2+{p_1}^2+{p_2}^2-{p_3}^2)-2p_0p_1p_2}{p_0({p_0}^2-{p_1}^2-{p_2}^2-{p_3}^2)+2p_1p_2p_3}$$

As noted above, four functions, (30)–(33), of three variables v_α determine the parametric form of mass shell. Let now obtain the algebraic form of the mass shell equation, i.e. $H^4(p_0,\, p_1,\, p_2,\, p_3) = 1$. The explicit form of the function $H^4(p_0,\, p_1,\, p_2,\, p_3)$ can be found as follows. First, four relations, which are valid in virtue of (30)–(33), are to be written:

$$\frac{p_0+p_1+p_2+p_3}{1+r_1+r_2+r_3} = \frac{ds}{dx_0}\left(\frac{dx_0}{ds_{B-M}}\right)^4 (1-v_1+v_2+v_3)\left[(1+v_1)^2-(v_2-v_3)^2\right], \quad (40)$$

$$\frac{p_0+p_1-p_2-p_3}{1+r_1-r_2-r_3} = \frac{ds}{dx_0}\left(\frac{dx_0}{ds_{B-M}}\right)^4 (1-v_1-v_2-v_3)\left[(1+v_1)^2-(v_2-v_3)^2\right], \quad (41)$$

$$\frac{p_0-p_1+p_2-p_3}{1-r_1+r_2-r_3} = \frac{ds}{dx_0}\left(\frac{dx_0}{ds_{B-M}}\right)^4 (1+v_1+v_2-v_3)\left[(1-v_1)^2-(v_2+v_3)^2\right], \quad (42)$$

$$\frac{p_0-p_1-p_2+p_3}{1-r_1-r_2+r_3} = \frac{ds}{dx_0}\left(\frac{dx_0}{ds_{B-M}}\right)^4 (1+v_1-v_2+v_3)\left[(1-v_1)^2-(v_2+v_3)^2\right]. \quad (43)$$

After that, examine the structure of the expressions in the right-hand parts of (40)–(43). Considering the structure demonstrated by formula (34) for the common factor $(dx_0/ds)\left(ds_{B-M}/dx_0\right)^4$, it can readily be noted that the right-hand parts of (40)–(43) are the products of different powers of four characteristic "brackets" $(1-v_1-v_2-v_3)$, $(1-v_1+v_2+v_3)$, $(1+v_1-v_2+v_3)$ and $(1+v_1+v_2-v_3)$. This observation suggests that the function $H^4(p_0,\, p_1,\, p_2,\, p_3)$ should be sought for as

$$\begin{aligned} H^4(p_0,\, p_1,\, p_2,\, p_3) \; &= \; \left(\frac{p_0+p_1+p_2+p_3}{1+r_1+r_2+r_3}\right)^a \left(\frac{p_0+p_1-p_2-p_3}{1+r_1-r_2-r_3}\right)^b \times \\ &\times \; \left(\frac{p_0-p_1+p_2-p_3}{1-r_1+r_2-r_3}\right)^c \left(\frac{p_0-p_1-p_2+p_3}{1-r_1-r_2+r_3}\right)^d . \end{aligned} \quad (44)$$

The first of the conditions to be imposed on the constants $a\,,b\,,c$ and d ensues from the physical meaning of the function $H(p_0,\, p_1,\, p_2,\, p_3)$ and consists in that the function must have a physical dimension that coincides with the dimension of momentum p_i. Therefore, the function (44) should be a homogeneous function of its own arguments of the fourth degree of homogeneity. This requirement means that

$$a+b+c+d=4\,. \quad (45)$$

The rest conditions to be imposed on the constants $a\,,b\,,c$ and d can be obtained in terms of the requirement that all the power exponents, which arise for four characteristic "brackets", after substituting the expressions (40)–(43) in (44), should equal zero. It is just in this case that we obtain the mass shell equation in the form $H^4(p_0,\, p_1,\, p_2,\, p_3) = 1$. Considering, however, that we have put $m=c=1$, the equation $H^4(p_0,\, p_1,\, p_2,\, p_3) = 1$ corresponds in the ordinary units to $H^4(p_0,\, p_1,\, p_2,\, p_3) = (mc)^4$.

So, if the proposed program is fulfilled, then we get the following four equations for the constants a, b, c and d to supplement the equation (45):

$$b + c + d - (3 - r_1 - r_2 - r_3)(a + b + c + d)/4 = 0\,, \tag{46}$$

$$a + c + d - (3 - r_1 + r_2 + r_3)(a + b + c + d)/4 = 0\,, \tag{47}$$

$$a + b + d - (3 + r_1 - r_2 + r_3)(a + b + c + d)/4 = 0\,, \tag{48}$$

$$a + b + c - (3 + r_1 + r_2 - r_3)(a + b + c + d)/4 = 0\,. \tag{49}$$

In virtue of (45), the set of five equations (45)–(49) can be rewritten as

$$a + b + c + d = 4\,, \tag{50}$$

$$b + c + d - (3 - r_1 - r_2 - r_3) = 0\,, \tag{51}$$

$$a + c + d - (3 - r_1 + r_2 + r_3) = 0\,, \tag{52}$$

$$a + b + d - (3 + r_1 - r_2 + r_3) = 0\,, \tag{53}$$

$$a + b + c - (3 + r_1 + r_2 - r_3) = 0\,. \tag{54}$$

Obviously, we get (50) by summing up equations (51)–(54). Therefore, (50) is not an independent equation, so four independent equations (51)–(54), or the respective set

$$b + c + d = (3 - r_1 - r_2 - r_3)\,, \tag{55}$$

$$a + c + d = (3 - r_1 + r_2 + r_3)\,, \tag{56}$$

$$a + b + d = (3 + r_1 - r_2 + r_3)\,, \tag{57}$$

$$a + b + c = (3 + r_1 + r_2 - r_3) \tag{58}$$

remain to determine four constants $a\,, b\,, c$ and d. The constants

$$a = 1 + r_1 + r_2 + r_3\,,\ \ b = 1 + r_1 - r_2 - r_3\,,$$
$$c = 1 - r_1 + r_2 - r_3\,,\ \ d = 1 - r_1 - r_2 + r_3$$

solve the set (55)–(58). This result means that the equation of mass shell in the entirely anisotropic momentum space is

$$\left(\frac{p_0 + p_1 + p_2 + p_3}{1 + r_1 + r_2 + r_3}\right)^{(1+r_1+r_2+r_3)} \left(\frac{p_0 + p_1 - p_2 - p_3}{1 + r_1 - r_2 - r_3}\right)^{(1+r_1-r_2-r_3)} \times$$
$$\times \left(\frac{p_0 - p_1 + p_2 - p_3}{1 - r_1 + r_2 - r_3}\right)^{(1-r_1+r_2-r_3)} \left(\frac{p_0 - p_1 - p_2 + p_3}{1 - r_1 - r_2 + r_3}\right)^{(1-r_1-r_2+r_3)} = 1. \tag{59}$$

Finally, we shall consider the relativistic symmetry group of the entirely anisotropic momentum space and show that the transformations of the 4-momenta that form the group leave the mass shell equation (59) invariant. From the general considerations it becomes clear that the transformations of relativistic symmetry of the entirely anisotropic momentum space are induced by the respective transformations (18) of the entirely anisotropic event space (2). The explicit form of the linear transformations of 4-momenta that represent the relativistic symmetry group will be constructed proceeding from the definition of the canonical 4-momentum (29).

Thus, the relations

$$p'_0 = -\frac{\partial S}{\partial x_i}\frac{\partial x_i}{\partial x'_0} = D^{-1}(\, L^{-1}_{00}\, p_0 - L^{-1}_{0\beta}\, p_\beta\,)\,, \tag{60}$$

$$p'_\alpha = \frac{\partial S}{\partial x_i}\frac{\partial x_i}{\partial x'_\alpha} = D^{-1}(\, -L^{-1}_{\alpha 0}\, p_0 + L^{-1}_{\alpha\beta}\, p_\beta\,) \tag{61}$$

are valid in virtue of (29) and (21). Considering the definition (22) of matrix L^{-1}_{ik}, we can unite the relations (60) and (61) into a single formula

$$p'_i = D^{-1}\,\mathcal{L}_{ik}\, p_k\,, \tag{62}$$

where

$$D^{-1} = e^{(\, r_1\,\alpha_1 + r_2\,\alpha_2 + r_3\,\alpha_3\,)}\,, \tag{63}$$

$$\mathcal{L}_{ik} = \begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} & \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \\ \tilde{\mathcal{B}} & \tilde{\mathcal{A}} & \tilde{\mathcal{D}} & \tilde{\mathcal{C}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} & \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{D}} & \tilde{\mathcal{C}} & \tilde{\mathcal{B}} & \tilde{\mathcal{A}} \end{pmatrix} \tag{64}$$

in virtue of (19) and (22). Here, α_1, α_2, α_3 are the group parameters; the matrix element $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ of the matrix $\mathcal{L}_{ik}$ are determined by formulas (23)–(26). Thus, we have constructed the explicit form (62) of the linear transformations of 4-momenta. The transformations give rise to the 3-parameter Abelian group of relativistic symmetry of the entirely anisotropic momentum space.

To verify that the form of the mass shell equation (59) is actually invariable, i.e. remains invariant, under transformations (62), it is expedient to find out first in what way four independent 1-forms entering equation (59) are transformed. The straightforward calculations by (62)–(64) and by (23)–(26) give

$$(p'_0 + p'_1 + p'_2 + p'_3) = e^{[\,(r_1-1)\,\alpha_1 + (r_2-1)\,\alpha_2 + (r_3-1)\,\alpha_3\,]}(p_0 + p_1 + p_2 + p_3)\,, \tag{65}$$

$$(p'_0 + p'_1 - p'_2 - p'_3) = e^{[\,(r_1-1)\,\alpha_1 + (r_2+1)\,\alpha_2 + (r_3+1)\,\alpha_3\,]}(p_0 + p_1 - p_2 - p_3)\,, \tag{66}$$

$$(p'_0 - p'_1 + p'_2 - p'_3) = e^{[\,(r_1+1)\,\alpha_1 + (r_2-1)\,\alpha_2 + (r_3+1)\,\alpha_3\,]}(p_0 - p_1 + p_2 - p_3)\,, \tag{67}$$

$$(p'_0 - p'_1 - p'_2 + p'_3) = e^{[\,(r_1+1)\,\alpha_1 + (r_2+1)\,\alpha_2 + (r_3-1)\,\alpha_3\,]}(p_0 - p_1 - p_2 + p_3)\,. \tag{68}$$

Thus, as should be expected, the transformations of the relativistic symmetry of the entirely anisotropic momentum space get much simplified in terms of 1-forms and reduce only to the scale transformations of those independent forms. Using (65)–(68), it can readily be verified that

$$\begin{aligned}
&(p'_0+p'_1+p'_2+p'_3)^{(1+r_1+r_2+r_3)}(p'_0+p'_1-p'_2-p'_3)^{(1+r_1-r_2-r_3)}\times\\
\times\ &(p'_0-p'_1+p'_2-p'_3)^{(1-r_1+r_2-r_3)}(p'_0-p'_1-p'_2+p'_3)^{(1-r_1-r_2+r_3)}=\\
=\ &(p_0+p_1+p_2+p_3)^{(1+r_1+r_2+r_3)}(p_0+p_1-p_2-p_3)^{(1+r_1-r_2-r_3)}\times\\
\times\ &(p_0-p_1+p_2-p_3)^{(1-r_1+r_2-r_3)}(p_0-p_1-p_2+p_3)^{(1-r_1-r_2+r_3)}\ .
\end{aligned}$$

It is this equality that proves that the mass shell equation (59) remains invariant under transformations (62).

Conclusion

We have but casually mentioned the relativistically symmetric Finslerian space with partially broken 3D isotropy and paid main attention to studying the relativistically symmetric Finslerian space with entirely broken 3D isotropy.

To avoid any misunderstanding, we wish to note that the relativistic symmetry is normally meant to be the symmetry with respect to the Lorentz boost or, in the wider sense, the symmetry with respect to the 6-parameter Lorentz group. Although any Lorentz group element can be presented to be a product of a Lorentz boost by 3D rotation, the nontrivial point is that the set of 3D rotations constitutes the 3-parameter subgroup of the Lorentz group, while the 3-parameter set of the Lorentz boosts does not constitute any group. In other words, by consecutively using two different Lorentz boosts we go over to the inertial reference frame, whose spatial axes are not parallel to the axes of the initial reference frame, but get an additional 3D rotation. It is this effect (which leads to the Thomas precession) that reflects the fact that the product of two arbitrary Lorentz boosts is not, generally speaking, a pure Lorentz boost. At the same time, it has been long known that the Lorentz group includes a single (up to isomorphism) 3-parameter noncompact subset that, like the compact 3D rotation subset, also constitutes a group. Since such 3-parameter group includes only the transformations that link the moving inertial reference frames, it is just that group, rather than the 6-parameter Lorentz group, that must be treated to be the relativistic symmetry group of the Minkowski space. This is justified especially as the Finslerian space with the relativistic symmetry group, which is locally isomorphic to the 3-parameter relativistic symmetry group of the Minkowski space, arises (instead of Minkowski space) under partial breaking of 3D isotropy. In the case of the entirely anisotropic Finslerian space of events, the group of the transformations for that space, which link different physically equivalent inertial reference frames, has also the meaning of the relativistic symmetry group. However, as shown above, such a group is the Abelian 3-parameter group.

References

[1] H. Rund, *The Differential Geometry of Finsler Spaces,* Springer, Berlin, 1959.

[2] Y. Takano, *Gravitational field in Finsler spaces,* Lett. Nuovo Cimento, 1974, v. 10, No 17, 747–750.

[3] Y. Takano, *On the theory of fields in Finsler spaces,* in Proc. Int. Symp. "Relativity and Unified Field Theory", Calcutta, Bose Insit. Phys. Sci, 1975–1976, 17–26.

[4] S. Ikeda, *On the theory of gravitational field in Finsler spaces,* Ann. d. Phys., 1987, v. 44, No 8, 558–562.

[5] S. Ikeda, *On the theory of the gravitational field nonlocalized by the internal variables,* Nuovo Cimento B, 1987, v. 98, No 2, 158–164.

[6] S. Ikeda, *On the field equations in the theory of the gravitational fields in Finsler spaces,* Tensor N. S., 1987, v. 44, 157–163.

[7] S. Ikeda, *Theory of fields in Finsler spaces,* Seminarul de mecanica, Univ. din Timisoara, Romania, 1988, No 8.

[8] D. A. Kirzhnits, V. A. Chechin, *Cosmic rays and elementary length,* Pis'ma Zh. Eksper. Teor. Fiz., 1971, v. 14, No 4, 261–262.

[9] D. A. Kirzhnits, V. A. Chechin, *Ultra-high energy cosmic rays and possible generalization of the relativistic theory,* Yadern. Fiz., 1972, v. 15, No 5, 1051–1059.

[10] K. Greisen, *End to the cosmic-ray spectrum?,* Phys. Rev. Lett., 1966, v. 16, No 17, 748–750.

[11] G. T. Zatsepin, V. A. Kuz'min, *On the upper bound of the cosmic-ray spectrum,* Pis'ma Zh. Eksper. Teor. Fiz., 1966, v. 4, No 3, 114–117.

[12] G. Yu. Bogoslovsky, *On a special relativistic theory of the anisotropic space-time,* Dokl. Akad. Nauk SSSR, 1973, v. 213, No 5, 1055–1058.

[13] G. Yu. Bogoslovsky, *A special relativistic theory of the locally anisotropic space-time,* Nuovo Cimento B, 1977, v. 40, No 1, 99–134.

[14] G. Yu. Bogoslovsky, *Theory of locally anisotropic space-time,* Moscow Univ. Press, M., 1992; in Russian.

[15] G. Yu. Bogoslovsky, *From the Weyl theory to a theory of locally anisotropic space-time,* Class. Quantum Grav., 1992, v. 9, 569–575.

[16] G. Yu. Bogoslovsky, *Finsler model of space-time,* Phys. Part. Nucl., 1993, v. 24, 354–379.

[17] G. Yu. Bogoslovsky, *A viable model of locally anisotropic space-time and the Finslerian generalization of the relativity theory,* Fortschr. Phys., 1994, v. 42, 2, 143–193.

[18] G. Yu. Bogoslovsky, H. F. Goenner, *On the generalization of the fundamental field equations for locally anisotropic space-time.* In Proc. of XXIV International Workshop "Fundamental Problems of High Energy Physics and Field Theory" (27–29 June, 2001, Protvino, Russia), Ed. V. A. Petrov, Protvino, 2001, 113–125.

[19] G. Yu. Bogoslovsky, H. F. Goenner, *Concerning the generalized Lorentz symmetry and the generalization of the Dirac equation,* Phys. Lett. A, 2004, v. 323, 40–47.

[20] G. Yu. Bogoslovsky, *Lorentz symmetry violation without violation of relativistic symmetry,* Phys. Lett. A, 2006, v. 350, 5–10.

[21] G. Yu. Bogoslovsky, *Subgroups of the group of generalized Lorentz transformations and their geometric invariants,* Symmetry, Integrability and Geometry: Methods and Applications (SIGMA), 2005, v. 1, PNo 17, 1–9.

[22] B. A. Arbuzov, *Muon g-2 anomaly and extra interaction of composite Higgs in a dynamically broken electroweak theory,* hep-ph/0110389.

[23] B. A. Arbuzov, *Spontaneous generation of effective interaction in a renormalizable quantum field theory model,* Teor. Mat. Fiz., 2004, v. 140, No 3, 367–387.

[24] D. Colladay, A. Kostelecký, *CPT violation and the standard model,* Phys. Rev. D, 1997, v. 55, No 11, 6760–6774.

[25] D. Colladay, A. Kostelecký, *Lorentz-violating extension of the standard model,* Phys. Rev. D, 1998, v. 58, 116002.

[26] *CPT and Lorentz symmetry II,* Ed. A. Kostelecký, Singapore, World Scientific, 2002.

[27] *CPT and Lorentz symmetry III,* Ed. A. Kostelecký, Singapore, World Scientific, 2005.

[28] H. F. Goenner, G. Yu. Bogoslovsky, *A class of anisotropic (Finsler–) space-time geometries,* Gen. Relativ. Gravit., 1999, v. 31, No 9, 1383–1394.

[29] A. Kostelecký, *Gravity, Lorentz violation, and the standard model,* Phys. Rev. D, 2004, v. 69, 105009.

[30] R. Tavakol, N. van den Bergh, *Finsler spaces and the underlying geometry of space-time,* Phys. Lett. A, 1985, v. 112, 23–25.

[31] R. Tavakol, N. van den Bergh, *Viability criteria for the theories of gravity and Finsler spaces,* Gen. Relativ. Gravit., 1986, v. 18, 849–859.

[32] I. W. Roxburgh, R. Tavakol, N. van den Bergh, *The geometry of space-time and Berwald spaces,* Tensor, 1992, v. 51, 72–77.

[33] G. Yu. Bogoslovsky, H. F. Goenner, *On the possibility of phase transitions in the geometric structure of space-time,* Phys. Lett. A, 1998, v. 244, 222–228.

[34] D. G. Pavlov, G. I. Garas'ko, *Notions of distance and velocity modulus in the linear Finslerian spaces,* Hypercomlex Numbes in Geometry and Physics, 2005, 1 (3), 3–15.

[35] L. Berwald, *Projective Krümmung allgemeiner affiner Räume und Finslersche Räume skalarer Krümmung,* Ann. Math. (2), 1947, v. 48, 755–781; und die Literaturhinweise darin.

[36] A. Moór, *Ergänzung,* Acta math., 1954, v. 91, 187–188.

[37] H. Weyl, *Gravitation und Elektrizität,* Sitzber. preuss Akad. Wiss., Physik–math. Kl., 1918, 465–480.

[38] H. Weyl, *Eine neue Erweiterung der Relativitätstheorie,* Ann. d. Phys., 1919, v. 59, No 10, 101–133.

[39] G. Yu. Bogoslovsky, H. F. Goenner, *Finslerian spaces possessing local relativistic symmetry,* Gen. Relativ. Gravit., 1999, v. 31, No 10, 1565–1603.

The Generalized Finslerian Metric Tensors

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Starting from a Finslerian metric function $F^2 = g_{ij}(x,y)\,y^i y^j,\ g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$, in this work the generalized three rank Finslerian metric tensors $G_{ijk} = \frac{1}{6}\frac{\partial^3 F^3}{\partial y^k \partial y^j \partial y^i} y^i y^j y^k$ and the generalized four rank Finslerian metric tensors $G_{ijkl} = \frac{1}{24}\frac{\partial^3 F^3}{\partial y^i \partial y^j \partial y^k \partial y^l}$, are studied. Taking into account the generalized rank five Christoffel symbols, the generalized differential equations of Finsler geodesics are determined and discussed.

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Introduction

The notion of metric tensor of Riemann and Finsler geometry is the central notion that determines the metric properties of space considered. The metric tensor is the well known notion and tensor analysis of metric space is hardly possible without it. It is usual to consider the metric tensor as a tensor of the second rank. Let us ask whether it is possible to generalize this fundamental notion of Finsler space not to restrict ourselves by the second rank's type of tensor. If this approach is possible mathematically it will permit to look for some applications in modern relativity and quantum physics. The more the rank of the metric tensor the more components it has and it gives possibility to look for, for example, correspondence between these components and fundamental physical interactions. This article is first attempt to consider the generalized metric tensor as a mathematical notion.

The Finslerian metric tensor is well known historically to be found by L. Berwald, G. L. Synge and J. H. Taylor at 1925 by analogy with the Riemannian metric tensor [1]. Although this analogy has helped to develop the Finsler space analysis it has its own boundary. The Riemannian metric tensor has fundamental role but it is not right for Finsler geometry because the Finslerian metric tensor of the second rank has special properties unlike its the Riemannian predecessor. Further consideration gives possibility to doubt universal role of the Finslerian metric tensor of second rank and therefore gives some background of its generalization.

1 Difference between the Finslerian metric tensor and the Riemannian one

The components of the Riemannian metric tensor appeared initially as the coefficients of the second order's expansion of the distance between near points, that is, we have:

$$ds^2 = g_{ij} dx^i dx^j \,. \tag{1}$$

Therefore the components in the fixed system of coordinates depend only on the point of Riemann space:

$$g_{ij} = g_{ij}(x).$$

Unlike Riemann space Finsler manifold is determined by set of axioms one of which represents the property of homogeneity of the Finslerian metric function. Owing to this important axiom the metric function has the next form analogous to (1):

$$F^2 = g_{ij}(x, y)y^i y^j\,. \tag{2}$$

Similarity between (1) and (2) is limited because the components in (2) depend not only on the point of base manifold x, but also on the contravariant vector of tangent manifold y. This imparts new character to (2): this expansion is multiple and hence it has not universal nature.

There is the fundamental formula of the Finsler metric tensor components in the books of this geometry [2–4]:

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^j \partial y^i}\,. \tag{3}$$

However it should be noted that the expansion (1) with the aid of (3) is not unique.

To illustrate it we consider Finsler space associated with the commutative associative algebra H_3. This algebra is the product of three real number's algebras: $H_3 = R \times R \times R$. The metric function of it is [5]:

$$F^3 = y^1 y^2 y^3\,. \tag{4}$$

It isn't difficult to check up that the square of this very metric function can be expanded as the following (2) using not only the classical metric tensor but also some other matrix (6):

$$F^2 = g_{ij}y^i y^j = \tilde{g}_{ij}y^i y^j\,,$$

$$g_{ij} = \left\| \begin{matrix} -\frac{1}{9}\frac{(y^2y^3)^{2/3}}{(y^1)^{4/3}} & \frac{2}{9}\frac{(y^3)^{2/3}}{(y^1y^2)^{1/3}} & \frac{2}{9}\frac{(y^2)^{2/3}}{(y^1y^3)^{1/3}} \\ \frac{2}{9}\frac{(y^3)^{2/3}}{(y^1y^2)^{1/3}} & -\frac{1}{9}\frac{(y^1y^3)^{2/3}}{(y^2)^{4/3}} & \frac{2}{9}\frac{(y^1)^{2/3}}{(y^2y^3)^{1/3}} \\ \frac{2}{9}\frac{(y^2)^{2/3}}{(y^1y^3)^{1/3}} & \frac{2}{9}\frac{(y^1)^{2/3}}{(y^2y^3)^{1/3}} & -\frac{1}{9}\frac{(y^1y^2)^{2/3}}{(y^3)^{4/3}} \end{matrix} \right\|, \tag{5}$$

$$\tilde{g}_{ij} = \left\| \begin{matrix} 0 & \frac{3}{4}g_{12} & \frac{3}{4}g_{13} \\ \frac{3}{4}g_{12} & 0 & \frac{3}{4}g_{23} \\ \frac{3}{4}g_{13} & \frac{3}{4}g_{23} & 0 \end{matrix} \right\|. \tag{6}$$

Besides the components of the Finslerian metric tensor unlike Riemann space has another property. These components may have a singularity at the point $y = 0$ if ($y^i \to 0$) by some special way. For example in the Berwald-Moor space of the third order (4) a component will tend to infinity if the denominator tends to zero more fast the numerator does.

The possibility of such singularities may be considered as another disadvantage of two rank tensors. But the generalized metric tensor may has not this disadvantage. For example the generalized three rank metric tensor of the Berwald-Moor space associated with the algebra H_3 has constant components and consequently the notion of three rank metric tensor is more appropriate for this space.

2 The generalized three rank Finslerian metric tensors

Owing to the key property of homogeneity of metric tensor in the form

$$F(x, ky) = kF(x, y)\,.$$

we can determine a generalized metric tensor.

The Euler's theorem of homogenous function gives the next identities:

$$F^2 = \underbrace{\frac{1}{2}\frac{\partial F^2}{\partial y^i}}_{y_i} y^i = \underbrace{\frac{1}{2}\frac{\partial^2 F^2}{\partial y^j \partial y^i}}_{g_{ij}} y^i y^j\,. \tag{7}$$

It is usual way to determine the covariant components of tangent vector y_i and metric tensor g_{ij}. Their connection with the contravariant components y^i is expressed by a formula:

$$y_{ij} = g_{ij} y^i$$

Due to the Euler theorem for homogenous functions it is possible by analogy with (7) to expand not only the second one but also the higher powers of the Finslerian function to the sum of products of the contravariant components of the vector. Further expanding of the 3-d and 4-th power of this function is going over and as a consequence of which the generalized metric tensors are defined.

$$F^3 = \underbrace{\frac{1}{3}\frac{\partial F^3}{\partial y^i}}_{y_i^* = y_i F} y^i = \underbrace{\frac{1}{6}\frac{\partial^2 F^3}{\partial y^j \partial y^i}}_{y_{ij}^{(3)}} y^i y^j = \underbrace{\frac{1}{6}\frac{\partial^3 F^3}{\partial y^k \partial y^j \partial y^i}}_{G_{ijk}} y^i y^j y^k. \tag{8}$$

The components of the covariant vector y_i^* are appeared to be at the first step of (8), but they differ from the components of the covariant vector y_i just by the factor of F and that is why are out of any interest. The second step of this expansion gives the doubly covariant metric tensor $y_{ij}^{(3)}$. The tensor $\tilde{y}_{ij}$ is a result of this tensor division by the Finslerian function F, that is, we have:

$$\tilde{y}_{ij} = y_{ij}^{(3)} \Big/ F. \tag{9}$$

The tensor (9) is the tensor that takes part in the alternative expansion of the square of the Finslerian function and that is why can be considered as a partial analogue of the fundamental metric tensor g_{ij} (3). The relationship of these two tensors is expressed by the following formula:

$$\tilde{y}_{ij} = \left(g_{ij} + y_i y_j / F^2\right) / 2\,. \tag{10}$$

Easy to see that the tensor (10) has resemblance to the angular Finslerian metric tensor h_{ij} [4]:

$$h_{ij} = g_{ij} - y_i y_j / F^2.$$

Have a look at the tensor $\tilde{y}_{ij}$ properties.

1. As the fundamental metric tensor g_{ij}, the tensor $\tilde{y}_{ij}$ is homogeneous function of zero degree, that is:

$$\tilde{y}^{ij}(x, ky) = \tilde{y}^{ij}(x, y)\,.$$

2. As the fundamental metric tensor g_{ij}, the tensor $\tilde{y}_{ij}$ can be used for raising and lowering index of arbitrary tangent vector, that is:

$$y^i = \tilde{y}^{ij} y_j\,, \quad y_i = \tilde{y}_{ij} y^j\,.$$

3. Unlike the fundamental metric tensor g_{ij}, the tensor $\tilde{y}_{ij}$ does not allow to raise and lower indices of tensors of the second rank and the highest one. For example the result of lowering index of an arbitrary tensor of two rank T_{ij} by $\tilde{y}_{ij}$ if it had been raised by g_{ij} is expressed by the formula:

$$(\tilde{y}_{ij} g^{jk}) T_{kl} = \frac{1}{2}\left[T_{il} + \frac{y_l}{F}\left(\frac{y^j}{F} \cdot T_{jl}\right)\right]\,.$$

4. The internal product of $\tilde{y}_{ij}$ by the tensor g_{ij} is equal to 1:

$$\tilde{y}_{ij} g^{ij} = \tilde{y}^{ij} g_{ij} = 1\,.$$

5. If we construct the Christoffel symbols on the base of $\tilde{y}_{ij}$ by usual way, that is, we have:

$$\tilde{\gamma}_{ijk} = \frac{1}{2}\left[\frac{\partial \tilde{y}_{ij}}{\partial x^k} + \frac{\partial \tilde{y}_{jk}}{\partial x^i} - \frac{\partial \tilde{y}_{ik}}{\partial x^j}\right],$$

this geometrical object obeys the usual equations of the Finslerian geodesics:

$$\frac{d^2 x^i}{ds^2} + \tilde{\gamma}^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \qquad \tilde{\gamma}^i_{jk} = g^{il} \tilde{\gamma}_{ljk} = \tilde{y}^{il} \tilde{\gamma}_{ljk}\,. \tag{11}$$

The proof of this property is analogous to the proof of assertion (20) (see further).

At the last, third step of the expansion (8) we determine the third rank metric tensor G_{ijk}:

$$G_{ijk} = \frac{1}{6} \frac{\partial^3 F^3}{\partial y^k \partial y^j \partial y^i}\,. \tag{12}$$

It should be noted that the generalized metric tensors (9) and (12) are symmetrical by all their indices and so are all the metric tensors.

3 The generalized four rank Finslerian metric tensors

It is possible to expand the fourth degree of the Finslerian function by analogy with (8) to give the next set of identities:

$$F^4 = \underbrace{\frac{1}{4}\frac{\partial F^4}{\partial y^i}}_{y_i^* = y_i F^2} y^i = \underbrace{\frac{1}{12}\frac{\partial^2 F^4}{\partial y^j \partial y^i}}_{y_{ij}^{(4)}} y^i y^j = \underbrace{\frac{1}{24}\frac{\partial^3 F^4}{\partial y^k \partial y^j \partial y^i}}_{y_{ijk}} y^i y^j y^k = \underbrace{\frac{1}{24}\frac{\partial^4 F^4}{\partial y^l \partial y^k \partial y^j \partial y^i}}_{G_{ijkl}} y^i y^j y^k y^l \,. \tag{13}$$

At the first step of (13) we have the covariant vector y^* the components of which differ from the components y_i by the factor of F^2 while we have the doubly covariant tensor $y_{ij}^{(4)}$ at the second step. The trebly covariant tensor y_{ijk} and four times covariant tensor G_{ijkl} are appeared to be at the third and the last, fourth step accordingly.

Now, let us review the properties of the tensor G_{ijkl}.

First, on considering an indicatrix of Finsler space, tensor G_{ijkl} gives possibility to write down not only the equation of tangent plane to a indicatrix's point (14) but also the equations of tangent surfaces of two and three order (15)–(16):

$$G_{ijkl}\left(x^m, y^m_{(0)}\right) \cdot y^i_{(0)} y^j_{(0)} y^k_{(0)} y^l = 1\,, \tag{14}$$

$$G_{ijkl}\left(x^m, y^m_{(0)}\right) \cdot y^i_{(0)} y^j_{(0)} y^k y^l = 1\,, \tag{15}$$

$$G_{ijkl}\left(x^m, y^m_{(0)}\right) \cdot y^i_{(0)} y^j y^k y^l = 1\,. \tag{16}$$

Consequently the known classifications of surfaces of the second and third order permit us to classify the indicatrix's points with the aid of G_{ijkl}.

Secondly, the tensor G_{ijkl} allows to set the five rank geometrical object the components of which may be called the generalized Christoffel symbols. We define the components of this object as the following:

$$\gamma_{i_1 i_2 i_3 i_4 i_5} = \frac{1}{12}\left\{\frac{\partial G_{i_1 i_2 i_3 i_4}}{\partial x^{i_1}} - \frac{\partial G_{i_1 i_3 i_4 i_5}}{\partial x^{i_2}} + \frac{\partial G_{i_1 i_2 i_4 i_5}}{\partial x^{i_3}} - \frac{\partial G_{i_1 i_2 i_3 i_5}}{\partial x^{i_4}} + \frac{\partial G_{i_1 i_2 i_3 i_4}}{\partial x^{i_5}}\right\}. \tag{17}$$

The generalized 5-rank Christoffel symbols of the first kind have properties analogous to the properties of the symmetry of classic 3-rank symbols of Christoffel:

a) a symmetry property by 1, 3 and 5, and also by 2 and 4 indices:

$$\gamma_{i_1 i_2 i_3 i_4 i_5} = \gamma_{i_5 i_2 i_3 i_4 i_1} = \gamma_{i_3 i_2 i_1 i_4 i_5} = \gamma_{i_1 i_4 i_3 i_2 i_5}\,; \tag{18}$$

b) a property connected with the permutation of 1 and 2,4 and 5 indices:

$$\gamma_{i_1 i_2 i_3 i_4 i_5} + \gamma_{i_2 i_1 i_3 i_5 i_4} = \frac{1}{6}\partial G_{i_1 i_2 i_4 i_5} / \partial x^{i_3}\,;$$

c) a property connected with a shift $dx^i (x'^i = dx^i/ds)$ along curve with the natural parameter s:

$$\gamma_{i_1 i_2 i_3 i_4 i_5} x'^{i_1} x'^{i_2} x'^{i_4} x'^{i_5} = \frac{1}{12} \cdot \frac{\partial G_{i_1 i_2 i_4 i_5}}{\partial x^{i_3}} x'^{i_1} x'^{i_2} x'^{i_4} x'^{i_5} \,. \tag{19}$$

With the help of the generalized Christoffel symbols the following assertion can be formulated:

Assertion. *The following generalized form of equations for the Finslerian geodesics is fair:*

$$\frac{d^2 x^i}{ds^2} + \gamma^i_{jklm} \frac{dx^j}{ds} \frac{dx^k}{ds} \frac{dx^l}{ds} \frac{dx^m}{ds} = 0, \tag{20}$$

where

$$\gamma^i_{jklm} = \tilde{y}^{(4)in} \gamma_{jknlm}, \quad \tilde{y}^{(4)}_{in} = y_i y_n - y^{(4)}_{in} \,.$$

On the proving of this assertion we shall predicate upon the equation of Euler-Lagrange where the length along the curve s as natural parameter is used:

$$\frac{d}{ds} \left(\frac{\partial F}{\partial x'^i} \right) - \frac{\partial F}{\partial x^i} = 0 \,. \tag{21}$$

Transform the first item in (21):

$$\frac{d}{ds} \left(\frac{1}{4F^3} \frac{\partial F^4}{\partial x'^i} \right) = \frac{1}{F^6} \left(\frac{d}{ds} \left[\frac{1}{4} \frac{\partial F^4}{\partial x'^i} \right] F^3 - \frac{3}{4} F^2 \frac{\partial F}{\partial s} \cdot \frac{\partial F^4}{\partial x'^i} \right) . \tag{22}$$

Now transform the incoming into (22) derivatives:

$$\frac{d}{ds} \left(\frac{1}{4} \frac{\partial F^4}{\partial x'^i} \right) = \frac{dy^*_i}{ds} = \frac{d}{ds} \left(y^{(4)}_{ij} x'^j \right) = \frac{dy^{(4)}_{ij}}{ds} x'^j + y^{(4)}_{ij} x''^j ,$$

where $\frac{dy^{(4)}_{ij}}{ds} = \frac{\partial y^{(4)}_{ij}}{\partial x'^k} \cdot \frac{dx'^k}{ds} = 2 y_{ijk} ds \cdot x''^k$, while $\frac{dF}{ds} = \frac{\partial F}{\partial x'^k} \cdot \frac{\partial x'^k}{\partial s} = \frac{\partial F}{\partial x'^k} x''^k$.

Substituting into (22) we get the following expression:

$$\frac{d}{ds} \left(\frac{\partial F}{\partial x'^i} \right) = \frac{1}{F^6} \left\{ F^3 \left[2 y_{ijk} ds \cdot x'^j x''^k + y^{(4)}_{ij} x''^j \right] - 3F^5 \frac{\partial F}{\partial x'^k} \frac{\partial F}{\partial x'^i} x''^k \right\} .$$

Note that $F(x, x') = 1$ due to our choice of the length along the curve as a parameter. Besides it is evident from (13) that $y_{ijk} x'^j ds = y^{(4)}_{ik}$. As a result the first item in the Euler-Lagrange equation looks like the following simple form:

$$\frac{d}{ds} \left(\frac{\partial F}{\partial x'^i} \right) = 3 \left(y^{(4)}_{ij} - y_i y_j \right) \cdot x''^j .$$

Transforming the second item of the Euler-Lagrange equation is possible as well:

$$\frac{\partial F}{\partial x^i} = \frac{1}{4F^{3/4}} \cdot \frac{\partial G_{jklm}}{\partial x^i} \cdot x'^j x'^k x'^l x'^l .$$

Taking into account the property c) of the generalized Christoffel symbols (19) the equations of geodesics look like the following form:

$$\tilde{y}^{(4)}_{ij} x''^{j} + \gamma_{jkilm} x'^{j} x'^{k} x'^{l} x'^{m} = 0, \qquad \text{where} \quad \tilde{y}^{(4)}_{ij} = y_i y_j - y^{(4)}_{ij}.$$

Introducing matrix $\tilde{y}^{(4)ij}$, inversed to the matrix, $\tilde{y}^{(4)}_{ij}$ and denoting $\gamma^{i}_{jklm} = \tilde{y}^{(4)in} \gamma_{jknlm}$ we get the very equation (20).

4 Classification of the generalized Finslerian metric tensors

In conclusion to systematize the available concepts of generalized metric tensors we shall classify them.

Definition. *We'll say that the generalized metric tensor belongs to the class (m, n), if its rank is equal to m, and its components are the coefficients in expanding of n-power of the Finslerian function, i.e. equality holds:*

$$F^n = \sum_{i_1,..,i_m=1}^{n} G^{(n)}_{i_1 \ldots i_m} \cdot y^{i_1} \cdot \ldots . \cdot y^{i_m}. \tag{23}$$

According to this definition the components of the metric tensor of class (m, n) is determined by the formula:

$$G^{(n)}_{i_1 \ldots i_m} = \frac{m!}{n!} \frac{\partial^m F^n}{\partial y^{i_1} \ldots \partial y^{i_m}}, \quad (n \geq m > 2)\,. \tag{24}$$

Note that the fundamental metric tensor belongs to the class $(2, 2)$.

Conclusions

The generalized Finslerian metric tensor is determined in this paper, some their properties are investigated and their classification is proposed. Besides the generalized five rank Christoffel symbols are proposed; it gives possibility to generalize the differential equations of the Finslerian geodesics.

Acknowledgements

It should be noted that the idea to generalize the metric tensor was originated from Dr. D. G. Pavlov, and this article is the first effort to realize this idea. The author is also grateful to Dr. G. S. Asanov for helpful discussion.

References

[1] H. Rund, *The Differential Geometry of Finsler Spaces,* Springer, Berlin, 1959.

[2] D. Bao, S.- S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry,* N.-Y.: Springer, 2000.

[3] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaiseisha Press, 1986.

[4] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theories*, Dordrecht: Reidel, 1985.

[5] D. G. Pavlov, *Chronometry of the three-dimensional time,* Hypercomplex Numbers in Geometry and Physics, **1** (2004), 19–30.

Hamilton Canonical Equations and Berwald-Moor Metric (on the Formalism of Physical Theories)

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The specific features of Finsler geometry that could be used to construct a theory of space-time are discussed. The role of the algebraic approach providing the possibility to obtain the equations of the theoretical Physics prior to the introduction of geometry is underlined. Hamilton canonical equations are obtained on the base of the formal use of the function related to the Berwald-Moor metric. These equations can be used to construct the physical theory in Finsler space.

MSC2000: 53B40, 37J99, 70H05, 70S05, 83D05.

1 Introduction

Three different approaches can be used to investigate the Finsler spaces:

- The first of them is the pure mathematical approach. Its characteristic feature is the logical consideration of the self-consistent constructions originating on the base of the arbitrary axioms.
- The second approach belongs to the theoretical Physics. This means that we need a motivation, and the new theory is to be compared with the old one. Besides, the used mathematical objects and their properties are compared to the results of the measurements performed in the real world.
- The third approach is characteristic for philosophy or at any rate for some meta-theory. In it the very possibility to use this or that mathematical construction to describe the real world and the meaning of the used notions are discussed.

The second and the third approaches naturally correlate with the origination, use and achievements of Einstein relativity theory constructed for the isotropic Riemann space-time. The specific feature of the Finsler geometry is the dependence of the metric tensor not only on the coordinate of the point but on the direction of the vector in this point also. This means that in the second and third approaches one should deal with some experiments in which the anisotropy of the world and the possible consequences of such anisotropy might be observed and also with the possibility of such anisotropy.

The anisotropic Finsler generalization of the relativity theory as an order structure was performed by R. I. Pimenov in [1]. There the equations analogous to Einstein equation were obtained for the Finsler spaces and the examples for various metric functions were considered. Paper [1] has a general mathematical character, and the author proves a series of statements that are important for all the three approaches mentioned above. Let us give some of these statements.

1. Finsler geometry provides such a model of the space-time that can not be distinguished from the Schwartzschield solution by means of orbits observations within the same accuracy of measurements. At the same time this geometry excludes the possibility of the gravitational collapse (black holes formation) while the Finsler-Friedman scenario of the first 6 seconds could be absolutely different though the red shift is still present.

2. In the anisotropic world the energy and momentum are not obligatory conserved.

3. The anisotropy of the space-time does not affect the structure of Hamilton approach when constructing a physical theory.

4. To build the formal base of the theory on the Finsler manifold the Leibnitz rule (the derivative of a product) is essentially needed.

5. If the simultaneity is understood in the "radar sense" (as it was done in Einstein relativity theory), then no experiment of the Michaelson type would lead to the conclusion that the space is anisotropic even if it is such. That is, the anisotropy can not be observed by such an experiment. That's why it is not rational to take the radar definition as the only one. At the same time the metric tensor dependence on the direction leads to the ambiguity in the definition of the orthogonal (to the world line of the observer) hyper plane. And this in its turn leads to the fundamental problem: is the simultaneity the notion of the causal structure or is it the notion of the Lagrangian structure?

6. Through the construction of the Finsler theory of the anisotropic space-time the inevitability of the passage from the smooth functions to the functions of the wider class emerges.

The first two statements mean that it is hardly possible to discover the anisotropy of the Universe as it is (if any) by observations. At the same time the consequences of such anisotropy revealing themselves as the break of the conservation laws could be used in the interpretation of the experiments with particles (this possibility was mentioned in [2]) and could be found on the cosmological scale.

Statements 3 and 4 mean that the canonical equations methods and Lie algebra methods can be used while studying Finsler spaces.

Statement 5 poses a serious problem the solution of which is probably beyond not only mathematics but beyond physics as well.

Statement 6 is additionally discussed in the separate paper [3] in which the author proclaims that "he determinism had not been 'found in Nature', it had not been 'obtained logically' or 'proved mathematically'. We only trusted in determinism". In [3] the following idea is presented. The essentially non-differentiable structures play an important role in the empirical description of physical reality - the fractal objects have appeared in science [4]. But the general relativity theory and the physical cosmology based on it both deal only with well-differentiable structures. This means that the generalization of the general relativity theory that should be built must be free from the functions differentiability hypothesis.

As to the particular use of the Berwald-Moor metric

$$s(X) = \sqrt[4]{x^1x^2x^3x^4} = (x^1x^2x^3x^4)^{1/4} \tag{1}$$

one could mention paper [5] in which the author used the notion of 'volume' for the metrization and described the gravity field in Finsler space..

In [6–8] the authors also tried to use the Berwald-Moor metric on the base of the notion of scalar poly-product introduced in [6]. It should be underlined that though the axiomatics of poly-product is satisfactory, this object is a new one and up to now was never used for the interpretation of physical experiments which means that a special attention should be paid to its application. Notice also that the simultaneity in these papers was understood in the radar sense and this can also lead to problems while interpreting the results.

2 Kauffman's algebraic approach

The circumstances mentioned in statements 3, 4 and 6 of paper [1] draw attention to Kauffman's paper [9]. In [9] the Lie algebras are proclaimed to be the possible foundation of the physical reality description, and this stresses the role of formalism in the construction of a physical theory. The latter and also the role of determinism in the physical theory were discussed in [10]. In paper [9] the author constructs the Lie algebra and starts with the introduction of the discrete derivative operation, this fact immediately broadening the class of functions in use. Then he passes to the commutators and introduces the shift operators to obtain the new algebra elements conjugated to the initial elements – all this being to provide the validity of Leibnitz rule. Using the Jacoby identity several formal algebraic structures are obtained and the forms of these structures appear to coincide with the forms of several equations of the theoretical physics, particularly, with Hamilton canonical equations. This was performed with the help of Legendre transformation which is characteristic to this approach.

In order to get a space with "reasonable curvature", several limitations on the commutators of the variables (and on the corresponding shift operators) are imposed. This results in the algebraic structures in which some other equations of theoretical physics are easily recognized. Particularly, if the algebra elements $\{X_0, X_1, ...X_n\}$ correspond to the commutation operators $\{H, P_1, ...P_n\}$, then the introduced definitions provide the following equations

$$\frac{dP_i}{dX_0} = -\frac{\partial H}{\partial X_i}; \qquad \frac{dX_i}{dX_0} = \frac{\partial H}{\partial P_i}; \qquad i = 1, 2, ...n \tag{2}$$

the form of which coincides with Hamilton canonical equations. In the same way the curvature operator could be formally introduced to define for the given elements

X and Y the "non-commutativity rate" of the corresponding operators ∇_X and ∇_Y with regard to the "non-commutativity rate" of X and Y. Then there appear the commutator $g_{ij} = [X_i, \dot{X}_j]$ which can be naturally correlated with the metric tensor and the Levi-Chivita connection $\Gamma_{ijk} = \frac{1}{2}(\nabla_i g_{jk} + \nabla_j g_{ik} - \nabla_k g_{ij})$ which appears in [9] from the calculation of the commutators and from Jacoby identity and has no initial correlation with geometry. In a similar formal way there appear the expressions coinciding with diffusion equation, Schroedinger equation, Maxwell equation, gauge theories equations.

Thus, the structure of several fundamental equations of theoretical physics, and particularly of Hamilton equations does not presume the prior choice of geometry that would be used to model the space-time. This provides the possibility to use the metric function in a formal way convenient to construct the canonical equations.

3 Canonical variables

Let us use this possibility to involve the Berwald-Moor metric into the theory. Let us regard the vector space with vectors $X = \{x^i\}$, $i = 1, 2, ...n$, choose the appropriate coordinate system, consider all $x^i > 0$, and introduce the following scalar function

$$s^2(X) = \sqrt{x^1 x^2 x^3 x^4}. \tag{3}$$

On the one hand, it is obviously related to the Berwald-Moor metric (1) and on the other hand, it provides the possibility to use of the Legendre transformation and to define vector P conjugated to vector X as the gradient of the introduced scalar

$$P \equiv \nabla \frac{1}{2} s^2(X) = s(X) \nabla s(X). \tag{4}$$

(Usually the further use of function s2(X) leads to the construction of the Cartan metric tensor $h_{ik} = \frac{1}{2}\, \partial_i \partial_k s^2(X)$.)

If $X = \{x^i\}$ are the contra-variant components of vector X, then the co-variant components of vector $P = \{p_i\}$ in n-dimensional case can be obtained with help of formula (3) analogue and Eq. (4)

$$p_i = \frac{s^2(X)}{n x^i} = \frac{(x^1 x^2 ... x^n)^{2/n}}{n x^i}. \tag{5}$$

Then for $n = 4$ these components have the form

$$p_1 = \frac{1}{4}\sqrt{\frac{x^2 x^3 x^4}{x^1}}; \quad p_2 = \frac{1}{4}\sqrt{\frac{x^1 x^3 x^4}{x^2}}; \quad p_3 = \frac{1}{4}\sqrt{\frac{x^1 x^2 x^4}{x^3}}; \quad p_4 = \frac{1}{4}\sqrt{\frac{x^1 x^2 x^3}{x^4}}. \tag{6}$$

Let us define the (pseudo) scalar product (X, Y) in the following way

$$(X, Y) = Y \frac{1}{2} \nabla s^2(X) = y^i x_i\,, \tag{7}$$

where the summation over repeating indexes rule is assumed. It is essential that the scalar product depends on the order of its terms. With regard to Eqs.(5, 7) the scalar product (X, Y) obtains the form

$$(X,Y) = y^i x_i = Y\frac{1}{2}\nabla s^2(X) = \frac{s^2(X)}{4}\sum_{k=1}^{4}\frac{y^k}{x^k}\,. \tag{8}$$

One can easily see that

$$(X,X) = s^2(X)\,, \tag{9}$$

which corresponds to the regular correlation between the metric and the norm defined as the square of a scalar product.

(The angle φ between vectors Y and X (from Y to X) can be also introduced if needed. It will be given by the formula $ch\,\varphi = \frac{(X,Y)}{s(X)s(Y)}$. Of course, the angle from X to Y is not equal to the angle from Y to X because the scalar product is not commutative. The sum of such angles in the 2-plane is not additive).

If the scalar product $(X, Y) = 0$, then vector Y may be called orthogonal to X (notice that generally speaking, vector $X$is not orthogonal to Y). Then the hyper plane of all such Y's may be called the hyper plane orthogonal to λX line. According to Eq.(8), this plane is given by the expression

$$\sum_{k=1}^{4}\frac{y^k}{x^k} = 0\,. \tag{10}$$

This expression can be used as the definition of the surface of relative simultaneity of the inertial observer λX. In other words it is the observer's proper space where the trajectories (i.e. the dependencies of space coordinates over space coordinates) of the moving bodies can be constructed. The dimensionality of this space is equal to $n - 1$.

Introducing the notion of action, we see that the usual expression for momentum takes place

$$S = -\frac{1}{2}s^2(X) \;\Rightarrow\; p_i = \frac{\partial S}{\partial x^i}\,; \quad i = 2,3,4\,. \tag{11}$$

The first component (or the zeroth component in the usual notation) in Eq.(6) may be considered to be the 'Hamiltonian', thus, we define

$$p_1 \equiv H = -\frac{\partial S}{\partial x^1} = \frac{1}{4}\sqrt{\frac{x^2x^3x^4}{x^1}}\,. \tag{12}$$

Then

$$\frac{\partial p_i}{\partial x^1} = -\frac{\partial H}{\partial x^i}\,; \quad i = 2,3,4\,, \tag{13}$$

which coincides with the first equation in (2).

To define the components of the three-dimensional velocity let us express p_1 with the help of $p_i$for $(i = 2, 3, 4)$

$$p_1 = H = \frac{16 p_2 p_3 p_4}{(x^1)^2} \tag{14}$$

Then the three-dimensional velocity components will be

$$\frac{dx^i}{dx^1} = \frac{\partial H}{\partial p_i}\,; \quad i = 2, 3, 4 \tag{15}$$

– these values being equal to v^i/c if we consider $x^1 = ct$. The expression (15) coincides with the second equation in (2).

Taking the mentioned definitions of "momentum" and "Hamiltonian", one can try to insert the physical meaning into them. Then the obtained canonical equation can be immediately used to describe the dynamics in the space with the Berwald-Moor metric.

On this way one may notice that the definition of action

$$S = -\alpha \int ds\,,$$

in which ds is the Berwald-Moor metric (1) includes a constant.

In the regular case of the pseudo Euclidean metric of special relativity the space time was isotropic and the light velocity was constant and did not depend on direction. It was natural to define the constant in such a way that the Lagrangian corresponded to the classical Lagrangian of the free particle $L = \frac{mv^2}{2}$ in the limit case $c \to \infty$. That's why the constant was taken $\alpha = mc$. But now we can't act in this way. The space-time is not isotropic and the Lagrangian of the free particle won't have such a simple form. The velocity of light can depend on the direction and the limit transition should be defined more accurately. These circumstances will be regarded in more detail later.

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References

[1] R. I. Pimenov, *Anisotropic Finsler Generalization of the Relativity Theory as an Order Structure.* Syktyvkar, 1987.

[2] G. Yu. Bogoslovskiy, *On the special relativity theory of the anisotropic space-time.* Doklady USSR, v. 213, p. 1055, 1973.

[3] R. I. Pimenov, *Differential equations – are they justified?* http://www.chronos.msu.ru/Public/pimenov_diffury.html

[4] B. B. Mandelbrot, *The Fractal Geometry of Nature.* W. H. Freeman, NY, 1982.

[5] G. S. Asanov, *Finsler space gravity field based on the notion of volume,* Vestnik Moscow University, v. 17, p. 288, 1976.

[6] D. G. Pavlov, *Generalization of the scalar product axioms,* Hypercomplex numbers in Geometry and Physics, v. 1, p. 5, 2004.

[7] S. V. Lebedev, *Properties of spaces associated with the commutative-associative algebras H_3 and H_4,* Hypercomplex numbers in Geometry and Physics, v. 1, p. 68, 2004.

[8] G. I. Garas'ko, D. G.Pavlov, *The notion of distance and velocity modulus in linear Finsler spaces,* Hypercomplex Numbers in Geometry and Physics, v. 3, p. 3, 2005.

[9] L. Kauffman, *The non-commutative worlds,* arXiv: quant-ph/0403012 v. 3

[10] S. Siparov, *The Physical World as a Function of Observer's Consciousness.* In SS&T, vol. 5, p. 193, 1997.

On Defining Equations for the Elements of Associative and Commutative Algebras and on Associated Metric Forms

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The paper deals with elements of associative and commutative finite algebras and with the equations, these elements satisfy. We consider the relations between automorphisms of algebra and these defining equations. The coefficients of the defining equations are calculated: these coefficients are homogeneous forms on the coordinates of elements from the algebra and may be associated with well-known Minkowski and Berwald-Moor metrics.

MSC2000: 13A99, 53B40.

1 Introduction

1.1 Statement of work

The fact that theory of finite-dimensional algebras and its methods can be usefully employed (see [1], [2]) in geometry, physics, and computer science can be explained, from author's point of view, as follows .

Composition algebras found the widest distribution in applications. We recall that these algebras are unital algebras with the non-degenerate quadratic forms $N(x)$ *(norms)* which are defined in the vector spaces over these algebras and which satisfy $N(xy) = N(x)\,N(y)$. The recursive algorithm for construction and classification of these algebras over various fields is tightly related to the fact that in algebras, obtained at every step of the recursion, there exist (anti)automorphisms $x \mapsto \bar{x}$ of order 2. These (anti)automorphisms, which can be recursively continued on the next steps of the algorithm for algebra construction, induce the above-mentioned forms $N(x)$.

However, to construct algebras, different from R and C, but with analogues of the real or complex norm, one has to pay a certain fee: the constructed algebras will be non-commutative and/or non-associative. Moreover, the Cayley-Dickson recursive algorithm for construction of composition algebras already at the third step comes to a non-associative structure and cannot be continued [3]. Notwithstanding this fact, for example, a commutative 4D quaternion algebra is usefully employed for solving the problems of mechanics, in machine vision, in physics. This can be explained by at least two reasons: (1) in this algebra there is a norm and, for example, (2) using this algebra 3D orthogonal transforms can be 'elegantly' written not in terms of the 'external' matrix language, bug in terms of 'internal' operations of the quaternion algebra, i.e. written in the 'coordinate-free' form. Author shares E. Artin's opinion, that, '...mathematical training stills suffers from the enthusiasm caused by discovery of the isomorphism [1]. As a result, the geometrical reasoning were virtually abandoned and geometry was replaced by calculations. Instead of very illustrative space transforms, preserving addition of vectors and multiplication of vectors by scalars ¡. . . ¿, operation on matrices are used. According to my own experience, proofs, involving operations matrix calculations can be abridged to less then half of the original length, if matrices are avoided' [2] [4]. For example, consider a four-dimensional algebra isomorphic to the algebra of (2×2)-matrices $M_2(R)$. In this algebra consider the 'Clifford basis' with the multiplication rules for basis elements given by

$$e_0^2 = e_0,\ e_1^2 = e_0,\ e_2^2 = e_0,\ e_3^2 = -e_0;$$
$$e_1 e_2 = -e_2 e_1,\ e_1 e_3 = -e_3 e_1,\ e_2 e_3 = -e_3 e_2;\ e_1 e_2 = e_3,$$

Further, consider involutive morphism in this algebra (a so called symplectic involution) given by

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \bar{X} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and a 'natural' embedding $R \to M_2(R)$ given by $y \mapsto y e_0,\ y \in R,$

Then, the coefficients of the defining equation (characteristic equation) for an element $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from this algebra can be represented in a coordinate-free form in terms of the norm $N(X) = X \cdot \bar{X} = \det X$ and the trace $Tr(X) = X + \bar{X} = a + d$, thus the defining equation can be written in a form

$$X^2 - Tr(X)\, e_0 X + N(X)\, e_0 = 0, \tag{1.1}$$

This is interesting to note, that the relation (1.1) may be considered as a special (two-dimensional) case of the Cayley-Hamilton theorem. From this it follows that, after appropriate interpretation, all the results of 'linear' *geometry in 2D plane* may be obtained from the *algebraic characteristics of the 4D Clifford algebra* $M_2(R)$.

One of the most remarkable examples of such an interpretation for Clifford algebras was proposed by D. Hestenes. He proposed to construct an algebra, isomorphic to the original one, via redefining certain algebra operations in such a way that elements of the constructed algebra (so called *'geometric algebra'*) have properties of scalars or vectors, or bivectors, etc. Thus, in this algebra the above-mentioned objects are 'equivalent' with respect to the redefined operations, despite the fact

[1] which takes a linear transform in the vector space to a certain matrix – V. C.

[2] Further, however, E. Artin notes with some humour: "Sometimes it is impossible; for example, when it is necessary to compute determinant".

that if canonical interpretations of linear geometric objects are used, these objects are elements of sets of different structure [5].

This is also important to note, that 'Clifford's approach' to geometry despite of its generality is closely related to analysis of symmetries associated with automorphisms (or antiautomorphisms) of *order equal to 2 only* (i.e. only with involutions). In contrast to composition algebras, associative-commutative finite-dimensional algebras are not quadratic algebras over the field R. Therefore, defining algebraic equations for elements of these algebras are naturally associated with automorphisms of higher orders. Dye to this fact, this appears promising to analyze the properties of geometric interpretations of these algebras, expressed in terms of the symmetry group of order greater than two.

To realize the ideas stated above so to understand better the role of high-order automorphisms in creation of geometric and/or physical models of time-space, from the author point of view, this is possible to start with analysis of the equations, which elements of associative-commutative algebras satisfy, analysis of invariant properties of these equations. Thus, as a first step, the analogues for the forms $N(X) = X \cdot \bar{X} = \det X$ and $Tr(X) = X + \bar{X} = a + d$ from the relation (1.1) are to be found. This paper can be considered as this initial step.

However paradoxical it is, the idea to use additional (or redefined) operations (associated with the automorphisms of order higher than 2) in certain finite-dimensional algebra so to employ the power 'algebraic methods' for solving geometric problems has a long history. For example, there was proposed an exotic algebraic structure – the 'Zassenhaus finite quasifield' [6]. This structure is exotic enough and not very well known, thus, we provide the description of its construction.

Suppose, that in a finite field F_q, $q = p^m$, (p is prime) there is an operation introduced $x * y = y$ $(x, y \in F_q)$, which is given by the following equality: $x * y = y \cdot \eta(x)$. Here $(\cdot)$ is multiplication in the field F_q; η is a Frobenius automorphism of a specific form. By Wedderburn's theorem, the introduced operation is non-associative and non-commutative. However, a certain $(*)$-power $((((x * x) * x\ ) * ...) * x)$ of the element x is an element of the prime field F_p and can be interpreted as a peculiar 'norm' of an element from the algebra F_q over the field F_p, induced by certain 'multilinear scalar product'. Being appropriately interpreted, the results obtained by the author, which are cited in [6] and are related to the finite geometries, can be reformulated and reproved in terms of the 'multilinear scalar product'. This is quite evident, but necessary to note, that the fact that the fields F_q are finite (and, as a result, that the corresponding Zassenhaus quasifields are finite), restricts the range of problems, which can be solved applying this technique, to configuration problems of the finite geometry.

The idea, to consider 'multilinear' (in contrast to classical 'bilinear') scalar products in associative-commutative algebras so to create adequate geometric-physical models was most likely introduced by D. G. Pavlov [7]. Below are certain results of this work provided, which, from author's point of view, are of certain methodological importance.

1. Corollary 2.1. The coefficients of the defining polynomial $\Phi(\xi; w)$ for an element w are invariant with respect to the 24-element automorphism group S_4. Thus, the set of roots of the *quartic* polynomial $\Phi(\xi; w)$ contains not only the root w but also automorphic images of this root with respect to automorphisms $\sigma \in S_4$.

2. Corollary 2.2. Four automorphisms used in the proof of Theorem 2.1 to construct the defining polynomial form the subgroup of the group S_4; this subgroup is isomorphic to the cyclic group C_4 of order 4. However, for construction of the same polynomial four other automorphisms can be used, which also form the subgroup of the group S_4, but this subgroup of order 4 is isomorphic to another group, namely to the Cartesian product $C_2 \times C_2$ of 2 cyclic groups of order 2.

3. Corollary 2.3. Moreover, the same defining polynomial can be generated by another four automorphisms, which do not form a group with respect to composition of functions. This statement allows us to make a 'cautious suggestion' that in general case the fact that generating automorphisms form the group is not a necessary condition.

Terminology. The author does not consider intentionally any possible geometric and/or physical interpretations of the obtained results, leaving this for specialists in these fields. That is why, throughout this paper the 'neutral' (with respect to interpretation) concepts are used (e.g. 'defining equation', 'defining forms') Although, for example, these 'defining forms' (obtained as a result of examining *algebraic* structural properties of appropriate algebras) may be easily interpreted as *metric* Minkowski or Berwald-Moor forms (see relations (2.6)).

Notation. Without any additional remarks, in this paper the symbols R, C denote the fields of real and complex numbers respectively. The direct sum of algebras is denoted by the symbol $\dot{+}$, and the symbol $\oplus$ is reserved for bitwise addition ($mod\,2$) of the integers. By the symbol $H(k)$ the direct sum of k instances of the field R is denoted (sometimes with certain contextual remarks).

1.2 Weierstrass theorem

Exhaustive classification of associative-commutative algebras that do not contain nilpotent elements is given in the *Weierstrass theorem* [8]:

Theorem 1.1. Any associative-commutative finite-dimensional algebra that does not contain nilpotent elements over is isomorphic to the direct sum of the algebras R and C.

From this theorem it immediately follows that there exist no more than three non-isomorphic 4D algebras of this class:

$$R\dot{+}R\dot{+}R\dot{+}R \cong \big(R\dot{+}R\big) \dot{+} \big(R\dot{+}R\big) \cong H\,(2)\,\dot{+}H\,(2) \cong H\,(4)\,, \tag{1.2}$$

$$R\dot{+}R\dot{+}C \cong H\,(2)\,\dot{+}C, \tag{1.3}$$

$$C\dot{+}C. \tag{1.4}$$

Let us prove show that these algebras are not isomorphic. Suppose that in each of these algebras, a basis is chosen, which is associated with representation of the algebra in a form of one of direct sums (1.2)–(1.4), and multiplication rules for basis elements are set:

- for algebra $R\dot{+}R\dot{+}R\dot{+}R$, the basis $\{E_1, E_2, E_3, E_4\}$ is chosen, and the multiplication rules for elements of the basis are given by the following Cayley table:

Table 1.1

	E_1	E_2	E_3	E_4
E_1	E_1	0	0	0
E_2	0	E_2	0	0
E_3	0	0	E_3	0
E_4	0	0	0	E_4

- for the algebra $R\dot{+}R\dot{+}C$, the basis $\{E_1, E_2, E_3, E_4\}$ is chosen, and multiplication rules for elements of the basis given by the following Cayley table:

Table 1.2.

	E_1	E_2	E_3	E_4
E_1	E_1	0	0	0
E_2	0	E_2	0	0
E_3	0	0	E_3	E_4
E_4	0	0	E_4	$-E_3$

- for the algebra $C\dot{+}C$, the basis $\{E_1, E_2, E_3, E_4\}$ is chosen, and multiplication rules for elements of the basis are given by the Cayley table:

Table 1.3.

	E_1	E_2	E_3	E_4
E_1	E_1	E_2	0	0
E_2	E_2	$-E_1$	0	0
E_3	0	0	E_3	E_4
E_4	0	0	E_4	$-E_3$

Using these Cayley tables, one may easily notice that to calculate the product of the 'constant' element $(aE_1 + bE_2 + cE_3 + dE_4)$ and the 'variable' element $(tE_1 + xE_2 + yE_3 + zE_4)$ this is sufficient to apply the linear operator onto the column-vector $(t, x, y, z)^T$ which consists of coordinates of the element

$(tE_1 + xE_2 + yE_3 + zE_4)$ in the basis $\{E_1, E_2, E_3, E_4\}$, i.e. this is sufficient to calculate the matrix product

$$M_1 \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}; \; M_2 : \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}; \; M_3 : \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{1.5}$$

for algebras $R\dot{+}R\dot{+}R\dot{+}R$, $R\dot{+}R\dot{+}C$ or $C\dot{+}C$, respectively. None of the (4×4)-matrices in (1.5) may be a matrix, associated with multiplication in another algebra if in that algebra another basis is selected, which may be obtained from using a linear transform. In fact, the characteristic equations of the matrices (1.5), being invariant with respect to linear transforms of the basis, correspond to the different the set of quartic polynomial root of different structure: namely (1) to the set of roots consisting of 4 real roots, (2) to the set of roots consisting of 2 real roots and 2 complex-conjugate roots, and (2) to the set of roots consisting of 2 pairs of complex conjugate roots respectively.

2 Algebra $R\dot{+}R\dot{+}R\dot{+}R \simeq H(4)$, its automorphisms and metric forms

2.1 'Isotropic' basis

Consider the algebra $R\dot{+}R\dot{+}R\dot{+}R \cong H(2) \dot{+} H(2) \cong H(4)$ with the basis $\{E_1, E_2, E_3, E_4\}$ and the multiplication rules for elements of the basis given by the Table 1.1. (In the sequel, this basis is referred to as the *'isotropic basis'*).

The multiplicative identity element (algebra unit) in this basis is the element $I = E_1 + E_2 + E_3 + E_4$ and the field R is canonically embedded into the algebra $R\dot{+}R\dot{+}R\dot{+}R$:

$$R \to R\dot{+}R\dot{+}R\dot{+}R \cong H(4), \quad x \mapsto xI, \quad x \in R.$$

Theorem 2.1. The algebra $R\dot{+}R\dot{+}R\dot{+}R$ is an algebra of order 4 over R, i.e. every element $w \in R\dot{+}R\dot{+}R\dot{+}R$ satisfies the algebraic equation with real coefficients of degree not greater than four.

Proof. Let $w = aE_1 + bE_2 + cE_3 + dE_4 \leftrightarrow (a, b, c, d)$. Consider 4 maps of algebra $R\dot{+}R\dot{+}R\dot{+}R$ into itself (it is evident that these maps are 4 automorphisms):

$$\begin{aligned} \tau_0 &: w = aE_1 + bE_2 + cE_3 + dE_4 \mapsto aE_1 + bE_2 + cE_3 + dE_4, \\ \tau_1 &: w = aE_1 + bE_2 + cE_3 + dE_4 \mapsto bE_1 + cE_2 + dE_3 + aE_4, \\ \tau_2 &: w = aE_1 + bE_2 + cE_3 + dE_4 \mapsto cE_1 + dE_2 + aE_3 + bE_4, \\ \tau_3 &: w = aE_1 + bE_2 + cE_3 + dE_4 \mapsto dE_1 + aE_2 + bE_3 + cE_4. \end{aligned} \tag{2.1}$$

In other words, these maps are cyclic permutations of the components (a, b, c, d) of the algebra element w. One can easily obtain that the element w is a root of the polynomial

$$\Phi(\xi; w) = (\xi - \tau_0(w))(\xi - \tau_1(w))(\xi - \tau_2(w))(\xi - \tau_3(w)), \tag{2.2}$$

and the coefficients of the polynomial $\Phi(\xi; w)$ are real. In fact, from (2.2) via direct calculations, we get

$$\Phi(\xi; w) = \xi^4 - s_1(w) I\xi^3 + s_2(w) I\xi^2 - s_3(w) I\xi^1 + s_4(w) I, \tag{2.3}$$

where real coefficients $s_\nu(w)$ are homogeneous symmetric forms of the components (a, b, c, d) of the algebra element w:

$$\begin{gathered} s_1(w) = a + b + c + d, \\ s_2(w) = ab + ac + ad + bc + bd + cd, \\ s_3(w) = bcd + acd + abd + abc, \\ s_4(w) = abcd. \qquad \square \end{gathered} \tag{2.4}$$

Definition 2.1. The polynomial $\Phi(\xi; w)$ of minimal degree with real coefficients and with the leading coefficient equal to 1 such that $\Phi(\xi; w)|_{\xi=w} = 0$ is called the *defining polynomial of the element* w and its coefficients are called the *defining forms* of this element.

This is quite evident, that the same element w, being represented in different basis of the same algebra may have different defining forms, i.e. these forms differ as functions of the element coordinates. In particular, forms (2.4) are written as functions of components of the element w, represented in basis $\{E_1, E_2, E_3, E_4\}$ with the multiplication rules for the basis elements given by Table 1.1. etc.

Despite the triviality of Theorem 2.1, this theorem has several interesting corollaries, which are not typical for the 'classical' theory of polynomials over the field.

Corollary 2.1. The forms (2.4) are invariant with respect to any permutation $\sigma \in S_4$ of four components (a, b, c, d) of the algebra element w. As a result, since $\sigma \in S_4$ is an automorphism of algebra $R \dot{+} R \dot{+} R \dot{+} R$ over R, then the equality $\Phi(\xi; w) = 0$ yields that

$$\begin{aligned} &0 = \sigma\left(\Phi(\xi; w)|_{\xi=w}\right) = \\ &= \sigma(w)^4 - \sigma(s_1(w)) I\sigma(w)^3 + \sigma(s_2(w)) I\sigma(w)^2 - \sigma(s_3(w)) I\sigma(w)^1 + \sigma(s_4(w)) I = \\ &= \sigma(w)^4 - s_1(\sigma(w)) I\sigma(w)^3 + s_2(\sigma(w)) I\sigma(w)^2 - s_3(\sigma(w)) I\sigma(w)^1 + s_4(\sigma(w)) I = \\ &= \Phi\left(\xi; \sigma(w)|_{\xi=\sigma(w)}\right) = 0 \end{aligned}$$

That is the *quartic* polynomial $\Phi(\xi; w)$ in addition to the root w has at least 23 other roots $\sigma(w)$, $\sigma \in S_4$, i.e. all the automorphic images of the element with respect to the maps $\sigma \in S_4$. This is interesting to note, that applying the automorphisms σ to both the parts of the equality $0 = \sigma\big(\Phi(\xi; w)|_{\xi=w}\big)$ in the form

(2.2) in order to obtain the relation $\Phi\big(\xi;\sigma(w)|_{\xi=\sigma(w)}\big)=0$ via the following chain of computations

$$
\begin{aligned}
&0=\sigma(\Phi(w;w))= \\
&=(\sigma(w)-(\sigma\circ\tau_0)(w))(\sigma(w)-(\sigma\circ\tau_1)(w))(\sigma(w)-(\sigma\circ\tau_2)(w))(\sigma(w)-(\sigma\circ\tau_3)(w)) \\
&=(\sigma(w)-\tau_0(\sigma(w)))(\sigma(w)-\tau_1(\sigma(w)))(\sigma(w)-\tau_2(\sigma(w)))(\sigma(w)-\tau_3(\sigma(w))) \\
&=\Phi(\sigma(w);\sigma(w))
\end{aligned}
$$

is not correct, as the automorphism group S_4 is non-commutative. The fact that the forms $s_1(w), s_2(w), s_3(w), s_4(w)$ are real is significant, and this fact is based on specific choice of automorphisms $\{\tau_0,\tau_1,\tau_2,\tau_3\}$.

Corollary 2.2. Four automorphisms $\{\tau_0,\tau_1,\tau_2,\tau_3\}$, used in the proof of Theorem 2.1, form the subgroup of the group S_4; this subgroup is isomorphic to the cyclic group C_4 of order 4. However, one may easily prove (via direct calculations) that the *same polynomial* (2.3) can be generated by four other automorphisms $\{\lambda_0,\lambda_1,\lambda_2,\lambda_3\}$:

$$
\begin{aligned}
\lambda_0: &\quad w=aE_1+bE_2+cE_3+dE_4\mapsto\ aE_1+bE_2+cE_3+dE_4, \\
\lambda_1: &\quad w=aE_1+bE_2+cE_3+dE_4\mapsto\ cE_1+dE_2+aE_3+bE_4, \\
\lambda_2: &\quad w=aE_1+bE_2+cE_3+dE_4\mapsto\ bE_1+aE_2+dE_3+cE_4, \\
\lambda_3: &\quad w=aE_1+bE_2+cE_3+dE_4\mapsto\ dE_1+cE_2+bE_3+aE_4,
\end{aligned}
$$

which form the subgroup of order 4 of the group S_4, and this subgroup is isomorphic to *another group*, namely to the Cartesian product $C_2\times C_2$ of 2 cyclic groups of order 2.

Corollary 2.3. Moreover, the *same polynomial* (2.3) can be generated by four other automorphisms $\{\nu_0,\nu_1,\nu_2,\nu_3\}$:

$$
\begin{aligned}
\nu_0: &\quad w=aE_1+bE_2+cE_3+dE_4\mapsto\ dE_1+aE_2+bE_3+cE_4, \\
\nu_1: &\quad w=aE_1+bE_2+cE_3+dE_4\mapsto\ bE_1+cE_2+dE_3+aE_4, \\
\nu_2: &\quad w=aE_1+bE_2+cE_3+dE_4\mapsto\ cE_1+bE_2+aE_3+dE_4, \\
\nu_3: &\quad w=aE_1+bE_2+cE_3+dE_4\mapsto\ aE_1+dE_2+cE_3+bE_4,
\end{aligned}
$$

which *do not form the group* with respect to composition of functions.

2.2 'Hyperbolic' basis

The set $\{E,I,J,K\}$ is called a *basis of hyperbolic units* (or, for short, an *hyperbolic basis*), if $\{E,I,J,K\}$ is a basis of the algebra $H(4)\cong R\dot{+}R\dot{+}R\dot{+}R$ and multiplication rules for elements of the basis elements are given by the following Cayley table:

Table 2.1

	E	I	J	K
E	E	I	J	K
I	I	E	K	J
J	J	K	E	I
K	K	J	I	E

The element $w \in H(4)$ can be represented in the form $w = tE + xI + yJ + zK$ $(t, x, y, z \in R)$, where E is a multiplicative identity (unit) of the algebra. The isotropic basis $\{E_1, E_2, E_3, E_4\}$ and the hyperbolic basis $\{E, I, J, K\}$ are bound with the linear transform with the orthogonal Hadamard matrix

$$had_4 = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

where the symbol $\otimes$ denotes Kronecker multiplication. The author found this possible to omit the computations, as one can easily obtain the exact formulas that bind coordinates of the element $w \in H(4)$ being represented in these two bases (this is a trivial linear-algebraic problem).

This is also quite evident, that for every automorphism from the group S_4 (this automorphism permutes the coordinates of the element $w \in H(4)$ represented in the basis $\{E_1, E_2, E_3, E_4\}$), there exists an appropriate linear transform of the coordinates t, x, y, z of the same element in the basis $w = tE + xI + yJ + zK$. This transform also represents a certain automorphism of the algebra $H(4)$. In particular, such automorphisms are:

$$\begin{aligned} \mu_0 : &\quad w \mapsto \mu_0(w) = tE + xI + yJ + zK, \\ \mu_1 : &\quad w \mapsto \mu_1(w) = tE + xI - yJ - zK, \\ \mu_2 : &\quad w \mapsto \mu_2(w) = tE - xI + yJ - zK, \\ \mu_3 : &\quad w \mapsto \mu_3(w) = tE - xI - yJ + zK. \end{aligned} \tag{2.5}$$

Suppose, that the element $w \in H(4)$ is represented in the basis $\{E_1, E_2, E_3, E_4\}$, then there is a subgroup of order 4 (isomorphic to the Cartesian product $C_2 \times C_2$ of 2 cyclic groups of order 2) of the group S_4 such that action of automorphisms from this subgroup onto components of the element is equivalent to four transforms (2.5).

For the defining polynomial, via direct calculations, we obtain:

$$\begin{aligned} &\Phi(\xi; w) = (\xi - \mu_0(w))(\xi - \mu_1(w))(\xi - \mu_2(w))(\xi - \mu_3(w)) = \\ &= (\xi^4 - S_1(w)\xi^3 + S_2(w)\xi^2 - S_3(w)\xi^1 + S_4(w))E, \end{aligned}$$

where

$$
\begin{gathered}
S_1(\omega) = 4t \\
S_2(\omega) = 6t^2 - 2x^2 - 2y^2 - 2z^2 \\
S_3(\omega) = -4tx^2 - 4y^2t - 4z^2t + 4t^3 + 8yzx = 4t(t^2 - x^2 - y^2 - z^2) + 8yzx \\
S_4(\omega) = x^4 + y^4 + z^4 - 2t^2x^2 - 2t^2y^2 - 2t^2z^2 + t^4 + 8txyz - 2x^2y^2 - 2x^2z^2 - 2y^2z^2
\end{gathered}
\tag{2.6}
$$

Remark 2.1. This is necessary to note, that the statements of Corollaries 2.1−2.3 still hold when the basis $\{E, I, J, K\}$ is used, but only if representations of the automorphisms are appropriately interpreted.

2.3 'Compound' basis

The set $\{B, AB, D, CD\}$ is called a *compound basis* of the algebra $H(4) \cong R\dot{+}R\dot{+}R\dot{+}R$, if is a basis of this algebra which is compatible with its isomorphic representation in the form of a direct sum $H(2)\dot{+}H(2) \cong H(4)$ (a direct sum of two 2D algebras of 'double' numbers) and if multiplication rules for elements of the basis are given by the following Cayley table.

Table 2.2

	B	AB	D	CD
B	B	AB	0	0
AB	AB	B	0	0
D	0	0	D	CD
CD	0	0	CD	D

When the compound basis is used, the element $w \in H(2)\dot{+}H(2) \cong H(4)$ can be represented in the form $w = (t + xA)B + (y + zC)D$ and the element $I = B + D$ is the multiplicative identity of the algebra $H(2)\dot{+}H(2)$.

Suppose, that four automorphisms are selected,

$$
\begin{aligned}
\zeta_0 : \quad & w \to \zeta_0(w) = (t + xA)B + (y + zC)D, \\
\zeta_1 : \quad & w \to \zeta_1(w) = (y + zA)B + (t + xC)D, \\
\zeta_2 : \quad & w \to \zeta_2(w) = (y - zC)B + (t - xC)D, \\
\zeta_3 : \quad & w \to \zeta_3(w) = (t - xC)B + (y - zC)D,
\end{aligned}
\tag{2.7}
$$

then we obtain the defining polynomial $\Phi(\xi; w)$ of the form

$$
\Phi(\xi; w) = \xi^4 - \Sigma_1(w)I\xi^3 + \Sigma_2(w)I\xi^2 - \Sigma_3(w)I\xi^1 + \Sigma_4(w)I,
$$

where

$$\begin{gathered} \Sigma_1(w) = 2(t+y) \\ \Sigma_2(w) = 4ty + t^2 + y^2 - x^2 - z^2 = (t+2y)^2 - 3y^2 - x^2 - z^2, \\ \Sigma_3(w) = 2ty^2 + 2t^2y - 2tz^2 - 2x^2y, \\ \Sigma_4(w) = t^2\,y^2 - t^2z^2 - x^2\,y^2 + x^2z^2. \end{gathered} \tag{2.8}$$

The algebra $H(4)$, its different bases, and different representations of the defining polynomials and defining forms, associated with different bases are used in the examples and considered in details, as for this algebra there exists a variety of the isomorphic representations:

$$R\dot{+}R\dot{+}R\dot{+}R \cong H(2)\dot{+}H(2) \cong H(4)$$

and the automorphism group over R for this algebra consists of 24 elements and is isomorphic to S_4.

For the algebras $C\dot{+}C$ and in particular $H_2\dot{+}C$, the automorphism groups over R consist of less number of elements. For construction the defining polynomial with real coefficients this property results in more restrictive constraints on the choice of four automorphisms and bases.

3 Algebra $C\dot{+}C$

The set $\{E, I, J, K\}$ is called a basis consisting of *elliptic-hyperbolic units* (or, for short, an *EH-basis*) if is a basis of the algebra $C\dot{+}C$ and multiplication rules for elements of the basis are given by the following Cayley table.

Table 3.1

	E	I	J	K
E	E	I	J	K
I	I	$-E$	K	J
J	J	K	$-E$	I
K	K	J	I	E

The element $w \in C\dot{+}C$ may be represented in the form $w = tE + xI + yJ + zK$ $(t, x, y, z \in R)$, where E is the multiplicative identity (unit) of the algebra.

The set $\{B, AB, D, CD\}$ is called a *compound basis* in the algebra $C\dot{+}C$ if is a basis that is compatible with its isomorphic representation as a direct sum of two 2D algebras of complex numbers and multiplication rules for elements of the basis are given by the following Cayley table.

Table 3.2

	B	AB	D	CD
B	B	AB	0	0
AB	AB	$-B$	0	0
D	0	0	D	CD
CD	0	0	CD	$-D$

The element $w \in C\dot{+}C$ can be represented in the form $w = (t + xA)\, B + (y + zC)\, D$, the multiplicative identity of the algebra $C\dot{+}C$ is the element $I = B + D$.

In the case of the EH-basis, selecting the four automorphisms in the form

$$\begin{aligned} \mu_0 : &\quad w \mapsto \mu_0\,(w) = tE + xI + yJ + zK, \\ \mu_1 : &\quad w \mapsto \mu_1\,(w) = tE + xI - yJ - zK, \\ \mu_2 : &\quad w \mapsto \mu_2\,(w) = tE - xI + yJ - zK, \\ \mu_3 : &\quad w \mapsto \mu_3\,(w) = tE - xI - yJ + zK. \end{aligned} \tag{3.1}$$

and in the case of the compound basis in the form

$$\begin{aligned} \zeta_0 : &\quad w \to \zeta_0\,(w) = (t + xA)\, B + (y + zC)\, D, \\ \zeta_1 : &\quad w \to \zeta_1\,(w) = (y + zA)\, B + (t + xC)\, D, \\ \zeta_2 : &\quad w \to \zeta_2\,(w) = (y - zC)\, B + (t - xC)\, D, \\ \zeta_3 : &\quad w \to \zeta_3\,(w) = (t - xC)\, B + (y - zC)\, D, \end{aligned}$$

we obtain the following defining forms:

$$\begin{gathered} S_1\,(\omega) = 4t, \\ S_2\,(\omega) = 6t^2 + 2x^2 - 2y^2 + 2z^2, \\ S_3\,(\omega) = +4tx^2 - 4y^2t + 4z^2t + 4t^3 + 8yzx = 4t(t^2 + x^2 - y^2 + z^2) + 8yzx, \\ S_4\,(\omega) = x^4 + y^4 + z^4 + 2t^2x^2 - 2t^2y^2 + 2t^2z^2 + t^4 - 8txyz + 2x^2y^2 - 2x^2z^2 + 2y^2z^2 \end{gathered} \tag{3.2}$$

and

$$\begin{gathered} \Sigma_1\,(w) = 2\,(t + y)\,, \\ \Sigma_2\,(w) = 4ty + t^2 + y^2 + x^2 + z^2 = (t + 2y)^2 - 3y^2 + x^2 + z^2, \\ \Sigma_3\,(w) = 2ty^2 + 2t^2y + 2tz^2 + 2x^2y, \\ \Sigma_4\,(w) = t^2\, y^2 + t^2z^2 + x^2\, y^2 + x^2z^2, \end{gathered} \tag{3.3}$$

respectively. This is interesting to note, that the forms (3.3) may be obtained from the forms (2.8), and the forms (3.2) – from the forms (2.6) via the formal change of variable $x \mapsto ix,\ z \mapsto iz$.

4 Algebra $H_2\dot{+}C$

The set $\{B, AB, D, CD\}$ is called a *compound basis* in the algebra $H_2\dot{+}C$ if the is a basis that is compatible with isomorphic representation of the algebra as a direct sum of two 2D algebras (the algebra of 'double' numbers and the algebra of complex numbers) and multiplication rules for elements of the basis given by the following Cayley table.

Table 4.1.

	B	AB	D	CD
B	B	AB	0	0
AB	AB	B	0	0
D	0	0	D	CD
CD	0	0	CD	$-D$

Similar to the previous sections, the element $w \in H_2\dot{+}C$ is represented in the form $w = (t + xA)\, B + (y + zC)\, D$, the multiplicative identity of the algebra $H_2\dot{+}C$ is the element $I = B + D$. However, in contrast to the algebras $C\dot{+}C$ and $H\,(2)\,\dot{+}H\,(2) \cong H\,(4)$, the summands of the direct sum $H_2\dot{+}C$ do not have the explicit symmetry over the field R. Here we omit the detailed argumentation for the 'complex' change of variables, but one may easily notice, that in this case the characteristic forms can be obtained from the forms (2.8) via formal change of variables $x \mapsto ix$ and can be represented in the form:

$$\begin{gathered}
\Sigma_1\,(w) = 2\,(t + z)\,, \\
\Sigma_2\,(w) = 4ty + t^2 + y^2 + x^2 - z^2 = (t + 2y)^2 - 3y^2 - x^2 + z^2, \\
\Sigma_3\,(w) = 2ty^2 + 2t^2y + 2tz^2 - 2x^2y, \\
\Sigma_4\,(w) = t^2\,y^2 + t^2z^2 - x^2\,y^2 - x^2z^2 = (t^2\ - x^2)\,(\,y^2 + z^2)\,.
\end{gathered} \tag{4.1}$$

5 Some generalization for algebras of higher dimensionality

The approach to metric structures in finite-dimensional associative-commutative algebras, which is considered in this work, would not be general enough, if the author didn't try to apply it to algebras of dimensionality higher than 4.

Certainly, the Weierstrass theorem holds in the algebras of any number of dimensions, however the number of the non-isomorphic algebras grows with the increase of the dimensionality and this complicates detailed analysis of the all possible metric structures, generated by the coefficients of the defining polynomials for the elements of these algebras. Nevertheless, we will examine in detail the structure of the automorphism group generating defining polynomials for a particular class of algebras, obtained via recursive application of the dimensionality doubling algorithm proposed by Grassman and Clifford [9] – [11].

5.1 Grassman-Clifford dimensionality doubling algorithm

Examine the algebra A_1 over the field R. Let:

$$z_1 = z_0 + z_0'\varepsilon_1 \in A_1, \tag{5.1}$$

where

$$\varepsilon_1^2 = \beta_1, \quad z_0, z_0', \beta_1 \in R.$$

Grassman-Clifford dimensionality doubling algorithm consists in the successive 'execution' of the following steps:

Step 1. Suppose that multiplication of the elements $z_1 = a_0 + b_0\varepsilon_1$, $z_1' = a_0' + b_0'\varepsilon_1 \in A_1$, be given by :

$$z_1 z_1' = (a_0 a_0' + \beta_1 b_0 b_0') + (a_0' b_0 + a_0 b_0')\,\varepsilon_1. \tag{5.2}$$

Note, that depending on the value of the parameter β_1, from (5.2) we can obtain 2D algebras of complex, double or dual numbers.

Step 2. On the second step, consider the algebra A_2 consisting of the elements:

$$z = z_1 + z_1'\varepsilon_2 \in A_2,$$

where

$$\varepsilon_2^2 = \beta_2, \quad \beta_2 \in R,\ z_1, z_1' \in A_1,$$

with the multiplication rule for elements of the basis given by

$$\varepsilon_1\varepsilon_2 = \alpha_{12}\varepsilon_2\varepsilon_1, \quad \alpha_{12} \in R.$$

Interpreting the product $\varepsilon_1\varepsilon_2$ as a basis element, we obtain the algebra of higher dimensionality.

Step 3. Continuing inductively the above-stated algorithm, on the n-th step we obtain the algebra A_n which consists of elements of the form

$$z_n = z_{n-1} + z_{n-1}'\varepsilon_n \in A_n,$$

where z_{n-1} , z_{n-1}' are the elements of the algebra constructed at the $(n-1)$-th step, and ε_n is a new generating element.

This is evident, that the typical element z_n of the new algebra A_n is of the form

$$z_n = \sum_{\substack{(\alpha_1,\dots,\alpha_d)\\ \alpha_j \in \{0,1\}}} C_{\alpha_1\dots\alpha_d}\varepsilon_1^{\alpha_1}\cdot\ldots\cdot\varepsilon_d^{\alpha_d}, \tag{5.3}$$

where $C_{\alpha_1\dots\alpha_d} \in R$.

Using this process, we can obtain a wide range of algebras used in applications; in particular, in physics for construction of the mathematical models. Certain examples [9] of these algebras follow:

- Clifford algebra of the dimensionality 2^n with $\varepsilon_s^2 = \pm 1$, $\varepsilon_s \varepsilon_l = -\varepsilon_l \varepsilon_s$, $1 \leq s, l \leq n$;
- Grassman algebra of the dimensionality 2^n with $\varepsilon_s^2 = 0$, $\varepsilon_s \varepsilon_l = -\varepsilon_l \varepsilon_s$, $1 \leq s, l \leq n$;
- Pauli algebra $n = 3$ with $\beta_s = 1$, $\alpha_{ls} = -1$;
- Dirac algebra $n = 4$ with $\beta_1 = 1$, $\beta_2 = \beta_3 = \beta_4 = -1$, $\alpha_{ls} = -1$;
- Kalutza algebra $n = 4$ with $\beta_1 = \beta_2 = 1$, $\beta_3 = \beta_4 = -1$, $\alpha_{ls} = -1$.

5.2 Classification for algebras of dimensionality 2^d, constructed using Grassman-Clifford algorithm

The algebras which were under analysis in the previous sections ($R\dot{+}R\dot{+}R\dot{+}R \cong H_R(4)$ and $C\dot{+}C \cong H_C(2)$) can be obtained on the second step of the Grassman-Clifford dimensionality doubling algorithm. This is interesting to note, that the algebra $R\dot{+}R\dot{+}C \cong H_R(2)\dot{+}C$ cannot be obtained as a result of this algorithm. This appears, that the following fact is typical for any dimensionality 2^d of algebras, obtained using Grassman-Clifford dimensionality doubling algorithm: there exist only two 'Grassman-Clifford' R-algebras. However, according to the Weierstrass theorem there exist much more commutative-associative algebras of the mentioned dimensionality. Let us prove the classification theorem for 'Grassman-Clifford' R-algebras.

Denote by V a d-dimensional space over R with the basis ε_1, ε_2, $\ldots \varepsilon_d$.

Definition 5.1. The commutative-associative hypercomplex algebra B_d is called the 2^d-dimensional R-algebra if the set Λ given by

$$\Lambda = \Big\{ \prod_{i \in I} \varepsilon_i^{\alpha_i}, \quad \alpha_i \in \{0, 1\} \ ; I = \{1, \ldots, d\} \Big\}, \tag{5.4}$$

where $\varepsilon_i^0 = 1$, $\varepsilon_i^1 = \varepsilon_i$, from the basis of Λ this algebra and multiplication rules for elements of are induced by the following relations on the basis elements of the space V:

$$\varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i, \quad \varepsilon_i^2 = \beta_i, \ i, j \in I. \tag{5.5}$$

The algebras B_d may be obtained as a result of the Grassman-Clifford dimensionality doubling algorithm. In fact, assigning for every binary vector of indexes $(\alpha_1, \ldots, \alpha_d)$ the corresponding integers

$$t = \alpha_1 + \alpha_2 2 + \ldots + \alpha_d 2^{d-1}, \quad \alpha_j \in \{0, 1\}, \tag{5.6}$$

where $t \in T = \big\{0,\ 1, \ldots, 2^d\ -\ 1\big\}$, the elements of the set Λ can be numerated:

$$E_t = \varepsilon_1^{\alpha_1} \cdot \ldots \cdot \varepsilon_d^{\alpha_d}. \tag{5.7}$$

Then an arbitrary element $g \in B_d$ can be represented in the form

$$g = \xi_0 E_0 + \ldots + \xi_{2^d - 1} E_{2^d - 1} = \sum_{t \in T} \xi_t E_t. \tag{5.8}$$

Addition in the algebra B_d can be performed on the element components. Let the elements are given by

$$g = \sum_{t \in T} \xi_t E_t, \quad h = \sum_{t \in T} \eta_t E_t, \quad g, h \in B_d, \tag{5.9}$$

then

$$(g + h) = \sum_{t \in T} (\xi_t + \eta_t) E_t. \tag{5.10}$$

Multiplication in the algebra B_d, represented in the form (5.8), is defined by the rules (5.5) for multiplication of the basis elements of the space$V$, as the elements of the algebra.

The following lemma establishes the relation between enumeration (5.6) of the multiplicands and numeration (5.4) of the products.

Lemma 5.1. Denote by $\oplus$ – the component-wise addition modulo 2:

$$\oplus : T \times T \to T, \quad t \oplus \tau = \sum_{i \in I} \left(\left(\alpha_i + \alpha_i^{'} \right) mod\, 2 \right) 2^{i-1}, \qquad \text{where} \tag{5.11}$$

$$t = (\alpha_1, \dots, \alpha_d), \quad \tau = (\alpha_1', \dots, \alpha_d'); \quad t, \tau \in T; \quad \alpha_i, \alpha_i' = 0, 1; \quad i \in I. \tag{5.12}$$

Further, let the function $h_i : T \times T \to \{0, 1\}$ be given by the equality

$$h_i (t, \tau) = \alpha_i \alpha_i', \quad i \in I, \tag{5.13}$$

and the function $\Psi : T \times T \to \{-1, 1\}$ be given by the equality

$$\Psi (t, \tau) = \prod_{i \in I} \beta_i^{h_i(t,\tau)}, \quad \beta_i = \{-1, 1\}. \tag{5.14}$$

Then multiplication rules for the elements of the basis Λ can be written in the form:

$$E_t E_\tau = \Psi (t, \tau) E_{t \oplus \tau}, \quad \forall\, t, \tau \in T. \tag{5.15}$$

Proof. By (5.7), let

$$E_t = \prod_{i \in I} \varepsilon_i^{\alpha_i}, \quad E_\tau = \prod_{i \in I} \varepsilon_i^{\alpha_i'},$$

then

$$E_t \cdot E_\tau = \prod_{i \in I} \varepsilon_i^{\alpha_i + \alpha_i' \pm 2 h_i \left(\alpha_i, \alpha_i' \right)}.$$

Taking into consideration, that $\alpha_i + a_i' - 2 h_i (\alpha_i, \alpha')_i \equiv \alpha_i + a_i'$ $(mod\, 2)$, from the previous relation we obtain

$$\prod_{i \in I} \varepsilon_i^{2 \cdot h_i \left(\alpha_i, \alpha_i' \right)} E_{t \oplus \tau} = \Psi (t, \tau) E_{t \oplus \tau}. \qquad \square$$

By Definition 5.1, this does not follow immediately that the algebra B_d with the basis Λ and multiplication given by (5.5) is unique. Classification of the algebras, introduced by this definition is provided in a form of the Theorem 5.1.

The following supplementary lemma is needed for the sequel:

Lemma 5.2. If for a certain index $l \in I$ in (5.5) the relation $\beta_l = -1$ holds, then

$$\sum_{t\in T} E_t^2 = 0. \tag{5.16}$$

Proof. Let the sum in the left side of the equality (5.16) be split into 2 sums in a way such that the first sum will consists of the basis elements containing ε_l^2, and the second sum – of the basis elements not containing ε_l^2. Then we obtain:

$$\sum_{t\in T} E_t^2 = \sum_{\substack{t\in T\\ h_l(t,t)\neq 0}} E_t^2 + \sum_{\substack{t\in T\\ h_l(t,t)=0}} E_t^2.$$

This is easy to see, that number of items in both the sums is equal:

$$\sum_{\substack{t\in T\\ h_l(t,t)\neq 0}} 1 = \sum_{\substack{t\in T\\ h_l(t,t)=0}} 1 = 2^{d-1}$$

As $\beta_l = -1$, from the last relation we get

$$(1+\beta_l) \sum_{\substack{t\in T\\ h_l(t,t)\neq 0}} E_t^2 = 0. \qquad \square$$

Corollary 5.1. If there exists $t \in T$ such that $E_t^2 = -1$, then among basis elements of the algebra B_d there exist 2^{d-1} elements with the second power equal to (-1). This is quite clear that if $\beta_i = 1$, for every $i \in I$, then for all the second powers of the elements of the following relation holds:

$$E_t^2 = +1$$

For consistency of notation, in this section the algebra B_d, with $\beta_i = 1$ for all $i \in I$, will be denoted by B_d^+. One may easily get that

$$B_d^+ \cong \underbrace{R\dot{+}R\dot{+}\ldots\dot{+}R}_{2^d} \cong H_R\left(2^d\right)$$

The principle result of this section is the following theorem, stating that the structure of the commutative-associative algebra with the basis (5.4) (i.e. an algebra, constructed as a result of Grassman-Clifford algorithm) depends on the fact if there exists at least one basis element $\varepsilon_j \in V$ (an element 'generating' the vector space V) with the square equal to (-1).

Theorem 5.1. For every $d \geq 1$ there exist only two non-isomorphic 2^d-dimensional algebras with operations, given by (5.10), (5.15), namely:

$$B_d^+ \cong \underbrace{R\dot{+}R\dot{+}\ldots\dot{+}R}_{2^d} \cong H_R\left(2^d\right) \cong \underbrace{H_R\left(2\right)\dot{+}H_R\left(2\right)\dot{+}\ldots\dot{+}H_R\left(2\right)}_{2^{d-1}}, \tag{5.17}$$

$$B_d^- \cong \underbrace{C\dot{+}C\dot{+}\ldots\dot{+}C}_{2^{d-1}} \cong H_C\left(2^{d-1}\right). \tag{5.18}$$

Proof. By Corollary 5.1, if $\beta_i = 1$ for all the indexes $i \in I$, then the corresponding algebra is

$$B_d^+ \cong \underbrace{R\dot{+}R\dot{+}\ldots\dot{+}R}_{2^d} \cong H_R\left(2^d\right).$$

Thus, suppose that there exists at least one element ε_l of the space V, for which the following relation holds: $\varepsilon_l^2 = -1$, $l \in I$. Without loss of generality, we assume that $l = 1$ (in the basis Λ to this element the element E_1 corresponds). Take E_t such that $E_t^2 = 1$. Note, that this is possible, i.e. such an element exists, as there either exists $\beta_k = 1$ or exists the linear combination $\varepsilon_1\varepsilon_k = E_{1\oplus k}$, where $\beta_k = -1$, $k \neq 1$. Then, every element $h \in B_d^-$ can be represented in the form

$$h = \sum_{i\in T}\eta_i E_i = \sum_{\substack{i\in T\\ h_k(i,i)=0}}\eta_i E_i + \sum_{\substack{i\in T\\ h_i(i,i)=1}}\eta_i E_i.$$

From Lemma 5.1 it follows that

$$E_i = \frac{1}{\Psi\left(t,t\right)}E_iE_tE_t$$

or, in the case under consideration, $E_i = E_iE_tE_t$. From the last relation, one may easily get the following

$$h = \sum_{\substack{i\in T\\ h_k(i,i)=0}}\eta_i E_i + \left(\sum_{\substack{i\in T\\ h_k(i,i)=1}}\eta_i E_iE_t\right)E_t.$$

Note, that all the possible products E_iE_t do not contain the generating element ε_k, therefore $h = a + bE_t$, where a, $b \in B_{d-1}$.

This can be verified directly, that the map $\Theta : B_d \to B_{d-1}\dot{+}B_{d-1}$, given by $h = \alpha + \gamma E_t \mapsto (\alpha + \gamma, \alpha + \Psi\left(k,k\right)\gamma) \in B_{d-1}\dot{+}B_{d-1}$, is an isomorphism. Applying successively Θ to B_{d-1}^-, by induction, we obtain

$$B_d \cong \Big(\underbrace{R\dot{+}R\dot{+}\ldots\dot{+}R}_{2^{(d-1)}}\Big) + \varepsilon_1 \cdot \Big(\underbrace{R\dot{+}R\dot{+}\ldots\dot{+}R}_{2^{(d-1)}}\Big). \tag{5.19}$$

Since $\varepsilon_1^2 = -1$, then

$$B_d \cong \underbrace{C\dot{+}C\dot{+}\ldots\dot{+}C}_{2^{d-1}} = B_d^-.$$

From the last relation it follows that any algebra for which there exists an element of the basis of the vector space V, with the second power equal to -1 is isomorphic to the algebra B_d^-. This completes the proof. □

Taking into consideration the classification theorem proven above, in the sequel by B_d^- we denote the algebra with the following multiplication rules for the basis elements of the space V:

$$\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i, \quad \varepsilon_i^2 = -1, \quad i \in I, \tag{5.20}$$

and by B_d^+ the algebra with the multiplication rule of the basis elements of the vector space V:

$$\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i, \quad \varepsilon_i^2 = +1, \quad i \in I, \tag{5.21}$$

This can be easily verified, that for an arbitrary index l, $l \in T$, the following relations hold

$$0 = 0 \oplus 0 = l \oplus l, \quad l = l \oplus 0 = 0 \oplus l. \tag{5.22}$$

For the function Ψ and for an arbitrary $l \in T$ the following relations hold

$$\Psi\left(0,0\right) = \Psi\left(0,l\right) = \Psi\left(l,0\right) = \tfrac{1}{\beta_l}\Psi\left(l,l\right) \tag{5.23}$$

and, by (5.20), the following equalities hold

$$\Psi\left(0,0\oplus 0\right) = \tfrac{1}{\beta_l}\Psi\left(l,l\oplus 0\right) = \Psi\left(0,l\oplus 0\right) = \Psi\left(l,0\oplus 0\right). \tag{5.24}$$

5.3 Automorphism in algebras of 2^d dimensions, obtained applying Grassman-Clifford algorithm

The following theorem generalizes the statement, related to the automorphic nature of the maps (2.5) and (3.1) for the case of the arbitrary dimensionality 2^d.

Theorem 5.2. Let $\psi:\ T \times T \to \{-1,1\}$ be given by

$$\psi\left(j,t\right) = \prod_{i\in I}\left(-1\right)^{h_i(j,t)}. \tag{5.25}$$

Then the set of 2^d maps $\sigma_j:\ B_d \to B_d$ such that

$$\sigma_j\left(\chi\right) = \sum_{i\in T} c_i\psi\left(j,i\right)E_i, \tag{5.26}$$

where $\chi = (c_0,\ldots,c_{2^d-1}) \in B_d$, $c_i \in R$, $j \in T$, is the set of automorphisms for the algebra B_d, regardless of what algebra (algebra B_d^+ or algebra B_d^-) is under consideration.

Proof. Let us prove that the maps from the set (5.25) are bijective and preserve the multiplication and addition. In fact, among elements of the basis Λ there are no zero-divisors and maps σ_j are linear. Thus, maps σ_j are bijections. Now, we show that maps σ_j preserve the addition and multiplication. Let the elements $g, h \in B_d$ be represented in the from (5.8). Then the following relations hold

$$\sigma_j\left(g+h\right)=\sigma_j\left(g\right)+\sigma_j\left(h\right), \tag{5.27}$$

$$\sigma_j\left(gh\right)=\sigma_j\left(g\right)\cdot\sigma_j\left(h\right). \tag{5.28}$$

In fact, (5.27) is yielded by the following relations

$$\sigma_j\left(g+h\right)=\sum_{i\in T}\left(\xi_i+\eta_i\right)\psi\left(j,i\right)E_i=\sum_{i\in T}\xi_i\psi\left(j,i\right)E_i+\sum_{i\in T}\eta_i\psi\left(j,i\right)E_i=\sigma_j\left(g\right)+\sigma_j\left(h\right).$$

Now, we will prove (5.28). This is evident that the following equality holds

$$\begin{aligned}\sigma_j\left(gh\right)&=\sum_{i\in T}\sum_{t\in T}\Psi\left(t,t\oplus i\right)\xi_{i\oplus t}\eta_t\psi\left(j,i\right)E_i=\\&=\sum_{t\in T}\eta_t\sum_{i\in T}\Psi\left(t,t\oplus i\right)\xi_{i\oplus t}\psi\left(j,i\right)E_i\,.\end{aligned} \tag{5.29}$$

Next, combining (5.15), (5.22) and $i=t\oplus i\oplus t$, the relation (5.29) can be converted to the form:

$$\sigma_j\left(gh\right)=\sum_{t\in T}\eta_t\psi\left(j,t\right)\Psi\left(t,t\right)E_t\sum_{i\in T}\xi_{i\oplus t}\psi\left(j,i\oplus t\right)\Psi\left(t,t\oplus i\right)E_{i\oplus t}. \tag{5.30}$$

Since the index i runs through all the values from the set T, then the index $i\oplus t$ will also run through all the elements from the set T. Thus, taking in (5.29) $\tau=t\oplus i$, we get:

$$\sigma_j\left(gh\right)=\sum_{t\in T}\eta_t\psi\left(j,t\right)E_t\sum_{\tau\in T}\xi_\tau\psi\left(j,\tau\right)E_\tau=\sigma_j\left(g\right)\cdot\sigma_j\left(h\right).$$

Provided that there exist the maps $\sigma_j\left(\chi\right)=\sigma_p\left(\chi\right)$, where $j\neq p$, we obtain the chain of equalities:

$$\sigma_j\left(\chi\right)=\sigma_p\left(\chi\right),$$

$$\prod_{i\in T}\left(-1\right)^{h_i(j,l)}=\prod_{i\in T}\left(-1\right)^{h_i(p,l)},\quad l=\overline{0,\ 2^d-1},$$

$$\sum_{i\in T}h_i\left(j,l\right)=\sum_{i\in T}h_i\left(p,l\right),\quad l=\overline{0,\ 2^d-1},$$

chain yields that $j=p$. The last relation is a controversy.

Thus, if $j\neq p$, then $\sigma_j\left(\chi\right)\neq\sigma_p\left(\chi\right)$. Therefore, all the automorphisms σ_j, numerated with the elements of the set T with $card\,T=2^d$, are different. This completes the proof. $\square$

Consider the set of 2^d maps $\sigma_j : B_d \to B_d$, defined in the Theorem 5.2. The following statement provides the condition which being fulfilled guarantees that the coefficients of the defining polynomial (for the element of algebra B_d) are real if the automorphisms considered in Theorem 5.2 are used for construction of this polynomial. The bulky, but transparent proof by induction of the following theorem is omitted.

Theorem 5.3. Let for the element $w \in B_d$ the polynomial $\Phi(\xi, w)$ be given by

$$\Phi(\xi, w) = \prod_{\sigma_j} (\xi - \sigma_j(w)) = \sum_{k=0}^{2^d} \xi^k (-1)^{2^d - k} S_{2^d-k}(w).$$

Then the following statements are true:

a) the element w and its automorphic images $\sigma_j(w)$ are the roots of the polynomial $\Phi(\xi, w)$;

b) the values of the function $S_{2^d-k}(w)$ are real numbers;

c) the functions $S_{2^d-k}(w)$ of the real components $(c_0, c_1, ..., c_{2^d-1})$ of the element w are homogeneous functions of degree $\left(2^d - k\right)$. □

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References

[1] E. Bayro-Corrochano, G. Sobczyk (Eds), *Advances in Geometric Algebra with Applications in Science and Engineering.* Birkhauser, Boston, 2001.

[2] G. Sommer (Ed.), *Geometric Computing with Clifford Algebras.* Springer Verlag, 2000.

[3] *General algebra* (Ed. L. A. Skornyakov). Nauka, M., 1990 (in Russian).

[4] E. Artin, *Geometric Algebra*, Interscience Publ., 1957.

[5] A. N. Lasenby, C. J. L. Doran and S. F. Gull (1996), *Lectures in Geometric Algebra,* In: W. E. Baylis, Ed., *Clifford (Geometric) Algebras with Applications to Physics, Mathematics and Engineering*, Birkhauser, Boston, 1996.

[6] M. Hall, *The Theory of Groups,* MacMillan, 1959.

[7] D. G. Pavlov, *The generalisation of scalar product axioms, Hypercomplex number in geometry and physics*, 1, 2004, 5–19.

[8] R. B. J. T. Allenby. *Rings, Fields and Groups: An Introduction to Abstract Algebra*, 2nd ed., 1991.

[9] F. A. Berezin *Introduction to algebra and analysis with anticommutative variables.* MSU Publ., M., 1983 (in Russian).

[10] Burlakov M. P. *Clifford structures on manifolds.* In: *Summa of science and technology. Modern mathematics an its applications.* VINITI, M., 1995. T. 30: Geometry 3, 205–257 (in Russian).

[11] I. L. Kantor, A. S. Solodovnikov, *Hypercomplex Numbers.* Springer, Berlin, 1989.

Generalized n-ary Composition Laws in the Algebra H_4 and their Relation to Associated Metric Forms

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The paper deals with multi-linearization of norms in the H_4 algebra. New binary, ternary and quaternary operations are introduced in the H_4 algebra with the isotropic basis ('Zassenhaus multiplication'). It is shown that quadratic Minkowski norm of the algebra element, Berwald-Moor norm, associated with the form of degree 4 as well as the norm, introduced in the earlier author's paper, associated with the cubic form, are equal to the values of the newly defined binary, quaternary and ternary operations (respectively), if all the operands (Zassenhaus factors) are equal.

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1 Introduction

1.1 Classical compositional algebras and algebras H_n

In physics, mechanics, computer science, other applications, *composition algebras* (unital algebras without zero divisors) are the most widely spread. In the vector spaces over these algebras, non-degenerate *quadratic* forms $N(x)$ *(norms)* satisfying the condition $N(xy) = N(x)N(y)$ are defined. Classification of these algebras and the recursive procedure for their construction over various fields is closely related to the fact that in structures, obtained at every step of the recursion, there exists an (anti) automorphism $x \mapsto \bar{x}$ of order 2. This anti(automoprhism), which can be recursively continued on the next step of the construction procedure, induces the above-mentioned forms $N(x)$. However, to construct algebras different from R and C but with an analogue of the real or complex norm available, one has to pay a certain fee: constructed algebras will be non-commutative and/or non-associative. Moreover, the Cayley-Dickson recursive process for construction of composition algebras already at the third step results in a non-associative structure and cannot be continued ([1]–[3]). In addition to composition algebras, using the Cayley-Dickson process this is possible to obtain a range of algebras with zero-divisors (e.g., an algebra of dual numbers, isomorphic to $R \oplus R$; an algebra of $(2{\times}2)$-matrices $M_2(R)$); over vector space of these algebras there is also possible to define a quadratic multiplicative form $N(x)$, but this form will be degenerate [1].

Non-commutative 4D quaternion algebra is usefully employed in solving various problems of mechanics, in machine vision, in physics. To explain this, there are at least two reasons: (1) in this algebra a norm is defined and, for example, (2) using this algebra the 3D orthogonal transforms can be 'elegantly' written not in terms of the 'external' matrix language, but in terms of 'internal' operations of the quaternion algebra, i.e. in the 'coordinate-free' form.

Further, consider, for example, an expansion of the element X from the $4D$ algebra, isomorphic to the algebra of (2×2)-matrices $M_2(R)$, in the 'Clifford basis' with the rules for multiplication of the basis elements given by the following relations.

$$e_0^2 = e_0, \quad e_1^2 = e_0, \quad e_2^2 = e_0, \quad e_3^2 = -e_0;$$
$$e_1e_2 = -e_2e_1, \quad e_1e_3 = -e_3e_1, \quad e_2e_3 = -e_3e_2; \quad e_1e_2 = e_3,$$

Let in this algebra an involutive map (a so called *symplectic involution*) be defined by

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \bar{X} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and, further, the 'natural' embedding $R \to M_2(R)$ with $y \mapsto ye_0,\ y \in R$ be defined.

Then the defining (characteristic) quadratic equation for the element X of this algebra with the coefficients being represented in a coordinate-free form, may be written in terms of the norm $N(X) = X \cdot \bar{X} = \det X$ and the trace $Tr\,(X) = X + \bar{X} = a + d$:

$$X^2 - Tr\,(X)\, e_0 X + N(X) e_0 = 0. \tag{1.1}$$

The relation (1.1) may be considered as a special (two-dimensional) case of the Cayley-Hamilton theorem. From this it follows that, after appropriate interpretation, all the results of 'linear' *geometry in the 2D plain* may be derived from the *algebraic properties of the 4D algebra* $M_2(R)$.

In contrast to the composition algebras, arbitrary associative-commutative finitedimensional algebras are not quadratic algebras over the field R. Thus, characteristic equations for elements from these algebras may be associated with the automorphisms of higher orders, and this suggests that this is viable to investigate properties of geometric interpretations of these algebras in terms of the 'symmetry' groups of the order greater than two.

To understand the role of the higher order automorphisms in construction of the geometric-physical models of the space-time, from the author's point of view, this is necessary to start with determination of invariant characteristic of the defining equations for the elements of the associative-commutative algebras; i.e. to find the counterparts of the forms $N(X) = X \cdot \bar{X} = \det X$ and $Tr\,(X) = X + \bar{X} = a + d$ in the relation (1.1). This invariant characteristic were considered in paper by the author [4], where (in particular) the following result was obtained (Theorem 2.1).

Consider the algebra $R \oplus R \oplus R \oplus R \cong H_4$ with the basis $\{E_1, E_2, E_3, E_4\}$ and the multiplication rule for the basis elements given by the Cayley table 1.1. (the basis with these multiplication rules will be called *isotropic*).

Table 1.1.

$\times$	E_1	E_2	E_3	E_4
E_1	E_4	0	0	0
E_2	0	E_2	0	0
E_3	0	0	E_3	0
E_4	0	0	0	E_4

The multiplicative identity element (algebra unit) in this basis is the element $I = E_1 + E_2 + E_3 + E_4$, and the field R is canonically embedded into the algebra $R \oplus R \oplus R \oplus R$ via the map:

$$R \to R \oplus R \oplus R \oplus R \cong H_4, \quad x \mapsto xI, \quad x \in R.$$

Theorem. The algebra $R \oplus R \oplus R \oplus R$ is the algebra of the degree four over R, i.e. any element $w \in R \oplus R \oplus R \oplus R$ satisfies an algebraic equation of degree not greater than 4 with real coefficients.

In fact, let $w = aE_1 + bE_2 + cE_3 + dE_4 \leftrightarrow (a, b, c, d)$. Consider four mappings of the algebra $R \oplus R \oplus R \oplus R$ into itself (quite obviously these maps are automorphisms):

$$\begin{aligned}
\tau_0 : \quad & w = aE_1 + bE_2 + cE_3 + dE_4 \ \mapsto \ aE_1 + bE_2 + cE_3 + dE_4, \\
\tau_1 : \quad & w = aE_1 + bE_2 + cE_3 + dE_4 \ \mapsto \ bE_1 + cE_2 + dE_3 + aE_4, \\
\tau_2 : \quad & w = aE_1 + bE_2 + cE_3 + dE_4 \ \mapsto \ cE_1 + dE_2 + aE_3 + bE_4, \\
\tau_3 : \quad & w = aE_1 + bE_2 + cE_3 + dE_4 \ \mapsto \ dE_1 + aE_2 + bE_3 + cE_4.
\end{aligned} \tag{1.2}$$

Clearly, the mappings (1.2) perform cyclic permutation of the components (a, b, c, d) of the expansion of the algebra element w in the isotropic basis. One may easily note, that the element w is a root of the polynomial

$$\Phi(\xi; w) = (\xi - \tau_0(w))(\xi - \tau_1(w))(\xi - \tau_2(w))(\xi - \tau_3(w)), \tag{1.3}$$

and the coefficients of the polynomial $\Phi(\xi; w)$ are real. In fact, via direct calculations we obtain:

$$\Phi(\xi; w) = \xi^4 - s_1(w) I\xi^3 + s_2(w) I\xi^2 - s_3(w) I\xi^1 + s_4(w) I, \tag{1.4}$$

where the real coefficients $s_\nu(w)$ are homogeneous symmetric forms of the components (a, b, c, d) of the element w from the algebra:

$$\begin{gathered}
s_1(w) = s_{14}(w) = a + b + c + d, \\
s_2(w) = s_{24}(w) = ab + ac + ad + bc + bd + cd, \\
s_3(w) = s_{34}(w) = bcd + acd + abd + abc, \\
s_4(w) = s_{44}(w) = abcd.
\end{gathered} \tag{1.5}$$

The polynomial $\Phi(\xi; w)$ of the minimal degree with the real coefficients and with the coefficients at the highest power of ξ equal to 1, such that $\Phi(\xi; w)|_{\xi=w} = 0$, will be called a *defining polynomial of the element* w, and its coefficients will be called *defining forms*.

Remark 1.1. The forms (1.5) are invariant with respect to any permutation $\sigma \in S_4$ of four components (a, b, c, d) of the algebra element w. Thus, as $\sigma \in S_4$ is an automorphism of the algebra $R\oplus R\oplus R\oplus R$ over the field R, then the polynomial $\Phi(\xi; w)$ of degree *four*, in addition to the root w will have at least 23 more roots. These roots are of the form $\sigma(w)$, $\sigma \in S_4$, i.e. all the automorphic images of the root w with respect to the automorphism group $\sigma \in S_4$ will be also the roots of $\Phi(\xi; w)$.

Remark 1.2. Consider the basis $\{E, I, J, K\}$ of the algebra $H_4 \cong R\oplus R\oplus R\oplus R$ with the multiplication rule for the basis elements given by the following Cayley table:

Table 1.2.

$\times$	E	I	J	K
E	E	I	J	K
I	I	E	K	J
J	J	K	E	I
K	K	J	I	E

In this basis, an element $\omega \in H_4$ may be represented in the form $\omega = tE + xI + yJ + zK$ $(t, x, y, z \in R)$, E is a multiplicative identity element (unit) of the algebra. Transformation between the isotropic basis $\{E_1, E_2, E_3, E_4\}$ and the basis $\{E, I, J, K\}$ is a linear transformation with the orthogonal Hadamard matrix

$$had_4 = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Obviously, for every automorphism associated with a certain element of the permutation group S_4, i.e. permuting the components of the element $\omega \in H_4$ in the basis $\{E_1, E_2, E_3, E_4\}$, there exist a linear transform of the components t, x, y, z of the element $\omega = tE + xI + yJ + zK$, which also implements a certain automorphism of the algebra H_4. In particular, these automorphisms may be:

$$\begin{aligned} \mu_0 : \quad & \omega \mapsto \mu_0(\omega) = tE + xI + yJ + zK, \\ \mu_1 : \quad & \omega \mapsto \mu_1(\omega) = tE + xI - yJ - zK, \\ \mu_2 : \quad & \omega \mapsto \mu_2(\omega) = tE - xI + yJ - zK, \\ \mu_3 : \quad & \omega \mapsto \mu_3(\omega) = tE - xI - yJ + zK. \end{aligned} \tag{1.6}$$

If the expansion of the element $\omega \in H_4$ in the basis $\{E_1, E_2, E_3, E_4\}$ is considered, this quadruple of transforms correspond to action on the components of the element expansion of a certain element of the subgroup (of order 4, isomorphic to the direct product $C_2 \times C_2$ of two cyclic groups of order 2) of the group S_4.

For the defining polynomial (in this particular case), via direct computations, we obtain:

$$\Phi(\xi;\omega) = (\xi - \mu_0(\omega))(\xi - \mu_1(\omega))(\xi - \mu_2(\omega))(\xi - \mu_3(\omega)) =$$
$$= \left(\xi^4 - S_1(\omega)\xi^3 + S_2(\omega)\xi^2 - S_3(\omega)\xi^1 + S_4(\omega)\right)E, \qquad \text{where}$$
$$S_{14}(\omega) = 4t,$$
$$S_{24}(\omega) = 6t^2 - 2x^2 - 2y^2 - 2z^2,$$
$$S_{34}(\omega) = -4tx^2 - 4y^2t - 4z^2t + 4t^3 + 8yzx = 4t(t^2 - x^2 - y^2 - z^2) + 8yzx,$$
$$S_{44}(\omega) = x^4 + y^4 + z^4 - 2t^2x^2 - 2t^2y^2 - 2t^2z^2 + t^4 + 8txyz - 2x^2y^2 - 2x^2z^2 - 2y^2z^2, \tag{1.7}$$

and $S_{24}(\omega)$ up to the norming factors is the same as the (pseudo)metric Minkowski form, and $S_{44}(\omega)$ has a 'standard Berwald-Moor' form.

1.2 Main ideas and definitions

This may seem paradoxical, but the idea to use additional (or redefined) operations (associated with the automorphisms of order higher than 2) in certain finite-dimensional algebra so to employ the power 'algebraic methods' for solving geometric problems has a long history. For example, there was proposed an exotic (and not very well-known) algebraic structure – the 'Zassenhaus finite quasifield', [5] (also [6], Chapter 20). Namely in the field F_q, $q = p^m$, (where p is prime) an operation is defined $x*y$ $(x, y \in F_q)$, which can be expressed in terms of multiplication in the field F_q in the form $x * y = y \cdot \eta(x)$, where η is a Frobenius automorphism of a specific form. This operation is non-commutative and non-associative (by Wedderburn's theorem, [6], [7]). Naturally, the fact that the fields F_q are finite (and, consequently, the Zassenhaus quasi-fields are also finite) limits the range of problems which may be solved applying this technique to configuration problems of the finite geometry [6].

The idea to consider 'multilinear' (in contrast to classical 'bilinear') scalar products in associative-commutative algebras in order to create adequate geometric-physical models was most likely first introduced by D. G. Pavlov [8].

For example, consider the algebra $R \oplus R$ with the basis $\{E_1, E_2\}$. Let multiplication rules for basis elements be given by the Cayley table 1.3 and the multiplicative identity element be $I = E_1 + E_2 = (1, 1)$.

Table 1.3.

$\times$	E_1	E_2
E_1	E_4	0
E_2	0	E_2

A typical element $x = x_1E_1 + x_2E_2$ will be denoted (for brevity) by $x = (x_1, x_2)$.

Example 1.1. Consider two permutations

$$\sigma^1 \leftrightarrow \sigma_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \sigma^2 \leftrightarrow \sigma_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}. \tag{1.8}$$

Let define action of operators $\sigma^1,\ \sigma^2$ (associated with the permutations $\sigma_1,\ \sigma_2$) on the elements $x = (x_1, x_2)$ of the algebra $R \oplus R$ by:

$$\sigma^1 (x) = (x_1, x_2), \quad \sigma^2 (x) = (x_2, x_1),$$

That is the operators $\sigma^1,\ \sigma^2$ permute the components of the element $x = (x_1, x_2)$ in accordance with the lower rows of the permutations $\sigma_1,\ \sigma_2$.

Let introduce in $R \oplus R$ a new binary operation $[x,\ y]$ ('Zassenhaus multiplication'):

$$[x,\ y] = \sigma^1 (x) \bullet\ \sigma^2 (y),$$

where the symbol $(\bullet)$ denotes conventional 'component-wise' multiplication of elements in $R \oplus R$.

The following relations may be verified via direct calculations.

$$[x,\ y] = (x_1, x_2) \bullet (y_2, y_1) = (x_1y_2, x_2y_1) \doteq (\xi_2, \xi_1), \tag{1.9}$$

$$[x,\ x] = (x_1, x_2) \bullet (x_2, x_1) = (x_1x_2, x_2x_1) = x_1x_2I \doteq N(x)I. \tag{1.10}$$

Note, that the function $N(x)$ given by (1.10) *coincides with the conventional norm of the element from the algebra $R \oplus R$, represented in terms of the isotropic coordinates (components).*

Further, the following relations hold:

$$\begin{gathered} N([x, y]) = [[x, y], [x, y]] = \sigma^1 ([x, y]) \bullet\ \sigma^2 ([x, y]) = \\ = (x_1y_2, x_2y_1) \bullet (x_2y_1, x_1y_2) = (x_1y_2x_2y_1, x_1y_2x_2y_1) = \\ = (x_1y_2, x_1y_2) \bullet (x_2y_1, x_2y_1) = (x_1y_2)\, I \bullet (x_2y_1)\, I = \xi_1\xi_2 I = \\ = (x_1x_2)\, I \bullet (y_1y_2)\, I = N(x)N(y). \end{gathered} \tag{1.11}$$

Example 1.2. Let consider one more illustration of the 'Zassenhaus multiplication'. Consider the set of $2D$ indexed $(n \times n)$ arrays with the component-wise multiplication. Let define a new 'multiplication' operation $*$, different from the 'conventional' matrix multiplication. Let $A = \{a_{ij}\}$, $B = \{b_{ij}\}$, $D = \{d_{ij}\} = A * B$. Then, by definition:

$$d_{ij} = \sum_{k=1}^{n} a_{ik}b_{jk},$$

or, informally, ¡¡(i, j)-th element of '$*$-product' is equal to the scalar product of the i-th row by the j-th column of the multiplicands¿¿. Transformation of the arrays

$\tau : A \to \tau(A)$ given by τ: $a_{ij} \mapsto a_{ji}$, represents a 'conventional' matrix transpose, but this transformation is *an automorphism of the second order* with respect to the operation $*$: $\tau(A * B) = \tau(A) * \tau(B)$, although with respect to the conventional matrix multiplication rules, taking the transpose represents an anti-automorphism: $(AB)^t = B^t A^t$. In terms of the operation $*$ conventional matrix multiplication is simply 'Zassenhaus multiplication' with respect to the operation $*$: $AB = A * \tau(B)$. If we consider, for example, $3D$ arrays $A = \{a_{ijk}\}$, $B = \{b_{ijk}\}$, $C = \{c_{ijk}\}$ and the automorphism of the order three $\tau : a_{ijk} \mapsto a_{jki}$, similarly we can obtain 'Zassenhaus multiplication' of 3D arrays $X = [A, B, C]$, where

$$x_{pqr} = \sum_{i,j=1}^{n} a_{pij} b_{iqj} c_{ijr}$$

and so on.

Thus, from the relation (1.11) it follows that:

- the equality $N([x, y]) = N(x) N(y)$ holds, i.e. the norm $N(x)$ with respect to the newly defined operation $[x, y]$ is 'multiplicative', and this norm is equal to the conventional norm of the element from the algebra $R \oplus R$, represented in terms of isotropic coordinates (components);
- the following equality holds

$$\sigma^1([x, y]) \bullet \sigma^2([x, y]) = \xi_1 \xi_2 I, \tag{1.12}$$

  where (ξ_1, ξ_2) are components of the 'Zassenhaus product' $[x, y]$, and the ordered pair of the upper indices of the operators σ^1, σ^2 coincides with the ordered pair of the lower indexes of the components (ξ_1, ξ_2).

The latter idea justifies generalization of composition laws for algebras with n-ary operations, and the former one (multiplicative property of the norm) motivates selection of this particular principle of generalization.

Let introduce necessary definitions and notation.

Let $\sigma_* \in S_n$ be a permutation:

$$\sigma_* = \begin{pmatrix} 1 & 2 & ... & n \\ \sigma_*(1) & \sigma_*(2) & ... & \sigma_*(n) \end{pmatrix}.$$

Let call by operator $\sigma^* : H_n \to H_n$ *associated with the permutation* $\sigma_* \in S_n$, the operator, permuting the isotropic coordinates of the algebra H_n:

$$\sigma^* : x = (x_1, x_2, ..., x_n) \ \mapsto \ \sigma^*(x) = \left(x_{\sigma_*(1)}, x_{\sigma_*(2)}, ..., x_{\sigma_*(n)}\right).$$

Let $(\sigma^1, \sigma^2, ..., \sigma^m)$ be an ordered set of associated operators $(m \le n)$. Let the action of the m-family $(\sigma^1, \sigma^2, ..., \sigma^m)$ on the m-family of the elements $(x_1, x_2, ..., x_m) \subset H_n$ be defined in the 'tensor sense':

$$\left(\sigma^1 \otimes \sigma^2 \otimes ... \otimes \sigma^m\right)(x_1, x_2, ..., x_m) = \sigma^1(x_1) \bullet \sigma^2(x_2) \bullet ... \bullet \sigma^m(x_m).$$

Let $A \subset Z^m$ be a set of indices, $\tilde{S}_n^m = \{(\sigma^{a_1}, \sigma^{a_2}, ..., \sigma^{a_m})\, ;\ (a_1, ..., a_m) \in \mathrm{A}\}$ be a certain set of m-families of associated operators 'marked' with the index set $\mathrm{A} \subset Z^m$.

Definition 1.1. Let denote by an m-ary operation in the algebra $H_n\,(m \le n)$, *induced by the family* $\tilde{S}_n^m = \{(\sigma^{a_1}, \sigma^{a_2}, ..., \sigma^{a_m})\, ;\ (a_{1,...,}a_m) \in \mathrm{A}\}$, an operation, given by the relation (here and in the following $\lambda_m \in R$)

$$\begin{aligned}
&[x_1, x_2, ..., x_m] = \\
&= \lambda_m \sum_{(\sigma^{a_1}, \sigma^{a_2}, ..., \sigma^{a_m}) \in \tilde{S}_n^m} (\sigma^{a_1} \otimes \sigma^{a_2} \otimes ... \otimes \sigma^{a_m})\,(x_1, x_2, ..., x_m) = \\
&= \lambda_m \sum_{(\sigma^{a_1}, \sigma^{a_2}, ..., \sigma^{a_m}) \in \tilde{S}_n^m} \sigma^{a_1}(x_1) \bullet \sigma^{a_2}(x_2) \bullet ... \bullet \sigma^{a_m}(x_m)
\end{aligned}.$$

Definition 1.2. An m-ary operation on the algebra $H_n\,(m \le n)$, induced by the family $\tilde{S}_n^m = \{(\sigma^{a_1}, \sigma^{a_2}, ..., \sigma^{a_m})\, ;\ (a_{1,...,}a_m) \in \mathrm{A}\}$, will be called *normable,* if

$$N(x) = \lambda_m \sum_{(\sigma^{a_1}, \sigma^{a_2}, ..., \sigma^{a_m}) \in \tilde{S}_n^m} (\sigma^{a_1} \otimes \sigma^{a_2} \otimes ... \otimes \sigma^{a_m})\,(x, x, ..., x) \in R. \qquad (1.13)$$

The family of the associated operators $\tilde{S}_n^m$ in this case will be called *norming family,* and the function $N(x) \in R$ will be called $\tilde{S}_n^m$- *norm* (or simply *norm,* if this doesn't introduce any confusion on which family $\tilde{S}_n^m$ is implied).

Definition 1.3. Let $S_n^m = \{(\sigma_{a_1}, \sigma_{a_2}, ..., \sigma_{a_m})\, ;\ (a_{1,...,}a_m) \in \mathrm{A}\}$ be a set of the permutations, associated with the norming operator family $\tilde{S}_n^m = \{(\sigma^{a_1}, \sigma^{a_2}, ..., \sigma^{a_m})\, ;\ (a_{1,...,}a_m) \in \mathrm{A}\}$, $[x_1, x_2, ..., x_m]$ is an m-ary operation, induced by the family $\tilde{S}_n^m$. In the isotropic coordinates let the element $[x_1, x_2, ..., x_m]$ have coordinates $(\xi_1, \xi_2, ..., \xi_m)$:

$$[x_1, x_2, ..., x_m] = (\xi_1, \xi_2, ..., \xi_m)\,.$$

Let say that *the family* $\tilde{S}_n^m$ *induces a generalized m-ary composition law,* if the following equality holds

$$\begin{aligned}
&N\,([x_1, x_2, ..., x_m]) = \underbrace{[[x_1, x_2, ..., x_m]\,, ..., [x_1, x_2, ..., x_m]]}_{m \quad \text{times}} = \\
&= \lambda_m \sum_{(\sigma^{a_1}, \sigma^{a_2}, ..., \sigma^{a_m}) \in \tilde{S}_n^m} (\sigma^{a_1} \otimes \sigma^{a_2} \otimes ... \otimes \sigma^{a_m}) \left(\underbrace{[x_1, x_2, ..., x_m]\,, ..., [x_1, x_2, ..., x_m]}_{m \quad \text{times}} \right) = \\
&= \lambda_m \sum_{(\sigma_{a_1}, \sigma_{a_2}, ..., \sigma_{a_m}) \in S_n^m} \xi_{\sigma_{a_1}(1)} \xi_{\sigma_{a_2}(2)} ... \xi_{\sigma_{a_m}(m)} I\,.
\end{aligned}$$

Example 1.3. Using the notation of the Example 1.1, consider a one-element set S_2^2, consisting of one pair of permutations $\{(\sigma_1,\ \sigma_2)\}$, defined by (1.8).

The equality (1.12) means that, the binary operation induced by the family $\tilde{S}_2^2$ of the associated operators $\tilde{S}_2^2 = \{(\sigma^1,\ \sigma^2)\}$, yields a binary composition law $\sigma^1([x,y]) \bullet\ \sigma^2([x,y]) = \xi_1\xi_2 I$ which is the same (up to notation) as the multiplicative composition law.

$$N([x,y]) = [[x,y],[x,y]] = N(x)N(y) \qquad \text{and} \quad \lambda_2 = 1.$$

Example 1.4. Not all the sets $\tilde{S}_n^m$ are norming sets of operators. One may easily note that using notation of the example 1.1, a one-element set $\tilde{S}_2^2 = \{(\sigma^1,\ \sigma^1)\}$ is not a norming set of operators, as $N(x) = [x,x] \notin R$. In this context, a Question 1 arises: what are necessary and sufficient conditions (formulated in number theoretic terms) for the set of operators, associated with $S_n^m = \{(\sigma_{a_1}, \sigma_{a_2}, ..., \sigma_{a_m})\}$, to be norming?

Remark 1.5. Naturally, the norming sets $\tilde{S}_4^m$ of associated operators from the set S_4^m, for which $N(w)$ for all $w = (a,b,c,d) \in H_4$ coincides with one of the forms (1.5). In this context, a Question 2 arises: what are necessary and sufficient conditions (formulated in number-theoretic terms) for the norm, induced by the set of operators $\tilde{S}_n^m$ to coincide with one of the forms (1.5) for H_4?

This paper is intended to answer the Question 2 in the part of *sufficient conditions;* i.e. to determine binary, ternary or quaternary operation in the algebra H_4, which the following relation holds for

$$N(x) = \underbrace{[x,x,...,x]}_{m \quad \text{times}} \in \{s_{24}(w), s_{34}(w), s_{44}(w)\}.$$

Remark 1.6. The proofs of the proposition in the next two sections of this paper may be reduced to the routinous and laborious verification of the identities, in the way similar to the Example 1.1. Thus, in this paper, the considered theorems are only formulated.

2 Generalized n-ary composition laws in the algebra $R \oplus R \oplus R \oplus R$

2.1 Generalized quaternary composition law in the algebra $R \oplus R \oplus R \oplus R$

Let the quaternary operation in the algebra $R \oplus R \oplus R \oplus R$ be given by

$$[x_1,x_2,x_3,x_4] = \lambda_4\left(\sigma^1 \otimes \sigma^2 \otimes \sigma^3 \otimes \sigma^4\right)(x_1,x_2,x_3,x_4) = \lambda_4\sigma^1(x_1)\bullet\sigma^2(x_2)\bullet\sigma^3(x_3)\bullet\sigma^4(x_4) \tag{2.1}$$

where $\lambda_4 = 1$ and

$$\sigma_1 = \begin{pmatrix} 1\,2\,3\,4 \\ 1\,2\,3\,4 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,3\,4\,1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,4\,1\,2 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,1\,2\,3 \end{pmatrix}. \tag{2.2}$$

Theorem 2.1. A one-element family of the quadruple of operators $\tilde{S}_4^4 = \{(\sigma^1,\ \sigma^2,\sigma^3,\sigma^4)\}$, associated with the one-element set S_4^4, consisting of four (cyclic) permutations $\{(\sigma_1,\ \sigma_2, \sigma_3, \sigma_4)\}$ is a norming set, inducing the generalized quaternary composition law in the algebra $R \oplus R \oplus R \oplus R$ in the (multiplicative) form

$$\begin{aligned} &N_4\left([x_1, x_2, x_3, x_4]\right) = \left[[x_1, x_2, x_3, x_4], [x_1, x_2, x_3, x_4], [x_1, x_2, x_3, x_4], [x_1, x_2, x_3, x_4]\right] = \\ &= \left(\sigma^1 \otimes \sigma^2 \otimes \sigma^3 \otimes \sigma^4\right)\left([x_1, x_2, x_3, x_4], [x_1, x_2, x_3, x_4], [x_1, x_2, x_3, x_4], [x_1, x_2, x_3, x_4]\right) = \\ &= \xi_{\sigma_1(1)}\xi_{\sigma_2(2)}\xi_{\sigma_3(3)}\xi_{\sigma_4(4)} I = \\ &= \left(x_{11}x_{12}x_{13}x_{14}\right)\left(x_{21}x_{22}x_{23}x_{24}\right)\left(x_{31}x_{32}x_{33}x_{34}\right)\left(x_{41}x_{42}x_{43}x_{44}\right) I = \\ &= N_4(x_1)\, I \cdot N_4(x_2)\, I \cdot N_4(x_3)\, I \cdot N_4(x_4)\, I, \end{aligned} \tag{2.3}$$

where $(\xi_1, \xi_2, \xi_3, \xi_4)$ is a quadruple of coordinates of the value of the quaternary operation $[x_1, x_2, x_3, x_3]$ defined above in the isotropic basis: $(\xi_1, \xi_2, \xi_3, \xi_4) = [x_1, x_2, x_3, x_4]$.

Moreover, the following equality holds:

$$\begin{aligned} &N_4(x) = [x, x, x, x] = \left(\sigma^1 \otimes \sigma^2 \otimes \sigma^3 \otimes \sigma^4\right)(x, x, x, x) = \\ &= \xi_{\sigma_1(1)}\xi_{\sigma_2(2)}\xi_{\sigma_3(3)}\xi_{\sigma_4(4)} I = \left(x_1x_2x_3x_4\right) I = s_{44}(x) \bullet I, \end{aligned} \tag{2.4}$$

Remark 2.1. From the Theorem 2.1 it follows, in particular, that the concept of the norm, given in the Definition 1.2 by relation (1.14) is in fact a multiplicative Berwald-Moor (pseudo) norm (1.5) $s_{44}(x)$, expressed in terms of the isotropic basis expansions.

2.2 Generalized ternary composition law in the algebra $R \oplus R \oplus R \oplus R$

Let the ternary operation in the algebra $R \oplus R \oplus R \oplus R$ be defined by

$$[x_1, x_2, x_3] = \left(\sigma^2 \otimes \sigma^3 \otimes \sigma^4 + \sigma^4 \otimes \sigma^1 \otimes \sigma^3 + \sigma^2 \otimes \sigma^4 \otimes \sigma^1 + \sigma^3 \otimes \sigma^1 \otimes \sigma^2\right)(x_1, x_2, x_3), \tag{2.5}$$

where $\lambda_3 = 1$ and

$$\sigma_1 = \begin{pmatrix} 1\,2\,3\,4 \\ 1\,3\,4\,2 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,2\,1\,3 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,4\,3\,1 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,1\,2\,4 \end{pmatrix}. \tag{2.6}$$

Theorem 2.2. A 4-element family of the triads of the operators

$$\tilde{S}_4^3 = \left\{\left(\ \sigma^2, \sigma^3, \sigma^4\right), \left(\sigma^4, \sigma^1,\ \sigma^3\right), \left(\ \sigma^2, \sigma^4, \sigma^1\right), \left(\sigma^3, \sigma^1,\ \sigma^2\right)\right\},$$

associated with the family S_4^4 of the triads of permutations

$$\sigma_1 = \begin{pmatrix} 1\,2\,3\,4 \\ 1\,3\,4\,2 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1\,2\,3\,4 \\ 4\,2\,1\,3 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,4\,3\,1 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,1\,2\,4 \end{pmatrix}, \tag{2.7}$$

is a norming set, inducing a generalized ternary composition law in the algebra $R \oplus R \oplus R \oplus R$ in the form

$$N_3([x_1,x_2,x_3]) = [[x_1,x_2,x_3],[x_1,x_2,x_3],[x_1,x_2,x_3]] =$$
$$= \left(\sigma^2\otimes\sigma^3\otimes\sigma^4+\sigma^4\otimes\sigma^1\otimes\sigma^3+\sigma^2\otimes\sigma^4\otimes\sigma^1+\sigma^3\otimes\sigma^1\otimes\sigma^2\right)([x_1,x_2,x_3],...,[x_1,x_2,x_3]) =$$
$$= \left(\xi_{\sigma_2(1)}\xi_{\sigma_3(2)}\xi_{\sigma_4(3)}+\xi_{\sigma_4(1)}\xi_{\sigma_1(2)}\xi_{\sigma_3(3)}+\xi_{\sigma_2(1)}\xi_{\sigma_4(2)}\xi_{\sigma_1(3)}+\xi_{\sigma_3(1)}\xi_{\sigma_1(2)}\xi_{\sigma_2(3)}\right)I, \tag{2.8}$$

where $(\xi_1,\xi_2,\xi_3,\xi_4)$ is a quadruple of the coordinates of the element $[x_1,x_2,x_3]$, the result element of the ternary operation defined above (in the isotropic basis):

$$(\xi_1,\xi_2,\xi_3,\xi_4) = [x_1,x_2,x_3],$$

and, moreover, the following relations hold

$$N_3(x) = [x,x,x] =$$
$$= (\sigma^2\otimes\sigma^3\otimes\sigma^4+\sigma^4\otimes\sigma^1\otimes\sigma^3+\sigma^2\otimes\sigma^4\otimes\sigma^1+\sigma^3\otimes\sigma^1\otimes\sigma^2)(x,x,x), \tag{2.9}$$

$$N_3(x) = s_{34}(x)\,I. \tag{2.10}$$

2.3 Generalized binary composition law in the algebra $R\oplus R\oplus R\oplus R$

Let the binary operation in the algebra $R\oplus R\oplus R\oplus R$ be defined by

$$[x_1,x_2] = \lambda_2\left(\sigma^1\otimes\sigma^2+\sigma^1\otimes\sigma^3+\sigma^1\otimes\sigma^4+\sigma^2\otimes\sigma^3+\sigma^2\otimes\sigma^4+\sigma^3\otimes\sigma^4\right)(x_1,x_2), \tag{2.11}$$

where $\lambda_2 = 1/2$ and

$$\sigma_1 = \begin{pmatrix}1\,2\,3\,4\\1\,2\,3\,4\end{pmatrix},\quad \sigma_2 = \begin{pmatrix}1\,2\,3\,4\\2\,3\,4\,1\end{pmatrix},\quad \sigma_3 = \begin{pmatrix}1\,2\,3\,4\\3\,4\,1\,2\end{pmatrix},\quad \sigma_4 = \begin{pmatrix}1\,2\,3\,4\\4\,1\,2\,3\end{pmatrix}. \tag{2.12}$$

Theorem 2.3. A six-element family of the pair of operators

$$\tilde{S}_4^2 = \left\{\left(\sigma^1,\sigma^2\right),\left(\sigma^1,\ \sigma^3\right),\left(\sigma^1,\sigma^4\right),\left(\sigma^2,\sigma^3\right),\left(\sigma^2,\sigma^4\right),\left(\sigma^3,\sigma^4\right)\right\},$$

associated with the family S_4^2 of pairs of permutations

$$S_4^2 = \{(\sigma_1,\sigma_2),(\sigma_1,\ \sigma_3),(\sigma_1,\sigma_4),(\sigma_2,\sigma_3),(\sigma_2,\sigma_4),(\sigma_3,\sigma_4)\} \tag{2.13}$$

is a norming set, inducing a generalized binary composition law in the algebra $R\oplus R\oplus R\oplus R$ in the form

$$N_2([x_1,x_2]) = [[x_1,x_2],[x_1,x_2]] =$$
$$= \tfrac{1}{2}(\sigma^1\otimes\sigma^2+\sigma^1\otimes\sigma^3+\sigma^1\otimes\sigma^4+\sigma^2\otimes\sigma^3+\sigma^2\otimes\sigma^4+\sigma^3\otimes\sigma^4)([x_1,x_2],[x_1,x_2]) =$$
$$= \tfrac{1}{2}\left(\xi_{\sigma_1(1)}\xi_{\sigma_2(2)}+\xi_{\sigma_1(1)}\xi_{\sigma_3(2)}+\xi_{\sigma_1(1)}\xi_{\sigma_4(2)}+\xi_{\sigma_2(1)}\xi_{\sigma_3(2)}+\xi_{\sigma_2(1)}\xi_{\sigma_4(2)}+\xi_{\sigma_3(1)}\xi_{\sigma_4(2)}\right)I, \tag{2.14}$$

where $(\xi_1, \xi_2, \xi_3, \xi_4)$ is a quadruple of coordinates of the element $[x_1, x_2]$, the result element of the binary operation defined above (in the isotropic basis): $(\xi_1, \xi_2, \xi_3, \xi_4) = [x_1, x_2]$.

Moreover, the following equalities hold

$$
\begin{aligned}
&N_2(x) = [x, x] = \\
&= 1/2\left(\sigma^1 \otimes \sigma^2 + \sigma^1 \otimes \sigma^3 + \sigma^1 \otimes \sigma^4 + \sigma^2 \otimes \sigma^3 + \sigma^2 \otimes \sigma^4 + \sigma^3 \otimes \sigma^4\right)(x, x)\ ,
\end{aligned} \tag{2.15}
$$

$$
N_2(x) = s_{24}(x)\,. \tag{2.16}
$$

Remark 2.3. From the Theorem 2.3 it follows (in particular) that the concept of the norm, introduced in the Definition 1.2 by the relation (1.14) in fact represents a (pseudo)norm, corresponding to the Minkowski metrics in the form $s_{44}(x)$ 'represented' in the isotropic basis.

Remark 2.4. In contrast to the set (2.4) of permutations in the Theorem 2.2 (this set is not a group), the set (2.8) of permutations in the Theorem 2.3 is a four-element non-cyclic group, and in the Theorem 2.1, the set of permutations is a cyclic group.

3 Generalizations and open problems

1. In context of Questions 1 and 2 of the Section 1 the following problem arises: how should generalized composition laws for the space H_n of arbitrary number of dimensions be classified?

Problem 1. What are necessary and sufficient conditions (in number-theoretic or combinatorial terms) for the set of operators, associated with the set of permutations $S_n^m = \{(\sigma_{a_1}, \sigma_{a_2}, ..., \sigma_{a_m})\}$, to be norming; how does the induced norm depend on the coefficients of the defining equation of the element from the algebra H_n?

Note that group structure of the set of permutations S_n^m is likely to be a sufficient condition for the associated set of operators to be norming (Theorems 2.1 and 2.3). However, this condition may be not necessary, as results from the Theorem 2.2. This appears possible, that the problem of classification of the sets S_n^m in the Problem 1 is a combinatorial problem and not just a group-theoretic one.

2. The thorough classification of the associate-commutative algebras without nilpotent elements is given in the Weierstrass theorem [2]: any associative-commutative algebra without nilpotent elements is isomorphic to the direct sum of algebras R and C.

From this theorem, it follows easily that there exist at most 3 non-isomorphic 4D algebras from this class, namely: $H_4, H_2 \oplus C, C \oplus C$. In this context, a problem to extrapolate the Theorems 2.1 − 2.3 to the above-mentioned $4D$ algebras and to an arbitrary associative-commutative finite-dimensional algebra A_n arises:

Problem 2. What are necessary and sufficient conditions for the set of operators associated with the family of automorphisms of the algebra A_n to be norming;

how is the induced norm related to the coefficients of the defining equation of the algebra A_n?

3. From the relation (2.3) and Remark 2.1 the following (purely algebraic) fact follows: the Berwald-Moor metrics in isotropic basis is multiplicative. Certainly, the corresponding (induced) relations on the invariance of the form $S_{44}(x)$ hold in the 'physical' basis $\{E, I, J, K\}$ with the Cayley table 2.1 for multiplication of the basis elements. But the multiplicative property of the form $s_{44}(x)$, $s_{44}(x_1 \bullet x_2 \bullet x_3 \bullet x) = s_{44}(x_1)\, s_{44}(x_2)\, s_{44}(x_3)\, s_{44}(x_4)$ may be likely interpreted as the 'scaling invariance' of the properties of the 4D space-time with the Berwald-Moor metrics and/or as its 'space-time isotropy'. In contrast to the Berwald-Moor metrization of the 4D space time, a 4D space with the Minkowski metrics, which is the same (up to scaling) as H_4, equipped with the metric form $S_{24}(x)$, has only the property of the 'spatial', but not 'space-time' isotropy. The group of (linear) isometries of the space H_4 with the Minkowski metrics is well-known without any particular relation to the Theorem 2.3.

Problem 3. May Lorenz transforms be obtained from the relations (2.9) and (2.10), i.e. as direct consequences of the Theorem 2.3 explicit relations?

4. Let $\{e_1, e_2, e_3, e_4\}$ be an 'arbitrary ' basis in H_4, e.g. an isotropic one. Let further B be an operator (may be linear) acting from H_4 into H_4, so that

$$Bx = y = y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 \in H_4.$$

The constraint on B to preserve the form of $s_{24}(x)$, i.e. the norm $N_2(x)$, may be written in the form

$$N_2(x) = [x, x] = [Bx, Bx] = [y, y] = N_2(y) = N_2(Bx)\,. \tag{3.1}$$

But, due to linearity of the operation $[w_1, w_2]$, the relation (3.1) maybe rewritten in the form

$$[x, x] = \sum_{i,j=1}^{4} x_ix_j\,[e_i, e_j] = [y, y] = \sum_{i,j=1}^{4} y_iy_j\,[e_i, e_j]\,.$$

If the operator B is linear, then the array $\{b_{ij} = [e_i, e_j]\,;\; i, j = 1, 2, 3, 4\}$ will (this is well-known) define completely the action of the operator B in the whole space H_4. Moreover, for linear isometric operations the conditions on the numbers b_{ij} may be obtained from the Theorem 2.3. If we require the operator B to preserve the form $s_{34}(x)$, i.e. the norm $N_3(x)$, then, due to linearity of the operation $[w_1, w_2, w_3]$, we obtain the conditions for the (non-linear) operator B to be isometric, these conditions will be expressed in terms of the *three-dimensional array* $\{b_{ijk} = [e_i, e_j, e_k]\,;\; i, j, k = 1, 2, 3, 4\}$ and *explicit relations* of the Theorem 2.2.

Problem 4. Applying the Theorem 2.2, describe operators B, preserving the form $s_{34}(x)$, i.e. the norm $N_3(x)$.

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References

[1] *Algebra,* (L. A. Skornyakov, ed), Science, M., 1990, (in Russian).

[2] R.B.J.T. Allenby, *Rings, Fields and Groups: An Introduction to Abstract Algebra,* 2nd edition, 1991.

[3] I. L. Kantor, A. S. Solodovnikov, *Hypercomplex Numbers.* Springer, Berlin, 1989

[4] V. M. Chernov, *About the defining equations for the elements of the associative-commutative finitedimensional algebras and associated metric forms.* Hypercomplex numbers in Geometry and Physics, 4, 2005, 57–74.

[5] H. Zassenhaus, *Uber endliche Fastkorper,* Abh. Math. Sem. Hamburg, 11 (1936), 187–220.

[6] M. Hall, *The Theory of Groups,* MacMillan, 1959.

[7] S. Lang, *Algebra*, Addison-Wesley, Reading, Mass., 1965.

[8] D. G. Pavlov, *Generalization of scalar product axioms,* Hypercomplex numbers in Geometry and Physics. 1, 2004, 5–19.

The Prolongations of a Finsler Metric to the Tangent Bundle $T^k(M)$ $(k > 1)$ of the Higher Order Accelerations

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An old problem in differential geometry is that of prolongation of a Riemannian structure $g(x)$ on a real n-dimensional C^∞-manifold $M, x \in M$, to the bundle of k-jets $\left(J_0^k M, \pi^k, M\right)$ or, equivalently the tangent bundle $\left(T^k M, \pi^k, M\right)$ of the higher order accelerations. The problem belongs to so-called geometry of higher order. It was solved in [18] for $k = 1$ and partially in [19] for $k = 2$. The same problem of prolongation can be considered for a Finslerian structure $F\left(x, y^{(1)}\right)$. In the paper [15] are given these solutions in the general cases, using the Sasaki-Matsumoto N-lift (for $k = 2$, see [3] and [6]).

But, the terms of Sasaki-Matsumoto prolongation of a Riemannian metric (or Finslerian metric) to $T^k M$ have not the same physical dimensions because these prolongations is not homogeneous on the fibres of the tangent bundle of order k. This is a disavantage in the study of the geometry of $T^k M$ using the Riemannian metrics determined by these prolongations.

In this paper, only for a Finsler space $F^n = \left(M, F\left(x, y^{(1)}\right)\right)$, we correct this disavantage introducing a new kind of prolongation $\overset{\circ}{\mathbf{G}}$ of the Finsler metric $g_{ab}\left(x, y^{(1)}\right) = \partial^2 F / \partial y^{(1)a} \partial y^{(1)b}$ given by (2.1), which is 0−homogeneous. Some properties of the Riemannian space $\left(\widetilde{T^k M}, \overset{\circ}{G}\right)$ are studied. The almost $(k-1)\,n-$contact structure $\overset{\circ}{\mathbf{F}}$ from (2.13) is introduced. It has the property of homogeneity and $\left(\overset{\circ}{\mathbf{G}}, \overset{\circ}{\mathbf{F}}\right)$ is a metrical almost $(k-1)\,n-$contact structure on $T^k M$. It depend only on the fundamental function $F\left(x, y^{(1)}\right)$ of the Finsler space F^n. The space $\left(\widetilde{T^k M}, \overset{\circ}{\mathbf{G}}, \overset{\circ}{\mathbf{F}}\right)$ is **the geometrical model** of the Finsler space $F^n = \left(M, F\left(x, y^{(1)}\right)\right)$.

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The Sasaki-Matsumoto N-lift of a Finsler metric

Let M be a real n-dimensional C^∞-manifold and $\left(T^k M, \pi^k, M\right)$ its tangent bundle of order k (or k-jet bundle, or tangent bundle of the higher order accelerations).

Let us consider the Finsler space $F^n = (M, F)$ with the fundamental function $F(x, y^{(1)}), F : T^1 M \to R$, and the fundamental tensor $g_{ab}(x, y^{(1)})$ on $T^1 M$ given by

$$g_{ab}\left(x, y^{(1)}\right) = \frac{1}{2} \frac{\partial^2 F}{\partial y^{(1)a} \partial y^{(1)b}}, \tag{1.1}$$

where $g_{ab}\left(x,y^{(1)}\right)$ is positively defined on T^1M.

The indices $a,b,...$ run over set $\{1,2,...,n\}$ and Einstein convention of summaring is adopted all over this work.

Let $\gamma^a_{bc}\left(x,y^{(1)}\right)$ be the formal Christoffel symbols of the $g_{ab}\left(x,y^{(1)}\right)$, *i.e.* :

$$\gamma^a_{bc}\left(x,y^{(1)}\right)=\frac{1}{2}g^{ad}\left(\frac{\partial g_{bd}}{\partial x^c}+\frac{\partial g_{dc}}{\partial x^b}-\frac{\partial g_{bc}}{\partial x^d}\right). \tag{1.2}$$

Then, the canonical semispray of F^n is given by

$$\frac{d^2x^a}{dt^2}+2\underset{(1)}{G}{}^a\left(x,\frac{dx}{dt}\right)=0, \tag{1.3}$$

where

$$\underset{(1)}{G}{}^a=\frac{1}{2}\gamma^a_{bc}\left(x,y^{(1)}\right)y^{(1)b}y^{(1)c}. \tag{1.3'}$$

The canonical nonlinear connection (determined only by the function F of the Finsler space F^n) is the Cartan nonlinear connection with the coefficients

$$G^a_b\left(x,y^{(1)}\right)=\frac{\partial \underset{(1)}{G}{}^a}{\partial y^{(1)b}}. \tag{1.4}$$

Then, on the domain of chart $\left(\pi^k\right)^{-1}(U)\subset T^kM, U\subset M$, we can consider the functions

$$\begin{aligned} F^*\left(x,y^{(1)},...,y^{(k)}\right) &= \left(F\circ\pi^k_1\right)\left(x,y^{(1)},...,y^{(k)}\right), \\ g^*_{ab}\left(x,y^{(1)},...,y^{(k)}\right) &= \left(g_{ab}\circ\pi^k_1\right)\left(x,y^{(1)},...,y^{(k)}\right), \\ & \forall\left(x,y^{(1)},...,y^{(k)}\right)(U), \end{aligned}$$

where $\pi^k_1:T^kM\to TM, \pi^k_1\left(x,y^{(1)},...,y^{(k)}\right)=\left(x,y^{(1)}\right)$ is the natural projection. For simplicity, F^* and g^*_{ab} will be denote by the same letters F and g_{ab}.

We have

1^0. The canonical nonlinear connection N on $\widetilde{T^kM}=T^kM\setminus\{0\}$ has the dual coefficients

$$\begin{aligned} &\underset{1}{M}{}^a_b=G^a_b, \\ &\underset{2}{M}{}^a_b=\tfrac{1}{2}\left(C\underset{1}{M}{}^a_b+\underset{1}{M}{}^a_c\underset{1}{M}{}^c_b\right), \\ &\dots\dots\dots\dots \\ &\underset{k}{M}{}^a_b=\tfrac{1}{k}\left(C\underset{k-1}{M}{}^a_b+\underset{1}{M}{}^a_c\underset{k-1}{M}{}^c_b\right), \end{aligned} \tag{1.5}$$

where C is the operator

$$C=y^{(1)a}\frac{\partial}{\partial x^a}+2y^{(2)a}\frac{\partial}{\partial y^{(1)a}}+...+ky^{(k)a}\frac{\partial}{\partial y^{(k-1)a}} \tag{1.6}$$

2^0. The Liouville d-vector field $z^{(k)}$ corresponding to the canonical nonlinear connection N is given by

$$kz^{(k)a} = ky^{(k)a} + (k-1)\, y^{(k-1)b} \underset{1}{M}{}^a_{\ b} + ... + y^{(1)b} \underset{k-1}{M}{}^a_{\ b}. \tag{1.7}$$

3^0. The following Lagrangian

$$L\left(x, y^{(1)}, ..., y^{(k)}\right) = g_{ab}\left(x, y^{(1)}\right) z^{(k)a} z^{(k)b}, \tag{1.8}$$

is a regular Lagrangian on $\widetilde{T^kM}$, determined only by $F\left(x, y^{(1)}\right)$ because g_{ab} and $z^{(k)}$ have this property.

4^0. Its fundamental tensor field coincide with the fundamental tensor field on Finsler space F^n, namely on $\widetilde{T^kM}$ we have

$$\frac{1}{2}\frac{\partial^2 L}{\partial z^{(k)a}\partial z^{(k)b}} = g_{ab}\left(x, y^{(1)}\right). \tag{1.9}$$

5^0. N determines the direct decomposition

$$T_u T^k M = N_0(u) \oplus N_1(u) \oplus ... \oplus N_{k-1}(u) \oplus V_k(u), \quad \forall u \in T^k M. \tag{1.10}$$

6^0. The adapted cobasis $\left\{dx^a, \delta y^{(1)a}, ..., \delta y^{(k)a}\right\}$ and the adapted basis $\left\{\frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^{(1)a}}, ..., \frac{\delta}{\delta y^{(k-1)a}}, \frac{\delta}{\delta y^{(k)a}}\right\}$ to N are depending only on fundamental function $F\left(x, y^{(1)}\right)$ of Finsler space F^n, where

$$\begin{aligned}
\delta y^{(1)a} &= dy^{(1)a} + \underset{1}{M}{}^a_{\ c} dx^c,\\
\delta y^{(2)a} &= dy^{(2)a} + \underset{1}{M}{}^a_{\ c} dy^{(1)c} + \underset{2}{M}{}^a_{\ c} dx^c,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\delta y^{(k)a} &= dy^{(k)a} + \underset{1}{M}{}^a_{\ c} dy^{(k-1)c} + ... + \underset{k-1}{M}{}^a_{\ c} dy^{(1)c} + \underset{k}{M}{}^a_{\ c} dx^c,
\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}
\frac{\delta}{\delta x^a} &= \frac{\partial}{\partial x^a} - \underset{1}{N}{}^c_{\ a} \frac{\partial}{\partial y^{(1)c}} - \underset{2}{N}{}^c_{\ a} \frac{\partial}{\partial y^{(2)c}} - ... - \underset{k}{N}{}^c_{\ a} \frac{\partial}{\partial y^{(k)c}},\\
\frac{\delta}{\delta y^{(1)a}} &= \frac{\partial}{\partial y^{(1)a}} - \underset{1}{N}{}^c_{\ a} \frac{\partial}{\partial y^{(2)c}} - ... - \underset{k-1}{N}{}^c_{\ a} \frac{\partial}{\partial y^{(k)c}},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\frac{\delta}{\delta y^{(k-1)a}} &= \frac{\partial}{\partial y^{(k-1)c}} - \underset{1}{N}{}^c_{\ a} \frac{\partial}{\partial y^{(k)c}}.
\end{aligned} \tag{1.11'}$$

We know that

$$\begin{aligned}
&\underset{1}{N}{}^a_{\ b} = \underset{1}{M}{}^a_{\ b}, \underset{2}{N}{}^a_{\ b} = \underset{2}{M}{}^a_{\ b} - \underset{1}{M}{}^c_{\ b} \underset{1}{M}{}^a_{\ c}, ...,\\
&\underset{k}{N}{}^a_{\ b} = \underset{k}{M}{}^a_{\ b} - \underset{1}{M}{}^c_{\ b} \underset{k-1}{N}{}^a_{\ c} - ... - \underset{k-2}{M}{}^c_{\ b} \underset{2}{N}{}^a_{\ c} - \underset{k-1}{M}{}^c_{\ b} \underset{1}{N}{}^a_{\ c},
\end{aligned} \tag{1.12}$$

and conversely

$$\begin{aligned}&\underset{1}{M}{}^a_b=\underset{1}{N}{}^a_b,\ \underset{2}{M}{}^a_b=\underset{2}{N}{}^a_b+\underset{1}{N}{}^a_c\underset{1}{M}{}^c_b,\ ...,\\&\underset{k}{M}{}^a_b=\underset{k}{N}{}^a_b+\underset{k-1}{N}{}^a_c\underset{1}{M}{}^c_b+...+\underset{2}{N}{}^a_c\underset{k-2}{M}{}^c_b+\underset{1}{N}{}^a_c\underset{k-1}{M}{}^c_b.\end{aligned}\tag{1.12'}$$

Then, the Sasaki-Matsumoto $N-$lift of $g_{ab}\left(x,y^{(1)}\right)$ to T^kM is defined by

$$\mathbf{G}\left(u\right)=g_{ab}\left(x,y^{(1)}\right)dx^a\otimes dx^b+\sum_{\beta=1}^{k}g_{ab}\left(x,y^{(1)}\right)\delta y^{(\beta)a}\otimes\delta y^{(\beta)b},\ \forall u\in\widetilde{T^kM}.\tag{1.13}$$

The following properties hold:

7^0. $\mathbf{G}$ is globally defined on T^kM.

8^0. $\mathbf{G}$ is a Riemannian structure on T^kM determined only by the Finsler space F^n.

9^0. $\mathbf{G}$ is not homogeneous on the fibres of T^kM.

Namely, for the homothety $h_t:\left(x,y^{(1)},...,y^{(k)}\right)\rightarrow\left(x,ty^{(1)},...,t^ky^{(k)}\right),\forall t\in R^+_*$, we get

$$\left(G\circ h_t\right)\left(u\right)=g_{ab}\left(x,y^{(1)}\right)dx^a\otimes dx^b+\sum_{\beta=1}^{k}t^{2\beta}g_{ab}\left(x,y^{(1)}\right)\delta y^{(\beta)a}\otimes\delta y^{(\beta)b}\neq G\left(u\right).$$

Let us consider the $\mathcal{F}\left(T^kM\right)-$ linear mapping $\mathbf{F}:\chi\left(T^kM\right)\rightarrow\chi\left(T^kM\right)$ given in the adapted basis (1.11') by

$$\begin{gathered}\mathbf{F}\left(\frac{\delta}{\delta x^a}\right)=-\frac{\partial}{\partial y^{(k)a}},\\ \mathbf{F}\left(\frac{\delta}{\delta y^{(1)a}}\right)=...=\mathbf{F}\left(\frac{\delta}{\delta y^{(k-1)a}}\right)=0,\\ \mathbf{F}\frac{\partial}{\partial y^{(k)a}}=\frac{\delta}{\delta x^a}.\end{gathered}\tag{1.14}$$

It follows that:

10^0. $\mathbf{F}$ is globally defined on T^kM and it is a $d-$ tensor field of type $(1,1)$.

11^0. $\mathbf{F}$ is an $(k-1)\,n-$contact structure : $\mathbf{F}^3+\mathbf{F}=0$.

12^0. $\mathbf{F}$ depend only on the fundamental function $F\left(x,y^{(1)}\right)$ of Finsler space F^n.

13^0. The pair $(\mathbf{G},\mathbf{F})$ is a Riemannian almost $(k-1)\,n$-contact structure on T^kM:

$$\mathbf{G}\left(\mathbf{F}X,Y\right)=-\mathbf{G}\left(X,\mathbf{F}Y\right),\forall X,Y\in\chi\left(T^kM\right).$$

Consequently, we get

Theorem 1.1 *The space $\left(T^kM,\mathbf{G},\mathbf{F}\right)$ is a Riemannian almost $(k-1)\,n$- contact space depending only on the fundamental function $F\left(x,y^{(1)}\right)$ of the Finsler space $F^n=(M,F)$.*

The previous space, called **"the geometrical model on T^kM of the Finsler space"** (M,F) is important in the study of the geometry of the initial Finsler space $F^n=(M,F)$.

The homogeneous prolongation to T^kM of a Finsler metric

We define a new prolongation $\overset{\circ}{G}$ on T^kM of the fundamental tensor field $g_{ab}\left(x,y^{(1)}\right)$ of a Finsler space $F^n=(M,F)$, which satisfies the following conditions:

1^0. $\overset{\circ}{G}$ is $0-$ homogeneous with respect to $y^{(1)a}, y^{(2)a}$, ...,and $y^{(k)a}$.

2^0. It depends only on the fundamental function $F\left(x,y^{(1)}\right)$.

3^0. In the mechanical meaning the terms of $\overset{\circ}{G}$ have the same physical dimensions.

Definition 2.1. *We call the homogeneous prolongation to T^kM of the fundamental tensor field $g_{ab}\left(x,y^{(1)}\right)$ of a Finsler space $F^n=(M,F)$,the following tensor field on T^kM:*

$$\overset{\circ}{G}(u)=g_{ab}\left(x,y^{(1)}\right)dx^a\otimes dy^b+\sum_{\beta=1}^{k}\frac{\mathbf{a}^{2\beta}}{\left\|y^{(1)}\right\|^{2\beta}}g_{ab}\left(x,y^{(1)}\right)\delta y^{(\beta)a}\otimes\delta y^{(\beta)b}, \forall u\in\widetilde{T^kM}, \tag{2.1}$$

where $\mathbf{a}>\mathbf{0}$ is a constant imposed by application in order to preserve the physical dimension of the components of $\overset{\circ}{G}$, and $\left\|y^{(1)}\right\|^2$ is the square of the norm of the first Liouvill vector field

$$\left\|y^{(1)}\right\|^2=g_{ab}\left(x,y^{(1)}\right)y^{(1)a}y^{(1)b}. \tag{2.2}$$

We get, without difficulties:

Theorem 2.1. 1.*The pair $\left(\widetilde{T^kM},\overset{\circ}{G}\right)$is a Riemann space.*

2. *$\overset{\circ}{G}$ is a 0-homogeneous tensor field with respect to $y^{(\beta)a}, (\beta=1,...,k)$.*

3. *$\overset{\circ}{G}$ depends only on the fundamental function $F\left(x,y^{(1)}\right)$ of Finsler space F^n.*

4. *The distributions $N_0,N_1,...,N_{k-1}$ and V_k are orthogonal, in pairs, with respect to $\overset{\circ}{G}$.*

We can write $\overset{\circ}{G}$ in the form

$$\overset{\circ}{\mathbf{G}}=\overset{\circ}{\mathbf{G}}^H+\overset{\circ}{\mathbf{G}}^{V_1}+...+\overset{\circ}{\mathbf{G}}^{V_k}, \tag{2.3}$$

where

$$\overset{\circ}{\mathbf{G}}^H=g_{ab}\left(x,y^{(1)}\right)dx^a\otimes dx^b, \overset{\circ}{\mathbf{G}}^{V_\beta}=g_{ab}\left(x,y^{(1)}\right)dy^{(\beta)a}\otimes dy^{(\beta)b} \tag{2.4}$$

and

$$\underset{(\beta)}{g}{}_{ab}\left(x,y^{(1)}\right)=\frac{\mathbf{a}^{2\beta}}{\left\|y^{(1)}\right\|^{2\beta}}g_{ab}\left(x,y^{(1)}\right), \quad (\beta=1,...,k). \tag{2.5}$$

As usually, let us denote

$$\partial_a = \frac{\partial}{\partial x^a}, \dot{\partial}_{1a} = \frac{\partial}{\partial y^{(1)a}}, ..., \dot{\partial}_{ka} = \frac{\partial}{\partial y^{(k)a}},$$

and from now on we denote the adapted basis (1.11') by

$$\left\{\delta_a, \delta_{1a}, ..., \delta_{(k-1)a}, \delta_{ka}\right\}.$$

In order to study the geometry of Riemann space $\left(\widetilde{T^kM}, \overset{\circ}{G}\right)$, we can apply the theory of the $(h, v_1, ..., v_k)-$ Riemannian metric given by author in [5] (for $k = 2$, see [2], [4]).

A linear connection D on T^kM is called a metrical N-linear connection with respect to $\overset{\circ}{\mathbf{G}}$ if $D_X\overset{\circ}{\mathbf{G}} = 0, \forall X \in \chi\left(T^kM\right)$ and it preserves by paralelism the horizontal and vertical distributions $N_0, N_1, ..., N_{k-1},\ V_k$.

We can easily prove the existence of the metrical N-linear connections in the adapted basis. To this aim we represent a linear connection D in the adapted basis in the following form:

$$\begin{gathered}
D_{\delta_c}\delta_b = \overset{0}{\underset{(00)}{L}}{}^a{}_{bc}\delta_a + \sum_{\beta=1}^{k}\overset{\beta}{\underset{(00)}{L}}{}^a{}_{bc}\delta_{\beta a},\\
D_{\delta_c}\delta_{\gamma b} = \overset{0}{\underset{(\gamma 0)}{L}}{}^a{}_{bc}\delta_a + \sum_{\beta=1}^{k}\overset{\beta}{\underset{(\gamma 0)}{L}}{}^a{}_{bc}\delta_{\beta a}, \quad \left(\gamma = 1, ..., k; \delta_{ka} = \dot{\partial}_{ka}\right),\\
D_{\delta_{1c}}\delta_b = \overset{0}{\underset{(01)}{C}}{}^a{}_{bc}\delta_a + \sum_{\beta=1}^{k}\overset{\beta}{\underset{(01)}{C}}{}^a{}_{bc}\delta_{\beta a},\\
D_{\delta_{1c}}\delta_{\gamma b} = \overset{0}{\underset{(\gamma 1)}{C}}{}^a{}_{bc}\delta_a + \sum_{\beta=1}^{k}\overset{\beta}{\underset{(\gamma 1)}{C}}{}^a{}_{bc}\delta_{\beta a}, \quad \left(\gamma = 1, ..., k; \delta_{ka} = \dot{\partial}_{ka}\right),\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
D_{\delta_{kc}}\delta_b = \overset{0}{\underset{(0k)}{C}}{}^a{}_{bc}\delta_a + \sum_{\beta=1}^{k}\overset{\beta}{\underset{(0k)}{C}}{}^a{}_{bc}\delta_{\beta a},\\
D_{\delta_{kc}}\delta_{\gamma b} = \overset{0}{\underset{(\gamma k)}{C}}{}^a{}_{bc}\delta_a + \sum_{\beta=1}^{k}\overset{\beta}{\underset{(\gamma k)}{C}}{}^a{}_{bc}\delta_{\beta a}, \quad \left(\gamma = 1, ..., k; \delta_{ka} = \dot{\partial}_{ka}\right).
\end{gathered} \tag{2.6}$$

The system of functions

$$\left(\overset{\alpha}{\underset{(00)}{L}}{}^a{}_{bc}, \overset{\alpha}{\underset{(\beta 0)}{L}}{}^a{}_{bc}, \overset{\alpha}{\underset{(01)}{C}}{}^a{}_{bc}, \overset{\alpha}{\underset{(\beta 1)}{C}}{}^a{}_{bc}, ..., \overset{\alpha}{\underset{(0k)}{C}}{}^a{}_{bc}, \overset{\alpha}{\underset{(\beta k)}{C}}{}^a{}_{bc}\right), \quad (\alpha = 0, 1, ..., k; \quad \beta = 1, ..., k),$$

are the coefficients of D and

$$\left(\overset{0}{\underset{(00)}{L}}{}^a{}_{bc}, \overset{\beta}{\underset{(\beta 0)}{L}}{}^a{}_{bc}, \overset{0}{\underset{(01)}{C}}{}^a{}_{bc}, \overset{\beta}{\underset{(\beta 1)}{C}}{}^a{}_{bc}, ..., \overset{0}{\underset{(0k)}{C}}{}^a{}_{bc}, \overset{\beta}{\underset{(\beta k)}{C}}{}^a{}_{bc}\right), \quad (\beta = 1, ..., k),$$

are the coefficients of an N-linear connection $D\Gamma(N)$ on T^kM.

Also, we will denote the coefficients of $D\Gamma(N)$ with

$$\left(\underset{(00)}{\overset{H}{L}}{}^{a}{}_{bc}, \underset{(\beta 0)}{\overset{V_\beta}{L}}{}^{a}{}_{bc}, \underset{(01)}{\overset{H}{C}}{}^{a}{}_{bc}, \underset{(\beta 1)}{\overset{V_\beta}{C}}{}^{a}{}_{bc}, ..., \underset{(0k)}{\overset{H}{C}}{}^{a}{}_{bc}, \underset{(\beta k)}{\overset{V_\beta}{C}}{}^{a}{}_{bc}\right), \quad (\beta = 1, ..., k).$$

It is not difficult to prove

Theorem 2.2. *There exist metrical N-linear connection $D\Gamma(N)$ on T^kM with respect to the homogeneous prolongation $\overset{\circ}{\mathbf{G}}$, which depend only of the fundamental function $F\left(x, y^{(1)}\right)$ of the Finsler space F^n. One of them has the* **"horizontal"** *coefficients:*

$$\begin{aligned} \underset{(00)}{\overset{H}{L}}{}^{a}{}_{bc} &= \frac{1}{2} g^{ad}\left(\delta_b g_{dc} + \delta_c g_{bd} - \delta_d g_{bc}\right), \\ \underset{(\beta 0)}{\overset{V_\beta}{L}}{}^{a}{}_{bc} &= \frac{1}{2} \underset{(\beta)}{g}{}^{ad}\left(\delta_b \underset{(\beta)}{g}{}_{dc} + \delta_c \underset{(\beta)}{g}{}_{bd} - \delta_d \underset{(\beta)}{g}{}_{bc}\right), \quad (\beta = 1, ..., k), \end{aligned} \tag{2.7}$$

the **"v$_1$-vertical"** *coefficients:*

$$\begin{aligned} \underset{(\beta 1)}{\overset{V_\beta}{C}}{}^{a}{}_{bc} &= \frac{1}{2} g^{ad}\left(\delta_{1b} g_{dc} + \delta_{1c} g_{bd} - \delta_{1d} g_{bc}\right), \\ \underset{(\beta 1)}{\overset{V_\beta}{C}}{}^{a}{}_{bc} &= \frac{1}{2} \underset{(\beta)}{g}{}^{ad}\left(\delta_{1b} \underset{(\beta)}{g}{}_{dc} + \delta_{1c} \underset{(\beta)}{g}{}_{bd} - \delta_{1d} \underset{(\beta)}{g}{}_{bc}\right), \quad (\beta = 1, ..., k), \end{aligned} \tag{3.2}$$

and the **"v$_\gamma$-vertical"** *coefficients vanish:*

$$\underset{(0\gamma)}{\overset{H}{C}}{}^{a}{}_{bc} = \underset{(1\gamma)}{\overset{V_1}{C}}{}^{a}{}_{bc} = ... = \underset{(k\gamma)}{\overset{V_k}{C}}{}^{a}{}_{bc} = 0, \quad (\gamma = 2, ..., k). \tag{2.9}$$

Let us remark the particular form of the metrical N-linear connection $D\Gamma(N)$ in (2.7), (2.8) and (2.9). Because it depends only on the fundamental function $F\left(x, y^{(1)}\right)$ *of the Finsler space* F^n, $D\Gamma(N)$ from the Theorem 2.2 will be called the **canonical** metrical N-linear connection of the space $\left(\widetilde{T^kM}, \overset{\circ}{\mathbf{G}}\right)$.

Let us denote

$$\sigma_c = -\frac{1}{2}\frac{1}{F^2}\delta_c F^2, \; \tau_c = -\frac{1}{2}\frac{1}{F^2}\delta_{1c} F^2. \tag{2.10}$$

We obtain

Theorem 2.3. *The coeffients of the metrical N-linear connection $D\Gamma(N)$ with respect to $\overset{\circ}{\mathbf{G}}$ given by* (2.1)*, satisfy the following equations:*

$$\begin{aligned} \underset{(\beta 0)}{\overset{V_\beta}{C}}{}^{a}{}_{bc} &= \underset{(00)}{\overset{H}{L}}{}^{a}{}_{bc} + \beta\left(\delta^a_b \sigma_c + \delta^a_c \sigma_b - g_{bc} g^{ad} \sigma_\alpha\right), \\ \underset{(\beta 1)}{\overset{V_\beta}{C}}{}^{a}{}_{bc} &= \underset{(01)}{\overset{V_\beta}{C}}{}^{a}{}_{bc} + \beta\left(\delta^a_b \sigma_c + \delta^a_c \sigma_b - g_{bc} g^{ad} \sigma_\alpha\right), \quad (\beta = 1, ..., k). \end{aligned} \tag{2.11}$$

Indeed, substituting the tensors $\underset{(1)}{g}_{ab}, \underset{(2)}{g}_{ab}, ..., \underset{(k)}{g}_{ab}$ given by (2.5) in (2.7) and (2.8) and using (2.10), one obtains (2.11).

It is not difficult to prove

Theorem 2.4. *The coefficients of the canonical metrical N-linear connection* $D\Gamma(N) = \left(\underset{(00)}{\overset{H}{L}}{}^a{}_{bc}, \underset{(\beta 0)}{\overset{V_\beta}{L}}{}^a{}_{bc}, \underset{(01)}{\overset{H}{C}}{}^a{}_{bc}, \underset{(\beta 1)}{\overset{V_\beta}{C}}{}^a{}_{bc}, ..., \underset{(0k)}{\overset{H}{C}}{}^a{}_{bc}, \underset{(\beta k)}{\overset{V_\beta}{C}}{}^a{}_{bc} \right)$, $(\beta = 1, ..., k)$ *satisfy the equations*

$$
\begin{aligned}
&\underset{(10)}{\overset{V_1}{L}}{}^a{}_{bc} = \underset{(00)}{\overset{H}{L}}{}^a{}_{bc} + \delta^a_b \sigma_c + \delta^a_c \sigma_b - g_{bc} g^{ad} \sigma_\alpha, \\
&\underset{(20)}{\overset{V_2}{L}}{}^a{}_{bc} = \underset{(10)}{\overset{V_2}{L}}{}^a{}_{bc} + \delta^a_b \sigma_c + \delta^a_c \sigma_b - g_{bc} g^{ad} \sigma_\alpha, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\underset{(k0)}{\overset{V_k}{L}}{}^a{}_{bc} = \underset{(k-10)}{\overset{V_{k-1}}{L}}{}^a{}_{bc} + \delta^a_b \sigma_c + \delta^a_c \sigma_b - g_{bc} g^{ad} \sigma_\alpha, \\
&\underset{(11)}{\overset{V_1}{C}}{}^a{}_{bc} = \underset{(01)}{\overset{H}{C}}{}^a{}_{bc} + \delta^a_b \sigma_c + \delta^a_c \sigma_b - g_{bc} g^{ad} \sigma_\alpha, \\
&\underset{(21)}{\overset{V_2}{C}}{}^a{}_{bc} = \underset{(11)}{\overset{V_1}{C}}{}^a{}_{bc} + \delta^a_b \sigma_c + \delta^a_c \sigma_b - g_{bc} g^{ad} \sigma_\alpha, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\underset{(k1)}{\overset{V_k}{C}}{}^a{}_{bc} = \underset{(k-11)}{\overset{V_{k-1}}{C}}{}^a{}_{bc} + \delta^a_b \sigma_c + \delta^a_c \sigma_b - g_{bc} g^{ad} \sigma_\alpha, \\
&\underset{(0\gamma)}{\overset{H}{C}}{}^a{}_{bc} = \underset{(1\gamma)}{\overset{V_1}{C}}{}^a{}_{bc} = ... = \underset{(k\gamma)}{\overset{V_k}{C}}{}^a{}_{bc} = 0, \quad (\gamma = 2, ..., k).
\end{aligned}
\tag{2.12}
$$

The particular form (2.12) of the canonical metrical N-linear connection shows that the curvature of the v_k-connection $\left(\underset{(k0)}{\overset{V_k}{L}}{}^a{}_{bc}, \underset{(k1)}{\overset{V_k}{C}}{}^a{}_{bc}, ..., \underset{(kk)}{\overset{V_k}{C}}{}^a{}_{bc} \right)$ lead to the Weyl's conformal curvature tensor with respect to the curvature of the v_{k-1}-connection $\left(\underset{(k-10)}{\overset{V_{k-1}}{L}}{}^a{}_{bc}, \underset{(k-11)}{\overset{Vk_{-1}}{C}}{}^a{}_{bc}, ..., \underset{(k-1k)}{\overset{Vk-1}{C}}{}^a{}_{bc} \right)$, ...,and the curvature of the v_1-connection $\left(\underset{(10)}{\overset{V_1}{L}}{}^a{}_{bc}, \underset{(11)}{\overset{V_1}{C}}{}^a{}_{bc}, ..., \underset{(1k)}{\overset{V_2}{C}}{}^a{}_{bc} \right)$ lead to the Weyl's conformal curvature of the h-connection $\left(\underset{(00)}{\overset{H}{L}}{}^a{}_{bc}, \underset{(01)}{\overset{H}{C}}{}^a{}_{bc}, ..., \underset{(0k)}{\overset{H}{C}}{}^a{}_{bc} \right)$.

This property shows the necessity to construct a gauge theory in the Asanov sense, [1], for the Riemannian metric given on $\widetilde{T^kM}$ by the prolongation $\overset{\circ}{\mathbf{G}}$, from (4.1).

Now, we remark that the almost $(k-1)\,n$-contact structure $\mathbf{F}$ defined in (1.14) has not the property of homogeneity. The $\mathcal{F}\left(\widetilde{T^kM}\right)$-linear mapping

$\mathbf{F} : \chi\left(\widetilde{T^kM}\right) \to \chi\left(\widetilde{T^kM}\right)$, applies the 1-homogeneous vector field δ_a into the $(1-k)$-homogeneous vector field $\delta_{ka} = \dot{\partial}_{ka}, (a = 1, ..., n)$.

Therefore, we consider the $\mathcal{F}\left(\widetilde{T^kM}\right)$-linear mapping $\overset{\circ}{\mathbf{F}} : \chi\left(\widetilde{T^kM}\right) \to \chi\left(\widetilde{T^kM}\right)$, given in the adapted basis by

$$\begin{aligned} \overset{\circ}{\mathbf{F}}(\delta_a) &= -\frac{\left\|y^{(1)}\right\|^k}{\mathbf{a}^k}\dot{\partial}_{ka}, \\ \overset{\circ}{\mathbf{F}}(\delta_{1a}) = ... = \overset{\circ}{\mathbf{F}}(\delta_{k-1a}) &= 0, \\ \overset{\circ}{\mathbf{F}}\left(\dot{\partial}_{ka}\right) &= \frac{\mathbf{a}^k}{\left\|y^{(1)}\right\|^k}\delta_a. \end{aligned} \tag{2.13}$$

By direct calculus, we can prove:

Theorem 2.5. $\overset{\circ}{\mathbf{F}}$ *has the following properties:*

1. $\overset{\circ}{\mathbf{F}}$ *is a tensor field of type* (1.1) *on* $\left(\widetilde{T^kM}\right)$.
2. $\overset{\circ}{\mathbf{F}}$ *is an almost* $(k-1)\,n$*-contact structure on* $\widetilde{T^kM} : \mathbf{F}^3 + \mathbf{F} = 0$.
3. $\overset{\circ}{\mathbf{F}}$ *depends only the fundamental function* $F\left(x, y^{(1)}\right)$ *of the Finsler space* F^n.
4. $\overset{\circ}{\mathbf{F}}$ *is homogeneous on the fibres on* $\widetilde{T^kM}$.
5. *The pair* $\left(\overset{\circ}{\mathbf{G}}, \overset{\circ}{\mathbf{F}}\right)$ *is a metrical* $(k-1)\,n$*-contact structure on* $\widetilde{T^kM}$:

$$\overset{\circ}{\mathbf{G}}\left(\overset{\circ}{\mathbf{F}}X, Y\right) = -\overset{\circ}{\mathbf{G}}\left(X, \overset{\circ}{\mathbf{F}}Y\right), \quad \forall X, Y \in \chi\left(\widetilde{T^kM}\right).$$

The space $\left(\widetilde{T^kM}, \overset{\circ}{\mathbf{G}}, \overset{\circ}{\mathbf{F}}\right)$ is **the geometrical model** of the Finsler space $F^n = (M, F)$, with respect to the homogeneous lift $\overset{\circ}{\mathbf{G}}$ given by (2.1). It can be used for studying the Finslerian higher order gauge theory and, in general, the geometry of the Finsler space $F^n = (M, F)$.

References

[1] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theory,* D. Reidel Publ. Co., Dordrecht, 1985.

[2] Gh. Atanasiu, *New aspects in the Differential Geometry of the second order,* Sem. de Mecanica, Univ. de Vest din Timisoara, Nr. 82, 2001, 1–81.

[3] Gh. Atanasiu, *The Homogeneous Prolongation to the Second Order Tangent Bundle T^2M of a Riemannian Metric,* Libertas Mathematica, Texas, Vol.XXI4, 2004, 11–20.

[4] Gh. Atanasiu, *Linear Connections in the Differential Geometry of Order two,* in vol. *Lagrange and Hamilton Geometries and Their Applications* (R. Miron-Ed.) Handbooks. Treatises. Monographs. Fair Partners Publ., Bucureşti, 49, 2004, 11–30.

[5] Gh. Atanasiu, *Linear Connections in the Higher-Order Differential Geometry,* Proc. of Int. Conf. "Physical Interpretations of Relativity Theory", 04–07. July 2005, Dep of Physics, Bauman Moscow State Tech. Univ.

[6] Gh. Atanasiu, *The Homogeneous Prolongation to the Second Order Tangent Bundle T^2M of a Finsler Metric,* The $7^{\underline{th}}$ Int. Workshop on Diff. Geom. and Top. and its Appl., Inst. de Mat. Acad. Română, 4–11 sept., 2005

[7] I. Čomić, *The Curvature Theory of Generalized Connection in Osc^2M,* Balkan J. Geom. and Its Appl., Vol. 1, No. 1, 1996, 21–29.

[8] I. Čomić, Gh. Atanasiu, E. Stoica, *The Generalized Connection in Osc^3M,* Annales Univ. Sci. Budapest, 41, 1998, 39–54.

[9] I. Čomić, H. Kawaguchi, *Different structures in the geometries of Osc^3M introduced by R. Miron and Gh. Atanasiu,* The 37-th Symposium on Finsler Geometry at Tsukuba; The 6-th Int. Conf. of Tensor Society on Diff. Geom & its Appl., 5–9 Aug., 2002, 73–81.

[10] R. Miron, *The Geometry of Higher Order Lagrange Spaces.* Applicatios to Mechanics and Physics, Kluwer Acad. Publ., FTPH, No. 82, 1997.

[11] R. Miron and Gh. Atanasiu, *Compendium on the Higher-Order Lagrange Spaces: The Geometry of k-osculator Bundles. Prolongation of the Riemannian, Finslerian and Lagrangian Structures. Lagrange Spaces.* Tensor N. S. 53, 1993, 39–57.

[12] R. Miron et Gh. Atanasiu, *Compendium sur les espaces Lagrange d'ordre supérieur: La géométrie du fibré k-osculateur; Le prolongement des structures Riemanniennes, Finsleriennes et Lagrangiennes; Les espaces Lagrange $L^{(k)n}$,* Univ. Timisoara, Seminarul de Mecanica, no. 40, 1994.

[13] R. Miron and Gh. Atanasiu, *Lagrange Geometry of Second Order,* Math. Comput. Modelling vol. 20, no. 4, 1994, 41–56.

[14] R. Miron and Gh. Atanasiu, *Differential Geometry of the k-osculator Bundle,* Rev. Roumaine Math. Pures et Appl., T. 41, No. 3–4, 1996, 205–236.

[15] R. Miron and Gh. Atanasiu, *Prolongation of the Riemannian, Finslerian and Lagrangian Structures,* Rev. Roumaine Math. Pures et Appl., T. 41, No. 3–4, 1996, 237–249.

[16] R. Miron and Gh. Atanasiu, *Higher-order Lagrange Spaces,* Rev. Roumaine Math. Pures et Appl., T. 41, No. 3–4, 1996, 251–263.

[17] R. Miron and Gh. Atanasiu, *Geometrical Theory of Gravitational and Electromagnetic Fields in the Higher-order Lagrange Spaces,* Tsukuba J. Math., Vol. 20, No. 1, 1996, 137–149.

[18] A. Morimoto, *Prolongations of Geometric Stuctures,* Nagoya Math.Jour., 32, 1968, 67–108.

[19] K. Yano, S. Ishihara, *Tangent and Cotangent Bundles. Differential Geometry,* M. Dekker, Inc., New-York, 1973.

The Berwald-Moor Metric in the Tangent Bundle of the Second Order

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As an application of the results of the first author obtained in the papers [1] and [2], the geometry of the second order tangent bundle T^2M (or second order jet bundle J_0^2M) endowed with two special types of metrics compatible with the 2-contact structures is studied. The particularity of these two models is that the horizontal and the $v^{(1)}$-part of the metric are both given by the same Riemannian metric (respectively, its horizontal part is Riemannian), while its $v^{(2)}$-part is given by the flag-Finsler Berwald-Moor metric (respectively, $v^{(1)}$ and $v^{(2)}$-parts are given by the flag-Finsler Berwald-Moor metric [5]).

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1 The 2-Tangent Bundle T^2M

Let M be a real 4-dimensional manifold of class $\mathcal{C}^\infty$, (T^2M, π^2, M) its second order tangent bundle, [1], and let $\widetilde{T^2M}$ be the space T^2M without its null section. For a point $u \in T^2M$, let $(x^i, y^{(1)i}, y^{(2)i})$ be its coordinates in a local chart.

Let N be a nonlinear connection, [3], [8]-[13], and denote its coefficients by $\left(\underset{1}{N}{}^i_j, \underset{2}{N}{}^i_j\right)$, i, $j = 1, ..., 4$. Then, N determines the direct decomposition

$$T_uT^2M = N_0(u) \oplus N_1(u) \oplus V_2(u), \ \forall u \in T^2M. \tag{1}$$

The adapted basis to (1) is $(\delta_i, \delta_{1i}, \delta_{2i})$ and its dual basis is $(dx^i, \delta y^{(1)i}, \delta y^{(2)i})$, where

$$\begin{cases} \delta_i = \dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - \underset{1}{N}{}^k_i \dfrac{\partial}{\partial y^{(1)k}} - \underset{2}{N}{}^k_i \dfrac{\partial}{\partial y^{(2)k}} \\ \delta_{1i} = \dfrac{\delta}{\delta y^{(1)i}} = \dfrac{\partial}{\partial y^{(1)i}} - \underset{1}{N}{}^k_i \dfrac{\partial}{\partial y^{(2)k}} \\ \delta_{2i} = \dfrac{\partial}{\partial y^{(2)i}} = \dot{\partial}_{2i}, \end{cases} \tag{2}$$

respectively,

$$\begin{cases} \delta y^{(1)i} = dy^{(1)i} + \underset{1}{M}{}^i_k dx^k \\ \delta y^{(2)i} = dy^{(2)i} + \underset{1}{M}{}^i_k dy^{(1)k} + \underset{2}{M}{}^i_k dx^k, \end{cases} \tag{3}$$

where $\underset{1}{M}{}^i_k, \underset{2}{M}{}^i_k$ are the dual coefficients of the nonlinear connection N.

Then, a vector field $X \in \mathcal{X}\left(T^2M\right)$ is represented in the local adapted basis as

$$X = X^{(0)i}\delta_i + X^{(1)i}\delta_{1i} + X^{(2)i}\delta_{2i}, \tag{4}$$

with the three right terms (called *d-vector fields*) belonging to the distributions N, N_1 and V_2 respectively.

A 1-form $\omega \in \mathcal{X}^*\left(T^2M\right)$ will be decomposed as

$$\omega = \omega_i^{(0)}dx^i + \omega_i^{(1)}\delta y^{(1)i} + \omega_i^{(2)}\delta y^{(2)i}.$$

Similarly, a tensor field $T \in \mathcal{T}_s^r\left(T^2M\right)$ can be split with respect to (1) into components, which will be called *d-tensor fields.*

2 N-linear connections. d-tensors of curvature

An *N-linear connection* D, [1], [2], is a linear connection on T^2M, which preserves by parallelism the distributions N, N_1 and V_2.

An N-linear connection is locally given by its coefficients

$$D\Gamma(N) = \left(\underset{(00)}{L}{}^i{}_{jk}, \underset{(10)}{L}{}^i{}_{jk}, \underset{(20)}{L}{}^i{}_{jk}, \underset{(01)}{C}{}^i{}_{jk}, \underset{(11)}{C}{}^i{}_{jk}, \underset{(21)}{C}{}^i{}_{jk}, \underset{(02)}{C}{}^i{}_{jk}, \underset{(12)}{C}{}^i{}_{jk}, \underset{(22)}{C}{}^i{}_{jk}\right), \tag{5}$$

where

$$\begin{cases} D_{\delta_k}\delta_j = \underset{(00)}{L}{}^i{}_{jk}\delta_i, \ D_{\delta_k}\delta_{1j} = \underset{(10)}{L}{}^i{}_{jk}\delta_{1i}, \ D_{\delta_k}\delta_{2j} = \underset{(20)}{L}{}^i{}_{jk}\delta_{2i} \\ D_{\delta_{1k}}\delta_j = \underset{(01)}{C}{}^i{}_{jk}\delta_i, \ D_{\delta_{1k}}\delta_{1j} = \underset{(11)}{C}{}^i{}_{jk}\delta_{1i}, \ D_{\delta_{1k}}\delta_{2j} = \underset{(21)}{C}{}^i{}_{jk}\delta_{2i} \\ D_{\delta_{2k}}\delta_j = \underset{(02)}{C}{}^i{}_{jk}\delta_i, \ D_{\delta_{2k}}\delta_{1j} = \underset{(12)}{C}{}^i{}_{jk}\delta_{1i}, \ D_{\delta_{2k}}\delta_{2j} = \underset{(22)}{C}{}^i{}_{jk}\delta_{2i} \end{cases}. \tag{6}$$

The curvature of the N-linear connection D,

$$R(X,Y)Z = D_XD_YZ - D_YD_XZ - D_{[X,Y]}Z,$$

is completely determined by its components (which are d-tensors) $R\left(\delta_{\gamma l}, \delta_{\beta k}\right)\delta_{\alpha j}$. Namely, the 2-forms of curvature of an N- linear connection are, [1], [2],

$$\begin{aligned} \underset{(\alpha)}{\Omega}{}^i{}_j = {} & \frac{1}{2}\underset{(0\alpha)}{R}{}_j{}^i{}_{kl}dx^k \wedge dx^l + \underset{(1\alpha)}{P}{}_j{}^i{}_{kl}dx^k \wedge \delta y^{(1)l} + \\ & + \underset{(2\alpha)}{P}{}_j{}^i{}_{kl}dx^k \wedge \delta y^{(2)l} + \frac{1}{2}\underset{(1\alpha)}{S}{}_j{}^i{}_{kl}\delta y^{(1)k} \wedge \delta y^{(1)l} + \\ & + \underset{(2\alpha)}{Q}{}_j{}^i{}_{kl}\delta y^{(1)k} \wedge \delta y^{(2)l} + \frac{1}{2}\underset{(2\alpha)}{S}{}_j{}^i{}_{kl}\delta y^{(2)k} \wedge \delta y^{(2)l}, \end{aligned} \tag{7}$$

$\alpha = 0, 1, 2$, where the coefficients $\underset{(0\alpha)}{R}{}_j{}^i{}_{kl}, \underset{(\beta\alpha)}{P}{}_j{}^i{}_{kl}, \underset{(2\alpha)}{Q}{}_j{}^i{}_{kl}, \underset{(\beta\alpha)}{S}{}_j{}^i{}_{kl}$ are d-tensors, named the *d-tensors of curvature* of the N-linear connection D.

3 Metric structures on T^2M

A *Riemannian metric* on T^2M is a tensor field G of type $(0,2)$, which is non-degenerate in each $u \in T^2M$ and positively defined on T^2M.

In this paper, we shall consider only metrics in the form

$$G = \underset{(0)}{g}_{ij} dx^i \otimes dx^j + \underset{(1)}{g}_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j} + \underset{(2)}{g}_{ij} \delta y^{(2)i} \otimes \delta y^{(2)j}, \tag{8}$$

where $\underset{(\alpha)}{g}_{ij} = \underset{(\alpha)}{g}_{ij}(x, y^{(1)}, y^{(2)})$; this is, such that the distributions N, N_1 and V_2 generated by the nonlinear connection N be orthogonal in pairs with respect to G.

Let also

$$F = \sqrt[4]{y^{(1)1} y^{(1)2} y^{(1)3} y^{(1)4}}$$

be the Berwald-Moor Finsler function, [14]–[14], and the generalized Lagrange metrics on M, given by

$$h_{ij} = \frac{1}{12F^4} \frac{\partial^2 F^4}{\partial y^i \partial y^j}, \quad \widetilde{h}_{ij} = \frac{1}{12F^6} \frac{\partial^2 F^4}{\partial y^i \partial y^j}. \tag{9}$$

(h defined above is the same as the one in [5], with the only difference that here we have divided by F^4 or F^6 instead of F^2, in order that the obtained tensors be homogeneous of degree 2, respectively, 4).

In the following, we shall use two particular kinds of metrics on $\widetilde{T^2M}$, namely:

1. $\underset{(0)}{g}_{ij} = \underset{(1)}{g}_{ij} = g_{ij}(x)$, $\underset{(2)}{g}_{ij} = \widetilde{h}_{ij}(y^{(1)})$,
2. $\underset{(0)}{g}_{ij} = g_{ij}(x)$, $\underset{(1)}{g}_{ij} = \underset{(2)}{g}_{ij} = h_{ij}(y^{(1)})$,

$g_{ij}(x)$ being a Riemannian metric on M, and h_{ij}, $\widetilde{h}_{ij}$ as above.

These two examples have an important property, namely, they are compatible to the almost contact structures $\mathbb{F}$ introduced in [1].

An N-linear connection D is called *metrical* if $D_X G = 0$, $\forall X \in \mathcal{X}(T^2M)$. The local expression of this equality is given in [1].

4 The Ricci tensor $Ric\,(D)$

If we consider the Ricci tensor $Ric\,(D)$, as the trace of the linear operator

$$V \mapsto R\,(V, X)\,Y,\ \forall V = V^{(0)i}\delta_i + V^{(1)i}\delta_{1i} + V^{(2)i}\delta_{2i} \in \mathcal{X}\left(T^2M\right), \tag{10}$$

then, [3], the Ricci tensor $Ric\,(D)$ has the following components:

$$Ric\,(D)\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = \underset{(00)}{R}{}_{i\ jl}^{\ l} =: R_{ij};$$

$$
\begin{aligned}
Ric\,(D)\left(\frac{\delta}{\delta y^{(1)j}},\frac{\delta}{\delta x^i}\right) &= -\underset{(10)}{P}{}_{i\ lj}^{\ l} =: -\overset{2}{\underset{(10)}{P}}{}_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(2)j}},\frac{\delta}{\delta x^i}\right) &= -\underset{(20)}{P}{}_{i\ lj}^{\ l} =: -\overset{2}{\underset{(20)}{P}}{}_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta y^{(1)i}}\right) &= \underset{(11)}{P}{}_{i\ jl}^{\ l} =: \overset{1}{\underset{(11)}{P}}{}_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(1)j}},\frac{\delta}{\delta y^{(1)i}}\right) &= \underset{(11)}{S}{}_{i\ jl}^{\ l} =: \underset{(1)}{S}{}_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(2)j}},\frac{\delta}{\delta y^{(1)i}}\right) &= -\underset{(21)}{Q}{}_{i\ lj}^{\ l} =: -\overset{2}{\underset{(21)}{Q}}{}_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta y^{(2)i}}\right) &= \underset{(22)}{P}{}_{i\ jl}^{\ l} =: \overset{1}{\underset{(22)}{P}}{}_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(1)j}},\frac{\delta}{\delta y^{(2)i}}\right) &= \underset{(22)}{Q}{}_{i\ jl}^{\ l} =: \overset{1}{\underset{(22)}{Q}}{}_{ij};\\
Ric\,(D)\left(\frac{\delta}{\delta y^{(2)j}},\frac{\delta}{\delta y^{(2)i}}\right) &= \underset{(22)}{S}{}_{i\ jl}^{\ l} =: \underset{(2)}{S}{}_{ij}.
\end{aligned}
$$

5 Canonical structures

Let $(M,\ g)$ be a Riemannian manifold and T^2M, its second order tangent bundle. **The canonical nonlinear connection** N is defined (cf. with R. Miron and Gh. Atanasiu, [13]) by its dual coefficients

$$
\underset{(1)}{M}{}^i_{\ j} = \gamma^i_{jk}y^{(1)k}, \quad \underset{(2)}{M}{}^i_{\ j} = \frac{1}{2}\left\{\mathbb{C}\left(\gamma^i_{jk}y^{(1)k}\right) + \underset{(1)}{M}{}^i_{\ k}\underset{(1)}{M}{}^k_{\ j}\right\}, \tag{11}
$$

$\gamma^i_{jk} = \gamma^i_{jk}(x)$ being the Christoffel symbols of g and $\mathbb{C} = y^{(1)i}\dfrac{\partial}{\partial x^i} + 2y^{(2)i}\dfrac{\partial}{\partial y^{(1)i}}$.

Let

$$
\underset{(1)}{N}{}^i_{\ j} = \underset{(1)}{M}{}^i_{\ j},\ \underset{(2)}{N}{}^i_{\ j} = \underset{(2)}{M}{}^i_{\ j} + \underset{(1)}{M}{}^i_{\ k}\underset{(1)}{M}{}^k_{\ j}
$$

be its (direct) coefficients. Then, the coefficients of the Lie brackets, [1],

$$
\begin{gathered}
[\delta_{0j},\delta_{0k}] = \underset{(01)}{R}{}^{\ i}_{jk}\delta_{1i} + \underset{(02)}{R}{}^{\ i}_{jk}\delta_{2i}, \quad [\delta_{0j},\delta_{1k}] = \underset{(11)}{B}{}^{\ i}_{jk}\delta_{1i} + \underset{(12)}{B}{}^{\ i}_{jk}\delta_{2i}\\
[\delta_{0j},\delta_{2k}] = \underset{(21)}{B}{}^{\ i}_{jk}\delta_{1i} + \underset{(22)}{B}{}^{\ i}_{jk}\delta_{2i}, \quad [\delta_{1j},\delta_{1k}] = \underset{(12)}{R}{}^{\ i}_{jk}\delta_{2i}\\
[\delta_{1j},\delta_{2k}] = \underset{(21)}{B}{}^{\ i}_{jk}\delta_{2i}, \quad [\delta_{2j},\delta_{2k}] = 0
\end{gathered} \tag{12}
$$

have the property that

$$
\underset{(11)}{B}{}^{\ i}_{jk} = \underset{(22)}{B}{}^{\ i}_{jk} = \gamma^i_{jk},\ \underset{(21)}{B}{}^{\ i}_{jk} = \underset{(12)}{R}{}^{\ i}_{jk} = \underset{(22)}{R}{}^{\ i}_{jk} = 0. \tag{13}
$$

In this paper, we shall use the metrical N-linear connection introduced by the first author, [1], given by the coefficients:

$$\begin{aligned}
\underset{(00)}{L}{}^{i}{}_{jk} &= \frac{1}{2}\underset{(0)}{g}{}^{il}(\delta_k \underset{(0)}{g}{}_{jl} + \delta_j \underset{(0)}{g}{}_{lk} - \delta_l \underset{(0)}{g}{}_{jk}) \\
\underset{(\beta 0)}{L}{}^{i}{}_{jk} &= \underset{(\beta\beta)}{B}{}^{i}{}_{kj} + \frac{1}{2}\underset{(\beta)}{g}{}^{il}(\delta_k \underset{(\beta)}{g}{}_{jl} - \underset{(\beta\beta)}{B}{}^{m}{}_{kj}\underset{(\beta)}{g}{}_{ml} - \underset{(\beta\beta)}{B}{}^{m}{}_{kl}\underset{(\beta)}{g}{}_{jm}) \\
\underset{(\delta 1)}{C}{}^{i}{}_{jk} &= \frac{1}{2}\underset{(\delta)}{g}{}^{il}\delta_{1k}\underset{(\delta)}{g}{}_{jl},\ (\delta = 0,2), \\
\underset{(\varepsilon 2)}{C}{}^{i}{}_{jk} &= \frac{1}{2}\underset{(\varepsilon)}{g}{}^{il}\dot{\partial}_{2k}\underset{(\varepsilon)}{g}{}_{jl},\ (\varepsilon = 0,1), \\
\underset{(\beta\beta)}{C}{}^{i}{}_{jk} &= \frac{1}{2}\underset{(\beta)}{g}{}^{il}(\delta_{\beta k}\underset{(\beta)}{g}{}_{jl} + \delta_{\beta j}\underset{(\beta)}{g}{}_{lk} - \delta_{\beta l}\underset{(\beta)}{g}{}_{jk}),\ \delta_{2i} = \partial_{2i},
\end{aligned} \tag{14}$$

where $\beta = 1, 2$.

Then, we have to remark that, taking into account the relations (13), two of the coefficients of the torsion tensor vanish, namely

$$\underset{(21)}{P}{}^{i}{}_{jk} = \underset{(12)}{S}{}^{i}{}_{jk} = 0, \tag{15}$$

where $\underset{(21)}{P}{}^{i}{}_{jk}\delta_{1i} = v_1 T(\delta_{2k}, \delta_j)$, $\underset{(12)}{S}{}^{i}{}_{jk}\dot{\partial}_{2i} = v_2 T(\delta_{1k}, \delta_{1j})$.

6 The case of the $g-h-h-$metric

Let the metric structure of $\widetilde{T^2M}$ be given by

$$G = g_{ij}(x)dx^i \otimes dx^j + h_{ij}(y^{(1)})\,\delta y^{(1)i} \otimes \delta y^{(1)j} + h_{ij}(y^{(1)})\delta y^{(2)i} \otimes \delta y^{(2)j},$$

where g is a Riemannian metric on M and h is as in (9). Then, G is h-Riemannian and v_1-, v_2-locally Minkovski. In this case, the detailed expressions of the coefficients $D\Gamma(N)$ of the canonical N-linear connection and of its curvatures and torsions are given in [1].

By applying the results in the cited paper and the relation (15), we obtain by a direct computation the following result:

Proposition 1. *The only nonvanishing components of the Ricci tensor $Ric(D)$ of the canonical-linear conenction are*

$$\begin{aligned}
Ric\,(D)\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) &= \underset{(00)}{R}{}_{i}{}^{k}{}_{jk} =: r_{ij}; \\
Ric\,(D)\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta y^{(1)i}}\right) &= \underset{(11)}{P}{}_{i}{}^{k}{}_{jk} =: \underset{(11)}{\overset{1}{P}}{}_{ij}; \\
Ric\,(D)\left(\frac{\delta}{\delta y^{(1)j}}, \frac{\delta}{\delta y^{(1)i}}\right) &= \underset{(11)}{S}{}_{i}{}^{k}{}_{jk} =: \underset{(1)}{S}{}_{ij},
\end{aligned}$$

where $r_{ij} = r^k_{i\,jk}$ *denotes the Ricci tensor of Levi-Civita connection attached to* g.

By applying the results in [3], we can state:

Proposition 2. *The Einstein equations associated to the metrical* N*-linear connection* D *are*

$$\begin{aligned}
&R_{ij} - \tfrac{1}{2}(r + \underset{(1)}{S})g_{ij} = \kappa \underset{(00)}{\mathcal{T}}_{ij};\\
&\underset{(11)}{\overset{1}{P}}{}_{ij} = \kappa \underset{(10)}{\mathcal{T}}{}_{ij};\\
&\underset{(1)}{S}{}_{ij} - \tfrac{1}{2}(r + \underset{(1)}{S})h_{ij} = \kappa \underset{(11)}{\mathcal{T}}{}_{ij},\ \alpha = 1, 2;\\
&\underset{(20)}{\mathcal{T}}{}_{ij} = \underset{(01)}{\mathcal{T}}{}_{ij} = \underset{(21)}{\mathcal{T}}{}_{ij} = \underset{(02)}{\mathcal{T}}{}_{ij} = \underset{(12)}{\mathcal{T}}{}_{ij} = \underset{(22)}{\mathcal{T}}{}_{ij} = 0.
\end{aligned}$$

7 The case of the $g - g - \widetilde{h}$–metric

Proposition 3. *Now, let the metric structure of* $\widetilde{T^2M}$ *be given by*

$$G = g_{ij}(x)dx^i \otimes dx^j + g_{ij}(x)\,\delta y^{(1)i} \otimes \delta y^{(1)j} + \widetilde{h}_{ij}(y^{(1)})\delta y^{(2)i} \otimes \delta y^{(2)j},$$

where g *is a Riemannian metric on* M *and* $\widetilde{h}$ *is as in (9). Then,* G *is* h-, v_1-*Riemannian and* v_2-*locally Minkowski.*

In order to determine the components of the Ricci tensor, we first have to compute coefficients of the canonical N-linear connection in our case. We have:

$$\begin{aligned}
\underset{(00)}{L}{}^i_{jk} &= \gamma^i{}_{jk},\ \underset{(10)}{L}{}^i_{jk} = \underset{(10)}{L}{}^i_{jk}(x),\ \underset{(20)}{L}{}^i_{jk} = \underset{(20)}{L}{}^i_{jk}(x, y^{(1)})\\
\underset{(01)}{C}{}^i_{jk} &= \underset{(11)}{C}{}^i_{jk} = 0,\ \underset{(21)}{C}{}^i_{jk} = \frac{1}{2}\tilde{h}^{il}\delta_{1k}\tilde{h}_{jl}\\
\underset{(02)}{C}{}^i_{jk} &= \underset{(12)}{C}{}^i_{jk} = \underset{(22)}{C}{}^i_{jk} = 0.
\end{aligned}$$

Using the expressions above, we obtain

Proposition 4. *All the components of the Ricci tensor of the* N*-linear connection* D *vanish, except*

$$Ric\,(D)\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = \underset{(00)}{R}{}_i{}^l{}_{jl} =: r_{ij},$$

where r_{ij} *denotes the Ricci tensor of the Levi-Civita connection of metric* g *on* M.

As a consequence, the Einstein equations can be written in this case as:

$$r_{ij} - \frac{1}{2}rg_{ij} = \kappa \underset{(00)}{\mathcal{T}}{}_{ij},$$

the other components of the energy-momentum tensor being identically 0. The equations above are exactly the Einstein equations of the Levi-Civita connection ∇ of $g = g(x)$. Obviously, the energy conservation law is satisfied.

References

[1] Gh. Atanasiu, *New Aspects in the Differential Geometry of Second Order*, Sem. de Mecanică, Univ de Vest, Timişoara, no. 82, 2001, 1–81.

[2] Gh. Atanasiu, *Linear connections in the higher-order geometry*, Proc. of the Int. Meeting "Physical Interpretations of Relativity Theory", 4–7 July 2005, Bauman Moscow State Tech. University, Moscow.

[3] Gh. Atanasiu, N. Brinzei-Voicu, *Einstein equations in the geometry of second order*, Studia Mathematica, Cluj-Napoca, 3 (2005), 21–29.

[4] Gh. Atanasiu, N. Brinzei, *Einstein equations in the higher order differential geometry*, Proc. of the Int. Meeting "Physical Interpretations of Relativity Theory", 4–7 July 2005, Bauman Moscow State Tech. University, Moscow.

[5] V. Balan, N., Brinzei, *Einstein equations for the Berwald-Moor type Euclidean-locally Minkowski relativistic model*, to appear.

[6] R. Miron, *The Geometry of Higher Order Lagrange Spaces. Applications to Mechanics and Physics*, Kluwer Acad. Publ. FTPM no. 82, 1997.

[7] R. Miron, Gh Atanasiu, *Geometrical theory of gravitational and electromagnetic fields in Higher Order Lagrange Spaces*, Tsukuba J. of Math. 20, 1 (1996), 137–149.

[8] R. Miron, Gh. Atanasiu, *Compendium on the higher-order Lagrange spaces: The geometry of k-osculator bundles. Prolongation of the Riemannian, Finslerian and Lagrangian structures. Lagrange spaces*, Tensor N. S. 53 (1993), 39–57.

[9] R. Miron, Gh. Atanasiu, *Compendium sur les espaces Lagrange d'ordre superieur: La geometrie du fibre k-osculateur. Le prolongement des structures Riemanniennes, Finsleriennes et Lagrangiennes. Les espaces $L^{(k)n}$*, Univ. Timişoara, Seminarul de Mecanică, 40 (1994), 1–27.

[10] R. Miron, Gh. Atanasiu, *Lagrange geometry of second order*, Math. Comput Modelling, 20, 4 (1994), 41–56.

[11] R. Miron, Gh. Atanasiu, *Differential geometry of the k-osculator bundle*, Rev. Roumaine Math. Pures et Appl. 41, 3/4 (1996), 205–236.

[12] R. Miron, Gh. Atanasiu, *Higher-order Lagrange spaces*, Rev. Roumaine Math. Pures et Appl., 41, 3/4 (1996), 251–262.

[13] R. Miron, Gh. Atanasiu, *Prolongations of the Riemannian, Finslerian and Lagrangian structures*, Rev. Roumaine Math. Pures et Appl., 41, 3/4 (1996), 237–249.

[14] D. G. Pavlov, *Chronometry of the three-dimensional time*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 19–30.

[15] D. G. Pavlov, *Four-dimensional time*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 31–39.

[16] D. G. Pavlov, *Generalization of scalar product axioms*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 5–18.

[17] N. Voicu (Brînzei), *Deviations of geodesics in the geometry of second order*, Ph. D. Thesis (in Romanian), "Babes-Bolyai" Univ., Cluj-Napoca, 2003.

The 2-Cotangent Bundle with Berwald-Moor Metric

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On the total space of the dual bundle $(T^{*2}M, \pi^{*2}, M)$ of the 2-tangent bundle (T^2M, π^2, M), the paper develops results related to the notions: of nonlinear connection, distinguished tensor fields, almost contact structure, Riemannian structures, N-linear connections and associated convariant derivations. The Ricci identities are derived and the local expressions of the corresponding d-tensors of torsion and curvature are provided. Further, the metric structures and the metric N-linear connections are studied, and the obtained results are specialized to the case when the metric tensor field is of Berwald-Moor type.

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1 The dual bundle $(T^{*2}M, \pi^{*2}, M)$ of the 2-tangent bundle (T^2M, π^2, M)

Let M be a real differentiable manifold of dimension n. A point of M will be denoted by x and its local coordinates in a chart (U, φ), as $\varphi(x) = (x^a)$. The indices $a, b, ...$ will further run over the set $\{1, ..., n\}$ and the Einstein convention of transvection will be adapted all over this work. Let (TM, π, M) be the tangent bundle of the manifold M and let (T^*M, π^*, M) be its cotangent bundle ([7],[9]).

Definition 1.1. We call *the dual bundle* of the 2−tangent bundle (T^2M, π^2, M), the differentiable bundle $(T^{*2}M, \pi^{*2}, M)$ whose total space is

$$T^{*2}M = TM \ \times_M \ T^*M \tag{1.1}$$

Sometimes we shall denote $(T^{*2}M, \pi^{*2}, M)$ briefly by $T^{*2}M$. A point $u \in T^{*2}M$ will be denoted by $u = (x, y, p)$, having the local coordinates (x^a, y^a, p_a). The projection is given by $\pi^{*2}(u) = \pi^{*2}(x, y, p) = x$. Evidently, we take the projections on the factors of the fibered product of (1.1): $\pi_1^{*2} : T^{*2}M \to TM$, $\pi : TM \to M$ as being $\pi_1^{*2}(x, y, p) = (x, y)$ and $\pi^*(x, y) = x$; also, $\overline{\pi}^* : T^{*2}M \longrightarrow T^*M$ is given by $\overline{\pi}^*(u) = \overline{\pi}^*(x, y, p) = (x, p)$.

The change of local coordinates on the manifold $T^{*2}M$ is:

$$\begin{cases} \tilde{x}^a = \tilde{x}^a(x^1, ..., x^n), \quad \det\left(\dfrac{\partial \tilde{x}^a}{\partial x^b}\right) \neq 0, \\ \tilde{y}^a = \dfrac{\partial \tilde{x}^a}{\partial x^b} y^b, \\ \widetilde{p}_a = \dfrac{\partial x^b}{\partial \tilde{x}^a} p_b. \end{cases} \tag{1.2}$$

The dimension of the manifold $T^{*2}M$ is $3n$.

The null section $O : M \to T^{*2}M$ of the projection π^{*2} is defined by $O : (x) \in M \to (x, 0, 0) \in T^{*2}M$, where we denote $\widetilde{T^{*2}M} = T^{*2}M \setminus \{0\}$.

Let us consider the tangent bundle of the differentiable manifold $T^{*2}M$, $(TT^{*2}M, \tau^{*2}, T^{*2}M)$, where τ^{*2} is the canonical projection and the vertical distribution $V : u \in T^{*2}M \longrightarrow V(u) \subset T_u T^{*2}M$ generated by the vector fields $\left\{ \dfrac{\partial}{\partial y^a}|_u, \dfrac{\partial}{\partial p_a}|_u \right\}$, $\forall u \in T^{*2}M$. We shall denote the natural basis as

$$\partial_a = \frac{\partial}{\partial x^a}, \quad \dot{\partial}_a = \frac{\partial}{\partial y^a}, \quad \dot{\partial}^a = \frac{\partial}{\partial p_a}.$$

By means of (1.2), we can consider the following subdistributions of V :

$$V_1 : u \in T^{*2}M \longrightarrow V_1(u) \subset T_u T^{*2}M,$$

and

$$W_2 : u \in T^{*2}M \longrightarrow W_2(u) \subset T_u T^{*2}M,$$

locally generated by the vector fields $\left\{ \dot{\partial}_a \mid_u, \ u \in T^{*2}M \right\}$ and $\left\{ \dot{\partial}^a \mid_u, u \in T^{*2}M \right\}$ respectively. Clearly,we have

$$V(u) = V_1(u) \oplus W_2(u), \quad \forall u \in T^{*2}M. \tag{1.3}$$

Let us consider the following forms

$$\omega = p_a dx^a \text{ (Liouville 1-form), and} \quad \theta = d\omega = dp_a \wedge dx^a.$$

Theorem 1.1 1°. *The differential forms ω and θ are globally defined on the manifold $T^{*2}M$.*
2°. *The 2-form θ is closed and the rank of the form θ is $2n$.*
3°. *The form θ provides a is a presymplectic structure on $T^{*2}M$.*

We note that the following $\mathcal{F}(T^{*2}M)$-linear mapping

$$J : \mathcal{X}(T^{*2}M) \to \mathcal{X}(T^{*2}M),$$

defined by

$$J(\partial_a) = \dot{\partial}_a, \quad J(\dot{\partial}_a) = 0, \quad J(\dot{\partial}^a) = 0, \quad \forall u \in \widetilde{T^{*2}M},$$

has geometrical meaning. It is not difficult to prove the following result:

Theorem 1.2 1°. *J is a tensor field of type $(1,1)$ on the manifold $T^{*2}M$.*
2°. *J is a tangent structure on $T^{*2}M$, i.e., $J_0 J = 0$.*
3°. *J is an integrable structure.*
4°. $J_0 J = J^2 = 0$.
5°. $Ker\ \ J = V_1 \oplus W_2,\ \mathrm{Im}\, J = V_1$.

With these object fields we can construct the geometry of the manifold $T^{*2}M$.

2 Nonlinear connections on $T^{*2}M$

We extend the classical definition of the nonlinear connection ([11]) to the total space of the dual bundle $(T^{*2}M, \pi^{*2}, M)$.

Definition 2.1 A nonlinear connection of the manifold $T^{*2}M$ is a regular distribution N on $T^{*2}M$, supplimentary to the vertical distribution V, i.e.,

$$T_u T^{*2}M = N(u) \oplus V(u), \forall u \in T^{*2}M. \tag{2.1}$$

Taking into account (1.3), it follows that the distribution N has the property

$$T_u T^{*2}M = N(u) \oplus V_1(u) \oplus W_2(u), \forall u \in T^{*2}M. \tag{2.2}$$

Therefore, the main geometrical objects on $T^{*2}M$ will be reported to the direct sum (1.6) of vector spaces.

We denote by

$$\left\{ \frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^a}, \frac{\partial}{\partial p_a} \right\}, \quad (a = 1, ..., n), \tag{2.3}$$

a local basis adapted to N, V_1, W_2. Clearly, we have

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - N^b_{\ a} \frac{\partial}{\partial y^b} + N_{ab} \frac{\partial}{\partial p_b}. \tag{2.4}$$

The system of functions $(N^b_{\ a}(x,y,p), N_{ab}(x,y,p))$ form the *coefficients* of the nonlinear connection N.

With respect to the coordinate transformations (1.2), we have the rule of change:

$$\frac{\delta}{\delta x^a} = \frac{\partial \tilde{x}^b}{\partial x^a} \frac{\delta}{\delta \tilde{x}^b}, \quad \frac{\partial}{\partial y^a} = \frac{\partial \tilde{x}^b}{\partial x^a} \frac{\partial}{\partial \tilde{y}^b}, \quad \frac{\partial}{\partial p_a} = \frac{\partial x^a}{\partial \tilde{x}^b} \frac{\partial}{\partial \widetilde{p}_b} \tag{2.5}$$

Theorem 2.1 *With respect to (1.2), the coefficients* $(N^a_{\ b}, N_{ab})$ *of a nonlinear connection* N *on* $T^{*2}M$ *obey the rule*

$$\begin{aligned} \widetilde{N}^a_{\ c}\frac{\partial \tilde{x}^c}{\partial x^b} &= N^c_{\ b}\frac{\partial \tilde{x}^a}{\partial x^c} - \frac{\partial \tilde{y}^a}{\partial x^b}, \\ \widetilde{N}_{ab} &= \frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial x^d}{\partial \tilde{x}^b}N_{cd} + p_c\frac{\partial^2 x^c}{\partial \tilde{x}^a \partial \tilde{x}^b}. \end{aligned} \tag{2.6}$$

Conversely, if the system of functions $(N^a_{\ b}, N_{ab})$ *are given on the every domain of local chart of the manifold* $T^{*2}M$*, such that the equations (2.6 hold, then* $(N^a_{\ b}, N_{ab})$ *are the coefficients of a nonlinear connection on* $T^{*2}M$*.*

Assuming that the manifold M is paracompact it follows that the manifold $T^{*2}M$ is paracompact, too. Let $\gamma_{ab}(x),\ x \in M$ be a Riemannian metric on M and $\gamma^a_{bc}(x)$ be its Christoffel symbols. Setting

$$f_b = \gamma^a_{bc}(x)p_a y^c.$$

Then, the system of functions

$$N^a_{\ b} = \dot{\partial}^a f_b, \quad N_{ab} = \dot{\partial}_b f_a, \tag{2.7}$$

are geometrical object fields on $T^{*2}M$, having the rules of transformations (2.6), with respect to the change of local coordinates (1.2). Hence we get the following

Theorem 2.2 *If the base manifold* M *is paracompact, then there exists a nonlinear connection on the manifold* $T^{*2}M$*.*

We shall further denote the basis (1.7) by:

$$\left\{\delta_a, \dot{\partial}_a, \dot{\partial}^a\right\}.$$

The dual basis of the adapted basis (1.7) is given by

$$\left\{dx^a, \delta y^a, \delta p_a\right\}, \tag{2.8}$$

where

$$\delta y^a = dy^a + N^a_{\ b}dx^b,\ \delta p_a = dp_a - N_{ba}dx^b.$$

With respect to (1.2), the covector fields (2.8) are transformed by the rules:

$$d\tilde{x}^a = \frac{\partial \tilde{x}^a}{\partial x^b}dx^b,\ \delta\tilde{y}^a = \frac{\partial \tilde{x}^a}{\partial x^b}\delta y^b,\ \delta\widetilde{p}_a = \frac{\partial x^b}{\partial \tilde{x}^a}\delta p_b.$$

3 Distinguished vector and covector fields. The Algebra of distinguished tensor fields.

Let N be a nonlinear connection on $T^{*2}M$. Let h, v_1, w_2 be the projectors defined by the distributions N, V_1, W_2 of the direct decomposition (1.6). We have

$$h + v_1 + w_2 = I,\ h^2 = h,\ v_1^2 = v_1,\ w_2^2 = w_2, \tag{3.1}$$
$$h \circ v_1 = v_1 \circ h = 0,\ h \circ w_2 = w_2 \circ h = 0,\ v_1 \circ w_2 = w_2 \circ v_1 = 0.$$

If $X \in \chi(\widetilde{T^{*2}M})$, then we denote

$$X^H = hX,\ X^{V_1} = v_1 X,\ X^{W_2} = w_2 X.$$

Therefore we have the unique decomposition

$$X = X^H + X^{V_1} + X^{W_2}. \tag{3.2}$$

Each of the components X^H, X^{V_1}, X^{W_2} are called *d-vector fields* on $\widetilde{T^{*2}M}$.

In the adapted basis (1.7) we get

$$X^H = X^{(0)a}\delta_a, \quad X^{V_1} = X^{(1)a}\dot{\partial}_a, \quad X^{W_2} = \underset{(2)}{X}{}_a\dot{\partial}^a.$$

By means of (2.5) we have

$$\widetilde{X}^{(0)a} = \frac{\partial \tilde{x}^a}{\partial x^b}X^{(0)b}, \quad \widetilde{X}^{(1)a} = \frac{\partial \tilde{x}^a}{\partial x^b}X^{(1)b}, \quad \underset{(2)}{\widetilde{X}}{}_a = \frac{\partial x^b}{\partial \tilde{x}^a}\underset{(2)}{X}{}_b,$$

i.e., the classical rules of the transformations of the local coordinates of vector and covector fields on M. Therefore, $X^{(0)a}, X^{(1)a}$ are called *d-vector fields* and $\underset{(2)}{X}{}_a$ is called a *d-covector field* on the manifold $T^{*2}M$.

A similar theory can be done for distinguished 1-forms.

With respect to the direct decomposition (1.6) a 1-form $\omega \in \chi^*(T^{*2}M)$ can be uniquely written in the form:

$$\omega = \omega^H + \omega^{V_1} + \omega^{W_2},$$

where

$$\omega^H = \omega \circ h, \quad \omega^{V_1} = \omega \circ v_1, \quad \omega^{W_2} = \omega \circ w_2.$$

In the adapted cobasis (2.8), we have

$$\omega = \underset{(0)}{\omega}{}_a dx^a + \underset{(1)}{\omega}{}_a \delta y^a + \omega^{(2)a}\delta p_a.$$

The quantities $\omega^H, \omega^{V_1}, \omega^{W_2}$ are called *$d-1$-forms.* The coefficients $\underset{(0)}{\omega}_a, \underset{(1)}{\omega}_a, \omega^{(2)a}$ are transformed by (1.2) as follows:

$$\underset{(0)}{\omega}_a = \frac{\partial \tilde{x}^b}{\partial x^a} \underset{(0)}{\widetilde{\omega}}_a, \quad \underset{(1)}{\omega}_a = \frac{\partial \tilde{x}^b}{\partial x^a} \underset{(1)}{\widetilde{\omega}}_b, \quad \widetilde{\omega}^{(2)a} = \frac{\partial \tilde{x}^a}{\partial x^b} \omega^{(2)b}.$$

Hence $\underset{(0)}{\omega}_a$ and $\underset{(1)}{\omega}_a$ are called *d-covector fields* and $\omega^{(2)a}$ is called a *d-vector field.*

Definition 3.1 A distinguished tensor (briefly, *$d-$tensor field*) on the manifold $T^{*2}M$ is a d-tensor field T of type (r,s) on $T^{*2}M$, with the property

$$\begin{aligned} T(\overset{1}{\omega}, ..., \overset{r}{\omega}, \underset{1}{X}, ..., \underset{s}{X}) &= T(\overset{1}{\omega}^H, ..., \overset{r}{\omega}^{W_2}, \underset{1}{X}^H, ..., \underset{s}{X}^{W_2}), \\ &\forall \overset{1}{\omega}, ..., \overset{r}{\omega} \in \chi^*(T^{*2}M),\ \forall \underset{1}{X}, ..., \underset{s}{X} \in \chi(T^{*2}M). \end{aligned}$$

For instance, every set of components X^H, X^{V_1}, X^{W_2} of a vector field X forms a d-tensor field of type $(1,0)$, and every set of components $\omega^H, \omega^{V_1}, \omega^{W_2}$ of a 1-form ω is a d-tensor field of type $(0,1)$.

In the adapted basis $(\delta_a, \dot{\partial}_a, \dot{\partial}^a)$ and its dual basis $(dx^a, \delta y^a, \delta p_a)$, a d-tensor field T of type (r,s) can written in the form:

$$T = T^{\ a_1 ... a_r}_{\ b_1 ... b_s}(x,y,p)\delta_{a_1} \otimes ... \otimes \dot{\partial}^{b_s} \otimes dx^{b_1} \otimes ... \otimes \delta p_{a_r},$$

where

$$T^{\ a_1 ... a_r}_{\ b_1 ... b_s}(x,y,p) = T(dx^{b_1}, ..., \delta p_{a_r}, \delta_{a_1}, ..., \dot{\partial}^{b_s}).$$

It follows that the set $\left\{1, \delta_a, \dot{\partial}_a, \dot{\partial}^a\right\}$ generates *the algebra* of the d-tensor fields over the ring of functions $\mathcal{F}(T^{*2}M)$.

With respect to the transformation of the coordinates on $T^{*2}M$, the local coefficients $T^{\ a_1 ... a_r}_{\ b_1 ... b_s}$ of T are transformed by the classical rule

$$\tilde{T}^{\ c_1 ... c_r}_{\ d_1 ... d_s} = \frac{\partial \tilde{x}^{c_1}}{\partial x^{a_1}} ... \frac{\partial \tilde{x}^{c_r}}{\partial x^{a_r}} \frac{\partial x^{b_1}}{\partial \tilde{x}^{d_1}} ... \frac{\partial x^{b_s}}{\partial \tilde{x}^{d_s}} T^{\ a_1 ... a_r}_{\ b_1 ... b_s}.$$

4 Lie brackets

In applications, the Lie brackets of the vector fields $(\delta_a, \dot{\partial}_a, \dot{\partial}^a)$ of the basis adapted to the direct decomposition (1.6), are important. By a direct calculus, we have:

Proposition 4.1 *The Lie brackets of the vector fields of the adapted basis are given by*

$$\begin{aligned}
&[\delta_b, \delta_c] = \underset{(01)}{R}{}^a{}_{bc}\dot{\partial}_a + \underset{(02)}{R}{}_{abc}\dot{\partial}^a, \\
&\left[\delta_b, \dot{\partial}_c\right] = \underset{(11)}{B}{}^a{}_{bc}\dot{\partial}_a + \underset{(12)}{B}{}_{abc}\dot{\partial}^a, \\
&\left[\delta_b, \dot{\partial}^c\right] = \underset{(21)}{B}{}^a{}_b{}^c\dot{\partial}_a + \underset{(22)}{B}{}_{ab}{}^c\dot{\partial}^a, \\
&\left[\dot{\partial}_b, \dot{\partial}_c\right] = 0,\ \left[\dot{\partial}_b, \dot{\partial}^c\right] = 0,\ \left[\dot{\partial}^b, \dot{\partial}^c\right] = 0,
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
&\underset{(01)}{R}{}^a{}_{bc} = \delta_c N^a_b - \delta_b N^a_c, && \underset{(02)}{R}{}_{abc} = \delta_b N_{ca} - \delta_c N_{ba}, \\
&\underset{(11)}{B}{}^a{}_{bc} = \dot{\partial}_c N^a_b, && \underset{(12)}{B}{}_{abc} = -\dot{\partial}_c N_{ba}, \\
&\underset{(21)}{B}{}^a{}_b{}^c = \dot{\partial}^c N^a_b, && \underset{(22)}{B}{}_{ab}{}^c = -\dot{\partial}^c N_{ba}.
\end{aligned} \tag{4.2}$$

Let us consider the followings coefficients from (4.1):

$$\underset{(11)}{B}{}^a{}_{bc} = \dot{\partial}_c N^a_b,\ -\underset{(22)}{B}{}_{ab}{}^c = \dot{\partial}^c N_{ba}\ \left(= -\underset{(22)}{B}{}^c{}_{ab}\right).$$

By means of (2.6) it follows

Proposition 4.2 *The coefficients* $\underset{(11)}{B}{}^a{}_{cb} = \underset{(11)}{U}{}^a{}_{bc}$, $-\underset{(22)}{B}{}^a{}_{bc} = \underset{(22)}{U}{}^a{}_{bc}$ *have the same rule of transformation with respect to the local change of coordinates (1.2) on* $T^{*2}M$*. This is*

$$\underset{(\beta\beta)}{\widetilde{U}}{}^a{}_{df}\frac{\partial x^d}{\partial x^b}\frac{\partial x^f}{\partial x^c} = \frac{\partial \tilde{x}^a}{\partial x^d}\underset{(\beta\beta)}{U}{}^d{}_{bc} - \frac{\partial^2 \tilde{x}^a}{\partial x^b \partial x^c}, \quad (\beta = 1, 2). \tag{4.3}$$

We will see that these coefficients are the horizontal coefficients of an N-linear connection on $T^{*2}M$. By straightforward direct computation, we obtain

Proposition 4.3 *The coefficients* $\underset{(01)}{R}{}^a{}_{bc}$, $\underset{(02)}{R}{}_{abc}$ *and*

$$\underset{(21)}{B}{}^a{}_b{}^c = \dot{\partial}^c N^a_b,\ \underset{(12)}{B}{}_{abc} = -\dot{\partial}_c N_{ba},$$

are d-tensor fields on $T^{*2}M$*, of type* $(1,2), (0,3), (2,1)$ *and, respectively,* $(0,3)$*, i.e.,*

$$\underset{(01)}{\widetilde{R}}{}^d{}_{cf} = \frac{\partial \tilde{x}^d}{\partial x^a}\frac{\partial x^b}{\partial \tilde{x}^c}\frac{\partial x^c}{\partial \tilde{x}^f}\underset{(01)}{R}{}^a{}_{bc},\ \textit{etc.}$$

We will see that (4.4) can be the vertical coefficients of N-linear connection on $T^{*2}M$.

Also, we have

Proposition 4.4 *For the nonlinear connection* $N(N^a{}_b, N_{ab})$ given by (2.7):

$$N^a{}_b = \gamma^a_{bc}(x)y^b, \quad N_{ab} = \gamma^c_{ab}(x)p_c, \tag{4.4}$$

the coefficients (4.2) of Lie brackets have the following expressions:

$$\begin{aligned} &\underset{(01)}{R}{}^a{}_{bc} = r_b{}^a{}_{cd}(x)y^d, && \underset{(02)}{R}{}_{abc} = r_a{}^d{}_{bc}(x)p_d, \\ &\underset{(11)}{B}{}^a{}_{bc} = \gamma^a_{bc}(x), && \underset{(12)}{B}{}_{abc} = 0, \\ &\underset{(21)}{B}{}^a{}_b{}^c = 0, && \underset{(22)}{B}{}_{ab}{}^c = -\gamma^c_{ab}(x). \end{aligned} \tag{4.5}$$

5 The almost contact structure $\mathbb{F}$.

The nonlinear connection N being fixed, we have the direct decomposition (1.5), (1.6) and the corresponding adapted basis (1.7).

Let us consider the $\mathcal{F}(T^{*2}M)$-linear mapping:

$$\mathbb{F} : \chi(T^{*2}M) \longrightarrow \chi(T^{*2}M),$$

determined by

$$\mathbb{F}(\delta_a) = -\dot{\partial}_a, \quad \mathbb{F}(\dot{\partial}_a) = \delta_a, \quad \mathbb{F}(\dot{\partial}^a) = 0. \tag{5.1}$$

Then, we obtain

Theorem 5.1 *The mapping* $\mathbb{F}$ *has the following properties:*

1°. *It is globally defined on* $\widetilde{T^{*2}M}$.
2°. $\mathbb{F}$ *is a tensor field of type* $(1,1)$.
3°. $Ker\,\mathbb{F} = W_2,\ \operatorname{Im}\mathbb{F} = N \oplus V_1$.
4°. $rank\,\mathbb{F} = 2n$.
5°. $\mathbb{F}^3 + \mathbb{F} = 0$.

Proof. For $1° - 5°$ see [7, p. 259].

We say that $\mathbb{F}$ is a *natural almost contact structure* determined by the nonlinear connection N.

6 The Riemann structures on $\widetilde{T^{*2}M}$.

Let us consider a Riemannian structure $\mathbb{G}$ on the manifold $\widetilde{T^{*2}M}$.

In the natural basis, $\mathbb{G}$ is given locally by

$$\mathbb{G} = \underset{(00)}{\bar{g}}{}_{ab}dx^a \otimes dx^b + \underset{(01)}{\bar{g}}{}_{ab}dx^a \otimes dy^b + \underset{(02)}{\bar{g}}{}_a{}^b dx^a \otimes dp_b + ... + \underset{(22)}{\bar{g}}{}^{ab}dp_a \otimes dp_b,$$

where the matrix $\| \underset{(\alpha\beta)}{\bar{g}} \|$ is positively defined.

Let $\{\delta_a\}$, $(a = 1, ..., n)$, be the basis adapted to N:

$$\delta_a = \partial_a - N^b{}_a \dot{\partial}_b + N_{ab} \dot{\partial}^b.$$

and let $\{dx^a, \delta y^a, \delta p_a\}$ be the cobasis adapted to N

$$\delta y^a = dy^a + N^a{}_b dx^b,\ \delta p_a = dp_a - N_{ba} dx^b.$$

Then, after a direct calculation, the Riemann structure $\mathbb{G}$ can be written in the adapted cobasis, in the form

$$\mathbb{G} = \underset{(00)}{g}{}_{ab} dx^a \otimes dx^b + \underset{(01)}{g}{}_{ab} dx^a \otimes \delta y^b + \underset{(02)}{g}{}_a{}^b dx^a \otimes \delta p_b + ... + \underset{(22)}{g}{}^{ab} dp_a \otimes \delta p_b, \quad (6.1)$$

where $\underset{(00)}{g}{}_{ab}$, $\underset{(01)}{g}{}_{ab}$, $\underset{(02)}{g}{}_a{}^b$, *etc.*, are expressed by $\underset{(00)}{\bar{g}}{}_{ab}$, $\underset{(01)}{\bar{g}}{}_{ab}$, $\underset{(02)}{\bar{g}}{}_a{}^b$, *etc.* and with the coefficients $N^a{}_b$ and N_{ab} of N given by (4.4).

Let $\mathbb{F}$ be the natural contact structure determined by the nonlinear connection N given by (4.4).

The following problem arises: when is the pair $(\mathbb{G}, \mathbb{F})$ a Riemannian almost contact structure?

For this, it is obviously necessary to have:

$$\mathbb{G}(\mathbb{F}X, Y) = -\mathbb{G}(X, \mathbb{F}Y), \quad \forall X, Y \in \chi(\widetilde{T^{*2}M}).$$

Consequently, we get

Theorem 6.1 *The pair $(\mathbb{G}, \mathbb{F})$ is a Riemannian almost contact structure if and only if in the adapted basis determined by N and V the tensor $\mathbb{G}$ has the form*

$$\mathbb{G} = g_{ab} dx^a \otimes dx^b + g_{ab} \delta y^a \otimes \delta y^b + h^{ab} \delta p_a \otimes \delta p_b. \quad (6.2)$$

Corollary 6.1 *With respect to the Riemannian structure (2.3), the distributions N, V_1, W_2 are orthogonal in pairs respectively.*

7 N-linear connections on $T^{*2}M$

A linear connection on $T^{*2}M$ is an mapping

$$D : \chi(T^{*2}M) \times \chi(T^{*2}M) \to \chi(T^{*2}M), \quad (X, Y) \longmapsto D_X Y,$$

with the properties:

1. $D_{X_1+X_2} Y = D_{X_1} Y + D_{X_2} Y,$

$D_{fX}Y = fD_XY, \quad \forall f \in \mathcal{F}(T^{*2}M), \quad \forall X, X_1, X_2, Y \in \chi(T^{*2}M).$

2. $D_X(Y_1 + Y_2) = D_XY_1 + D_XY_2, \quad \forall X, Y_1, Y_2 \in \chi(T^{*2}M).$

3. $D_X(fY) = (Xf)Y + fD_XY, \quad \forall X, Y \in \chi(T^{*2}M), \quad \forall f \in \mathcal{F}(T^{*2}M).$

We consider $X, Y \in \chi(T^{*2}M)$. With respect to the decomposition of type (3.2), we have

$$D_XY = \sum_{\alpha=0}^{2}(D_{X^H}Y^{V_\alpha} + D_{X^{V_1}}Y^{V_\alpha} + D_{X^{W_2}}Y^{V_\alpha}),$$

where $V_0 = H$ and $V_2 = W_2$.

The components $D_{X^H}Y^{V_\alpha},\ D_{X^{V_1}}Y^{V_\alpha},\ D_{X^{W_2}}Y^{V_\alpha},\ (V_0 = H,\ V_2 = W_2)$, are (not necessarily distinguished) vector fields.

The linear connection D on $T^{*2}M$ is uniquely determined by its 27 sets of coefficients, written in the adapted basis. To work with these 27 sets of coefficients is not impossible, but is laborious. We shall further use N-linear connections whose coefficients are much easier to determine and operate with.

Let N be a nonlinear connection on $T^{*2}M$.

Definition 7.1 A linear connection D on $T^{*2}M$ is called an *N-linear connection* if it preserves by parallelism the horizontal and the vertical distributions N, V_1 and W_2 on $T^{*2}M$.

By the general theory of connections on manifolds, the horizontal and vertical distributions are preserved by parallelism if for any $X \in \chi(T^{*2}M)$, D_X carries the horizontal vector fields to horizontal vector fields and the vertical vector fields to vertical vector fields. Thus D_XY^H is always an horizontal vector field, and $D_XY^{V_\beta}$ are vertical ones, $(\beta = 1, 2;\ V_2 = W_2)$.

Obviously, the local description of an N-linear connection $D\Gamma(N)$ on $T^{*2}M$ is given by *nine* unique sets of adapted coefficients:

$$D\Gamma(N) := (\underset{(00)}{H}{}^a{}_{bc}, \underset{(10)}{H}{}^a{}_{bc}, \underset{(20)}{H}{}_a{}^b{}_c, \underset{(01)}{C}{}^a{}_{bc}, \underset{(11)}{C}{}^a{}_{bc}, \underset{(21)}{C}{}_a{}^b{}_c, \underset{(02)}{C}{}^a{}_b{}^c, \underset{(12)}{C}{}^a{}_b{}^c, \underset{(22)}{C}{}_a{}^{bc}),$$

We have

Theorem 7.1 1°. *An N-linear connection D on $T^{*2}M$ can be uniquely represented in the adapted basis $(\delta_a, \dot{\partial}_a, \dot{\partial}^a)$ in the form*

$$\begin{cases} D_{\delta_c}\delta_b = \underset{(00)}{H}{}^a{}_{bc}\delta_a, & D_{\delta_c}\dot{\partial}_b = \underset{(10)}{H}{}^a{}_{bc}\dot{\partial}_a, \quad D_{\delta_c}\dot{\partial}^b = -\underset{(00)}{H}{}_a{}^b{}_c\dot{\partial}^a, \\ D_{\dot{\partial}_c}\delta_b = \underset{(01)}{C}{}^a{}_{bc}\delta_a, & D_{\dot{\partial}_c}\dot{\partial}_b = \underset{(11)}{C}{}^a{}_{bc}\dot{\partial}_a, \quad D_{\dot{\partial}_c}\dot{\partial}^b = -\underset{(21)}{C}{}_a{}^b{}_c\dot{\partial}^a, \\ D_{\dot{\partial}^c}\delta_b = \underset{(02)}{C}{}^a{}_b{}^c\delta_a, & D_{\dot{\partial}^c}\dot{\partial}_b = \underset{(12)}{C}{}^a{}_b{}^c\dot{\partial}_a, \quad D_{\dot{\partial}^c}\dot{\partial}^b = -\underset{(22)}{C}{}_a{}^{bc}\dot{\partial}^a. \end{cases} \tag{7.1}$$

2°. *With respect to the coordinate transformation (1.2), the coefficients* $\underset{(\alpha 0)}{H}{}^{a}{}_{bc}$, $(\alpha = 0,1,2;\ \underset{(20)}{H}{}^{a}{}_{bc} := \underset{(20)}{H}{}_{b}{}^{a}{}_{c})$ *obey the rule of transformation:*

$$\underset{(\alpha 0)}{\widetilde{H}}{}^{a}{}_{de}\frac{\partial \tilde{x}^d}{\partial x^b}\frac{\partial \tilde{x}^e}{\partial x^c} = \frac{\partial \tilde{x}^a}{\partial x^e}\underset{(\alpha 0)}{H}{}^{e}{}_{bc} - \frac{\partial^2 \tilde{x}^a}{\partial x^b \partial x^c}.$$

3°. *The system of functions* $\underset{(\alpha 1)}{C}{}^{a}{}_{bc}$, $\underset{(\alpha 2)}{C}{}^{a}{}_{b}{}^{c}$, $(\alpha = 0,1,2;\ \underset{(21)}{C}{}^{a}{}_{bc} := \underset{(21)}{C}{}_{b}{}^{a}{}_{c};\ \underset{(22)}{C}{}^{a}{}_{b}{}^{c} := \underset{(22)}{C}{}_{b}{}^{ac}$ *are d-tensor fields of type* $(1,2)$ *and* $(2,1)$, *respectively.*

We have the following theorem of existence of N-linear connections on $T^{*2}M$.

Theorem 7.2 *If the manifold M is paracompact and N is a nonlinear connection on $T^{*2}M$ with coefficients N^a_b, N_{ab}, then there exists an N-linear connection on $T^{*2}M$.*

Proof. Since M is paracompact, then there exists a linear connection on M of local coefficients, say $\Gamma^a_{bc}(x)$. Let $N^a_b(x,y,p)$ and $N_{ab}(x,y,p)$ be the local coefficients of the nonlinear connection N. We set $\underset{(00)}{H}{}^{a}{}_{bc} = \Gamma^a_{bc}(x)$, $\underset{(10)}{H}{}^{a}{}_{bc} = \dot{\partial}_b N^a_c$, $\underset{(20)}{H}{}^{a}{}_{bc} = \dot{\partial}^a N_{bc}$. Thus, taking into account the previous results, we obtain three sets of functions which transform, with respect to (1.2), by (7.1). It results that $D\Gamma(N)$ given by

$$D\Gamma(N) = (\Gamma^a_{bc}(x), \underset{(11)}{B}{}^{a}{}_{cb}, -\underset{(22)}{B}{}^{a}{}_{bc}, 0,0,0,0,0,0),$$

defines an N-linear connection on $T^{*2}M$.

In applications, we use the N-linear connection of the form

$$B\Gamma(N) = (\underset{(00)}{L}{}^{a}{}_{bc}, \underset{(11)}{B}{}^{a}{}_{cb}, -\underset{(22)}{B}{}^{a}{}_{bc}, 0, \underset{(11)}{C}{}^{a}{}_{bc}, 0,0,0, \underset{(22)}{C}{}_{b}{}^{ac})$$

called *N-linear connection of Berwald type on $T^{*2}M$.*

8 The h_α-, $v_{1\alpha}$- and $w_{2\alpha}$-covariant derivatives in the local adapted basis, $(\alpha = 0,1,2)$

The N-linear connection $D\Gamma(N)$ induces a linear connection on the d-tensors set of the 2-cotangent bundle $(T^{*2}M, \pi^{*2}, M)$, in a natural way. Thus, starting with a d-vector field X and a d-tensor field T, locally expressed by

$$X = X^{(0)a}\delta_a + X^{(1)a}\dot{\partial}_a + \underset{(2)}{X}{}_a\dot{\partial}^a,$$

$$T = T^{\,a_1 \dots a_r}_{\,b_1 \dots b_s}(x,y,p)\delta_{a_1} \otimes \dots \otimes \dot{\partial}^{b_s} \otimes dx^{b_1} \otimes \dots \otimes \delta p_{a_r},$$

we can define the covariant derivative

$$D_X T = \Big\{ X^{(0)d} T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s \shortmid \alpha d} + \\ + X^{(1)d} T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} |_{\alpha d} + \underset{(2)}{X}_d T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} |^{\alpha d} \Big\} \delta_{a_1} \otimes \dots \otimes \delta p_{a_r},$$

where

$$T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s \shortmid \alpha d} = \delta_d T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} + \underset{(\alpha 0)}{H}{}^{a_1}{}_{cd} T^{\ c a_2 \dots a_r}_{\ b_1 \dots b_s} + \dots + \underset{(\alpha 0)}{H}{}^{a_r}{}_{cd} T^{\ a_1 \dots c}_{\ b_1 \dots b_s} - \\ - \underset{(\alpha 0)}{H}{}^{c}{}_{b_1 d} T^{\ a_1 \dots a_r}_{\ c b_2 \dots b_s} - \dots - \underset{(\alpha 0)}{H}{}^{c}{}_{b_s d} T^{\ a_1 \dots a_r}_{\ b_1 \dots c},$$

$$T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} |_{\alpha d} = \dot{\partial}_d T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} + \underset{(\alpha 1)}{C}{}^{a_1}{}_{cd} T^{\ c a_2 \dots a_r}_{\ b_1 \dots b_s} + \dots + \underset{(\alpha 1)}{C}{}^{a_r}{}_{cd} T^{\ a_1 \dots c}_{\ b_1 \dots b_s} - \\ - \underset{(\alpha 1)}{C}{}^{c}{}_{b_1 d} T^{\ a_1 \dots a_r}_{\ c b_2 \dots b_s} - \dots - \underset{(\alpha 1)}{C}{}^{c}{}_{b_s d} T^{\ a_1 \dots a_r}_{\ b_1 \dots c},$$

$$T^{\ a_1 \dots a_r}_{\ b_1 \dots b_s} |^{\alpha d} = \dot{\partial}^d T^{\ a_1 \dots a_r}_{\ b_1 \dots b_r} + \underset{(\alpha 2)}{C}{}_c{}^{a_1 d} T^{\ c a_2 \dots a_r}_{\ b_1 \dots b_s} + \dots + \underset{(\alpha 2)}{C}{}_c{}^{a_r d} T^{\ a_1 \dots c}_{\ b_1 \dots b_s} - \\ - \underset{(\alpha 2)}{C}{}_{b_1}{}^{cd} T^{\ a_1 \dots a_r}_{\ c b_2 \dots b_s} - \dots - \underset{(\alpha 2)}{C}{}_{b_s}{}^{cd} T^{\ a_1 \dots a_r}_{\ b_1 \dots c}.$$

Definition 8.1 The local derivative operators "$_{\shortmid \alpha d}$", " $|_{\alpha d}$ " *and* " $|^{\alpha d}$ " are called the $\mathbf{h}_\alpha -, \mathbf{v}_{\mathbf{1}\alpha} -$ ***and*** $\mathbf{w}_{\mathbf{2}\alpha}-$covariant derivatives of $D\Gamma(N), (\alpha = 0, 1, 2)$.

Remark 8.1 (i) In the particular case when T is a function $f(x, y, p)$ on $T^{*2}M$, the preceding covariant derivatives reduce to

$$f_{\shortmid \alpha d} = \delta_d f = \partial_d f - N^c{}_d \dot{\partial}_c,$$

$$f\ |_{\alpha d} = \dot{\partial}_d f, \quad f\ |^{\alpha d} = \dot{\partial}^d f, \quad \forall f \in \mathcal{F}(T^{*2}M).$$

(ii) Considering the d-tensor $T = Y$ as a d-tensor on $T^{*2}M$, locally expressed by

$$Y = Y^{(0)a} \delta_a + Y^{(1)a} \dot{\partial}_a + \underset{(2)}{Y}_a \dot{\partial}^a,$$

the following expressions of local covariant derivatives of $D\Gamma(N)$ hold good:

$$Y^{(0)a}{}_{\shortmid \alpha c} = \delta_c Y^{(0)a} + Y^{(0)b} \underset{(\alpha 0)}{H}{}^a{}_{bc}, Y^{(1)a}{}_{\shortmid \alpha c} = \delta_c Y^{(1)a} + Y^{(1)b} \underset{(\alpha 0)}{H}{}^a{}_{bc},$$

$$\underset{(2)}{Y}_{b \shortmid \alpha c} = \delta_c \underset{(2)}{Y}_b - \underset{(2)}{Y}_a \underset{(\alpha 0)}{H}{}^a{}_{bc},$$

$$Y^{(0)a}\ |_{\alpha c} = \dot{\partial}_c\ Y^{(0)a} + Y^{(0)b} \underset{(\alpha 1)}{C}{}^a{}_{bc}, Y^{(1)a}\ |_{\alpha c} = \dot{\partial}_c\ Y^{(1)a} + Y^{(1)b} \underset{(\alpha 1)}{C}{}^a{}_{bc},$$

$$\underset{(2)}{Y}_b\ |_{\alpha c} = \dot{\partial}_c \underset{(2)}{Y}_b - \underset{(2)}{Y}_a \underset{(\alpha 1)}{C}{}^a{}_{bc},$$

$$Y^{(0)a}\ |^{\alpha b} = \dot{\partial}^b\ Y^{(0)a} + Y^{(0)c} \underset{(\alpha 2)}{C}{}_c{}^{ab}, Y^{(1)a}\ |^{\alpha b} = \dot{\partial}^b\ Y^{(1)a} + Y^{(1)c} \underset{(\alpha 2)}{C}{}_c{}^{ab},$$

$$\underset{(2)}{Y}_b\ |^{\alpha b} = \dot{\partial}^b \underset{(2)}{Y}_c - \underset{(2)}{Y}_a \underset{(\alpha 2)}{C}{}_c{}^{ab}.$$

Proposition 8.1 *The quantities* $T^{\,a_1\dots a_r}_{\,b_1\dots b_s\shortmid\alpha d}$, $T^{\,a_1\dots a_r}_{\,b_1\dots b_s}|_{\alpha d}$, $T^{\,a_1\dots a_r}_{\,b_1\dots b_s}|^{\alpha d}$, $(\alpha = 0,1,2)$ *are d-tensor fields on* $T^{*2}M$. *The first six are of type* $(r, s+1)$, *the last three are of type* $(r+1, s)$.

9 Ricci identities. Local expressions of d-tensors of torsion and curvature

Let $D\Gamma(N)$ be an N-linear connection with the coefficients

$$D\Gamma(N) = (\underset{(00)}{H}{}^a{}_{bc}, \underset{(10)}{H}{}^a{}_{bc}, \underset{(20)}{H}{}^a{}_{bc}, \underset{(01)}{C}{}^a{}_{bc}, \underset{(11)}{C}{}^a{}_{bc}, \underset{(21)}{C}{}^a{}_{bc}, \underset{(02)}{C}{}_a{}^{bc}, \underset{(12)}{C}{}_a{}^{bc}, \underset{(22)}{C}{}_a{}^{bc}), \tag{9.1}$$

By a straightforward calculation we obtain:

Theorem 9.1 *For any N-linear connection D and any d-vector field $X \in \chi(T^{*2}M)$, the following Ricci formulae hold:*

$$X^a{}_{\shortmid\alpha b\shortmid\alpha c} - X^a{}_{\shortmid\alpha c\shortmid\alpha b} = X^f \underset{(\alpha00)}{R}{}_f{}^a{}_{bc} - \overset{0}{\underset{(\alpha0)}{T}}{}^f{}_{bc} X^a{}_{\shortmid\alpha f} - \underset{(01)}{R}{}^f{}_{bc} X^a|_{\alpha f} - \underset{(02)}{R}{}_{fbc} X^a|^{\alpha f},$$

$$X^a{}_{\shortmid\alpha b}|_{\alpha c} - X^a|_{\alpha c\shortmid\alpha b} = X^f \underset{(\alpha01)}{R}{}_f{}^a{}_{bc} - \underset{(\alpha1)}{C}{}^f{}_{bc} X^a{}_{\shortmid\alpha f} - \underset{(\alpha1)}{P}{}^f{}_{bc} X^a|_{\alpha f} - \underset{(12)}{B}{}_{fbc} X^a|^{\alpha f},$$

$$X^a{}_{\shortmid\alpha b}|^{\alpha c} - X^a|^{\alpha c}{}_{\shortmid\alpha b} = X^f \underset{(\alpha02)}{R}{}_f{}^a{}_b{}^c - \underset{(\alpha2)}{C}{}_b{}^{fc} X^a{}_{\shortmid\alpha f} - \underset{(21)}{B}{}^f{}_b{}^c X^a|_{\alpha f} - \underset{(\alpha2)}{P}{}_{fb}{}^c X^a|^{\alpha f},$$

$$X^a|_{\alpha b}|_{\alpha c} - X^a|_{\alpha c}|_{\alpha b} = X^f \underset{(\alpha11)}{R}{}_f{}^a{}_{bc} - \underset{(\alpha1)}{S}{}^f{}_{bc} X^a|_{\alpha f},$$

$$X^a|_{\alpha b}|^{\alpha c} - X^a|^{\alpha c}|_{\alpha b} = X^f \underset{(\alpha12)}{R}{}_f{}^a{}_b{}^c - \underset{(\alpha2)}{C}{}_b{}^{fc} X^a|_{\alpha f} - \underset{(\alpha1)}{C}{}^c{}_{fb} X^a|^{\alpha f},$$

$$X^a|^{\alpha b}|^{\alpha c} - X^a|^{\alpha c}|^{\alpha b} = X^f \underset{(\alpha22)}{R}{}_f{}^{abc} - \underset{(\alpha2)}{S}{}_f{}^{bc} X^a|^{\alpha f}, \quad (\alpha = 0,1,2).$$

where all the terms in $\underset{(01)}{R}{}^a{}_{bc}$, $\underset{(02)}{R}{}_{abc}$, $\underset{(12)}{B}{}_{abc}$, $\underset{(21)}{B}{}^a{}_b{}^c$ *are known from the Lie brackets (4.1), and the coefficients* $D\Gamma(N)$ *are given in (9.1).*

We further denote

$$\overset{0}{\underset{(\alpha0)}{T}}{}^a{}_{bc} = \underset{(\alpha0)}{H}{}^a{}_{bc} - \underset{(\alpha0)}{H}{}^a{}_{cb}, \; \underset{(\alpha1)}{P}{}^a{}_{bc} = \underset{(11)}{B}{}^a{}_{bc} - \underset{(\alpha0)}{H}{}^a{}_{cb}, \; \underset{(\alpha2)}{P}{}_{ab}{}^c = \underset{(22)}{B}{}_{ab}{}^c + \underset{(\alpha0)}{H}{}^c{}_{ab},$$

$$\underset{(\alpha1)}{S}{}^a{}_{bc} = \underset{(\alpha1)}{C}{}^a{}_{bc} - \underset{(\alpha1)}{C}{}^a{}_{cb}, \; \underset{(\alpha2)}{S}{}_a{}^{bc} = -(\underset{(\alpha2)}{S}{}_a{}^{bc} - \underset{(\alpha2)}{C}{}_a{}^{cb}),$$

and $\overset{0}{\underset{(00)}{T}}{}^a{}_{bc}$, $\overset{1}{\underset{(00)}{T}}{}^a{}_{bc}$, $\overset{2}{\underset{(00)}{T}}{}^a{}_{bc}$, $\overset{0}{\underset{(01)}{P}}{}^a{}_{bc}$, $\overset{1}{\underset{(01)}{P}}{}^a{}_{bc}$, $\overset{2}{\underset{(01)}{P}}{}^a{}_{bc}$, $\overset{0}{\underset{(02)}{P}}{}^a{}_{bc}$, $\overset{1}{\underset{(02)}{P}}{}^a{}_{bc}$, $\overset{2}{\underset{(02)}{P}}{}^a{}_{bc}$, $\overset{1}{\underset{(11)}{S}}{}^a{}_{bc}$, $\overset{1}{\underset{(12)}{Q}}{}^a{}_{bc}$, $\overset{2}{\underset{(12)}{Q}}{}_{ab}{}^c$, $\overset{2}{\underset{(22)}{S}}{}_a{}^{bc}$ are called $d-$*tensors of torsion* of D. These are given by:

$$\begin{cases} \overset{0}{\underset{(00)}{T}}{}^a{}_{bc} = \underset{(00)}{H}{}^a{}_{bc} - \underset{(00)}{H}{}^a{}_{cb}, \; \overset{1}{\underset{(00)}{T}}{}^a{}_{bc} = \underset{(01)}{R}{}^a{}_{bc}, \; \overset{2}{\underset{(00)}{T}}{}_{abc} = \underset{(02)}{R}{}_{abc}, \\ \overset{0}{\underset{(01)}{P}}{}^a{}_{bc} = \underset{(01)}{C}{}^a{}_{bc}, \; \overset{1}{\underset{(01)}{P}}{}^a{}_{bc} = \underset{(11)}{B}{}^a{}_{bc} - \underset{(10)}{H}{}^a{}_{cb}, \; \overset{2}{\underset{(01)}{P}}{}_{abc} = \underset{(12)}{B}{}_{abc}, \\ \overset{0}{\underset{(02)}{P}}{}^a{}_b{}^c = \underset{(02)}{C}{}^a{}_b{}^c, \; \overset{1}{\underset{(02)}{P}}{}^a{}_b{}^c = \underset{(21)}{B}{}^a{}_b{}^c, \; \overset{2}{\underset{(02)}{P}}{}_{ab}{}^c = \underset{(22)}{B}{}_{ab}{}^c + \underset{(20)}{H}{}^c{}_{ab}, \end{cases}$$

$$\begin{cases} \underset{(11)}{\overset{1}{S}}{}^{a}{}_{bc} = \underset{(11)}{C}{}^{a}{}_{bc} - \underset{(11)}{C}{}^{a}{}_{cb}, & \underset{(22)}{\overset{2}{S}}{}_{a}{}^{bc} = -(\underset{(22)}{C}{}_{a}{}^{bc} - \underset{(22)}{C}{}_{a}{}^{cb}) \\ \underset{(12)}{\overset{1}{Q}}{}^{a}{}_{b}{}^{c} = \underset{(12)}{C}{}^{a}{}_{b}{}^{c} =: \underset{(12)}{C}{}^{ac}{}_{b}, & \underset{(12)}{\overset{2}{Q}}{}_{ab}{}^{c} = \underset{(21)}{C}{}_{a}{}^{c}{}_{b} =: \underset{(21)}{C}{}^{c}{}_{ab}. \end{cases}$$

We remark that $\underset{(11)}{P}{}^{a}{}_{bc} = \underset{(01)}{\overset{1}{P}}{}^{a}{}_{bc}$, $\underset{(22)}{P}{}_{ab}{}^{c} = \underset{(02)}{\overset{2}{P}}{}_{ab}{}^{c}$, *etc.* Also, $\underset{(\alpha 00)}{R}, \ldots,$ are called *d-tensors of curvature* of D, and they are given by:

$$\begin{cases} \underset{(\alpha 00)}{R}{}_{b}{}^{a}{}_{cd} = \delta_d \underset{(\alpha 0)}{H}{}^{a}{}_{bc} - \delta_c \underset{(\alpha 0)}{H}{}^{a}{}_{bd} + \underset{(\alpha 0)}{H}{}^{f}{}_{bc} \underset{(\alpha 0)}{H}{}^{a}{}_{fd} - \underset{(\alpha 0)}{H}{}^{f}{}_{bd} \underset{(\alpha 0)}{H}{}^{a}{}_{fc} + \\ \qquad\qquad + \underset{(\alpha 1)}{C}{}^{a}{}_{bf} \underset{(01)}{R}{}^{f}{}_{cd} + \underset{(\alpha 2)}{C}{}_{b}{}^{af} \underset{(02)}{R}{}_{fcd}, \\ \underset{(\alpha 01)}{R}{}_{b}{}^{a}{}_{cd} = \dot{\partial}_d \underset{(\alpha 0)}{H}{}^{a}{}_{bc} - \underset{(\alpha 1)}{C}{}^{a}{}_{bd|\alpha c} + \underset{(\alpha 1)}{C}{}^{a}{}_{bf} \underset{(\alpha 1)}{P}{}^{f}{}_{bc} + \underset{(\alpha 2)}{C}{}_{b}{}^{af} \underset{(12)}{B}{}_{fcd}, \\ \underset{(\alpha 02)}{R}{}_{b}{}^{a}{}_{c}{}^{d} = \dot{\partial}^d \underset{(\alpha 0)}{H}{}^{a}{}_{bc} - \underset{(\alpha 2)}{C}{}_{b}{}^{ad}{}_{|\alpha c} + \underset{(\alpha 1)}{C}{}^{a}{}_{bf} \underset{(21)}{B}{}^{f}{}_{c}{}^{d} + \underset{(\alpha 2)}{C}{}_{b}{}^{af} \underset{(\alpha 2)}{P}{}_{fc}{}^{d}, \quad (\alpha = 0, 1, 2), \end{cases}$$

$$\begin{cases} \underset{(\alpha 11)}{R}{}_{b}{}^{a}{}_{cd} = \dot{\partial}_d \underset{(\alpha 1)}{C}{}^{a}{}_{bc} - \dot{\partial}_c \underset{(\alpha 1)}{C}{}^{a}{}_{bd} + \underset{(\alpha 1)}{C}{}^{f}{}_{bc} \underset{(\alpha 1)}{C}{}^{a}{}_{fd} - \underset{(\alpha 1)}{C}{}^{f}{}_{bd} \underset{(\alpha 1)}{C}{}^{a}{}_{fc}, \\ \underset{(\alpha 12)}{R}{}_{b}{}^{a}{}_{c}{}^{d} = \dot{\partial}^d \underset{(\alpha 1)}{C}{}^{a}{}_{bc} - \dot{\partial}_c \underset{(\alpha 2)}{C}{}_{b}{}^{ad} + \underset{(\alpha 1)}{C}{}^{f}{}_{bc} \underset{(\alpha 2)}{C}{}_{f}{}^{ad} - \underset{(\alpha 2)}{C}{}_{b}{}^{fd} \underset{(\alpha 1)}{C}{}^{a}{}_{fc}, \\ \underset{(\alpha 22)}{R}{}_{b}{}^{acd} = \dot{\partial}^d \underset{(\alpha 2)}{C}{}_{b}{}^{ac} - \dot{\partial}^c \underset{(\alpha 2)}{C}{}_{b}{}^{ad} + \underset{(\alpha 2)}{C}{}_{b}{}^{fc} \underset{(\alpha 2)}{C}{}_{f}{}^{ad} - \underset{(\alpha 2)}{C}{}_{b}{}^{fd} \underset{(\alpha 2)}{C}{}_{f}{}^{ac}, \quad (\alpha = 0, 1, 2). \end{cases}$$

10 Metric structures on the manifold $T^{*2}M$. Metric N-linear connections

Definition 10.1 A *metric structure* on the manifold $T^{*2}M$ is a symmetric covariant tensor field $\mathbb{G}$ of type (0,2) which is non-degenerate at each point $u \in T^{*2}M$ and of constant signature on $T^{*2}M$. If $\mathbb{G}$ is positive definite we say that it defines a Riemann structure on $T^{*2}M$.

Let us consider a metric structure $\mathbb{G}$ on $T^{*2}M$ for which the distributions N, V_1, W_2 are more general then (6.2), namely we have the decomposition:

$$\mathbb{G}(X,Y) = \mathbb{G}(X^H, Y^H) + \mathbb{G}(X^{V_1}, Y^{V_1}) + \mathbb{G}(X^{W_2}, Y^{W_2}), \quad \forall X, Y \in T^{*2}M. \tag{10.1}$$

In other words, $\mathbb{G}$ decomposes as a sum of three d-tensor fields,

(0) $\mathbb{G}^H$ of type $(0,2)$ defined by $\mathbb{G}^H(X,Y) = \mathbb{G}(X^H, Y^H)$,
(1) $\mathbb{G}^{V_1}$ of type $(0,2)$ defined by $\mathbb{G}^{V_1}(X,Y) = \mathbb{G}(X^{V_1}, Y^{V_1})$,
(2) $\mathbb{G}^{W_2}$ of type $(0,2)$ defined by $\mathbb{G}^{W_2}(X,Y) = \mathbb{G}(X^{W_2}, Y^{W_2})$.

Locally, these d-tensor fields can be written as

$$\mathbb{G}^H = \underset{(0)}{g}{}_{ab} dx^a \otimes dx^b, \quad \mathbb{G}^{V_1} = \underset{(1)}{g}{}_{ab} \delta y^a \otimes \delta y^b, \quad \mathbb{G}^{W_2} = \underset{(2)}{g}{}^{ab} \delta p_a \otimes \delta p_b,$$

where $\underset{(0)}{g}_{ab} = \mathbb{G}(\delta_a, \delta_b)$, $\underset{(1)}{g}_{ab} = \mathbb{G}(\dot{\partial}_a, \dot{\partial}_b)$, $\underset{(2)}{g}^{ab} = \mathbb{G}(\dot{\partial}^a, \dot{\partial}^b)$, and

$$rank \parallel \underset{(\alpha)}{g}_{ab} \parallel = n, \quad (\alpha = 0, 1, 2), \quad \parallel \underset{(2)}{g}_{ab} \parallel = \parallel \underset{(2)}{g}^{ab} \parallel^{-1}.$$

Thus the decomposition (10.1) looks locally as

$$\mathbb{G} = \underset{(0)}{g}_{ab} dx^a \otimes dx^b + \underset{(1)}{g}_{ab} \delta y^a \otimes \delta y^b + \underset{(2)}{g}^{ab} \delta p_a \otimes \delta p_b. \tag{10.2}$$

Definition 10.2 An N-linear connection D on $T^{*2}M$ endowed with a metric structure $\mathbb{G}$ is said to be a *metric N-linear connection* if $D_X \mathbb{G} = 0$ for every $X \in T^{*2}M$.

Let $\mathbb{G}$ be a metric structure on $T^{*2}M$ given by (10.2). We have

Proposition 10.2 *An N-linear connection on $T^{*2}M$ is a metric N-linear connection with respect to $\mathbb{G}$ given by (10.2) if and only if*

$$\underset{(\alpha)}{g}_{ab\shortmid\alpha c} = 0, \quad \underset{(\alpha)}{g}_{ab} \mid_{\alpha c} = 0, \quad \underset{(\alpha)}{g}^{ab} \mid^{\alpha c} = 0, \tag{10.3}$$

where $\parallel \underset{(\alpha)}{g}^{ab} \parallel = \parallel \underset{(\alpha)}{g}_{ab} \parallel^{-1}$, $(\alpha = 0, 1, 2)$.

Remark 10.1 The conditions (10.3) are equivalent with the conditions

$$\underset{(\alpha)}{g}^{ab}{}_{\shortmid\alpha c} = 0, \quad \underset{(\alpha)}{g}^{ab} \mid_{\alpha c} = 0, \quad \underset{(\alpha)}{g}_{ab} \mid^{\alpha c} = 0, \quad (\alpha = 0, 1, 2).$$

We shall now discuss the existence of metric N-linear connection on $T^{*2}M$. By straightforward calculation we get

Theorem 10.1 *If the manifold T^*M is endowed with the metric structure $\mathbb{G}$ given by (10.2), then there exists on $T^{*2}M$ a metric N-linear connection, depending only on $\mathbb{G}$, whose $h(hh)-$, $v_1(v_1v_1)-$and $w_2(w_2w_2)-$tensors of torsion, $\underset{(00)}{\overset{0}{T}}{}^a{}_{bc}$, $\underset{(11)}{\overset{1}{S}}{}^a{}_{bc}$, $\underset{(22)}{\overset{2}{S}}{}_a{}^{bc}$ vanish. Its local coefficients defined by*

$$D\Gamma(N) := (\underset{(00)}{H}{}^a{}_{bc}, \underset{(10)}{H}{}^a{}_{bc}, \underset{(20)}{H}{}_a{}^b{}_c, \underset{(01)}{C}{}^a{}_{bc}, \underset{(11)}{C}{}^a{}_{bc}, \underset{(21)}{C}{}_a{}^b{}_c, \underset{(02)}{C}{}^a{}_b{}^c, \underset{(12)}{C}{}^a{}_b{}^c, \underset{(22)}{C}{}_a{}^{bc}),$$

are as follows

$$
\begin{aligned}
&\overset{c}{\underset{(00)}{H}}{}^{a}{}_{bc} = \frac{1}{2}\underset{(0)}{g}{}^{ad}(\delta_c \underset{(0)}{g}{}_{bd} + \delta_b \underset{(0)}{g}{}_{dc} - \delta_d \underset{(0)}{g}{}_{bc}),\\
&\overset{c}{\underset{(10)}{H}}{}^{a}{}_{bc} = \underset{(11)}{B}{}^{a}{}_{cb} + \frac{1}{2}\underset{(1)}{g}{}^{ad}(\delta_c \underset{(1)}{g}{}_{bd} - \underset{(11)}{B}{}^{f}{}_{cb}\underset{(1)}{g}{}_{fd} - \underset{(11)}{B}{}^{f}{}_{cd}\underset{(1)}{g}{}_{bf}),\\
&\overset{c}{\underset{(20)}{H}}{}^{a}{}_{bc} = -\underset{(22)}{B}{}^{a}{}_{bc} + \frac{1}{2}\underset{(2)}{g}{}^{ad}(\delta_c \underset{(2)}{g}{}_{bd} + \underset{(22)}{B}{}^{f}{}_{bc}\underset{(2)}{g}{}_{fd} + \underset{(22)}{B}{}^{f}{}_{dc}\underset{(2)}{g}{}_{bf}),\\
&\overset{c}{\underset{(01)}{C}}{}^{a}{}_{bc} = \frac{1}{2}\underset{(0)}{g}{}^{ad}\dot{\partial}_c \underset{(0)}{g}{}_{bd}, \quad \overset{c}{\underset{(02)}{C}}{}_{a}{}^{bc} = -\frac{1}{2}\underset{(0)}{g}{}_{ad}\dot{\partial}_c \underset{(0)}{g}{}^{bd},\\
&\overset{c}{\underset{(11)}{C}}{}^{a}{}_{bc} = \frac{1}{2}\underset{(1)}{g}{}^{ad}(\dot{\partial}_c \underset{(1)}{g}{}_{bd} + \dot{\partial}_b \underset{(1)}{g}{}_{dc} - \dot{\partial}_d \underset{(1)}{g}{}_{bc}),\\
&\overset{c}{\underset{(21)}{C}}{}^{a}{}_{bc} = \frac{1}{2}\underset{(2)}{g}{}^{ad}\dot{\partial}_c \underset{(2)}{g}{}_{bd}, \quad \overset{c}{\underset{(12)}{C}}{}_{a}{}^{bc} = -\frac{1}{2}\underset{(1)}{g}{}_{ad}\dot{\partial}^c \underset{(1)}{g}{}^{bd},\\
&\overset{c}{\underset{(22)}{C}}{}_{a}{}^{bc} = -\frac{1}{2}\underset{(2)}{g}{}_{ad}(\dot{\partial}^c \underset{(2)}{g}{}^{bd} + \dot{\partial}^b \underset{(2)}{g}{}^{dc} - \dot{\partial}^d \underset{(2)}{g}{}^{bc}).
\end{aligned}
\tag{10.4}
$$

Definition 10.3 The metric N-linear connection given by (10.4) will be called the *canonical N-linear connection associated with* $\mathbb{G}$.

11 Berwald-Moor metrics on the manifold $T^{*2}M$

We further specialize the obtained results to the case when the base manifold is a Space-Time. Then $\dim M = 4$, $\dim T^*M = 8$ and $\dim T^{*2}M = 12$. The nonlinear connection $N = (N^a{}_b, N_{ab})$ given by (4.4), has the coefficients of the Lie brackets of the adapted basis satisfying the relations (4.5). We consider the Riemannian metric on $T^{*2}M$:

$$\mathbb{G} = g_{ab}(x)dx^a \otimes dx^b + g_{ab}(x)\delta y^a \otimes \delta y^b + h^{ab}(y)\delta p_a \otimes \delta p_b$$

where g_{ab} is a Riemannian metric on M and h^{ab} is the dual of the Berwald-Moor type metric

$$h_{ab} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^a \partial y^b}, \quad a, b = \overline{1,4}, \tag{11.1}$$

where $F(y) = \sqrt[4]{|y^1 y^2 y^3 y^4|}$. Then the structure $\mathbb{F}$ given in (5.1) satisfies the relation

$$\mathbb{G}(\mathbb{F}X, Y) = -\mathbb{G}(X, \mathbb{F}Y). \tag{11.2}$$

As well, the following results regarding the canonic d-linear connection hold true:

Theorem 11.1 1° *The canonic metrical linear $d-$connection $D\overset{c}{\Gamma}(N)$ has the components*

$$
\begin{cases}
\overset{c}{\underset{(00)}{H}}{}^{a}{}_{bc} = \gamma^a_{bc}(x), \ \overset{c}{\underset{(10)}{H}}{}^{a}{}_{bc} = \gamma^a_{bc}(x),\\
\overset{c}{\underset{(20)}{H}}{}^{a}{}_{bc} = \dfrac{1}{2}\left\{\gamma^a_{bc}(x) - \gamma^f_{cd}(x)h^{am}\left[y^d(\dot{\partial}_f h_{bm}) + \delta^d_m h_{bf}\right]\right\},
\end{cases}
$$

and

$$\underset{(01)}{\overset{c}{C}}{}^{a}{}_{bc}=0,\ \underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc}=0,\ \underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc}=\frac{1}{2}h^{ad}\dot{\partial}_c h_{bd},\ \underset{(02)}{\overset{c}{C}}{}_{a}{}^{bc}=0,\ \underset{(12)}{\overset{c}{C}}{}_{a}{}^{bc}=0,\ \underset{(22)}{\overset{c}{C}}{}_{a}{}^{bc}=0.$$

2°. *The d-tensors of torsion are given by*

$$\begin{array}{l}
\underset{(00)}{\overset{0}{T}}{}^{a}{}_{bc}=0,\ \underset{(01)}{R}{}^{a}{}_{bc}=r_b{}^a{}_{cd}(x)y^d,\ \underset{(02)}{R}{}_{abc}=r_a{}^d{}_{bc}(x)p_d,\\
\underset{(01)}{\overset{0}{P}}{}^{a}{}_{bc}=0,\ \underset{(01)}{\overset{1}{P}}{}^{a}{}_{bc}=0,\ \underset{(01)}{\overset{2}{P}}{}_{abc}=0,\\
\underset{(02)}{\overset{0}{P}}{}^{a}{}_{b}{}^{c}=0,\ \underset{(02)}{\overset{1}{P}}{}^{a}{}_{b}{}^{c}=0,\ \underset{(02)}{\overset{2}{P}}{}_{bc}{}^{a}=-\gamma^a_{bc}(x)+\underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc},
\end{array}$$

and $\underset{(11)}{\overset{1}{S}}{}^{a}{}_{bc}=0,\ \underset{(22)}{\overset{2}{S}}{}_{a}{}^{bc}=0\ \underset{(12)}{\overset{1}{Q}}{}^{a}{}_{b}{}^{c}=0,\ \underset{(12)}{\overset{2}{Q}}{}_{bc}{}^{a}=\frac{1}{2}h^{ad}\dot{\partial}_c h_{bd}.$

3° *The d-tensors of curvature are given in the adapted basis by*

$$\begin{cases}
\underset{(000)}{R}{}_{b}{}^{a}{}_{cd}=r_b{}^a{}_{cd}(x),\ \underset{(100)}{R}{}_{b}{}^{a}{}_{cd}=r_b{}^a{}_{cd}(x)\\
\underset{(200)}{R}{}_{b}{}^{a}{}_{cd}=\bar{\delta}_d\underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc}-\bar{\delta}_c\underset{(20)}{\overset{c}{H}}{}^{a}{}_{bd}+\underset{(20)}{\overset{c}{H}}{}^{f}{}_{bc}\underset{(20)}{\overset{c}{H}}{}^{a}{}_{fd}-\underset{(20)}{\overset{c}{H}}{}^{f}{}_{bd}\underset{(20)}{\overset{c}{H}}{}^{a}{}_{fc}+\\
\qquad +\frac{1}{2}h^{am}(\dot{\partial}_f h_{bm})r_c{}^f{}_{dm}y^m,
\end{cases}$$

where $\bar{\delta}_d=\partial_d-N^m{}_d\dot{\partial}_m$ *and*

$$\begin{cases}
\underset{(001)}{R}{}_{b}{}^{a}{}_{cd}=0,\ \underset{(101)}{R}{}_{b}{}^{a}{}_{cd}=0\\
\underset{(201)}{R}{}_{b}{}^{a}{}_{cd}=\dot{\partial}_d\underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc}-\underset{(21)}{\overset{c}{C}}{}^{a}{}_{bd|2c}+\underset{(21)}{C}{}^{a}{}_{bf}(\gamma^f_{cd}(x)-\underset{(20)}{\overset{c}{H}}{}^{f}{}_{dc}),
\end{cases}$$

$$\underset{(002)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=0,\ \underset{(102)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=0,\ \underset{(202)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=0,$$

$$\begin{cases}
\underset{(011)}{R}{}_{b}{}^{a}{}_{cd}=0,\ \underset{(111)}{R}{}_{b}{}^{a}{}_{cd}=0,\\
\underset{(211)}{R}{}_{b}{}^{a}{}_{cd}=\dot{\partial}_d\underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc}-\dot{\partial}_c\underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc}+\underset{(21)}{\overset{c}{C}}{}^{f}{}_{bc}\underset{(21)}{\overset{c}{C}}{}^{a}{}_{fd}-\underset{(21)}{C}{}^{f}{}_{bc}\underset{(21)}{C}{}^{a}{}_{fc}=c_b{}^a{}_{cd}(y),
\end{cases}$$

$$\underset{(012)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=0,\ \underset{(112)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=0,\ \underset{(212)}{R}{}_{b}{}^{a}{}_{c}{}^{d}=0,$$

$$\underset{(022)}{R}{}_{b}{}^{acd}=0,\ \underset{(122)}{R}{}_{b}{}^{acd}=0,\ \underset{(222)}{R}{}_{b}{}^{acd}=0$$

If we endow the space T^*M with the metric

$$\mathbb{G}=g_{ab}(x)dx^a\otimes dx^b+g_{ab}(x)\delta y^a\otimes\delta y^b+h^{ab}(p)\delta p_a\otimes\delta p_b,$$

where g_{ab} is a Riemannian metric on M and h^{ab} is the Berwald-Moor type metric

$$h^{ab} = \frac{1}{2}\frac{\partial^2 F^2}{\partial p_a \partial p_b}, \quad a, b = \overline{1,4}, \tag{11.3}$$

where $F(y) = \sqrt[4]{|p_1 p_2 p_3 p_4|}$, then the structure $\mathbb{F}$ given in (5.1) satisfies the relation (11.2). As well, we can state the following

Theorem 11.2 *1° The canonic metrical d−connection $D\overset{c}{\Gamma}(N)$ has the components:*

$$\begin{cases} \underset{(00)}{\overset{c}{H}}{}^a{}_{bc} = \gamma^a_{bc}(x), \ \underset{(10)}{\overset{c}{H}}{}^a{}_{bc} = \gamma^a_{bc}(x) \\ \underset{(20)}{\overset{c}{H}}{}^a{}_{bc} = \gamma^a_{bc}(x) + \dfrac{1}{2}h^{ad}(N_{cf}\dot{\partial}^f h_{bd} - \gamma^f_{bc}h_{fd} - \gamma^f_{dc}h_{bf}), \end{cases}$$

$$\underset{(01)}{\overset{c}{C}}{}^a{}_{bc} = 0, \ \underset{(11)}{\overset{c}{C}}{}^a{}_{bc} = 0, \ \underset{(21)}{\overset{c}{C}}{}^a{}_{bc} = 0, \ \underset{(02)}{\overset{c}{C}}{}_a{}^{bc} = 0, \ \underset{(12)}{\overset{c}{C}}{}_a{}^{bc} = 0,$$

$$\underset{(22)}{C}{}_a{}^{bc} = -\frac{1}{2}h_{ad}(\dot{\partial}^c h^{bd} + \dot{\partial}^b h^{dc} - \dot{\partial}^d h^{bc}) = \Gamma^{bc}_{a}(p).$$

2° *The following sets of components of the d-tensors of torsion are nontrivial:*

$$\underset{(01)}{R}{}^a{}_{bc} = r_b{}^a{}_{cd}(x)y^d, \ \underset{(02)}{R}{}_{abc} = r_a{}^d{}_{bc}(x)p_d, \ \underset{(02)}{\overset{2}{P}}{}_{ab}{}^c = -\gamma^c_{ab} + \underset{(20)}{\overset{c}{H}}{}^c{}_{ab}.$$

3° *The following sets of components of the d-tensors of curvature are nontrivial:*

$$\underset{(000)}{R}{}_b{}^a{}_{cd} = r_b{}^a{}_{cd}(x), \ \underset{(100)}{R}{}_b{}^a{}_{cd} = r_b{}^a{}_{cd}(x)$$

$$\underset{(200)}{R}{}_b{}^a{}_{cd} = \tilde{\delta}_d \underset{(20)}{\overset{c}{H}}{}^a{}_{bc} - \tilde{\delta}_c \underset{(20)}{\overset{c}{H}}{}^a{}_{bd} + \underset{(20)}{\overset{c}{H}}{}^f{}_{bc} \underset{(20)}{\overset{c}{H}}{}^a{}_{fd} - \underset{(20)}{H}{}^f{}_{bd} \underset{(20)}{H}{}^a{}_{fc} + \\ + \underset{(22)}{C}{}_b{}^{af} \underset{(02)}{R}{}_{fcd},$$

where $\tilde{\delta}_d = \partial_d - N_{df}\dot{\partial}^f$ *and* $\underset{(222)}{R}{}_b{}^{acd} = s_b{}^{acd}(p)$,

$$\underset{(202)}{R}{}_b{}^a{}_c{}^d = \dot{\partial}^d \underset{(20)}{H}{}^a{}_{bc} - \underset{(22)}{C}{}_b{}^{ad}{}_{|2c} + \underset{(22)}{C}{}_b{}^{af}(-\gamma^d_{fc} + \underset{(20)}{H}{}^d{}_{fc}).$$

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References

[1] G. S. Asanov, *Full anisotropic Finsler metric function. Relativistic aspects*, lecture at The Workshop "Geometry of Finsler spaces with the Berwald-Moor metric", 15–22 October 2005, Cairo, Egipt.

[2] Gh. Atanasiu, V. Balan, M. Neagu, *The 4-poliforms of momenta* $K(p) = \sqrt[4]{p_1 p_2 p_3 p_4}$ *and its applications in the Hamilton Geometry*, Hypercomplex Numbers in Geometry and Physics, **2 (4)**, Vol 2, 2005 and this book.

[3] Gh. Atanasiu, M. Târnoveanu, *New aspects in the Differential Geometry of the second order Cotangent Bundle*, Sem. de Mecanică, West Univ. of Timisoara, 2005.

[4] V. Balan, N. Brinzei, *Einstein equations for the Berwald-Moor type Euclidean-locally Minkowski relativistic model*, lecture at The 5-th Conference of Balkan Society of Geometers, Aug. 29-sept.2., 2005, Mangalia, Romania.

[5] I. Comic, Gh. Atanasiu, E. Stoica, *The Generalized connection in* Osc^3M, Annales Univ. Sci. Budapest (1998), 39–54.

[6] R. Miron, *The Geometry of Higher Order Hamilton Spaces. Applications to Hamilton Mechanics and Physics*, Kluwer Acad. Publ., 2003.

[7] R. Miron, H. Shimada, D. Hrimiuc, V.S. Sabau, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Acad. Publ., FTPH 118, 2001.

[8] D.G. Pavlov, *Chronometry of the three-dimensional time*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 19–30.

[9] D.G. Pavlov, *Four-dimensional time*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 31–39.

[10] D.G. Pavlov, *Generalization of scalar product axioms*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 5–18.

[11] D.J. Saunders, *The Geometry of Jet Bundles*, Cambridge Univ. Press, 1989.

[12] K. Yano, S. Ishihara, *Tangent and Cotangent Bundles. Differential Geometry*, M. Dekker, Inc., New-York, 1973.

Berwald-Moor-type (h, v)-metric physical models

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In the framework of vector bundles endowed with (h, v)-metrics several physical models for relativity are presented. A characteristic of these models is that the vertical part is provided by the flag-Finsler Berwald-Moor (fFBM) metric, while the horizontal part is specialized to the conformal and to Synge-relativistic optics metrics. As well, the particular case of h-Riemannian v-fFBM metric of Riemann-Minkowski type is examined, considering as nonlinear connection both the trivial canonical connection, and the one induced by the Lagrangian of electrodynamics. For all these models, basic properties are described and the extended Einstein and Maxwell equations are determined.

MSC: 53B40, 53C60, 53C07.

1 Introduction

The recent attempts of modeling relativity based on metrical structures include two notable trends: one originates in the theory of bundles endowed with Ehresmann connection (e.g. via osculating structures and their duals, R. Miron [7, 10, 8, 9]) and one based on a palette of physical models relying on the Berwald-Moor metric (D. G. Pavlov, G. S. Asanov [13, 12, 1]). The present work proposes several relativistic models of Miron type which emerge naturally from this metric. The basic geometric structure is an (h, v)-metric on a vector bundle (in particular the tangent bundle of a Space-Time), where the horizontal part is of Generalized Lagrange type ([8]) and the vertical one is of Finslerian Berwald-Moor type. For these models (h-conformal, h-relativistic optic, h-electromagnetic and h-classical Riemannian) the GR formalism is developed, and the Einstein and relativistic Maxwell equations are described.

2 The flag-Finsler Berwald-Moor metric

Let M be a 4-dimensional differential manifold of class $\mathcal{C}^\infty$, TM its tangent bundle and (x^i, y^a) the coordinates in a local chart on TM. If $F : TM \to \mathbf{R}$, $F = F(y)$ is a Finsler function, we denote by

$$h^*_{ab} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^a \partial y^b}, \quad a, b = \overline{1, 4},$$

the associated metric tensor. For $F(y) = \sqrt[4]{|y^1 y^2 y^3 y^4|}$, Pavlov has studied the "4-pseudoscalar product" related to the Berwald-Moor metric ([13])

$$(X, Y, Z, T) = G_{abcd} X^a Y^b Z^c T^d, \tag{2.1}$$

where

$$G_{abcd} = \frac{1}{4!} \frac{\partial^4 \mathcal{L}}{\partial y^a \partial y^b \partial y^c \partial y^d}, \tag{2.2}$$

and $\mathcal{L} = F^4$. We denote

$$\langle X, Y \rangle = \frac{1}{F^2}(X, Y, y, y,), \quad X, Y \in \mathcal{X}(M), \tag{2.3}$$

where $y = y^a \dfrac{\partial}{\partial y^a}$ is the Liouville vector field ([9]), the vector fields X, Y being considered at some $x \in M$. Then $\langle\ ,\ \rangle$ is a pseudo-scalar product; locally we have

$$\langle X, Y \rangle = \frac{1}{F^2} G_{abcd} X^a Y^b y^c y^d = \frac{G_{ab00}}{F^2} X^a Y^b, \tag{2.4}$$

where the null index represents transvection with y. The coefficients of the scalar product (2.4) are hence

$$h_{ab} = \frac{G_{ab00}}{F^2} = \frac{1}{12F^2} \frac{\partial^2 F^4}{\partial y^a \partial y^b}, \tag{2.5}$$

providing a tensor which coincides with the one $\widetilde{g}^{(4)}_{ij}$ proposed by Lebedev ([6]). Then, h_{ab} is a 2-covariant tensor field, and (M, h) thus becomes a generalized Lagrange space. Its absolute energy, $\mathcal{E} = h_{ab} y^a y^b$, is

$$\mathcal{E} = \frac{G_{ab00}}{F^2} y^a y^b = \frac{1}{4F^2} \frac{\partial F^4}{\partial y^b} y^b = \frac{F^4}{F^2} = F^2,$$

this is, $\mathcal{E} = F^2$. The Lagrange metric associated to h is exactly

$$\frac{1}{2} \frac{\partial^2 \mathcal{E}}{\partial y^a \partial y^b} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^a \partial y^b} = h^*_{ab},$$

and taking into account that F is a Finsler function, h^* is nondegenerate and of constant signature, which shows that $(M, \mathcal{E} = F^2)$ is a Lagrange space. From the homogeneity of F it also follows that

$$\frac{1}{2} \frac{\partial \mathcal{E}}{\partial y^a} = h_{ab} y^b. \tag{2.6}$$

Consequently, we can state

Theorem 1. *The space (M, h) with h given by (2.5) is a generalized Lagrange space with regular metric. The associated Lagrange metric h^*_{ab} coincides with the Finsler metric generated by F and the two metrics provide the same energy,*

$$\mathcal{E} = F^2 = h_{ab} y^a y^b = h^*_{ab} y^a y^b.$$

Remark. The considerations above hold true for an arbitrary Finsler space whose fundamental function is of locally Minkowski type.

3 A Riemann-locally Minkovski model

Let TM be endowed with a nonlinear connection N with coefficients $N^a_{\ i} = N^a_{\ i}(x, y)$ and let

$$\left\{ \delta_i = \frac{\delta}{\delta x^i}, \dot{\partial}_a = \frac{\partial}{\partial y^a} \;\middle|\; i, a = \overline{1,4} \right\}$$

denote the corresponding adapted basis, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^b_{\ i}\frac{\partial}{\partial y^b},\ i = \overline{1,4}.$$

We also denote the dual basis by $\{dx^i, \delta y^a \mid i, a = \overline{1,4}\}$, with $\delta y^a = dy^a + N^a_{\ j}dx^j$. If D is a linear d-connection on TM ([9]), then it is described by its adapted coefficients $D\Gamma(N) = \{L^i_{\ jk}, L^a_{\ bk}, C^i_{\ jc}, C^a_{\ bc}\}$, where:

$$D_{\delta_k}\delta_j = L^i_{\ jk}\delta_i,\ D_{\delta_k}\dot{\partial}_b = L^a_{\ bk}\dot{\partial}_a,$$

$$D_{\dot{\partial}_c}\delta_j = C^i_{\ jc}\delta_i,\ D_{\dot{\partial}_c}\dot{\partial}_b = C^a_{\ bc}\dot{\partial}_a.$$

We shall further denote by $_{|}$ and $|$ the h- and v- covariant derivatives induced by D respectively.

As well, the torsion T of the linear connection D has the adapted components

$$\begin{array}{ll} hT(\delta_k, \delta_j) = T^i_{\ jk}\delta_i, & vT(\delta_k, \delta_j) = R^a_{\ jk}\dot{\partial}_a, \\ hT(\dot{\partial}_c, \delta_j) = C^i_{\ jc}\delta_i, & vT(\dot{\partial}_c, \delta_j) = P^a_{\ jc}\dot{\partial}_a, \\ hT(\dot{\partial}_c, \dot{\partial}_b) = 0, & vT(\dot{\partial}_c, \dot{\partial}_b) = S^a_{\ bc}\dot{\partial}_a, \end{array}$$

while the adapted components of the curvature R are

$$\begin{array}{ll} R(\delta_l, \delta_k)\delta_j = R^i_{jkl}\delta_i, & R(\delta_l, \delta_k)\dot{\partial}_b = R^a_{bkl}\dot{\partial}_a, \\ R(\dot{\partial}_c, \delta_k)\delta_j = P^i_{jkc}\delta_i, & R(\dot{\partial}_c, \delta_k)\dot{\partial}_b = P^a_{bkc}\dot{\partial}_a, \\ R(\dot{\partial}_c, \dot{\partial}_b)\delta_j = S^i_{jbc}\delta_i, & R(\dot{\partial}_d, \dot{\partial}_c)\dot{\partial}_b = S^a_{bcd}\dot{\partial}_a. \end{array}$$

Now, let us consider on TM the following Riemann-locally Minkovski $(h, v)-$metric:

$$\mathcal{G} = g_{ij}(x)dx^i \otimes dx^j + h_{ab}(y)\delta y^a \otimes \delta y^b, \tag{3.1}$$

which we shall use in our further considerations. Together with N, this metric produces the *canonical* metrical $d-$connection $C\Gamma(N)$ ([9]),

$$\begin{cases} L^i{}_{jk} = \frac{1}{2} g^{ih} \left(\frac{\delta g_{hj}}{\delta x^k} + \frac{\delta g_{hk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h} \right), \\ L^a{}_{bk} = \frac{\partial N^a_k}{\partial y^b} + \frac{1}{2} h^{ac} \left(\frac{\delta h_{bc}}{\delta x^k} - \frac{\partial N^d_k}{\partial y^b} h_{dc} - \frac{\partial N^d_k}{\partial y^c} h_{bd} \right), \\ C^i{}_{jc} = \frac{1}{2} g^{ih} \frac{\partial g_{jh}}{\partial y^c}, \\ C^a{}_{bc} = \frac{1}{2} h^{ad} \left(\frac{\partial h_{db}}{\partial y^c} + \frac{\partial h_{dc}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d} \right). \end{cases} \tag{3.2}$$

For h given in (2.5), the $(h,v)-$metric $\mathcal{G}$ given in (3.1) is v-regular, which implies that the coefficients of the canonical (Kern [8, 9]) nonlinear connection $\tilde{N}$ vanish,

$$N^i_a(x,y) = 0, \qquad i, a = \overline{1,4}. \tag{3.3}$$

The canonical metrical linear d-connection $C\Gamma(\tilde{N})$ associated to $\mathcal{G}$, is given by ([9])

$$L^i{}_{jk} = \gamma^i{}_{jk},\ L^a{}_{bk} = 0,\ C^i{}_{jc} = 0,\ C^a{}_{bc} = \frac{1}{2} h^{ad} \left(\frac{\partial h_{db}}{\partial y^c} + \frac{\partial h_{dc}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d} \right),$$

where $\gamma^i{}_{jk}$ denote the Christoffel symbols of g. It is worth mentioning that, for the canonic d-linear connection in the Kern case (3.3), the torsion vanishes,

$$T^i{}_{jk} = 0,\ R^a{}_{jk} = 0,\ C^i{}_{jc} = 0,\ P^a{}_{jb} = 0,\ S^a{}_{bc} = 0.$$

4 Locally v-Minkovskian metrics

In general, an $(h,v)-$metric

$$\mathcal{G} = g_{ij}(x,y) dx^i \otimes dx^j + h_{ab}(x,y) \delta y^a \otimes \delta y^b \tag{4.1}$$

which has the property that in the neghborhood of any point $(x,y) \in TM$ there exists a local map in which $h(x,y) = h(y)$, is called *v-locally Minkovski.* A known result provides consequences specific to this case, as follows

Theorem 2. ([9]) *If $\mathcal{G}$ is a v-locally Minkovski metric and $h = h(y)$ is weakly regular, then the Kern nonlinear connection $\tilde{N}$ and the canonic linear d-connection D (3.2) given by $C\Gamma(\tilde{N}) = \{L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\}$ obey the properties*

1. $N^a_j = 0,\quad L^i{}_{jk} = \{^{\ i}_{jk}\},\quad L^a{}_{bk} = 0;$
2. $T^i{}_{jk} = 0,\quad S^a{}_{bc} = 0,\quad R^a{}_{jk} = 0,\quad P^a{}_{jb} = 0.$
3. $R^a_{b\ jk} = 0,\quad P^a_{b\ kc} = 0,$

where $\{^i{}_{jk}\}$ are the Christoffel symbols corresponding to $g = g(x,y)$.

Remark 1. In our case, the following consequences hold true:

1. The equality $N^a_j = 0$ yields $\dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i}$.
2. The torsion of the canonic linear d-connection has a single non-vanishing component, namely the coefficient $P^i{}_{jc} = C^i{}_{jc}$ of $hT(\dot{\partial}_c, \delta_j)$.
3. $C^a{}_{bc}$ are the Christoffel symbols of second kind associated to $h_{ab} = h_{ab}(y)$ and they depend on y only.

We shall assume further that $h = h(y)$ is the metric (2.5) from [5]; this satisfies

$$h_{ab} = \frac{1}{12\mathcal{E}} \frac{\partial^2 \mathcal{E}^2}{\partial y^a \partial y^b}.$$

In this case, the *deflection tensor fields* attached to the nonlinear connection above are

$$D^a{}_j = y^a{}_{|j} = \frac{\partial y^a}{\partial x^j} + y^b L^a{}_{bj} = 0, \quad d^a{}_b = y^a|_b = \delta^a_b + y^c C^a{}_{cb}.$$

From the definition of C^a_{cb}, (since h is $0-$homogeneous in y), it follows that

$$y^c C^a{}_{cb} = \frac{1}{2} h^{ad} \left(\frac{\partial h_{bd}}{\partial y^c} + \frac{\partial h_{dc}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d} \right) y^c = \frac{1}{2} h^{ad} \left(\frac{\partial h_{dc}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d} \right) y^c.$$

Taking into account the particular form (2.5) of h, and taking into account the homogeneity of $\mathcal{E}$, we get by deriving the product w.r.t. y^b that

$$\frac{\partial h_{dc}}{\partial y^b} y^c = \frac{1}{12} y^c \frac{\partial}{\partial y^b} \left(\frac{1}{\mathcal{E}} \frac{\partial^2 \mathcal{E}^2}{\partial y^c \partial y^d} \right) = -\frac{1}{2\mathcal{E}} \frac{\partial \mathcal{E}}{\partial y^b} \frac{\partial \mathcal{E}}{\partial y^d} + 2h_{bd},$$

is a geometric object symmetrical in the indices b and d, whence

$$y^c C^a{}_{cb} = 0 \Rightarrow d^a{}_b = \delta^a{}_b.$$

Hence, the canonic linear d-connection is *of Cartan type* ([9]) and the deflection tensors are

$$D_{ij} = 0, \quad d_{ab} = h_{ab},$$

where the indices were raised/lowered using the corresponding parts of the $(h, v)-$metric. We obtain subsequently that the *electromagnetic tensors identically vanish*,

$$\begin{cases} F_{ij} = \dfrac{1}{2}(D_{ij} - D_{ji}) = 0, \\ f_{ab} = \dfrac{1}{2}(d_{ab} - d_{ba}) = 0. \end{cases}$$

and, since D is of Cartan type, we have ([9])

$$S^a_{d\ bc} y^d = S^a{}_{bc} = 0, y^d R^a_{d\ jk} = R^a{}_{jk} = 0, \ y^d P^a_{d\ kc} = P^a{}_{kc} = 0.$$

5 Einstein equations for the Riemann-locally Minkovski model

The curvatures of the canonical metrical linear d-connection associated to $\mathcal{G}$ in (3.1) with (2.5) are, according to [9],

$$\begin{cases} R_j{}^i{}_{kh} = r_j{}^i{}_{kh}, R_b{}^a{}_{kh} = 0, \quad P_j{}^i{}_{kc} = 0, P_b{}^a{}_{kc} = 0, \; S_j{}^i{}_{bc} = 0, \\ S^a_{b\ cd} = \dfrac{\partial C^a{}_{bc}}{\partial y^d} - \dfrac{\partial C^a{}_{bd}}{\partial y^c} + C^f{}_{bc} C^a{}_{fd} - C^f{}_{bd} C^a{}_{fc}, \end{cases} \tag{5.1}$$

where $r^i_{j\ kh}$ are the components of the curvature tensor of the horizontal metric. Taking into account the relations (5.1), it follows, as in [9], that the Einstein equations of the canonical metrical linear d-connection $C\Gamma(\tilde{N})$ (3.2)-(3.3) can be written as

$$\begin{cases} r_{ij} - \dfrac{1}{2}(r+S) g_{ij} = T^H_{ij}, \\ T^{M_1}_{bj} = 0, \quad T^{M_2}_{jb} = 0, \\ S_{ab} - \dfrac{1}{2}(r+S) h_{ab} = T^V_{ab}, \end{cases} \tag{5.2}$$

where r_{ij} denotes the Ricci tensor $r_{ij} = r_i{}^h{}_{jh}$ attached to the Riemannian metric g, S_{ab} is the Ricci tensor attached to the vertical metric h_{ab}, r is the scalar curvature of r^i_{jkl} and $T_{\alpha\beta}$ are the components of the energy-momentum tensor field. If it is to compare (5.2) with the (classical) Einstein equations of the Riemannian manifold (M, g), we have to notice in the h-part of the above equations the "perturbation" introduced by the term $-\dfrac{1}{2} S g_{ij}$. According to [9], the energy conservation law is identically satisfied by $C\Gamma(\tilde{N})$.

6 The electrodynamic case

If we consider the Lagrangian of electrodynamics ([10]),

$$L_0(x, y) = mc\gamma_{ij}(x) y^i y^j + \frac{2e}{m} A_i(x) y^i, \tag{6.1}$$

where γ_{ij} is a Lorentz metric tensor, $A_i(x)$ is a covector field and m, c, e are physical constants, then, the attached Lagrange metric tensor is

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} = mc\gamma_{ij}.$$

On the other hand, from the variational problem associated to 6.1, there arises a nonlinear connection $\hat{N}$, whose coefficients are given by ([10])

$$N^a_{\ j} = \gamma^a{}_{jb}(x) y^b - \overset{\circ}{F}{}^a{}_j, \tag{6.2}$$

where F is the electromagnetic field

$$\overset{\circ}{F}{}^i{}_j = \frac{e}{2m} g^{ik}(A_{j;k} - A_{k;j}),$$

the symbol ";" denotes the covariant derivative defined by means of the Christoffel symbols $\gamma^i{}_{jk}(x)$ of the Lorentz metric tensor γ_{ij}, and we denoted for simplicity, $\gamma^a{}_{jb} = \delta^a_i \delta^k_b \gamma^i{}_{jk}$.

If we consider now TM endowed with the (h, v)-metric

$$\mathcal{G} = g_{ij}(x)dx^i \otimes dx^j + h_{ab}(y)\delta y^a \otimes \delta y^b,$$

then the canonical metrical linear $d-$connection $C\Gamma(\hat{N})$ associated to $\mathcal{G}$ is given by

$$\begin{cases} \hat{L}^i{}_{jk} = \gamma^i{}_{jk}, \\ \hat{L}^a{}_{bk} = \gamma^a{}_{bk} - \dfrac{1}{2} h^{ac}(N^d_k \dfrac{\partial h_{bc}}{\partial y^d} + \gamma^d{}_{kb} h_{dc} + \gamma^d{}_{kc} h_{bd}), \\ \hat{C}^i{}_{jc} = 0, \quad C^a{}_{bc} = \dfrac{1}{2} h^{ad} \left(\dfrac{\partial h_{db}}{\partial y^c} + \dfrac{\partial h_{dc}}{\partial y^b} - \dfrac{\partial h_{bc}}{\partial y^d} \right). \end{cases}$$

By direct computation, one obtains that this time, the torsions of $C\Gamma(\hat{N})$ are

$$T^i{}_{jk} = 0, \; C^i{}_{jc} = 0, \; S^a{}_{bc} = 0,$$

while $P^a{}_{jb}$ and $R^a{}_{jk}$ do not vanish. Its curvatures are

$$R_j{}^i{}_{kh} = r_j{}^i{}_{kh}, \; R_b{}^a{}_{kh}, \quad P_j{}^i{}_{kc} = 0, \; P_b{}^a{}_{kc}, \quad S_j{}^i{}_{bc} = 0, \; S^a_b{}_{cd},$$

where the expression of $S^a_b{}_{cd}$ is similar to that one in the previous section. The Ricci tensor has the properties

$$R_{ij} = r_{ij}, \overset{2}{P}_{jb} = P^h_j{}_{hb} = 0.$$

The Einstein equations take the particular form

$$\begin{cases} r_{ij} - \dfrac{1}{2}(r + S)g_{ij} = \overset{h}{T}_{ij}, \\ \overset{1}{T}_{bj} = \overset{1}{P}_{bj}, \; \overset{2}{T}_{jb} = 0, \\ S_{ab} - \dfrac{1}{2}(r + S)h_{ab} = \overset{v}{T}_{ab}, \end{cases}$$

while the energy conservation law writes as:

$$\begin{cases} \left(r^i{}_j - \dfrac{1}{2} r \delta^i{}_j \right)_{|i} + \overset{1}{P}{}^a{}_j |_a = 0, \\ \left(S^a{}_b - \dfrac{1}{2} S \delta^a{}_b \right) |_a = 0, \end{cases}$$

where $r^i_j = g^{ih} r_{hj}$, $S^a_b = h^{ac} S_{cb}$, $\overset{1}{P}{}^a_j = h^{ac} \overset{2}{P}_{cj}$.

We shall study further two particular cases of v-locally Minkowski metrics, by preserving $h = h(y)$ from (2.5) and particularizing $g = g(x, y)$. In these cases the results in Section 4 still hold true, and the nonlinear connection used throughover is according to Theorem 2, the trivial one.

7 The relativistic Miron-Kawaguchi optic h-metric case

Let $\gamma_{ij} = \gamma_{ij}(x)$ be a Riemannian metric on M. We denote

$$y_i = \gamma_{ij} y^j, \ \|y\|^2 = \gamma_{ij} y^i y^j.$$

We consider now the metric $\mathcal{G}$ from (4.1), in which the $h-$metric is given by

$$g_{ij} = \gamma_{ij} + c^{-2} y_i y_j,$$

where c is a nonzero real constant. The coefficients $C^i{}_{jd}$ of the linear $d-$connection are

$$C^i{}_{jd} = \frac{1}{2} g^{ih} \frac{\partial g_{jh}}{\partial y^d} = \frac{g^{ih}}{2c^2} (\gamma_{jd} y_h + \gamma_{hd} y_j),$$

and $C^a{}_{bc} = C^a{}_{bc}(y)$ are determined in [5].

From the theorem above, it results that the Ricci tensor field has the components

$$\begin{aligned} R_{ij} &= R^h_{i\ jh}, \ \overset{1}{P}_{bj} = P^a_{b\ ka} = 0, \\ \overset{2}{P}_{jb} &= P^h_{j\ hb}, \ S_{bc} = S^a_{b\ ca}. \end{aligned}$$

The Einstein equations write then

$$\begin{cases} R_{ij} - \dfrac{1}{2}(R + S) g_{ij} = \overset{h}{T}_{ij}, \\ \overset{1}{T}_{bj} = 0, \ \overset{2}{T}_{jb} = -\overset{2}{P}_{jb}, \\ S_{ab} - \dfrac{1}{2}(R + S) h_{ab} = \overset{v}{T}_{ab}, \end{cases}$$

and the energy conservation law is described by the system of PDEs

$$\begin{cases} \left(R^i{}_j - \dfrac{1}{2} R \delta^i{}_j \right)_{|i} = 0, \\ \left(S^a{}_b - \dfrac{1}{2}(R + S) \delta^a{}_b \right) |_a - \overset{2}{P}{}^i_{b|i} = 0, \end{cases}$$

where $R^i{}_j = g^{ih}R_{hj}$, $S^a{}_b = h^{ac}S_{cb}$, $\overset{2}{P}{}^i{}_b = g^{ij}\overset{2}{P}_{jb}$.

The first equality from above is identically satisfied (see [8]), since it coincides with the horizontal part of the energy conservation law for the canonical linear $d-$connection of the generalized Lagrange space (M, g) (which is infered straightforward by the Bianchi identity).

8 The h-conformal metric case

In the $h-$conformal metric case, i.e. for the horizontal metric given by

$$g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x),$$

the coefficients $L^i{}_{jk}$ are given by ([2])

$$L^i{}_{jk} = \gamma^i{}_{jk} + \delta^i_j\sigma_k + \delta^i_k\sigma_j - \gamma_{jk}\sigma^i,$$

where $\sigma^i = \gamma^{il}\sigma_l$, $\gamma^i{}_{jk}$ are the Christoffel symbols of $\gamma_{ij}(x)$ and for h given by (2.5) we have $\sigma_k = \dfrac{\delta\sigma}{\delta x^k} = \dfrac{\partial\sigma}{\partial x^k}$. Obviously, $L^a{}_{bk}$ and $C^a{}_{bc}$ are as in Theorem 2 and Remark 1. By direct computation, we get

$$C^i{}_{jc} = \frac{1}{2}g^{ih}\frac{\partial g_{jh}}{\partial y^c} = \delta^i_j\dot\sigma_c,$$

where $\dot\sigma_c$ denotes the derivative of σ w.r.t. y : $\dot\sigma_c = \dfrac{\partial\sigma}{\partial y^c}$. As well, the torsion components vanish, except $P^i{}_{jc} = C^i{}_{jc}$ and the curvature components are

$$\begin{cases} R^i_{j\ kl},\ R^a_{b\ jk} = 0, P^i_{j\ kc},\ P^a_{b\ kc} = 0,\ S^a_{b\ cd.} \\ S^i_{j\ bc} = \dfrac{\partial C^i{}_{jb}}{\partial y^c} - \dfrac{\partial C^i{}_{jc}}{\partial y^b} + C^h{}_{jb}C^i{}_{hc} - C^h{}_{jc}C^i{}_{hb} = 0 \\ P^i_{j\ kb} = \delta^i_k\sigma_{jb} - \gamma_{jk}\gamma^{il}\sigma_{lb}, \end{cases}$$

where $\sigma_{jb} = \dfrac{\partial^2\sigma}{\partial x^j\partial y^b}$, $\sigma_{lb} = \dfrac{\partial^2\sigma}{\partial x^l\partial y^b}$. The Ricci tensor has the properties: $\overset{1}{P}_{bj} = P^a_{b\ ka} = 0$ and

$$\overset{2}{P}_{jb} = P^h_{j\ hb} = \delta^h_h\sigma_{jb} - \gamma_{jh}\gamma^{hl}\sigma_{lb} = 4\sigma_{jb} - \delta^l_j\sigma_{lb} = 3\sigma_{jb}.$$

Then the Einstein equations are

$$\begin{cases} R_{ij} - \dfrac{1}{2}(R+S)g_{ij} = \overset{h}{T}_{ij}, \\ \overset{1}{T}_{bj} = 0,\ \overset{2}{T}_{jb} = -3\sigma_{jb}, \\ S_{ab} - \dfrac{1}{2}(R+S)h_{ab} = \overset{v}{T}_{ab}. \end{cases}$$

Taking into account that $S = S(y)$, the conservation law is described by

$$\begin{cases} \left(R^i{}_j - \dfrac{1}{2}R\delta^i{}_j\right)_{|i} = 0, \\ \left(S^a{}_b - \dfrac{1}{2}(R+S)\delta^a{}_b\right)|_a - 3\sigma^j{}_{b|j} = 0, \end{cases}$$

where the first equality is identically satisfied.

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References

[1] G. S. Asanov, *Finsleroid space supplemented by angle and scalar product*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 40–62.

[2] V. Balan, *Generalized Einstein-Yang Mills Equations for the Space* $GL^n = (M, g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x))$, Tensor N.S., 52 (1993), 199–203.

[3] V. Balan, N. Brinzei, *Einstein equations for the Berwald-Moor type Euclidean-locally Minkowski relativistic model*, preprint.

[4] J. Kern, *Lagrange geometry*, Arch. Math. 25 (1974), 483–443.

[5] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry I, II*, Interscience Publishers, New York, (1963), (1969).

[6] S. V. Lebedev, *Properties of spaces associated with commutative-associative H_3 and H_4 algebras*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 63–69.

[7] R. Miron, *The Geometry of Higher Order Finsler Spaces*, Hadronic Press, 1998.

[8] R. Miron, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publishers, 1994.

[9] R. Miron, M.Anastasiei, *Vector Bundles. Lagrange Spaces. Applications to Relativity*, Geometry Balkan Press, 1996.

[10] R. Miron, D. Hrimiuc, H. Shimada, S. V. Sabău, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer, FTPh. 198, 2001.

[11] Gh. Munteanu, V. Balan, *Lectures of Relativity Theory* (Romanian), Bren Eds., Bucharest, 2000.

[12] D. G. Pavlov, *Chronometry of the three-dimensional time*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 19–30.

[13] D.G. Pavlov, *Four-dimensional time*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 31–39.

[14] D.G. Pavlov, *Generalization of scalar product axioms*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 5–18.

The Horizontal and Vertical Semisymmetric Metrical D-Connections in the Relativity Theory

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Let E be the $(m+n)$-dimensional total space of a vector bundle (E,p,M), $dim\ M = n$, a given fixed nonlinear connection N on E and a given (h,v)-metrical structure $G \in \mathcal{T}_2^0(E)$. In the paper, we determine the Einstein equations of an h- and v-semisymmetric metrical distinguished connection on $E = TM$, if $n = 4$, for a Riemann – local Minkowski model.

1 Vector bundles. Distinguished linear connections ([11])

Let $\xi = (E,p,M)$ be a vector bundle with $\dim E = m+n,\ p : E \to M$, where M is a n-dimensional smooth differentiable manifold. If N is a nonlinear connection on E and V is a complementary vertical distribution of N then,

$$T_uE = H_uE \oplus V_uE, \quad \forall u \in E. \tag{1.1}$$

Definition 1.1 *A linear connection D on E is called distinguished linear connection or d-connection if the linear connection D preserves by parallelism the horizontal and vertical distributions:*

$$D_Z X \in HE,\ \ D_Z Y \in VE,\ \ \forall X \in HE,\ \ Y \in VE,\ \ Z \in \mathcal{X}(E). \tag{1.2}$$

For a d-connection D we have the unique decomposition

$$D = D^H + D^V. \tag{1.3}$$

where D^H and D^V are the h- and v-covariant derivatives on $\mathcal{X}(E)$

We denote by $X^H\left(X^V\right)$ and $\omega^H\left(\omega^V\right)$, the horizontal (vertical) components of $X \in \mathcal{X}(E)$ respectively $\omega \in \mathcal{X}^*(E)$

In the local coordinates (x^i, y^a) of point $u(u^\alpha) \in E,\ \alpha = \overline{1,m+n},\ i = \overline{1,n},\ a = \overline{1,m}$, we have $\left(\delta_i, \dot{\partial}_a\right)$, $(dx^i, \delta y^a)$ the adapted frames to $N\left(N^a{}_i(x,y)\right)$:

$$\delta_i\ =\ \partial_i - N^a{}_i(x,y)\,\dot{\partial}_a, \quad \delta y^a = dy^a + N^a{}_i(x,y)\,dx^i, \tag{1.4}$$

$$\left(\delta_i = \delta/\delta x^i,\ \partial_i = \partial/\partial x^i,\ \dot{\partial}_a = \partial/\partial y^a\right).$$

Then, $\left(L^i{}_{jk}(x,y),\ L^a{}_{bk}(x,y),\ C^i{}_{jc}(x,y),\ C^a{}_{bc}(x,y)\right)$ are the local components of a d-connection $D\Gamma(N)$.

$$\begin{aligned} D_{\delta_k}\delta_j &= L^i{}_{jk}(x,y)\,\delta_i, \quad D_{\dot\partial_c}\delta_j = C^i{}_{jc}\delta_i, \\ D_{\delta_k}\dot\partial_b &= L^a{}_{bk}(x,y)\,\dot\partial_a, \quad D_{\dot\partial_c}\dot\partial_b = C^a{}_{bc}\dot\partial_a. \end{aligned} \tag{1.5}$$

Also, we denote by: $T^i{}_{jk}, R^a{}_{jk}, P^a{}_{jc}, C^i{}_{jc}, S^a{}_{bc}$, the local components of five d-tensor fields of torsion of d-connection $D\Gamma(N)$, (1.4) and with: $R_j{}^i{}_{kh}, R_b{}^a{}_{jk}, P_j{}^i{}_{kd}, P_b{}^a{}_{kd}, S_j{}^i{}_{cd}, S_b{}^a{}_{cd}$, the local component of six d-tensors fields of curvature of d-connection $D\Gamma(N)$, (1.4).

The Algebra of d-tensor fields on E is locally generated by $\left\{1, \delta_i\dot\partial_a\right\}$ over the differentiable functions $\mathcal{F}(E)$.

2 Metrical structures and metrical d-connections on E ([11])

We will consider a given fixed nonlinear connection N on E with the local components $N^a{}_i(x,y)$ and a given (h,v)-metrical structure $\mathbb{G} \in \tau_2^0(E)$:

$$\mathbb{G} = g_{ij}(x,y)\,dx^i \otimes dx^j + h_{ab}(x,y)\,\delta y^a \otimes \delta y^b, \tag{2.1}$$

where

$$\begin{aligned} g_{ij}(x,y) &= g_{ji}(x,y), \quad rank\,\|g_{ij}(x,y)\| = n, \\ h_{ab}(x,y) &= h_{ba}(x,y), \quad rank\,\|h_{ab}(x,y)\| = m \end{aligned} \tag{2.2}$$

Obviously, we have

$$\mathbb{G}\left(X^H,\ Y^V\right) = 0, \quad \forall X \in HE, \quad Y \in VE, \tag{2.3}$$

in other words, the distributions H and V are orthogonal with respect to $\mathbb{G}$ given by (2.1).

Remark If $E = TM$, there exist metrics of type (2.1) wich satisfy (2.2). Indeed, we shall consider a Lagrange (Finsler) structure $g_{ij}(x,y)$ on TM and by Sasaki-Matsumoto lift:

$$\mathbb{G} = g_{ij}(x,y)\,dx^i \otimes dx^j + g_{ij}(x,y)\,\delta y^i \otimes \delta y^j \tag{2.4}$$

is obtained a metric of type (2.1) wich satisfy the relation (2.2).

Conversly, if $\mathbb{G} \in \tau_2^0(E)$ is a metric on E, then there exists a nonlinear connection $N\left(N^a{}_i(x,y)\right)$ given by $\mathbb{G}\left(X^H, Y^V\right) = 0$.

Definition 2.1 *A d-connexion D on E is called a* **metrical d-connexion** with *respect to* $\mathbb{G} \in \tau_2^0(E)$ *given by (2.1), if* $D_X\mathbb{G} = 0$, $\forall X \in \mathcal{X}(E)$.

Proposition 2.1 *A d-connexion D on E* it *is metrical if and only if*

$$D_X^H\mathbb{G}^H = 0, \quad D_X^H\mathbb{G}^V = 0, \quad D_X^V\mathbb{G}^H = 0, \quad D_X^V\mathbb{G}^V = 0, \quad \forall X \in \mathcal{X}(E), \tag{2.5}$$

where $\mathbb{G}^H = g_{ij}(x,y)\,dx^i \otimes dx^j$ *is the horizontal part and* $\mathbb{G}^V = h_{ab}(x,y)\,\delta y^a \otimes \delta y^b$ *is the vertical part of* $\mathbb{G}$ *given by (2.1).*

Proposition 2.2 *There exists a metrical d−connection on E which dependes only* $N^a{}_i(x,y), g_{ij}(x,y)$ *and* $h_{ab}(x,y)$. *This is given by*

$$\begin{aligned}
\overset{M}{L}{}^i{}_{jk}(x,y) &= 1/2g^{ih}\left(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}\right), \\
\overset{M}{L}{}^a{}_{bk}(x,y) &= \dot{\partial}_b N^a{}_k + 1/2h^{ad}\left(\delta_k h_{bd} - h_{bc}\dot{\partial}_d N^c{}_k - h_{cd}\dot{\partial}_b N^c{}_k\right), \\
\overset{M}{C}{}^i{}_{jc}(x,y) &= 1/2g^{ih}\dot{\partial}_c g_{jh}, \\
\overset{M}{C}{}^a{}_{bc}(x,y) &= 1/2h^{ad}\left(\dot{\partial}_b h_{dc} + \dot{\partial}_c h_{bd} - \dot{\partial}_d h_{bc}\right),
\end{aligned} \tag{2.6}$$

where $\|g^{ij}\| = \|g_{ij}\|^{-1}, \quad \|h^{ab}\| = \|h_{ab}\|^{-1}$.

The distinguished metrical d-connection (2.4) is said to be **Miron connection** of $\mathbb{G}$ and it will denoted by $MD\Gamma(N)$.

Proposition 2.3 *There exists an unique metrical d-connection* $D\Gamma(N) = \left(L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\right)$ *on E for which:*

$$L^a{}_{bk}(x,y) = \overset{M}{L}{}^a{}_{bk}(x,y), \quad C^i{}_{jc}(x,y) = \overset{M}{C}{}^i{}_{jc}(x,y) \tag{2.7}$$

and the d-tensor fields $T^i{}_{jk}, S^a{}_{bc}$ *are prescribed. This connection is given by (2.5) and*

$$\begin{aligned}
L^i{}_{jk}(x,y) &= \overset{M}{L}{}^i{}_{jk}(x,y) + 1/2g^{ir}\left(g_{rh}T^h{}_{jk} - g_{jh}T^h{}_{rk} + g_{kh}T^h{}_{jr}\right), \\
C^a{}_{bc}(x,y) &= \overset{M}{C}{}^a{}_{bc}(x,y) + 1/2h^{ad}\left(h_{df}S^f{}_{bc} - h_{bf}S^f{}_{dc} + h_{cf}S^f{}_{bd}\right).
\end{aligned} \tag{2.8}$$

The metrical distinguished connection given by (2.5) and (2.6) will be called **generalized Miron connection** of the metric $\mathbb{G}$ given by (2.1) and it will denoted by $GMD\Gamma(N)$.

We note

$$\varepsilon(x,y) = < y, y > = h_{ab}(x,y)\,y^a y^b \tag{2.9}$$

the **absolut energy** of vertical part G^V and

$$h^*_{ab}(x,y) = \frac{1}{2}\frac{\partial^2 \varepsilon}{\partial y^a \partial y^b}. \tag{2.10}$$

Definition 2.2 *The d-tensor field* $h_{ab}(x,y)\,\delta y^a \otimes \delta y^b$ *is said to be* **weakly regular** *if the d-tensor field with components* $h^*_{ab}(x,y)$ *given by (2.8) is nondegenerate, i.e.* $\det \|h^*_{ab}(x,y)\| \neq 0$, *where* $E = TM$.

Theorem 2.1 (R. Miron, [10]; *see also* [11] *pg. 127 and* [12]*) If* $h_{ab}(x,y)\,\delta y^a \otimes \delta y^b$ *is a weakly regular v-metric on* $E = TM$ *then the functions*

$$N^a{}_i(x,y) = \dot{\partial}_b G^a(x,y)\,\delta^b_i, \quad G^a = \frac{1}{2}h^{*ab}\left[\left(\dot{\partial}_b \partial_k \varepsilon\right)\delta^k_c y^c - (\partial_k \varepsilon)\,\delta^k_b\right], \tag{2.11}$$

are the coefficients of a nonlinear connection completely determined by $h_{ab}(x,y)$.

3 h- and v-semisymmetric metrical d-connections and their transformations

Definition 3.1 *Metrical d-connection on E is said to be* **h-semisymmetric** *if*

$$T^i{}_{jk} = \sigma_j \delta^i_k - \sigma_k \delta^i_j, \tag{3.1}$$

and **v-semisymmetric** *if*

$$S^a{}_{bc} = \tau_b \delta^a_c - \tau_c \delta^a_b, \tag{3.2}$$

where σ_i, τ_a *are d-covector fields on E.*

Theorem 3.1 *There exists on E an unique metrical d-connection* both *h-and v-semisymmetric,* $D\Gamma(N) = \left(L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\right)$, *with prescribed d-covector fields* σ_i, τ_a. *That d-connection is given by (2.5) and*

$$\begin{aligned} L^i{}_{jk} &= \tfrac{1}{2}g^{ih}\left(\delta_j g_{hk} + \delta_k g_{jh} - \delta_h g_{jk}\right) + \sigma_j \delta^i_k - g_{jk}\sigma^i, \\ C^a{}_{bc} &= \tfrac{1}{2}h^{ad}\left(\dot{\partial}_b h_{dc} + \dot{\partial}_c h_{bd} - \dot{\partial}_d h_{bc}\right) + \tau_b \delta^a_c - h_{bc}\tau^a, \end{aligned} \tag{3.3}$$

where $\sigma^i = g^{ij}\sigma_j$ *and* $\tau^a = h^{ab}\tau_b$.

Now, we have the following interesting transformations of h- and v-semisymmetric metrical d-connections.

Theorem 3.2 *The transformations of h-and v-semisymmetric metrical d-connections, which preserve the nonlinear connection* $N, D\Gamma(N) \longrightarrow D\bar{\Gamma}(N)$, *are given by*

$$\begin{aligned} \bar{L}^i{}_{jk} &= L^i{}_{jk} + p_j \delta^i_k - g_{jk}p^i, \\ \bar{L}^a{}_{bk} &= L^a{}_{bk}, \\ \bar{C}^i{}_{jc} &= C^i{}_{jc}, \\ \bar{C}^a{}_{bc} &= C^a{}_{bc} + q_b \delta^a_c - h_{bc}q^a, \end{aligned} \tag{3.4}$$

where $p^i = g^{ij}p_j, q^a = h^{ab}q_b$ *and* p_i, q_a *are arbitrary d-covector fields on E.*

We shall denote these transformations by $t(p,q)$.

Theorem 3.3 *The set of all transformations* $t(p,q)$ *given by* (3.4) *is a transformations group* $\mathcal{G}_N$ *of the set of all h- and v-semisymmetric metrical d-connections, with respect to* (2.1), *together with the mapping product*

$$t(p',q') \circ t(p,q) = t(p+p', q+q').$$

This group $\mathcal{G}_N$ is an Abelian group and acts on the set of all h-and v-semisymmetric metrical d-connections, having the same nonlinear connection, transitively.

If we investigate the influences for the torsion and curvature tensor fields, we have

Theorem 3.4 *The following d-tensor fields*

$$R^a{}_{jk},\ P^a{}_{jc},\ C^i{}_{jc}$$

$$T^i{}_{jk} - \tfrac{1}{n-1}\left(T_j\delta^i_k - T_k\delta^i_j\right), \quad S^a{}_{bc} - \tfrac{1}{m-1}\left(S_b\delta^a_c - S_c\delta^a_b\right), \tag{3.5}$$

$$\left(T_j = T^k{}_{jk},\ S_b = S^c{}_{bc}\right),$$

are invariants with respect to transformations of the group $\mathcal{G}_N$.

Theorem 3.5 *For $n > 2,\ m > 2$, the following d-tensor fields $H_j{}^i{}_{kl}, M_b{}^a{}_{cd}$ of h- and v-semisymmetric metrical d-connections, are invariants of the group $\mathcal{G}_N$:*

$$H_j{}^i{}_{kl} = R_j{}^i{}_{kl} + 2\underset{(k,l)}{\mathcal{A}}\left\{\underset{1}{\Omega}{}^{si}_{jk}\left[R_{sl} - Rg_{sl}/2\left(n-1\right)\right]\right\}/\left(n-2\right), \tag{3.6}$$

$$M_b{}^a{}_{cd} = S_b{}^a{}_{cd} + 2\underset{(c,d)}{\mathcal{A}}\left\{\underset{1}{\wedge}{}^{cd}_{bc}\left[S_{ed} - Sh_{ed}/2\left(m-1\right)\right]\right\}/\left(m-2\right), \tag{3.7}$$

where we denoted the alternation operator by $\mathcal{A}$, the Obata operators $\underset{1}{\Omega}$ and $\underset{1}{\wedge}$ of g_{ij} and h_{ab} respectively, by:

$$\underset{1}{\Omega}{}^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k\delta^j_l - g_{kl}g^{ij}\right), \quad \underset{1}{\wedge}{}^{ab}_{cd} = \frac{1}{2}\left(\delta^a_c\delta^b_d - h_{cd}h^{ab}\right),$$

and

$$R_{jk} = R_j{}^l{}_{kl}, \quad S_{bc} = S_b{}^d{}_{cd}, \quad R = g^{ij}R_{ij}, \quad S = h^{ab}S_{ab}.$$

Theorem 3.6 *We have*

$$H_j{}^i{}_{kl} = \overset{M}{H}_j{}^i{}_{kl}, \quad M_b{}^a{}_{cd} = \overset{M}{M}_b{}^a{}_{cd}, \tag{3.8}$$

where $\overset{M}{H}_j{}^i{}_{kl}, \overset{M}{M}_b{}^a{}_{cd}$ are construct by means of the Miron connection of $\mathbb{G}$, $MD\Gamma\left(N\right)$, given by (2.4).

Proof. We consider (3.3) as a transformation of h- and v-semisymmetric metrical d-connections $MD\Gamma\left(N\right) \longrightarrow D\Gamma\left(N\right)$ and we obtain (3.8), with respect to $(3.6), (3.7)$

By straightforward calculus, we get:

Theorem 3.7 *If the Miron connection, $MD\Gamma\left(N\right)$, (2.4), has the properties of h- and v-isotropy:*

$$\overset{M}{R}_j{}^i{}_{kl} = h\left(x,y\right)\left(g_{jk}\delta^i_l - g_{jl}\delta^i_k\right), \quad \overset{M}{S}_b{}^a{}_{cd} = v\left(x,y\right)\left(h_{bc}\delta^a_d - h_{bd}\delta^a_c\right) \tag{3.9}$$

then, we have

$$H_j{}^i{}_{kl} = 0, \quad M_b{}^a{}_{cd} = 0 \tag{3.10}$$

4 The Riemann-local Minkowski model of relativity with h- and v-semisymmetric torsions

In this Section, we consider $E = TM,\ \dim M = n$.
If $h_{ab}(x,y) = h_{ab}(y)$, the metric $\mathbb{G}$ given by (2.1) is called **v-local Minkowski**
We have

Theorem 4.1 *If the metric structure $\mathbb{G}$ given by (2.1) is h-Riemannian, v-locally Minkowski and $h_{ab}(y)$ is weakly regular, then:*

I) The h- and v-semisymmetric metrical d-connection, compatible with respect to $\mathbb{G}$, that corresponds to the 1-forms $\sigma_i(x,y) = \sigma_i(x)$, $\tau_a(x,y) = \tau_a(y)$ has the coefficients given by

$$\begin{aligned} \hat{L}^i{}_{jk} &= \gamma^i_{jk} + \sigma_j \delta^i_k - g_{jk}\sigma^i, \\ \hat{L}^a{}_{bk} &= 0, \\ \hat{C}^a{}_{jc} &= 0, \\ \hat{C}^a{}_{bc} &= \gamma^a_{bc} + \tau_b \delta^a_c - h_{bc}\tau^a, \end{aligned} \tag{4.1}$$

here γ^i_{jk} and γ^a_{bc} are the Levi-Civita connections corresponding to the $g_{ij}(x)$ and $h_{ab}(y)$, respectively.

II) d-tensor fields of (4.1) are

$$\begin{aligned} \hat{T}^i{}_{jk} &= \sigma_j \delta^i_k - \sigma_k \delta^i_j, \\ \hat{R}^a{}_{jk} &= 0, \quad \hat{C}^i{}_{jc} = 0, \quad \hat{P}^a{}_{jc} = 0, \\ \hat{S}^a{}_{bc} &= \tau_b \delta^a_c - \tau_c \delta^a_b. \end{aligned} \tag{4.2}$$

III) d-curvature fields of (4.1) are

$$\begin{aligned} \hat{R}_j{}^i{}_{kl} &= r_j{}^i{}_{kl} + 2 \underset{(k,l)}{\mathcal{A}} \left\{ \underset{1}{\Omega}{}^{si}_{jk} \sigma_{sl} \right\}, \\ \hat{R}_b{}^a{}_{kl} &= 0, \ \hat{P}_j{}^i{}_{kd} = 0, \quad \hat{P}_b{}^a{}_{kd} = 0, \quad \hat{S}_j{}^i{}_{cd} = 0, \\ \hat{S}_b{}^a{}_{cd} &= s_b{}^a{}_{cd} + 2 \underset{(c,d)}{\mathcal{A}} \left\{ \underset{1}{\wedge}{}^{fa}_{bc} \tau_{fd} \right\}, \end{aligned} \tag{4.3}$$

where we denoted $\mathcal{A}$, $\underset{1}{\Omega}$, $\underset{1}{\wedge}$, as in Theorem 3.6, by $r_j{}^i{}_{kl}$, $s_b{}^a{}_{cd}$ the tensor fields of curvatures of $\gamma^i_{jk}, \gamma^a_{bc}$ *respectively, and*

$$\begin{aligned} \sigma_{ij} &= \sigma_{i \hat{\restriction} j} - 2\sigma_i \sigma_j + g_{ij}\alpha, \quad 2\alpha = g^{ij}\sigma_i\sigma_j, \\ \tau_{ab} &= \tau_{a \hat{|} b} - 2\tau_a \tau_b + h_{ab}\beta, \quad 2\beta = h^{ab}\tau_a\tau_b; \end{aligned} \tag{4.4}$$

(here $\hat{\restriction}$ and $\hat{|}$ denote the h- and v-covariante derivatives with respect to $D\hat{\Gamma}$, (3.4)).

Remark 4.1 For d-connection (4.1), $h\,(h)$-torsion and $h\,(hh)$-curvature are internal, only and $v\,(v)$-torsion and $v\,(vv)$-curvature are external, only.

Let $\mathbb{G}$ be a metrical h-Riemannian, v-locally Minkowski on $E = TM$, v-weakly regular (Theorem 2.1) and we denote $r_{ij} = r_i{}^k{}_{jk}$, $r = g^{ij}r_{ij}$, $s_{ab} = s_a{}^c{}_{bc}$, $s = h^{ab}s_{ab}$, etc.

Taking into account the results of [1] and [2] (see, also [5] and [11] , pg.83), we obtain

Theorem 4.2 *The Einstein equations of d-connection $D\hat{\Gamma}$, (4.1) of Riemann-local Minkowski metric $\mathbb{G}$, (2.1), are given by*

$$\begin{aligned} &r_{jk} - \tfrac{1}{2}(r+s)\,g_{jk} - (n-2)\left(\sigma_{jk} - \tfrac{1}{2}\sigma g_{jk}\right) + \tfrac{1}{2}(m-1)\,\tau g_{jk} = \kappa \overset{1}{\mathcal{T}}_{jk}, \\ &s_{bc} - \tfrac{1}{2}(s+r)\,h_{bc} - (m-2)\left(\tau_{bc} - \tfrac{1}{2}\tau h_{bc}\right) + \tfrac{1}{2}(n-1)\,\sigma h_{bc} = \varkappa \overset{4}{\mathcal{T}}_{bc}, \end{aligned} \tag{4.5}$$

where κ is constant, $\overset{1}{\mathcal{T}}_{ij}, \overset{2}{\mathcal{T}}_{ij} = 0, \overset{3}{\mathcal{T}}_{ib} = 0, \overset{4}{\mathcal{T}}_{ab}$ are the components in the adapted basis of the energy-momentum tensor field

$$\mathcal{T} = \overset{1}{\mathcal{T}}_{ij}dx^i \otimes dx^j + \overset{4}{\mathcal{T}}_{ab}\delta y^a \otimes \delta y^b, \tag{4.6}$$

$$\sigma = 2g^{ij}\sigma_{ij}, \quad \tau = 2h^{ab}\tau_{ab}. \tag{4.7}$$

Theorem 4.3 *The conservation law in this model is given by*

$$\begin{aligned} &\left[r^i_j - \frac{1}{2}r\delta^i_j - (n-2)\left(\sigma^i_j - \frac{1}{2}\sigma\delta^i_j\right)\right]_{\widehat{\,|i}} = 0, \\ &\left[s^a_b - \frac{1}{2}s\delta^a_b - (m-2)\left(\tau^a_b - \frac{1}{2}\tau\delta^a_b\right)\right]_{\widehat{|a}} = 0 \end{aligned} \tag{4.8}$$

where

$$r^i_j = g^{ik}r_{kj}, \quad \sigma^i_j = g^{ik}\sigma_{kj}, \quad s^a_b = h^{ac}s_{cb}, \quad \tau^a_b = h^{ac}\tau_{cb}. \tag{4.9}$$

Theorem 4.4 *The divergence of energy-momentum tensor is as follows*

$$\left(Div\overset{1}{\mathcal{T}}\right)_j = \frac{1}{\kappa}U_j = 0, \quad \left(Div\overset{4}{\mathcal{T}}\right)_b = \frac{1}{\kappa}U_b = 0, \tag{4.10}$$

where

$$\left(Div\overset{1}{\mathcal{T}}\right)_j = \overset{1}{\mathcal{T}}{}^i_{j\widehat{\,|\,i}}, \quad \left(Div\overset{4}{\mathcal{T}}\right)_b = \overset{4}{\mathcal{T}}{}^a_{b\widehat{\,|\,a}}$$

and

$$\begin{aligned} U_j &= \frac{1}{2}\sigma_i\left(r^i_j - \frac{1}{2}\sigma\delta^i_j\right) + (n-2)\left[\sigma^i\left(\partial_i\sigma_j - \partial_j\sigma_i\right) - \frac{1}{2}\left(\partial_j\alpha - 3\alpha\sigma_j\right)\right], \\ U_b &= \frac{1}{2}\tau_a\left(s^a_b - \frac{1}{2}\tau\delta^a_b\right) + (m-2)\left[\tau^a\left(\dot{\partial}_a\tau_b - \dot{\partial}_b\tau_a\right) - \frac{1}{2}\left(\dot{\partial}_b\beta - 3\beta\tau_b\right)\right]. \end{aligned} \tag{4.11}$$

Generally, the equations (4.4) are not identically satisfied. Therefore, we need to find the conditions for 1-forms σ_i and τ_a, such that the conservation law to be satisfied.

In this aim, if we denote by $_{\shortparallel}$ the covariant derivative with respect to Levi-Civita connection γ^i_{jk} of $g_{ij}(x)$ and with $\|$ the covariant derivative with respect to Levi-Civita connection γ^a_{bc} of $h_{ab}(y)$, we obtain

Theorem 4.5 *The conservation law in the Riemann-local Minkowski model with h- and v-semisymmetric torsions is satisfied, if and only if the fields of* 1*-forms* σ_i *and* τ_a *satisfies the equations*

$$\begin{aligned} &\left(r^i_j - \tfrac{1}{2} r\delta^i_j\right)\sigma_i + (n-2)\left[\sigma_{j\shortparallel i}\sigma^i + \sigma\sigma_j + (n-4)\,\partial_j\alpha - 3\,(n-3)\,\sigma_j\alpha\right] = 0,\\ &\left(s^a_b - \tfrac{1}{2} s\delta^a_b\right)\tau_a + (m-2)\left[\tau_{b\|a}\tau^a + \tau\tau_b + (m-4)\,\dot{\partial}_b\beta - 3\,(m-3)\,\tau_b\beta\right] = 0. \end{aligned} \tag{4.12}$$

Now, we consider $\dim M = 4$. We have, also $m = 4$.

Taking into account the above notations, we obtain:

Theorem 4.6 Let $\mathbb{G}$ be a Riemannian-locally Minkowski structure on $E = TM$, $\dim M = 4$, v-weakly regular. Then:

(i) The Einstein equations of the d-connection (4.1) are given by:

$$\begin{aligned} r_{jk} - \frac{1}{2}\left(r + s - 2\sigma - 3\tau\right) g_{jk} - 2\sigma_{jk} &= \varkappa \overset{1}{T}_{jk}, \\ s_{bc} - \frac{1}{2}\left(r + s - 2\tau - 3\sigma\right) h_{bc} - 2\tau_{bc} &= \varkappa \overset{4}{T}_{bc}. \end{aligned} \tag{4.13}$$

(ii) The conservation law is given by:

$$\begin{aligned} \left[r^i_j - \frac{1}{2}\left(r - 2\sigma\right)\delta^i_j - 2\sigma^i_j\right]_{\widehat{\shortmid} i} &= 0, \\ \left[s^a_b - \frac{1}{2}\left(s - 2\tau\right)\delta^a_b - 2\tau^a_b\right]_{\widehat{\mid} a} &= 0. \end{aligned} \tag{4.14}$$

(iii) The conservation law is satisfied if and only if the fields of 1*-forms* $\sigma_i(x)$ *and* $\tau_a(y)$ *satisfies the equations*

$$\begin{aligned} \left(r^i_j - \frac{1}{2} r\delta^i_j\right)\sigma_i + 2\left[\sigma_{j\,\shortparallel\,i}\sigma^i + (\sigma - 3\alpha)\,\sigma_j\right] &= 0, \\ \left(s^a_b - \frac{1}{2} s\delta^a_b\right)\tau_a + 2\left[\tau_{b\,\|\,a}\tau^a + (\tau - 3\beta)\,\tau_b\right] &= 0. \end{aligned} \tag{4.15}$$

References

[1] M. Anastasiei, *Vector Bundles. Einstein Equations,* An. Şt. Univ. Iaşi, 32, s I-a Mat., 1986, 17–24.

[2] M. Anastasiei, *Conservation Laws in the* (V, H)*-Bundle Model of Relativity,* Tensor, N. S., 46, 1987, 323–328.

[3] A. K. Aringazin, G. S. Asanov, *Problems of Finslerian Theory of Gauge Fields and Gravitation,* Rep. Math. Phys., 25 (2), 1988, 35–93.

[4] G. S. Asanov, *Fibred Generalization of the Gauge Field Theory. Finslerian and Jet Gauge Fields,* Moskow Univ., 1989.

[5] Gh. Atanasiu, *Connexion semi-symétriques sur l'espace total d'un fibré vectoriel,* Memoriile Sect.Şt., IV, VIII, 1, 1985, Ed. Acad. Române, 1987, 33–41.

[6] Gh. Atanasiu, *Metrical Kawaguchi Connections on the Total Space of a Vector Bundle and Some Special Subordinated Cases,* Tensor, N. S., 46, 1987, 248–264.

[7] E. Cartan, *Sur les equations de la gravitation d'Einstein,* Gautier Villars, Paris, 1922.

[8] R. S. Ingarden, *Differential Geometry and Physics,* Tensor, N. S., 30, 1976, 201–209.

[9] A. Lichnerowitz, *Theories relativistes de la gravitation et de l'électromagnetisme,* Masson, Paris, 1955.

[10] R. Miron, *Metrical Finsler Structures and Metrical Finsler Connections,* J. Math. Kyoto Univ., 23, 1983, 219–224.

[11] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications,* Kluwer Acad. Publ., F. Th. Ph., 59, 1994.

[12] R. Miron, S. Watanabe, S. Ikeda, *Some Connections on Tangent Bundle and their Applications to the General Relativity,* Tensor, N. S., 46, 1987, 8–22.

[13] I. D. Novikov, V. P. Frolov, *Physics of Black Holes,* Kluwer Acad. Publ., F. Th. Ph., 27, 1989.

[14] E. Stoica, *A geometrical characterization of normal Finsler connections,* Analele Şt. Univ. "Al. I. Cuza" Iaşi, 30, sIa Mat., 1984, 3.

[15] E. Stoica, *G-structures of Finsler type* (in Romanian), Ph. D. Thesis, "Al. I. Cuza" Univ. of Iaşi, 1986, 1–144 + XXVI.

[16] K. Yano, S. Ishihara, *Tangent and cotangent bundle,* M. Dekker, New-York, 1973.

The Pavlov's 4-polyform of momenta $K(p) = \sqrt[4]{p_1p_2p_3p_4}$ and its applications in Hamilton geometry

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The aim of this paper is to associate a generalized Hamilton space to a 4-pseudoscalar product defined in a Cartan-Minkowski space. The components of the 4-pseudoscalar product $G^{ijkl}(x,p)$ are given in terms of Cartan metrical fundamental d-tensor $g^{*ij}(x,p)$. In the particular case of the Pavlov function $K(p) = \sqrt[4]{p_1p_2p_3p_4}$ the components of the v-covariant derivation of this generalized Hamilton space are derived.

MSC: 53B40, 53C60, 53C07.

1 4-pseudoscalar product in a Cartan-Minkowski space

Let M^n be an n-dimensional differential manifold of class C^∞, (T^*M, π^*, M) its cotangent bundle and (x^i, p_i) the local coordinates on T^*M.

Let $K: T^*M \to \mathbf{R}_+$, $K(x,p) = K(p) > 0$, be a locally Cartan-Minkowski metrical function. Note that the function $K(p)$ is 1-positive homogeneous in the argument p. Moreover, the Cartan-Minkowski function $K(p)$ produces the Cartan fundamental metrical d-tensor field

$$g^{*ij}(x,p) = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_j}.$$

Now, let us introduce the *"4-pseudoscalar product"*, given by

$$(\omega^1, \omega^2, \omega^3, \omega^4) = G^{ijkl}(x,p)\omega_i^1\omega_j^2\omega_k^3\omega_l^4,$$

where $\omega^1, \omega^2, \omega^3, \omega^4 \in \Gamma(T^*M)$ and

$$G^{ijkl}(x,p) = \frac{1}{4!}\frac{\partial^4 K^4}{\partial p_i \partial p_j \partial p_k \partial p_l}.$$

Remark 1. *In the particular case of a 4-dimensional manifold M^4 and of the Pavlov metrical function*

$$K(p) = \sqrt[4]{p_1p_2p_3p_4},$$

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where $p_1p_2p_3p_4 > 0$, *this 4-pseudoscalar product was studied by Pavlov [8]. In this case, we have*

$$G^{ijkl}(x,p) = \frac{1}{4!},$$

that is

$$(\omega^1, \omega^2, \omega^3, \omega^4) = \frac{1}{4!} \sum_{\tau \in \sigma_4} \omega^1_{\tau(1)} \omega^2_{\tau(2)} \omega^3_{\tau(3)} \omega^4_{\tau(4)}.$$

Taking into account the 1-homogeneity of the Cartan-Minkowski function $K(p)$, the following statements hold good:

- $G^{ijkl}(x,p)$ is totally symmetric with respect to the indices i, j, k, l;
- $G^{ijk0}(x,p) = G^{ijkl}(x,p)p_l = \dfrac{1}{4!} \dfrac{\partial^3 K^4}{\partial p_i \partial p_j \partial p_k}$ is 1-homogeneous in p;
- $G^{ij00}(x,p) = G^{ijkl}(x,p)p_k p_l = \dfrac{1}{12} \dfrac{\partial^2 K^4}{\partial p_i \partial p_j}$ is 2-homogeneous in p;
- $G^{i000}(x,p) = G^{ijkl}(x,p)p_j p_k p_l = \dfrac{1}{4} \dfrac{\partial K^4}{\partial p_i}$ is 3-homogeneous in p;
- $G^{0000}(x,p) = G^{ijkl}(x,p)p_i p_j p_k p_l = K^4$ is 4-homogeneous in p.

Let us define the *"pseudo-scalar product"*

$$< \omega^1, \omega^2 >_\omega = \frac{1}{K^2}(\omega^1, \omega^2, \omega, \omega),$$

where $\omega = p_i dx^i$ is the canonical Liouville 1-form of the Cartan-Minkowski space $(M^n, K(p))$ and $\omega^1, \omega^2 \in \Gamma(T^*M)$.

Remark 2. *For the Pavlov metric* $K(p) = \sqrt[4]{p_1p_2p_3p_4}$, *it is obvious that the entity* $<, >_\omega$ *is bilinear in the two arguments and it satisfies the axioms of a pseudo-scalar product [9].*

Note that we locally have

$$< \omega^1, \omega^2 >_\omega = \frac{1}{K^2} G^{ijkl}(x,p) \omega^1_i \omega^2_j p_k p_l = \frac{G^{ij00}(x,p)}{K^2} \omega^1_i \omega^2_j$$

and hence

$$g^{ij}(x,p) = \frac{G^{ij00}(x,p)}{K^2} = \frac{1}{12K^2} \frac{\partial^2 K^4}{\partial p_i \partial p_j}.$$

Supposing that the metrical d-tensor $g^{ij}(x,p)$ is non-degenerate, we can give the following important result:

Proposition 3. *The pair* $GH^n = (M^n, g^{ij}(x,p))$ *is a generalized Hamilton space. The absolute energy of this space is exactly* $\mathcal{E} = K^2$.

Proof. The absolute energy of the generalized Hamilton space GH^n is

$$\mathcal{E} = g^{ij}(x,p)p_ip_j = \frac{G^{ij00}(x,p)}{K^2}p_ip_j = \frac{G^{0000}(x,p)}{K^2} = \frac{K^4}{K^2} = K^2.$$

■

As a conclusion, the metrical d-tensor field produced by the absolute energy $\mathcal{E} = K^2$ is

$$\frac{1}{2}\frac{\partial^2 \mathcal{E}}{\partial p_i \partial p_j} = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_j} = g^{*ij}(x,p),$$

that is exactly the Cartan fundamental metrical d-tensor field of the Cartan-Minkowski space $(M^n, K(p))$.

Remark 4. *i) From the 1-homogeneity of the Cartan-Minkowski function K and the definition of $g^{ij}(x,p)$, it follows that*

$$\frac{1}{2}\frac{\partial \mathcal{E}}{\partial p_i} = g^{ij}(x,p)p_j = g^{*ij}(x,p)p_j.$$

ii) Note that we also have $\mathcal{E} = K^2 = g^{ij}p_ip_j = g^{*ij}p_ip_j$.

2 The local components of the 4-pseudoscalar product

In the sequel, we establish the relation between the generalized Hamilton metric $g^{ij}(x,p)$ and the Cartan-Minkowski one $g^{*ij}(x,p)$.

Theorem 5. *The following relation is true:*

$$3g^{ij} = g^{*ij} + 2\frac{g^{*i0}g^{*j0}}{g^{*00}},$$

where $g^{*i0} = g^{*ij}p_j$ and $g^{*00} = g^{*ij}p_ip_j$.

Proof. For a regular generalized Hamilton metric, we have the equality [5], [6]

$$g^{*ij} = g^{ij} + \frac{\partial g^{ik}}{\partial p_j}p_k. \tag{2.1}$$

Taking into account that $\mathcal{E} = K^2$, we can write $g^{ij}(x,p)$ in the more convenient form

$$g^{ij}(x,p) = \frac{1}{12\mathcal{E}}\frac{\partial^2 \mathcal{E}^2}{\partial p_i \partial p_j}.$$

Now, replacing $g^{ij}(x,p)$ into (2.1) and using the fact that $\mathcal{E}$ is 2-homogeneous, a straightforward computation leads to

$$g^{*ij} = 3g^{ij} - \frac{1}{2\mathcal{E}}\frac{\partial \mathcal{E}}{\partial p_i}\frac{\partial \mathcal{E}}{\partial p_j}. \tag{2.2}$$

Because we have
$$\frac{1}{2}\frac{\partial \mathcal{E}}{\partial p_i} = g^{ij}(x,p)p_j,$$
we get
$$g^{*ij} = 3g^{ij} - 2\frac{g^{i0}g^{j0}}{g^{00}},$$
where $g^{i0} = g^{ij}p_j$ and $g^{00} = g^{ij}p_ip_j = \mathcal{E}$.

The converse relation is immediate from (2.2), if we notice that
$$2g^{*i0} = \frac{\partial^2 \mathcal{E}}{\partial p_i \partial p_j}p_j = \frac{\partial \mathcal{E}}{\partial p_i}$$
and
$$\mathcal{E} = g^{*00} = g^{*ij}p_ip_j,$$
and the claim is proved. ■

Let us express now the coefficients $G^{ijkl}(x,p)$ in terms of the Cartan fundamental metrical d-tensor field $g^{*ij}(x,p)$. By a straightforward computation, we obtain
$$4!G^{ijkl} = 2S\mathcal{E}^{ijk}\mathcal{E}^{l} + 2(\mathcal{E}^{ij}\mathcal{E}^{kl} + \mathcal{E}^{ik}\mathcal{E}^{jl} + \mathcal{E}^{il}\mathcal{E}^{jk}) + 2\mathcal{E}\mathcal{E}^{ijkl},$$
where the sign S means a cyclic sum and the upper indices of $\mathcal{E}$ mean the derivation with the corresponding components of $p = (p_i)$. If in the above equality we replace
$$\mathcal{E} = g^{*00}, \mathcal{E}^{i} = 2g^{*i0}, \mathcal{E}^{ij} = 2g^{*ij}, \mathcal{E}^{ijk} = 2g^{*ij,k}, \mathcal{E}^{ijkl} = 2g^{*ij,kl},$$
then the required relation is
$$4!G^{ijkl} = 2Sg^{*ij,k}g^{*l0} + 2(g^{*ij}g^{*kl} + g^{*ik}g^{*jl} + g^{*il}g^{*jk}) + 2g^{*00}g^{*ij,kl}.$$
Let us denote $g^{*}_{\omega^1\omega^2} = g^{*ij}\omega^1_i\omega^2_j$, where $\omega^1, \omega^2 \in \Gamma(T^*M)$. By a simple computation, we obtain

Theorem 6. *The components of the 4-pseudoscalar product are expressed by*
$$(\theta,\theta,\eta,\eta) = 2\sum g^{*}_{\theta\theta,\eta}g^{*}_{\eta 0} + 2(g^{*}_{\theta\theta}g^{*}_{\eta\eta} + 2g^{*}_{\theta\eta}g^{*}_{\theta\eta}) + 2g^{*00}g^{*}_{\theta\theta,\eta\eta}$$
and
$$\begin{aligned}(\theta,\theta,\theta,\eta) + (\theta,\eta,\eta,\eta) \ = \ & 2\sum g^{*}_{\theta\theta,\theta}g^{*}_{\eta 0} + 6g^{*}_{\theta\theta}g^{*}_{\theta\eta} + 2g^{*00}g^{*}_{\theta\theta,\theta\eta} + \\ & +2\sum g^{*}_{\theta\eta,\eta}g^{*}_{\eta 0} + 6g^{*}_{\theta\eta}g^{*}_{\eta\eta} + 2g^{*00}g^{*}_{\theta\eta,\eta\eta},\end{aligned}$$
where $\theta, \eta \in \Gamma(T^*M)$.

The v-covariant derivation in the Pavlov case $K(p) = \sqrt[4]{p_1p_2p_3p_4}$

Let us consider that M^4 is a 4-dimensional manifold. Let

$$K(p) = \sqrt[4]{p_1p_2p_3p_4},$$

where $p_1p_2p_3p_4 > 0$, be the Berwald-Moor metric studied by Pavlov in [8]. The Hamiltonian absolute energy is then

$$\mathcal{E} = K^2 = \sqrt{p_1p_2p_3p_4}.$$

In this case, denoting $(a, b, c, d) = (p_1, p_2, p_3, p_4)$, the matrix $(g^{ij}(x, p))$ and its inverse $(g_{ij}(x, p))$ are

$$[g^{ij}(x,p)] = \frac{1}{12K^2}\begin{pmatrix} 0 & cd & bd & bc \\ cd & 0 & ad & ac \\ bd & ad & 0 & ab \\ bc & ac & ab & 0 \end{pmatrix}$$

and

$$[g_{ij}(x,p)] = 12K^2\begin{pmatrix} -\dfrac{2a}{3dcb} & \dfrac{1}{3cd} & \dfrac{1}{3db} & \dfrac{1}{3cb} \\ \dfrac{1}{3cd} & -\dfrac{2b}{3dca} & \dfrac{1}{3da} & \dfrac{1}{3ca} \\ \dfrac{1}{3db} & \dfrac{1}{3da} & -\dfrac{2c}{3dba} & \dfrac{1}{3ba} \\ \dfrac{1}{3cb} & \dfrac{1}{3ca} & \dfrac{1}{3ba} & -\dfrac{2d}{3cba} \end{pmatrix},$$

where $\det g = -3(abcd)^2 \neq 0$ for $abcd > 0$.

In other words, we have

$$g^{ii}(x,p) = 0, \qquad \forall i = \overline{1,4},$$
$$g^{i_1i_2}(x,p) = \frac{p_{i_3}p_{i_4}}{12\mathcal{E}},\ i_1 \neq i_2,$$

where $\mathcal{E} = K^2 = \sqrt{p_1p_2p_3p_4}$. The inverse matrix has the components

$$g_{ii}(x,p) = -\frac{8p_i^2}{\mathcal{E}}, \qquad \forall i = \overline{1,4},$$
$$g_{i_1i_2}(x,p) = \frac{4\mathcal{E}}{g^{i_1i_2}} = \frac{48\mathcal{E}^2}{p_{i_3}p_{i_4}} = 48p_{i_1}p_{i_2},\ i_1 \neq i_2,$$

where $\mathcal{E} = K^2 = \sqrt{p_1p_2p_3p_4}$. Moreover, we have

$$g^{*ii}(x,p) = -\frac{\mathcal{E}}{8p_i^2},\ g^{*i_1i_2}(x,p) = \frac{p_{i_3}p_{i_4}}{8\mathcal{E}},$$

where $g^{*ij}(x,p) = \dfrac{1}{2}\dfrac{\partial^2 K^2}{\partial p_i \partial p_j}$ and $\mathcal{E} = K^2$. Note that we have

$$g^{i_1 i_2}(x,p) = \frac{2}{3} g^{*i_1 i_2}(x,p), \ i_1 \neq i_2.$$

Let

$$C^{hjk} = g^{hi} C_i^{jk} = -\frac{1}{2}\left(\frac{\partial g^{hk}}{\partial p_j} + \frac{\partial g^{jh}}{\partial p_k} - \frac{\partial g^{jk}}{\partial p_h}\right)$$

be the components that determine the coefficients C_i^{jk} of the vertical covariant derivation [6] produced by the generalized Hamiltonian metric $g^{ij}(x,p)$.

Remark 7. *Note that the tensor field*

$$C_i^{jk} = -\frac{1}{2} g_{is}\left(\frac{\partial g^{sk}}{\partial p_j} + \frac{\partial g^{js}}{\partial p_k} - \frac{\partial g^{jk}}{\partial p_s}\right)$$

gives the v-coefficients of a v-covariant derivation with the metrical property [6]:

$$g^{ij}|^k = \frac{\partial g^{ij}}{\partial p_k} + C_s^{ik} g^{sj} + C_s^{jk} g^{is} = 0.$$

In the following, we maintain the same convention: we denote by i_1, i_2, i_3, i_4 the distinct values from 1 to 4 ($i_j \neq i_k$ for $j \neq k$). Then, for the distinct indices i_1, i_2, i_3, we have

$$\begin{aligned} C^{i_1 i_2 i_3} &= -\frac{1}{3}\left(\frac{\partial g^{*i_1 i_3}}{\partial p_{i_2}} + \frac{\partial g^{*i_2 i_1}}{\partial p_{i_3}} - \frac{\partial g^{*i_2 i_3}}{\partial p_{i_1}}\right) = \\ &= -\frac{1}{3}\left(\frac{1}{2}\mathcal{E}^{i_1 i_3 i_2} + \frac{1}{2}\mathcal{E}^{i_2 i_1 i_3} - \frac{1}{2}\mathcal{E}^{i_2 i_3 i_1}\right) = -\frac{1}{6}\mathcal{E}^{i_1 i_2 i_3}. \end{aligned}$$

In the same way, it follows that

$$\begin{aligned} C^{i_1 i_1 i_2} &= C^{i_1 i_2 i_1} = 0, \\ C^{i_2 i_1 i_1} &= -\frac{1}{3}\mathcal{E}^{i_1 i_1 i_2}, \\ C^{i_1 i_1 i_1} &= 0. \end{aligned}$$

We obtain now the coefficients $C_i^{jk} = g_{is} C^{sjk}$ in terms of the energy $\mathcal{E}$.

Theorem 8. *The v-coefficients of the v-covariant derivation of the generalized Hamilton space $GH^4 = (M^4, g^{ij}(x,p))$ are given by the formulas:*

$$\begin{aligned} C_{i_1}^{i_2 i_3} &= \frac{4}{3}\frac{p_{i_1}^2}{\mathcal{E}}\mathcal{E}^{i_1 i_2 i_3} - 8 p_{i_1} p_{i_4} \mathcal{E}^{i_2 i_3 i_4}, \\ C_{i_1}^{i_1 i_2} &= -8 p_{i_1} p_{i_3} \mathcal{E}^{i_1 i_2 i_3} - 8 p_{i_1} p_{i_4} \mathcal{E}^{i_1 i_2 i_4}, \\ C_{i_1}^{i_2 i_2} &= \frac{8}{3}\frac{p_{i_1}^2}{\mathcal{E}}\mathcal{E}^{i_1 i_2 i_2} - 16 p_{i_1} p_{i_3} \mathcal{E}^{i_2 i_2 i_3} - 16 p_{i_1} p_{i_4} \mathcal{E}^{i_2 i_2 i_4}, \\ C_{i_1}^{i_1 i_1} &= -16 p_{i_1} p_{i_2} \mathcal{E}^{i_2 i_1 i_1} - 16 p_{i_1} p_{i_3} \mathcal{E}^{i_3 i_1 i_1} - 16 p_{i_1} p_{i_4} \mathcal{E}^{i_4 i_1 i_1}. \end{aligned}$$

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References

[1] Gh. Atanasiu, *The invariant expression of Hamilton geometry*, Tensor N. S. 47, 3 (1988), 225–234.

[2] Gh. Atanasiu, M. Hashiguchi, *Semi-symmetric Miron connections in dual Finsler spaces*, Reports of the Faculty of Science, Kagoshima Univ., Japan, 20 (1987), 43–49.

[3] Gh. Atanasiu, F. C. Klepp, *Nonlinear connection in cotangent bundle*, Publ. Math., Debrecen, Hungary 39, f. 1–2 (1991), 107-111.

[4] Gh. Atanasiu, Gh. Munteanu, *New aspects in geometry of time dependent generalized metrics*, Tensor N. S. 50, 3 (1991), 248–255.

[5] V. Balan, N. Brînzei, *Einstein equations for the Berwald-Moor type Euclidian-locally Minkowski relativistic model*, (2005), preprint.

[6] R. Miron, *Hamilton geometry*, An. Şt. Univ. "Al. I. Cuza", Iaşi, Romania, 35 (1989), 33–85.

[7] R. Miron, *The geometry of Cartan spaces*, Prog. in Math., India, 22 (1988), 1–38.

[8] D. G. Pavlov, *Four-dimensional time*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 31–39.

[9] D. G. Pavlov, *Generalization of scalar product axioms*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 5–18.

The Lagrangian-Hamiltonian Formalism in Gauge Complex Field Theories

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An introduction in the study of gauge field theory in terms of complex Finsler geometry on the total space of a G-complex vector bundle E was made by us in [Mu2]. Here we briefly recall the obtained results and similar notions are investigated on the dual bundle E^* by complex Legendre transformation (the $\mathcal{L}$-dual process).

The complex field equations are determined with respect to a gauge complex vertical connections. The complex Hamilton equations are write for the general $\mathcal{L}$-dual Hamiltonian obtained as a sum of particle Hamiltonian, Yang-Mills and Hilbert-Einstein Hamiltonians.

1 Introduction

Gauge theory is called to use the differential geometric methods in order to describe the interactions of fields over a certain symmetry group G.

For initial Yang-Mills gauge theory the Lagrangians had strict local gauge symmetry. After introducing the spontaneously symmetry breaking and Higgs mechanism usually the gauge group is of complex matrices and the gauge Lagrangians are defined over a complexified G-bundle, for instance the Klein-Gordon Lagrangian, Higgs particle Lagrangian or complex fermion-gravitation, etc. These Lagrangians act on the first order jet manifold, which plays the role of a finite dimensional configuration space of fields. By Legendre morphism, intrinsically related to a Lagrange manifold is the multimomentum Hamiltonian ([Ar, Sa]...) which works on the corresponding phase manifold (the dual G-bundle). Although in Quantum Mechanics the Lagrangian and Hamiltonian formalism is a usual technique, in the gauge field theory it remains almost unknown, especially for the complex situation.

In the present paper, our goal is to introduce a gauge complex field theory in terms of complex Lagrange and Hamilton geometries, [Mu3], extended to an associated fiber of one complex bundle and respectively to its dual bundle.

In the first section, we briefly introduce the geometric machinery which characterize these geometries and then we study the gauge invariance of the main geometric presented objects.

In the next section we recall from [Mu2] the basic notions concerning the complex Euler-Lagrange field equations and the complex gauge invariant Lagrangian for field particle, complex Yang-Mills and Hilbert-Einstein Lagrangians are also written. In the final we translate by complex Legendre transformation the studied results on the dual bundle, and thus we obtain the complex Hamilton field equations and the $\mathcal{L}$-dual Hamiltonians.

2 The geometric background

In [Mu3], we make an exhaustive study of complex Lagrange (particularly Finsler) and Hamilton (Cartan) spaces, which have as a base manifold the holomorphic tangent respectively cotangent bundles of a complex manifold M.

Part of the notions studied in this book can extend to a G-complex vector bundle, and here we do this. By this way, since the extension is natural, we will omit the proofs. For more details in this part see the introductory paper [Mu2].

Let M be a complex manifold, $(z^k)_{k=\overline{1,n}}$ complex coordinates in a local chart $(U_\alpha, \varphi_\alpha)$, $\pi : E \to M$ a complex vector bundle of $\mathbb{C}^m$ fiber, and $\eta = \eta^a s_a$ a local section on E, $a = \overline{1,m}$. Consider G a closed m-dimensional Lie group of complex matrices, whose elements are holomorphic functions over M.

Definition 2.1. *A structure of G-complex vector bundle of E is a fibration with transition functions taking values in G.*

This means that if $z'^i = z'^i(z)$ is a local change of charts on M, then the section η changes by the rule

$$z'^i = z'^i(z) \ \ ; \ \ \eta'^a = M^a_b(z)\eta^b \ , \tag{2.1}$$

where $M^a_b(z) \in G$ and $\partial M^a_b(z)/\partial \bar{z}^k = 0$ for any $a, b = \overline{1,m}$ and $k = \overline{1,n}$.

E has a natural structure of $(n+m)-$complex manifold, a point of E is designed by $u = (z^k, \eta^a)$.

The geometry of E manifold (the total space), endowed with a Hermitian metric $g_{a\bar{b}} = \partial^2 L/\partial\eta^a\partial\bar{\eta}^b$ derived from a homogeneous Lagrangian $L : E \to \mathbb{R}^+$, was intensively studied by T. Aikou ([Ai1, Ai2, Ai3, Mu3]). Let us consider the vertical bundle $VE = \ker \pi^T \subset T'E$. A local base for its sections is $\{\dot{\partial}_a := \frac{\partial}{\partial\eta^a}\}_{a=\overline{1,m}}$ and from (1.1) we have the changes $\dot{\partial}_a = M^b_a(z)\dot{\partial}'_b$. The vertical distribution V_uE is isomorphic to the sections module of E in u.

A supplementary subbundle of VE in $T'E$, i.e. $T'E = VE \oplus HE$, is called a *complex nonlinear connection*, in brief (*c.n.c.*). A local base for the horizontal distribution H_uE, called *adapted* for the (*c.n.c.*), is $\{\delta_k := \frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^a_k \frac{\partial}{\partial\eta^a}\}_{k=\overline{1,n}}$, where $N^a_k(z,\eta)$ are the coefficients of the (*c.n.c.*). Locally $\{\delta_k\}$ defines an isomorphism of $\pi^T(T'M)$ with HE if and only if they are changed under the rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and hence N^a_k obey a certain rule of transformation.

Definition 2.2. *A gauge complex transformation on G-complex vector bundle E, is a pair $\Upsilon = (F_0, F_1)$, where locally $F_1 : E \to E$ is an F_0-holomorphic isomorphism which satisfies $\pi^T \circ F_1 = F_0 \circ \pi^T$.*

A gauge complex transformation $\Upsilon : u \to \widetilde{u}$ is locally given by a system of analytic functions:

$$\widetilde{z}^i = X^i(z) \, ; \qquad \widetilde{\eta}^a = Y^a(z,\eta) \tag{2.2}$$

with the regularity condition: $\det\left(\frac{\partial X^i}{\partial z^j}\right) \cdot \det\left(\frac{\partial Y^a}{\partial\eta^b}\right) \neq 0$.

Let be $X^i_j := \frac{\partial X^i}{\partial z^j}$ and $Y^a_b := \frac{\partial Y^a}{\partial \eta^b}$; and denote by $X^{\bar{i}}_{\bar{j}}$, $Y^{\bar{a}}_{\bar{b}}$ their conjugates.

Obviously, from the holomorphy requirements we have $X^i_{\bar{j}} = \frac{\partial X^i}{\partial \bar{z}^j} = 0$ and $Y^a_{\bar{j}} = \frac{\partial Y^a}{\partial \bar{z}^j} = 0, Y^a_{\bar{b}} = \frac{\partial Y^a}{\partial \bar{\eta}^b} = 0$.

A (*c.n.c.*) is said to be gauge, (*g.c.n.c*), if the adapted frames transforms into $d-$complex gauge fields, i.e. in addition to $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ we have

$$\delta_j = X^i_j \delta_{\widetilde{i}}; \quad \dot{\partial}_b = Y^a_b \dot{\partial}_{\widetilde{a}} \ , \tag{2.3}$$

where $\delta_{\widetilde{i}} = \frac{\delta}{\delta \widetilde{z}^k}$ and $\dot{\partial}_{\widetilde{a}} = \frac{\partial}{\partial \widetilde{\eta}^a}$.

Let us consider now the dual G-bundle $\pi^* : E^* \to M$ of the G-bundle E. Likewise as above, E^* has a natural structure of complex manifold, a point is denoted by $u^* = (z^k, \zeta_a)$, $k = \overline{1,n}$ and $a = \overline{1,m}$, with the following change of charts,

$$z'^i = z'^i(z)\,; \quad \zeta'_a = \overset{*}{M}{}^b_a (z)\zeta_b \tag{2.4}$$

where $\overset{*}{M}{}^b_a$ is the inverse of M^b_a from (1.1).

By a similar way as for E manifold, we consider $T'E^*$ the holomorphic tangent bundle of E^* and $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \zeta_a}\}$ a base for $T'_{u^*}E^*$. Then $\{\dot{\partial}^a := \frac{\partial}{\partial \zeta_a}\}_{a=\overline{1,m}}$ will be a base for the sections in the vertical bundle $VE^* = \ker \pi^{*T}$ and theret follows the changes $\dot{\partial}^a = \overset{*}{M}{}^a_b \, \dot{\partial}'^b$. A (*c.n.c*) on E^* is defined by a decomposition $T'E^* = VE^* \oplus HE^*$. The local base for the horizontal distribution $H_{u^*}E^*$ will be denoted by $\{\delta^*_k := \frac{\delta^*}{\delta z^k} = \frac{\partial}{\partial z^k} + N_{ak}\frac{\partial}{\partial \zeta_a}\}_{k=\overline{1,n}}$ and will be called adapted for the (*c.n.c.*) if $\delta^*_k = \frac{\partial z'^j}{\partial z^k}\delta^{*\prime}_j$.

A complex gauge transformation on E^* is defined by a pair $\overset{*}{\Upsilon} = (\overset{*}{F}_0, \overset{*}{F}_1)$, where locally $\overset{*}{F}_1$: $E^* \to E^*$ is an $\overset{*}{F}_0$-holomorphic isomorphism which satisfies $\pi^{*T} \circ \overset{*}{F}_1 = \overset{*}{F}_0 \circ \pi^{*T}$.

The local expression of a complex gauge transformation on E^* is:

$$\widetilde{z}^i = X^i(z)\,; \qquad \widetilde{\zeta}_a = Y_a(z, \zeta) \tag{2.5}$$

with the regularity isomorphism condition assumed.

Let be $X^i_j := \frac{\partial X^i}{\partial z^j}$ and $Y^b_a := \frac{\partial Y_a}{\partial \zeta_b}$; then obviously, from the holomorphy requirements, we have $X^i_{\bar{j}} = \frac{\partial X^i}{\partial \bar{z}^j} = 0$ and $Y_{a\bar{j}} = \frac{\partial Y_a}{\partial \bar{z}^j} = 0$; $Y^{\bar{b}}_a = \frac{\partial Y_a}{\partial \bar{\zeta}_b} = 0$.

The various $d-$geometric objects on E^* are defined in complete analogy with those defined by us on E.

A (*c.n.c.*) on E^* is gauge, in brief it is (*g.c.n.c.*), if its adapted frames transform by the rules

$$\delta^*_j = X^i_j \delta^*_{\widetilde{i}} \ ; \ \dot{\partial}^a = Y^a_b \dot{\partial}^{\widetilde{b}} \ , \tag{2.6}$$

where $\delta^*_{\widetilde{i}} = \frac{\delta^*}{\delta \widetilde{z}^i}$ and $\dot{\partial}^{\widetilde{a}} = \frac{\partial}{\partial \widetilde{\zeta} a}$.

Now, let us consider $L : E \to \mathbb{R}$ a complex regular Lagrangian, that is the function $L(z, \eta)$ defines a metric tensor $g_{a\bar{b}} = \partial^2 L / \partial \eta^a \partial \bar{\eta}^b$ which is Hermitian,

$g_{a\bar{b}} = \overline{g_{b\bar{a}}}$ and $\det(g_{a\bar{b}}) \neq 0$ in any point $u = (z, \eta)$ of E. By $g^{\bar{b}a}$ is denoted its inverse metric tensor. The following weighty result was proved in [Mu2]

Proposition 2.1. *If $L(z, \eta)$ is a gauge invariant Lagrangian on E, i.e. $L(z, \eta) = L(\tilde{z}, \tilde{\eta})$, then*

$$N_k^a = g^{\bar{b}a} \frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^b} \tag{2.7}$$

is a (g.c.n.c.).

A fundamental notion in our study is that of d-complex vertical connection on E. The metric tensor $g_{a\bar{b}}$ determines a metric Hermitian structure $\mathbf{G} = g_{a\bar{b}} d\eta^a \otimes d\bar{\eta}^b$ on the vertical bundle VE. The connection form of a d-complex vertical connection D is written according to (7.2.4) from [Mu3] as follows

$$\omega_b^a = L_{bk}^a dz^k + L_{b\bar{k}}^a d\bar{z}^k + C_{bc}^a \delta\eta^c + C_{b\bar{c}}^a \delta\bar{\eta}^c \,, \tag{2.8}$$

where $(dz^k, \delta\eta^c = d\eta^c + N_k^c dz^k)$ is the dual adapted base of the (*c.n.c.*) and $(L_{bk}^a,\ L_{b\bar{k}}^a,\ C_{bc}^a,\ C_{b\bar{c}}^a)$ are the coefficients of the vertical connection D.

From the general theory of Hermitian connection it result a unique metrical Hermitian connection with respect to $\mathbf{G}$ and of $(1,0)$-type, called the *Chern-Lagrange complex connection*, which can be obtained by the same technique as we did for the $T'M$ bundle (Corollary 5.1.1, [Mu3]):

$$\overset{CL}{L}{}_{bk}^a = g^{\bar{d}a} \delta_k g_{b\bar{d}}\,; \quad \overset{CL}{L}{}_{b\bar{k}}^a = 0\,; \quad \overset{CL}{C}{}_{bc}^a = g^{\bar{d}a} \dot{\partial}_c g_{b\bar{d}}\,; \quad \overset{CL}{C}{}_{b\bar{c}}^a = 0. \tag{2.9}$$

A simplification presents a special partial complex connection (cf. [Ai2, Ai3]), called the *complex Bott connection*, which is not metrical but has a very simple expression

$$D_X Y = v\,[X, Y]\ , \quad \forall X \in HE\,, \quad Y \in VE.$$

From the calculus of the Lie brackets, see (7.1.10) in [Mu3], it results that the connection form of the complex Bott connection is

$$\omega_b^a = \overset{B}{L}{}_{bk}^a\, dz^k\,, \quad \text{where} \quad \overset{B}{L}{}_{bk}^a = \frac{\partial N_k^a}{\partial \eta^b}.$$

The unique nonzero component of the complex Bott connection on E is

$$\Omega_b^a = R_{bi\bar{j}}^a\, dz^i \wedge d\bar{z}^j \quad \text{with} \quad R_{bi\bar{j}}^a = -\delta_{\bar{j}}\, \overset{B}{L}{}_{bi}^a\,, \tag{2.10}$$

while the nonzero components of complex Chern-Lagrange connection are more numerous. For this reason the complex Bott connection is an appropriate connection for our approach.

A complex vertical connection determines the following derivative laws on VE :

$$\overset{h}{D}_{\delta_k}\dot{\partial}_b = L^a_{bk}\dot{\partial}_a\,;\quad \overset{\bar{h}}{D}_{\delta_{\bar{k}}}\dot{\partial}_b = L^a_{b\bar{k}}\dot{\partial}_a\,;$$
$$\overset{v}{D}_{\dot{\partial}_c}\dot{\partial}_b = C^a_{bc}\dot{\partial}_a\,;\quad \overset{\bar{v}}{D}_{\dot{\partial}_{\bar{c}}}\dot{\partial}_b = C^a_{b\bar{c}}\dot{\partial}_a\,.$$

The covariant derivatives of a vertical field $\Phi = \Phi^a\frac{\partial}{\partial\eta^a}$ will be denoted with $\Phi^a{}_{\shortmid k}$, $\Phi^a{}_{\shortmid \bar{k}}$ and $\Phi^a{}_{|b}$, $\Phi^a{}_{|\bar{b}}$, where

$$\Phi^a{}_{\shortmid k} = \delta_k\Phi^a + L^a_{bk}\Phi^b\,;\quad \Phi^a{}_{\shortmid \bar{k}} = \delta_{\bar{k}}\Phi^a + L^a_{b\bar{k}}\Phi^b\,; \tag{2.11}$$
$$\Phi^a{}_{|c} = \dot{\partial}_c\Phi^a + C^a_{bc}\Phi^b\,;\quad \Phi^a{}_{|\bar{c}} = \dot{\partial}_{\bar{c}}\Phi^a + C^a_{b\bar{c}}\Phi^b.$$

If D is a gauge invariant connection, because $\delta_k, \dot{\partial}_c$ and $\delta_{\bar{k}}, \dot{\partial}_{\bar{c}}$ are gauge invariant, we may conclude that these covariant derivatives are gauge invariant as long as Φ is gauge invariant.

On E^* manifold we may introduce the similar d-complex connections with respect to a metric tensor derived from a regular Hamiltonian.

A regular complex Hamiltonian is a real valued function $H : E^* \to \mathbb{R}$ such that $h^{\bar{b}a} = \partial^2 H/\partial\zeta_a\partial\bar{\zeta}_b$ defines a Hermitian metric tensor on E^*, i.e. $\overline{h^{\bar{b}a}} = h^{\bar{a}b}$ and $\det(h^{\bar{b}a}) \neq 0$ on E^*. Let $h_{a\bar{b}}$ be its inverse. A regular complex Hamiltonian determines a metric Hermitian structure on the vertical bundle VE^*, defined by $\mathbf{H} = h^{\bar{b}a}\, d\zeta_a \otimes d\bar{\zeta}_b$. In completly analogy with the result on E we check

Proposition 2.2. *Let $H(z,\zeta)$ be a complex gauge invariant Hamiltonian on E^*, i.e. $H(z,\zeta) = H(\widetilde{z},\widetilde{\zeta})$. Then,*

$$N_{ak} = -h_{a\bar{b}}\frac{\partial^2 H}{\partial z^k\partial\bar{\zeta}_b} \tag{2.12}$$

is a (g.c.n.c.) on E^*.

With respect to adapted frames of (2.12) (*c.n.c.*) a d-vertical connection on VE^* is denoted by $\overset{*}{D}$ and has the following components,

$$\overset{h*}{D}_{\delta^*_k}\dot{\partial}^a = H^a_{bk}\dot{\partial}^b\,;\quad \overset{\bar{h}*}{D}_{\delta^*_{\bar{k}}}\dot{\partial}^a = H^a_{b\bar{k}}\dot{\partial}^b\,;$$
$$\overset{v*}{D}_{\dot{\partial}^c}\dot{\partial}^a = C^{ac}_b\dot{\partial}^b\,;\quad \overset{\bar{v}*}{D}_{\dot{\partial}^{\bar{c}}}\dot{\partial}^a = C^{a\bar{c}}_b\dot{\partial}^b$$

and their conjugates by $\overline{D_X Y} = D_{\bar{X}}\bar{Y}$.

It results that its connection form is

$$\omega^a_b = H^a_{bk}dz^k + H^a_{b\bar{k}}d\bar{z}^k + C^{ac}_b\delta\zeta_c + C^{a\bar{c}}_b\delta\bar{\zeta}_c\,, \tag{2.13}$$

with respect again to the dual adapted frame of the (2.12) (*c.n.c.*).

There exists a unique metric connection with respect to the Hermitian structure **H** on VE^* which is of $(1,0)$-type,

$$\overset{CH}{H^a_{bk}}= h^{\bar{d}a}\delta^*_k h_{b\bar{d}}\,; \quad \overset{CH}{H^a_{b\bar{k}}}= 0\,; \quad \overset{CH}{C^{ac}_{b}}= -h_{b\bar{d}}\dot{\partial}^c h^{\bar{d}a}\,; \quad \overset{CH}{C^{a\bar{c}}_{b}}= 0, \tag{2.14}$$

called the *complex Chern-Hamilton vertical connection.*

A partial vertical *connection of Bott type* on VE^* is given by the vertical part of the bracket, $\overset{*B}{D}_X\, Y = v\,[X,Y]\,,\ \forall X \in HE^*,\ Y \in VE^*$, and has the following connection form

$$\omega^a_b = \overset{B}{H^a_{bk}}\, dz^k\,,\quad \omega^a_b = \overset{B}{L^a_{bk}}\, dz^k\,,\quad \text{where}\quad \overset{B}{H^a_{bk}}= \frac{\partial N_{bk}}{\partial \zeta_a}\,. \tag{2.15}$$

The unique nonzero component of the complex Bott connection on E^* is

$$\Omega^a_b = \overset{*}{R^a_{bi\bar{j}}}\, dz^i \wedge d\bar{z}^j \quad \text{with}\quad \overset{*}{R^a_{bi\bar{j}}}= -\delta_{\bar{j}}\, \overset{B}{H^a_{bi}}\,. \tag{2.16}$$

If H is a gauge invariant Hamiltonian, then both complex Chern-Hamilton and Bott connection on VE^* are gauge invariant. The proof derives from the fact that $h^{\bar{b}a}$ and N_{bk} given by (2.12) are gauge invariant and δ^*_k, $\dot{\partial}^a$ are gauge adapted frames.

The sections of VE^* are 1−forms, $\Phi = \Phi_a(z,\zeta)\frac{\partial}{\partial \zeta_a} = \Phi_a\dot{\partial}^a$. Then a vertical connection $\overset{*}{D}$ on VE^* induces covariant derivatives which act under the section Φ as follows

$$\begin{aligned}
\Phi_{a\,\shortmid\, k} &= \delta^*_k\Phi_a + H^b_{ak}\Phi_b\,; \quad \Phi_{a\,\shortmid\, \bar{k}} = \delta^*_{\bar{k}}\Phi_a + H^b_{a\bar{k}}\Phi_b\,; \\
\Phi_{a\,|\,^c} &= \dot{\partial}^c\Phi_a - C^{bc}_{a}\Phi_b\,; \quad \Phi_{a\,|\,^{\bar{c}}} = \dot{\partial}^{\bar{c}}\Phi_a - C^{b\bar{c}}_{a}\Phi_b.
\end{aligned} \tag{2.17}$$

Now, we recall that in [Mu3] a Lagrangian-Hamiltonian formalism was introduced for the holomorphic tangent bundle $T'M$ by using a complex Legendre morphism. We proved that by complex Legendre transformation (the $\mathcal{L}$-dual process) the image of a complex Lagrange space is (at least locally) a complex Hamilton space. The complex Legendre transformation pushes-forward and its inverse pulls-back the various described geometric objects of a complex Lagrange space and complex Hamilton space, respectively.

Without more other details we can reproduce here, generalizing the $T'M$ case, the process of $\mathcal{L}$-duality for the pairs $(E, L\ )$ and (E^*, H). Let us consider L a local Lagrangian on $U \subset E$. Then the map $\Lambda : U \subset E \to \bar{U}^* \subset \overline{E^*}$

$$\Lambda : (z^k, \eta^a) \to \Big(z^k, \bar{\zeta}_a = \frac{\partial L}{\partial \eta^a}\Big) \tag{2.18}$$

is a local diffeomorphism. Since the sections of VE are identified with those of E, we can extend Λ to the open set of VE. By conjugation, the local diffeomorphism $\Lambda \times \overline{\Lambda}$

sends the sections of the complexified bundle $VE \times \overline{VE}$ into sections of $VE^* \times \overline{VE^*}$. This (local) morphism will be called the *complex Legendre transformation*, briefly $(c.L.t)$.

Then, locally the function

$$H = \zeta_a \eta^a + \bar{\zeta}_a \bar{\eta}^a - L \tag{2.19}$$

defines a regular (local) Hamiltonian on E^*.

By the inverse $\Lambda^{-1} : \bar{U}^* \to U$, $\Lambda^{-1} : (z^k, \bar{\zeta}_a) \to (z^k, \eta^a = \frac{\partial H}{\partial \zeta_a})$, from a Hamiltonian structure on E^* a Lagrangian structure on E is obtained.

The properties obtained by $(c.L.t)$ are called $\mathcal{L}$-dual one to other. Like in [Mu3], in the following with "$*$" will be designed the image of an object by Λ and with "o" their image by Λ^{-1}. Some of the assertions of § 6.7 from [Mu3] can be easily translated in our framework. For instance, in virtue of (2.19) we have

Proposition 2.3. *The unique pair of* (*c.n.c.*) *on* VE *and respective on* VE^* *which correspond by* $\mathcal{L}$*-duality is given by (2.7) and (2.12). Moreover, if* L *is gauge invariant Lagrangian then both of these* (*c.n.c.*) *are gauge invariant.*

Further, simple calculus proves that

Proposition 2.4. *The following equalities hold by* $\mathcal{L}$*-duality:*

i) $\left(\frac{\delta}{\delta z^k}\right)^* = \frac{\delta^*}{\delta z^k}\,; \quad \left(\frac{\partial}{\partial \eta^a}\right)^* = h_{a\bar{b}} \frac{\partial}{\partial \bar{\zeta}_b}\,; \quad \left(\frac{\delta^*}{\delta z^k}\right)^o = \frac{\delta}{\delta z^k}\,; \quad \left(\frac{\partial}{\partial \zeta_a}\right)^o = g_{a\bar{b}} \frac{\partial}{\partial \bar{\eta}_b}$

ii) $\left(dz^k\right)^* = d^* z^k\,; \quad (\delta \eta^a)^* = h^{\bar{b}a} \delta \bar{\zeta}_b\,; \quad \left(d^* z^k\right)^o = dz^k\,; \quad (\delta \zeta_a)^o = g_{a\bar{b}} \delta \bar{\eta}^b$

iii) $(\mathbf{G})^* = \mathbf{H}$ *and* $(\mathbf{H})^o = \mathbf{G}$.

If D is a metrical connection, then its dual $(D)^*$ is metrical too, moreover their curvatures correspond by $\mathcal{L}$-duality, $(R(X,Y)Z)^* = \overset{*}{R}(X^*, Y^*)Z^*$. We note that the image by $\mathcal{L}$-duality of the complex Bott connection is not the complex Bott connection on E^*. However, we shall use both of these connections for theirs simple expressions and convenience in calculus.

We end this section with a remark. With respect to adapted frames of the $\mathcal{L}$-dual (2.7) and (2.12) (*c.n.c.*) we can consider the almost simplectic forms ω and θ, $\mathcal{L}$-dual one to other, $\theta = (\omega)^*$,

$$\omega = g_{a\bar{b}}\ \delta\eta^a \wedge \delta\bar{\eta}^b\,; \quad \theta = h^{\bar{b}a}\ \delta\zeta_a \wedge \delta\bar{\zeta}_b. \tag{2.20}$$

3 The Euler-Lagrange complex field equations

Let E be a G-complex vector bundle over M. From physical point of view a section of E is treated as a field particle. The field particle dynamics assumes to consider the variation of a Lagrangian particle $L_p : E \to \mathbb{R}$, which is a first order differential operator over the sections of E. This is $L_p = L_p(j_1\Phi)$, where $\Phi = \Phi^a s_a$ is a section and $j_1\Phi$ its first jet. Enlarge this is, $\hat{L}_p(\Phi) = L_p(\Phi^a, \partial_i\Phi^a, \partial_{\bar{i}}\Phi^a, \dot{\partial}_b\Phi^a, \dot{\partial}_{\bar{b}}\Phi^a)$ where $\partial_i = \frac{\partial}{\partial z^i}$, $\dot{\partial}_b = \frac{\partial}{\partial \eta^b}$.

The field equations imply to find the particle Φ from the variational principle $\delta\mathcal{A} = \frac{d}{dt} |_{t=0} \mathcal{A}(\Phi + t\delta\Phi)$, where $\mathcal{A}(\Phi) = \int \hat{L}_p(\Phi)$ is the action integral. Actually, the action integral is defined on a compact subset $\theta \subset E$ and, for the independence of the integral at the changes of local charts, instead of $\hat{L}_p(\Phi)$ we consider the Lagrangian density $\mathcal{L}_p(\Phi) = \hat{L}_p(\Phi) \mid g \mid^2$, where $\mid g \mid = \mid \det g_{a\bar{b}} \mid$ and $g_{a\bar{b}} = \partial^2 L_p/\partial\eta^a\partial\bar{\eta}^b$ (since L_p depends on (z, η) by means of Φ). In the following the regularity condition for L_p will be assumed.

The problem of solutions for the field equations is one difficult, first because the chosen Lagrangian needs to be one gauge invariant (by means of Φ and its derivatives). Then the derivations in field equations are with respect to the natural frames ∂_i, $\dot{\partial}_b$ which, for a gauge invariant expression of the field equations, need to be replaced with the adapted frames of one $(g.c.n.c.)$, i.e. $\partial_i = \delta_i + N_i^a \; \dot{\partial}_a$. Such a way was followed in [Mu1] in order to obtain the gauge invariant field equations on $T'M$. The modern gauge field theories is based on the "minimal replacement" principle ([Bl, DM, Pa]...), which is nothing but a generalization of Einstein's covariance principle.

The minimal replacement principle consists in replacement in $L_p(\Phi^a, \partial_i\Phi^a,$ $\partial_{\bar{i}}\Phi^a, \dot{\partial}_i\Phi^a, \dot{\partial}_{\bar{i}}\Phi^a)$ partial derivatives with covariant derivatives of a gauge invariant vertical connection, possible the complex Bott connection. At the first glance this seems to be a notational process, but it is a more subtle idea. The connection becomes a dynamical variable which joints mechanics with the geometry of the space. Thus we will study the variation of the action for the Lagrangian $L_p(\Phi, D\Phi)$. But for the beginning let us introduce, as in standard theory, the (complex) currents on E :

$$J(\Phi, D\Phi) \wedge \delta\omega := \frac{d}{dt} |_{t=0} \mathcal{L}(\Phi, D\Phi + t\delta\omega) \tag{3.1}$$

where $\delta\omega$ is a variation for the connection form of D connection.

Direct calculus in (2.2) yields the following complex currents:

$$\overset{h}{J}{}^i_a = \frac{\partial\mathcal{L}}{\partial\Phi^a_{\;\shortmid i}} \;;\quad \overset{\bar{h}}{J}{}^{\bar{i}}_a = \frac{\partial\mathcal{L}}{\partial\Phi^a_{\;\shortmid \bar{i}}} \;;\quad \overset{v}{J}{}^b_a = \frac{\partial\mathcal{L}}{\partial\Phi^a_{\;\mid b}} \;;\quad \overset{\bar{v}}{J}{}^{\bar{b}}_a = \frac{\partial\mathcal{L}}{\partial\Phi^a_{\;\mid \bar{b}}} \tag{3.2}$$

which implicitly contain the following components

$$\overset{h}{J}{}^{ib}_a = \frac{\partial\mathcal{L}}{\partial L^a_{bi}} \;;\quad \overset{\bar{h}}{J}{}^{\bar{i}b}_a = \frac{\partial\mathcal{L}}{\partial L^a_{b\bar{i}}} \;;\quad \overset{v}{J}{}^{cb}_a = \frac{\partial\mathcal{L}}{\partial C^a_{bc}} \;;\quad \overset{\bar{v}}{J}{}^{\bar{c}b}_a = \frac{\partial\mathcal{L}}{\partial C^a_{b\bar{c}}} \;.$$

Now, let us focus attention to the variation of the action integral, $\delta\mathcal{A}(\Phi) = \frac{d}{dt} |_{t=0} \int_\theta \mathcal{L}(\Phi, D\Phi + t\delta\omega) = 0$. This implies

$$\int_\theta \left\{ \frac{\partial\mathcal{L}}{\partial\Phi^a}\delta\Phi^a + \frac{\partial\mathcal{L}}{\partial\Phi^a_{\;\shortmid i}}\delta(\Phi^a_{\;\shortmid i}) + \frac{\partial\mathcal{L}}{\partial\Phi^a_{\;\shortmid \bar{i}}}\delta(\Phi^a_{\;\shortmid \bar{i}}) + \frac{\partial\mathcal{L}}{\partial\Phi^a_{\;\mid b}}\delta(\Phi^a_{\;\mid b}) + \frac{\partial\mathcal{L}}{\partial\Phi^a_{\;\mid \bar{b}}}\delta(\Phi^a_{\;\mid \bar{b}}) \right\} = 0.$$

Further, for instance the calculus of the second term involves

$$\frac{\partial\mathcal{L}}{\partial\Phi^a_{\,|\,i}}\delta(\Phi^a_{\,|\,i}) = \frac{\partial\mathcal{L}}{\partial\Phi^a_{\,|\,i}}\frac{\partial}{\partial z^i}(\delta\Phi^a) + \frac{\partial\mathcal{L}}{\partial\Phi^a_{\,|\,i}}\delta(L^a_{bi}\Phi^b) =$$

$$= \frac{\partial}{\partial z^i}\left(\frac{\partial\mathcal{L}}{\partial\Phi^a_{\,|\,i}}\delta(\Phi^a)\right) - \frac{\partial}{\partial z^i}\left(\frac{\partial\mathcal{L}}{\partial\Phi^a_{\,|\,i}}\right)(\delta\Phi^a) + \frac{\partial\mathcal{L}}{\partial\Phi^a_{\,|\,i}}\delta(L^a_{bi}\Phi^b)$$

and analogously for the other terms. If we assume a nul variation on the boundary of θ, then finally for the variation of the integral action we obtain

$$\frac{\partial\mathcal{L}}{\partial\Phi^a} = \frac{\partial}{\partial z^i}(\frac{\partial\mathcal{L}}{\partial\Phi^a_{\,|\,i}}) + \frac{\partial}{\partial \bar{z}^i}(\frac{\partial\mathcal{L}}{\partial\Phi^a_{\,|\,\bar{\imath}}}) + \frac{\partial}{\partial \eta^b}(\frac{\partial\mathcal{L}}{\partial\Phi^a_{\,|\,b}}) + \frac{\partial}{\partial \bar{\eta}^b}(\frac{\partial\mathcal{L}}{\partial\Phi^a_{\,|\,\bar{b}}}) - < J, \delta\omega > ,$$

where, $$< J, \delta\omega >= \int_\theta \{ \overset{h}{J}{}^i_a\, \delta(L^a_{bi}\Phi^b) + \overset{\bar{h}}{J}{}^i_a\, \delta(L^a_{b\bar{\imath}}\Phi^b) + \overset{v}{J}{}^c_a\, \delta(C^a_{bc}\Phi^b) + \overset{\bar{v}}{J}{}^{\bar{c}}_a\, \delta(L^a_{b\bar{c}}\Phi^b)\}.$$

Taking into account the (2.3) expressions of the complex currents, in adapted frames of the (2.7) (*c.n.c.*) the previous field equations are written

$$\frac{\partial\mathcal{L}}{\partial\Phi^a} = \delta_i \overset{h}{J}{}^i_a + \delta_{\bar{\imath}} \overset{\bar{h}}{J}{}^{\bar{\imath}}_a + \dot{\partial}_b \overset{v}{J}{}^b_a + \dot{\partial}_{\bar{b}} \overset{\bar{v}}{J}{}^{\bar{b}}_a + N^b_i \dot{\partial}_b \overset{h}{J}{}^i_a + N^{\bar{b}}_{\bar{\imath}} \dot{\partial}_{\bar{b}} \overset{\bar{h}}{J}{}^{\bar{\imath}}_a - < J, \delta\omega > . \tag{3.3}$$

The (2.4) equations, for $a = \overline{1, m}$, will be called the *complex field equations* of the particle Φ.

The gauge invariance of the Lagrangian L_p, with respect to particle Φ and their covariant derivatives, implies the gauge invariance of the complex currents and consequently the gauge invariance of (2.4) complex field equations. Certainly, everywhere we take in discussion a gauge invariant vertical connection D, particularly the complex Chern-Lagrange or Bott connections.

For the existence of a such gauge invariant particle Lagrangian subsequently we propose a particle Lagrangian of Klein-Gordon type, quite generalized and adequate for various field applications. For this purpose we consider a pair of Hermitian metrics, one being the Lorentz metric $\gamma_{i\bar{j}}(z)$ on the complex world manifold M. The second is a mass Hermitian metric $\gamma_{a\bar{b}}(z,\eta)$ on E, derived from the matter field Lagrangian $L_m = m_{a\bar{b}}\Phi^a\bar{\Phi}^b$ ($m_{a\bar{b}}$ the Hermitian mass matrix). In the last period one Finsler-Minkowski metric kick up some interest in applications of Finsler geometry in relativity, namely the Berwald-Moor metric. We can propose instead of L_m the folowing complex version of Berwald-Moor metric $L_{BM} = \{\prod_a(\eta^a\bar{\eta}^a)\}^{\frac{1}{m}}$. If we wish to connect our field theory with other, a good choose instead of mass metric is one derived from an external Lagrangian with physical meaning, for instance an Antonelly-Shimada complex Lagrangian $L_{AS} = e^{2\sigma(z)}\{\sum_a(\eta^a\bar{\eta}^a)^m\}^{\frac{1}{m}}$ (see [Mu3]), with applications in biology and relativistic optics. However, each of these last complex Lagrangians could be of interest for complex Finsler geometry.

The Hermitian metric $\gamma_{a\bar{b}}$ determines the (2.7) (*c.n.c.*) and its adapted frames. Then, a gauge invariant Lagrangian with respect to a complex vertical connection D and a real valued potential function $V(\Phi)$ can be

$$L_p(\Phi, D\Phi) = \frac{1}{2}\sum_a \{\gamma^{\bar{j}i} D_{\delta_i}\Phi^a D_{\delta_{\bar{j}}}\bar{\Phi}^a + \gamma^{\bar{b}c} D_{\dot{\partial}_c}\Phi^a D_{\dot{\partial}_{\bar{b}}}\bar{\Phi}^a\} + V(\Phi). \tag{3.4}$$

Note that L_p contains informations about matter field by means of $\gamma_{a\bar{b}}$ and by covariant derivatives of the field. $V(\Phi)$ is a potetial fuctions which, for instance, can be considered as beeing $V(\Phi) = \mp m^2 \parallel \Phi \parallel^2 - \frac{1}{4} \parallel \Phi \parallel^4$, with $\parallel \Phi \parallel^2 = \sum_a \Phi^a \bar{\Phi}^a$, for the exact symmetry or for the broken symmetry, respectively.

As we already know from the classical field theory, this particle Lagrangian $L_p(\Phi, D\Phi)$ is not able, quite so in a generalized form, to offer a solid physical theory because it does not contain enough the geometrical aspects of the space (curvature, etc.). For this purpose, in the generalized Maxwell equations the total Lagrangian of electrodynamics is taken in the form:

$$L_e(\Phi, D\Phi) = L_p(\Phi, D\Phi) + L_{YM}(D), \tag{3.5}$$

where

$$L_{YM}(D) = -\frac{1}{2}\Omega \wedge {}^*\Omega \tag{3.6}$$

is a connection Lagrangian, Ω being the curvature form of D and ${}^*\Omega$ is its Hodge dual.

For the complex Bott connection on E we obtain

$$L_{YM}(\overset{B}{D}) = -\frac{1}{2}\sum_{a,b} \gamma^{\bar{j}i}\gamma^{\bar{k}l} R^a_{bi\bar{j}} R^a_{bl\bar{k}}\ .$$

The curvature form of Chern-Lagrange connection is a bit complicate hence we renounce to apply here.

Since, $\delta_D \mathcal{A}_e(\Phi, D\Phi) = \delta_D \mathcal{A}_p(\Phi, D\Phi) + \delta_D \mathcal{A}_{YM}(D)$, and $\delta_D \mathcal{A}_p(\Phi, D\Phi) = -< J, \delta\omega >= -< \delta\omega, {}^* J >$ (${}^*J$ is the dual form current), a computation like in [Pa], yields for the complex Bott connection that $\delta_D \mathcal{A}_{YM}(D) =< \delta\omega, {}^* D^*\Omega >$. Hence, for the complex Bott connection we have that $D^*\Omega = {}^* J$, or else

$$\delta_k \Omega^a_b + L^a_{ck}\Omega^c_b - L^c_{bk}\Omega^a_c = {}^* \overset{h}{J}{}^a_{kb}, \tag{3.7}$$

this generally being called the *complex Yang-Mills equation* on E.

Also we can check that $D^*J = 0$ (the same calculus like for formulae (6.7) from [DM]) and therefore the complex currents are conservative. We note that in this complex Y-M equation the curvature form of Bott connection contains implicitly the Hermitian metric tensor $g_{a\bar{b}} = \partial^2 L_p / \partial\eta^a \partial\bar{\eta}^b$ of the particle Lagrangian.

Finally, for coupling with gravity we again consider the Lorentz Hermitian metric $\gamma_{i\bar{j}}(z)$ on M, which now we assume it derives from a gravitational potential, and $\mathcal{G} = \gamma_{i\bar{j}} dz^i \wedge d\bar{z}^j + g_{a\bar{b}} \delta\eta^a \wedge \delta\bar{\eta}^b$ a metric structure on $T_C E$.

By $S_{i\bar{j}} = \sum S^k_{ki\bar{j}}$ and by $\rho(\gamma) = \gamma^{\bar{j}i} S_{i\bar{j}}$ we denote the Ricci curvature and scalar, respectively, with respect to L-C connection of $\gamma_{i\bar{j}}$ metric lifted on $T_C E$. Also by $R_{i\bar{j}} = \sum R^a_{ai\bar{j}}$ and $\rho(g) = \gamma^{\bar{j}i} R_{i\bar{j}}$ we have the Ricci curvature and scalar, respectively, with respect to Bott connection of the g metric. The sum $\rho = \rho(\gamma) + \rho(g)$ generates an Hilbert-Einstein type Lagrangian $L_G = -\frac{1}{\chi}\rho$, where χ is the universal constant.

The complex Einstein equations on E will be

$$S_{i\bar{j}} - \frac{1}{2}\rho(\gamma)\gamma_{i\bar{j}} = \chi T_{i\bar{j}}\,; \quad R_{i\bar{j}} - \frac{1}{2}\rho(g)\gamma_{i\bar{j}} = \chi T_{i\bar{j}} \tag{3.8}$$

where $T_{i\bar{j}}$ is the stress-energy tensor of the potential gravity $\gamma_{i\bar{j}}(z)$ on M.

The total Lagrangian for coupling gravity with electodynamics (complex inhomogeneus Maxwell equations) is

$$L_t(\Phi, D\Phi) = L_p(\Phi, D\Phi) + L_{YM}(D) + L_G. \tag{3.9}$$

4 Hamiltonian gauge complex theory

In the preview section, in fact a field particle was treated as section $\Phi = \Phi^a(z,\eta)s_a$ on E which induced naturally the section $\Phi = \Phi^a(z,\eta)\dot{\partial}_a$ on VE. The associated particle Lagrangian is a function of Φ and the covariant derivative $D\Phi$ is with respect to a complex vertical connection, particularly for simplicity the Bott connection. Indeed, L_p depends implicitly by the base point $u = (z,\eta) \in E$. Then by complex Legendre transformation (2.18), (2.19), the sections of VE (plus their conjugates) will be send into sections of VE^*. We obtain hereby the field particles on E^*:

$$\Phi_a(z,\zeta) = h_{a\bar{b}}\Phi^{\bar{b}}(z, \eta := \frac{\partial H_p}{\partial \zeta}) = \left(\frac{\partial L_p}{\partial \Phi^a}\right)^*. \tag{4.1}$$

Consequently, by (2.19) we obtain a Hamiltonian for the $\mathcal{L}$- dual particle $\Phi^* = \Phi_a \dot{\partial}^a$,

$$H_p(\Phi^*) = \Phi_a\Phi^a + \bar{\Phi}_a\bar{\Phi}^a - L_p(\Phi). \tag{4.2}$$

We note that H_p is gauge invariant with respect to the $\mathcal{L}$-dual gauge transformation $\overset{*}{\Upsilon}$ of Υ, forasmuch L_p is gauge with respect to Υ. As well, we proved that the $\mathcal{L}$-dual of a vertical connection on VE is a vertical connection VE^*, i.e. $(D)^* = \overset{*}{D}$, and moreover if one is gauge the other is gauge too. Hence, $L_p(\Phi^a, D\Phi^a)$ by (4.2) determines the $\mathcal{L}$-dual Hamiltonian $H_p(\Phi_a, \overset{*}{D}\,\Phi_a)$.

Now, by taking $\overset{*}{D}\,\Phi_a$ as an independent variable for the Hamiltonian, we can write down the following variation

$$\delta H = \frac{\partial H}{\partial \Phi_a}(\delta\Phi_a) + \frac{\partial H}{\partial \Phi_{a\,\shortmid\, i}}(\delta\Phi_{a\,\shortmid\, i}) + \frac{\partial H}{\partial \Phi_{a\,|\,b}}(\delta\Phi_{a\,|\,b}) + \ conjugates$$

By the same symbols ω and θ from (2.20) we denoting the $\mathcal{L}$-dual symplectic forms associated to the variations of field particle. Thus, we may write θ as being

$$\theta = h^{\bar{b}a}\{\delta\Phi_a \wedge \delta\bar{\Phi}_b + \sum_i \delta\Phi_{a\,|\,i} \wedge \delta\bar{\Phi}_{b\,|\,i} + \sum_c \delta\Phi_{a\,|\,\dot{c}} \wedge \delta\bar{\Phi}_{b\,|\,\dot{c}}\} \tag{4.3}$$

Let as associate to Φ^a, on the curve $t \to \Phi^a(z(t), \eta(t))$, the vector field X_{Φ^a}

$$X_{\Phi^a} = \frac{\delta\Phi^a}{dt}\frac{\delta}{\delta\Phi^a} + \sum_i \frac{\delta\Phi^a_{\,|\,i}}{dt}\frac{\delta}{\delta\Phi^a_{\,|\,i}} + \sum_b \frac{\delta\Phi^a_{\,|\,b}}{dt}\frac{\delta}{\delta\Phi^a_{\,|\,b}} + conjugates.$$

By $\mathcal{L}$-duality on the curve $t \to \Phi_a(z(t), \zeta(t))$ we obtain the vector field $\overset{*}{X}_{\Phi_a} = h^{\bar{b}a}\,(X_{\bar{\Phi}^b})^*$,

$$\overset{*}{X}_{\Phi^a} = \frac{\delta\Phi_a}{dt}\frac{\delta}{\delta\Phi_a} + \sum_i \frac{\delta\Phi_{a\,|\,i}}{dt}\frac{\delta}{\delta\Phi_{a\,|\,i}} + \sum_b \frac{\delta\Phi_{a\,|\,\dot{b}}}{dt}\frac{\delta}{\delta\Phi_{a\,|\,\dot{b}}} + conjugates.$$

The requirement $i_{\overset{*}{X}_{\Phi^a}}\theta = \delta H$ of integral curve for $\overset{*}{X}_{\Phi^a}$ yields

$$h^{\bar{b}a}\frac{\delta\bar{\Phi}_b}{dt} = -\frac{\partial H}{\partial\Phi_a}\ ;\ \ h^{\bar{b}a}\frac{\delta\bar{\Phi}_{b\,|\,i}}{dt} = -\frac{\partial H}{\partial\Phi_{a\,|\,i}}\ ;\ \ h^{\bar{b}a}\frac{\delta\bar{\Phi}_{b\,|\,\dot{c}}}{dt} = -\frac{\partial H}{\partial\Phi_{a\,|\,\dot{c}}}.$$

Tacking variations $\delta\Phi_a$ in (2.17), we easily can check that $(\delta\Phi_a)_{\,|\,i} = \delta(\Phi_{a\,|\,i})$ and $(\delta\Phi_a)_{\,|\,\dot{c}} = \delta(\Phi_{a\,|\,\dot{c}})$ and hence, from the above formulas is obtain

$$h^{\bar{b}a}\frac{\delta\bar{\Phi}_b}{dt} = -\frac{\partial H}{\partial\Phi_a}\ ;\ \ \left(\frac{\partial H}{\partial\Phi_a}\right)_{|\,i} = \frac{\partial H}{\partial(\Phi_{a\,|\,i})}\ ;\ \ \left(\frac{\partial H}{\partial\Phi_a}\right)_{|\,\dot{c}} = \frac{\partial H}{\partial(\Phi_{a\,|\,\dot{c}})} \tag{4.4}$$

called the *complex Hamilton field equations.*

By $\mathcal{L}$-duality let us obtain now from (2.5) the Klein-Gordon type Hamiltonian. Since $\gamma_{i\bar{j}}(z)$ is a Hermitian metric on the base manifold M, we identify it with $\left(\gamma_{i\bar{j}}(z)\right)^*$ on E^*. For the Hermitian mass metric $\gamma_{a\bar{b}}(z,\eta)$, (or eventually for one which comes from an external Lagrangian of Antonelli-Shimada type, for instance), we recall from [Mu3] that the $\mathcal{L}-$dual of a complex Lagrange (Finsler) space is a complex Hamilton (Cartan) space and their metrics correspond by $\mathcal{L}-$duality. So, let us setting $\tau_{a\bar{b}} := (\gamma_{a\bar{b}})^*$ and then $\tau^{\bar{b}a}$ its inverse. Then the associated Klein-Gordon Hamiltonian to Φ_a particle is

$$H_p(\Phi^*, \overset{*}{D}\,\Phi^*) = -\frac{1}{2}\sum_a\{\gamma^{\bar{j}i}\,\overset{*}{D}_{\delta_i^*}\,\Phi_a\,\overset{*}{D}_{\delta_{\bar{j}}^*}\,\bar{\Phi}_a + \tau^{\bar{b}c}\,\overset{*}{D}_{\dot{\partial}^c}\,\Phi_a D_{\dot{\partial}^b}\bar{\Phi}_a\} - (V(\Phi))^*. \tag{4.5}$$

Because its metric tensor is the $\mathcal{L}$-dual of the Lagrangian particle metric tensor, $h_{a\bar{b}} = (g_{a\bar{b}})^*$, the corresponding Hamiltonian density to the Lagrangian density $\mathcal{L}_p = L_p \mid g \mid^2$ will be $\mathcal{H}_p = H_p \mid g \mid^{-2}$.

For the Yang-Mills Hamiltonian we take into account the Proposition 2.6 and Proposition 2.7 and therefore we obtain a complex Hamilonian which contains only the curvature of a vertical connection on E^*. Although the Bott complex connections don't correspond by $\mathcal{L}$-duality, for applications is useful the following Y-M Hamiltonian,

$$H_{YM}(\overset{*B}{D}) = \frac{1}{2}\sum_{a,b}\gamma^{\bar{j}i}\gamma^{\bar{k}l}\ \overset{*}{R}{}^a_{bi\bar{j}}\overset{*}{R}{}^a_{\bar{b}l\bar{k}}\ . \tag{4.6}$$

Finally, if we consider for (4.5) its metric tensor $h_{a\bar{b}} = (g_{a\bar{b}})^*$, we may construct the Ricci curvatures for γ and h on VE^* and thereafter the Ricci scalars, $\overset{*}{\rho}\ (\gamma)$ and $\overset{*}{\rho}\ (h)$. We observe that $\overset{*}{\rho}\ (\gamma)$ is identified with $\rho(\gamma)$. Thus, the Hilbert-Einstein gravitational Hamiltonian is $H_{HE} = \frac{1}{\chi}\overset{*}{\rho}$, where $\overset{*}{\rho} = \rho(\gamma) + \overset{*}{\rho}\ (h)$.

References

[Ai1] T. Aikou, *A partial connection on complex Finsler bundle and its appl.*, Illinois J. of Math., **42** (1998), 481–492.

[Ai2] T. Aikou, *Applications of Bott connection in Finsler Geom.*, Steps in Diff. Geometry, Debrecen 2000, 3–13.

[Ai3] T. Aikou, *Finsler geometry on complex vector bundles,* Riemann Finsler Geometry, MSRI Publications, **50** (2004), 85–107.

[Ar] V. I. Arnold: *Mathemathical methods of classical Mechanics*, Springer-Verlag, 1978.

[As] G. S. Asanov, *Finsler geometry, Relativity and gauge theories*, D.Reidel Publ. Co., Dordrecht, 1985.

[As1] G. S. Asanov, *Finsleroid-relativistic time-asymmetric space and quantized fields*, Reports of Math. Physics, **57** (2006), 199–231.

[Be] A. Bejancu, *Finsler Geometry and Appl.*, Ellis Harwood, 1990.

[Bl] D. Bleeker, *Gauge theory and variational principles*, Addison-Wesley Publ. Co. Inc., 1984.

[CN] M. Chaichian, N.F. Nelipa, *Introduction to gauge field theories*, Springer-Verlag, 1984.

[CR] F. Cooperstock, N. Rosen, *A nonlinear gauge-invariant field theory of leptons*, Int. J. Theor. Physics, **28** (1989) 423–440.

[DM] W. Drechsler, M. E. Mayer, *Fiber bundle techniques in gauge theories,* Springer-Verlag, Lecture Note in Physics, **67** (1977).

[FM] R. Friedman, J. Morgan, *Gauge theory and topology of four-manifolds*, ed. IAS/ PARK CITY, Math. Series, AMS, **4** (1998).

[Ma] Y. I. Manin, *Gauge field theory and complex geometry*, Springer-Verlag, 1997.

[MA] R. Miron, M. Anastasiei, *The geometry of Lagrange spaces. Theory and appl.*, Kluwer Acad. Publ. FTPH **59** (1994).

[MS] R. Miron, D. Hrimiuc, H. Shimada, S. Sabau, *The Geomery of Hamilton and Lagrange Spaces,* Kluwer Acad. Publ., **118** (2001).

[Mu1] G. Munteanu, *Gauge transformations on holomorphic bundles*, Balkan J. of Geom and Appl. **6** (2001), nr. 2, 71-80.

[Mu2] G. Munteanu, B. Iordachiescu, *Gauge complex field theory*, Balkan J. of Geom and Appl. **12** (2005), nr.2, 55–64.

[Mu3] G. Munteanu, *Complex spaces in Finsler, Lagrange and Hamilton geometries*, Kluwer Acad. Publ., FTPH **141** (2004).

[Pa] R. Palais, *The geometrization of physics*, Lecture Notes in Math., Hsinchu, Taiwan, 1981.

[Sa] G. Sardanashvily, *Gravidity as Higgs field. II. Fermion-gravitation complex*, arXiv: gr-qc/9407032, 1994.

[Wu] N. Wu, *General gauge field theory*, arXiv: hep-ph/9805453, 1998.

Geodesics, Connections and Jacobi Fields for Berwald-Moor Quartic Metrics

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For Finsler spaces (M, F) with quartic metrics $F = \sqrt[4]{G_{ijkl}(x, y)y^i y^j y^k y^l}$, we determine the equations of geodesics and the corresponding arising geometrical objects-canonical spray, nonlinear Cartan connection, Berwald linear connection – in terms of the non-homogenized flag Lagrange metric $h_{ij} = G_{ij00}$. Further, are studied the geodesics and Jacobi fields of the tangent space TM for hv-metric models.

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1 The equations of geodesics in quartic Berwald-Moor spaces

Let (M, F) be an n-dimensional Finsler space. We shall denote by (x, y) the local coordinates on TM and by the signs ", " and "; " preceding an index, the partial derivative relative to the corresponding component of x and of the direction y, respectively. Let G_{ijkl} be the local components of the 0-homogeneous 4-metric

$$G_{ijkl}(x, y) = \frac{1}{4!}(F^4)_{;ijkl}. \tag{1.1}$$

We denote by h_{ij} the flag non-homogenized metric

$$h_{ij} = \frac{1}{12}(F^4)_{;ij} \tag{1.2}$$

which coincides with the tensor field $y^{(4)}_{ij}$ from [9]). We shall further prove that h_{ij} is nondegenerate. The link between the two tensors (1.1) and (1.2) is

$$h_{ij} = G_{ij00}, \quad G_{ijkl} = \frac{1}{2}h_{ij;kl}$$

where the index 0 means transvection by y. We consider the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial F}{\partial y^i}\right) - \frac{\partial F}{\partial x^i} = 0 \tag{1.3}$$

and we look for the solutions $c : t \in [0, 1] \to x(t) \in M$, parametrized by arclength, this is, $v(t) = 1, \forall t \in [0, 1]$, where

$$v(t) = F(x(t), y(t)), \quad y(t) = \frac{dx}{dt}(t), \quad \forall t \in [0, 1].$$

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Then we have the following

Proposition 1. *The system (1) is equivalent with*

$$\frac{d}{dt}\left(\frac{\partial F^\alpha}{\partial y^i}\right)-\frac{\partial F^\alpha}{\partial x^i}=0,\ \ \alpha\neq 0. \tag{1.4}$$

Proof. We have $\frac{\partial F^\alpha}{\partial x^i}=\alpha F^{\alpha-1}\frac{\partial F}{\partial x^i}$, $\frac{\partial F^\alpha}{\partial y^i}=\alpha F^{\alpha-1}\frac{\partial F}{\partial y^i}$, and since c is a unit-speed curve, it follows that $\frac{dv}{dt}=0\Rightarrow\frac{d}{dt}\left(\frac{\partial F^\alpha}{\partial y^i}\right)=\alpha F^{\alpha-1}\frac{d}{dt}(\frac{\partial F}{\partial y^i})$, which lead to the claim. □

Remark. In particular, for $\alpha=4$, (1.4) leads to

$$\frac{d}{dt}\left(\frac{\partial F^4}{\partial y^i}\right)-\frac{\partial F^4}{\partial x^i}=0. \tag{1.5}$$

Using $F^4=G_{mjkl}y^my^jy^ky^l$, it follows ([5]) $(F^4)_{;i}=4G_{i000}$, and further,

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial F^4}{\partial y^i}\right)&=4\frac{dG_{ijkl}}{dt}y^jy^ky^l+12G_{ijkl}\frac{dy^j}{dt}y^ky^l=\\&=12G_{ijkl}\frac{dy^j}{dt}y^ky^l+4\left(\frac{\partial G_{ijkl}}{\partial x^m}y^my^jy^ky^l+G_{ijkl;m}\frac{dy^m}{dt}y^jy^ky^l\right).\end{aligned}$$

Since G_{ijkl} is 0-homogeneous, using Euler's relation we infer

$$G_{ijkl;m}\frac{dy^m}{dt}y^jy^ky^l=(G_{imkl;j}y^j)\frac{dy^m}{dt}y^ky^l=0 \tag{1.6}$$

and hence

$$\frac{d}{dt}\left(\frac{\partial F^4}{\partial y^i}\right)=12G_{ijkl}\frac{dy^j}{dt}y^ky^l+4\frac{\partial G_{ijkl}}{\partial x^m}y^my^jy^ky^l.$$

Replacing (1.6) and the x^i-derivative $(F^4)_{,i}=G_{mjkl,i}y^my^jy^ky^l$ in the Euler-Lagrange equation (1.5), this rewrites

$$12G_{ijkl}y^ky^l\frac{dy^j}{dt}+(4G_{ijkl,m}-G_{mjkl,i})y^my^jy^ky^l=0, \tag{1.7}$$

where $y^i=\frac{dx^i}{dt}$. Using the notation $h_{ij}=y^{(4)}_{ij}=G_{ijkl}y^ky^l$ ([9]), (1.7) becomes

$$h_{ij}\frac{dy^j}{dt}+\frac{1}{12}(4G_{ijkl,m}-G_{jklm,i})y^my^jy^ky^l=0. \tag{1.8}$$

Denoting

$$\gamma^i_{jklm}=\frac{1}{12}h^{ip}\gamma_{p\,jklm},\quad \gamma_{p\,jklm}=(4G_{pjkl,m}-G_{jklm,p}), \tag{1.9}$$

we note that ${\gamma^i}_{jklm}$ is symmetric w.r.t. the first three lower indices and the equations of geodesics can be written as

$$\frac{dy^i}{dt}+{\gamma^i}_{jklm}y^jy^ky^ly^m=0. \tag{1.10}$$

As well, denoting $\tilde{\gamma}^i_{jklm} = h^{ip}\tilde{\gamma}_{p\,jklm}/12$, where

$$\tilde{\gamma}_{p\,jklm} = G_{pjkl,m} + G_{pmjk,l} + G_{plmj,k} + G_{pklm,j} - G_{mjkl,p},$$

we can easily see that (4) can be rewritten as

$$\frac{dy^i}{dt} + \tilde{\gamma}^i{}_{jklm} y^j y^k y^l y^m = 0. \tag{1.11}$$

Remarks. 1. The tensor with which we have raised the indices is $h_{ij} = y^{(4)}_{ij}$, not $\widetilde{y}^{(4)}_{ij} = F_{;i}F_{;j} - h_{ij}$ (cf. [9]), which is degenerate, as we shall further prove. The equations of geodesics (4) can be expressed only in terms of the non-homogenized flag 2-metric $h_{ij} = G_{ij00}$. Having in view that $G_{ijkl,m}y^k y^l = h_{ij,m}$, we rewrite (2) as

$$h_{is}\frac{dy^s}{dt} + \frac{1}{12}(4h_{ij,k} - h_{jk,i})y^j y^k = 0, \tag{1.12}$$

or, still

$$\frac{dy^i}{dt} + \frac{h^{is}}{12}(4h_{ij,k} - h_{jk,i})y^j y^k = 0. \tag{1.13}$$

Applying the variational principle to $F^4 = h_{ij}y^i y^j$ one gets the same equations of geodesics (3), which are the equations of geodesics of the Lagrange space (M, L) with the Lagrangian $L = F^4 = h_{ij}y^i y^j = G_{ijkl}y^i y^j y^k y^l$.

Unfortunately, the coefficients $\gamma^i_{jm00} = h^{ij}(4h_{ij,m} - h_{mj,i})/12$ can *not* stand for the coefficients of a linear connection on TM.

Last but not least, we point out several considerations regarding the used (0,2) tensor fields. We shall further skip for brevity the symbol ";" in the partials of F w.r.t. y (e.g., $F_i = F_{;i}, F_{ij} = F_{;ij}$, etc). Let $l_i = F^{-1}y_i = F_i$, where $y_i = g_{ij}y^j$ and $g_{ij} = (F^2)_{;ij}/2$ is the *fundamental Finsler metric tensor field.* Then we have:

Proposition 2. *Consider the following family of* $(0,2)$*-tensor fields*

$$\Theta_{ij} = \lambda g_{ij} + \mu l_i l_j, \ \ \lambda, \mu \in \mathcal{F}(M), \tag{1.14}$$

Denote by g^{ij} *the dual and by* δ *the determinant of* g_{ij}*. Then*

a) Θ_{ij} *is non-degenerate for* $\lambda(\lambda+\mu) \neq 0$ *on* TM*.*

b) The dual of Θ_{ij} *is*

$$\Theta^{ij} = \frac{1}{\lambda}g^{ij} + \frac{-\mu}{\lambda(\lambda+\mu)F^2}y^i y^j.$$

c) The determinant of Θ_{ij} *is*

$$\Delta = \lambda^{n-1}(\lambda+\mu)\cdot\delta.$$

Proof. From the 1-homogeneity of F follow $F_i y^i = F$, $F_{ij}y^j = 0$, $y_i = FF_i$. The claim follows using these relations and from straightforward calculation using properties of determinants.

Lemma. *Consider the matrix* $\tilde{\Gamma} = (\tilde{\gamma}_{ij})_{i,j\in\overline{1,n}}$, $\tilde{\gamma}_{ij} = \gamma_{ij} + u_i u_j$, *with* $\Gamma = (\gamma_{ij})_{i,j\in\overline{1,n}}$ *non-degenerate. Then:*

a) The inverse of $\tilde{\Gamma}$ *has the coefficients* $\tilde{\gamma}^{ij} = \gamma^{ij} - (1 + u_s u^s)^{-1} u^i u^j$*, where* $u^i = \gamma^{is} u_s$*.*
b) We have $\det(\tilde{\Gamma}) = \det(\Gamma) \cdot (1 + u_s u^s)$*.*

Particular cases.

1. Obviously, g_{ij} is part of the pencil (1.14), obtained for $\lambda = 1, \mu = 0$.

2. We note that $g_{ij} = FF_{ij} + F_i F_j$ which infers that

$$\Theta_{ij} = \lambda F F_{ij} + (\lambda + \mu) F_i F_j, \quad \lambda, \mu \in \mathcal{F}(M), \tag{1.15}$$

where both tensor fields F_{ij} and $F_i \cdot F_j$ are degenerate.

3. For $\lambda = 1, \mu = -1$ (1.14) provides *the angular metric*

$$\hat{g}_{ij} = g_{ij} - l_i l_j. \tag{1.16}$$

Its halved version-denoted by $\tilde{y}_{ij}$, is employed in [9, (10)].

4. From (1.15) we respectively obtain the tensor fields used in [9, (13), (20')], as particular cases:

$$\begin{aligned} h_{ij} = y^{(4)}_{ij} &= \tfrac{1}{12}(F^4)_{;ij}, \quad &&\text{for} \quad \lambda = F^2/3, \mu = 2F^2/3 \\ \tilde{y}^{(4)}_{ij} &= y_i y_j - y^{(4)}_{ij}, &&\text{for} \quad \lambda = -\mu = -F^2/3. \end{aligned} \tag{1.17}$$

We emphasize that the $(0,2)$-tensor field $\tilde{y}^{(4)}_{ij}$ satisfies the following equalities

$$\tilde{y}^{(4)}_{ij} = -F^3 F_{ij}/3 = -F^2 \hat{g}_{ij}/3,$$

and hence has the property of $\hat{g}_{ij}$ of being degenerate.

We note that the proposition above provides for $\lambda = 1+\alpha, \mu = -\alpha \in \mathbb{R}$ the following

Corollary ([1]). *The following* $(0,2)$ *Finsler tensor fields are 0-homogeneous and non-degenerate:*

$$g_{ij} + \alpha \hat{g}_{ij}, \alpha \in \mathbb{R}. \tag{1.18}$$

Regarding h_{ij}, this can be homogenized by dividing to F^2. According to the Corollary, the resulting (Generalized Lagrange) homogeneous metric is included in the family of metrics (1.18). More exactly, we have

$$\frac{h_{ij}}{F^2} = \frac{1}{12F^2}\left[2F^2(F^2)_{;i}\right]_{;j} = \frac{1}{6}\left[(F^2)_{;ij} + 4F^{-2} y_i y_j\right] = g_{ij} + \alpha \hat{g}_{ij}, \ \alpha = -2/3.$$

Definition 1. We call *generalized 4-index angular metric tensor*, the tensor field

$$\omega_{ijkl} \equiv G_{ijkl} - l_i l_j l_k l_l. \tag{1.19}$$

This definition may be easily extended to any number of indices. In analogy with [1] we have the following

Proposition 3. *The tensors of form* $\tilde{G}_{ijkl} = G_{ijkl} + \alpha \omega_{ijkl}$*,* $\alpha \in \mathbb{R}$ *are generalized metric tensors which share the same energy* F^4*.*

Proof. Using that $l_i y^i = F^{-1} y_i y^i = F$, we get $\tilde{G}_{0000} = G_{0000} + \alpha(G_{0000} - (l_s y^s)^4) = F^4$, whence the claim follows.

We should note as well the relation

$$\omega_{ij00} = G_{ij00} - F^2 l_i l_j = h_{ij} - F^2 F_i F_j = F^2(F_i F_j + F F_{ij}/3) - F^2 F_i F_j = -\tilde{y}^{(4)}_{ij}.$$

2 The nonlinear connection

Consider the semispray given by the second term in the equations of geodesics (1.13)

$$2G^i = \frac{h^{ip}}{12}(4G_{pjkl,m} - G_{mjkl,p})y^m y^j y^k y^l.$$

By taking into account (1.1) and the 1-homogeneity of F, we get $G_{mjkl}y^m y^j y^k y^l = F^4$, $G_{pjkl}y^j y^k y^l = G_{p000} = \frac{1}{4}(F^4)_{;p}$, and hence G^i can be written as

$$2G^i = \frac{h^{ip}}{12}\left(\frac{\partial^2 F^4}{\partial x^m \partial y^p}y^m - \frac{\partial F^4}{\partial x^j}\right). \tag{2.1}$$

Within the Lagrange structure $(M, L = F^4/6)$, where the classical Lagrange metric induced by L is $h_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$, (2.1) is exactly the Kern canonical semi-spray of L ([8], [11, Theorem 7.4.1, p. 113]),

$$G^i = \frac{h^{ip}}{4}\left(\frac{\partial^2 L}{\partial x^m \partial y^p}y^m - \frac{\partial L}{\partial x^j}\right). \tag{2.2}$$

and $N^i{}_j = \frac{\partial G^i}{\partial y^j}$ are the Kern coefficients of the canonical nonlinear connection attached to L on TM. Its autoparallel curves described by (1.13) are exactly the geodesics determined by L. Then the equations (4) can be written as

$$\frac{d^2x^i}{dt^2} + 2G^i = 0 \Leftrightarrow \frac{d^2x^i}{dt^2} + N^i{}_j y^j = 0,$$

or, denoting $\delta y^i = dy^i + N^i_s dx^s$,

$$\frac{\delta y^i}{dt} = 0.$$

Aiming to obtain a normal linear connection $(L^i{}_{jk}, C^i{}_{jk})$ on TM, one possible choice is, for example, $L^i{}_{jk} = \frac{\partial N^i{}_j}{\partial y^k}$ and $C^i_{jk} = 0$. Then the equations of geodesics rewrite

$$\frac{d^2x^i}{dt^2} + L^i{}_{jk}y^j y^k = 0.$$

Remark. The candidates for a nonlinear connection

$$\tilde{N}^i{}_l = \frac{h^{ip}}{12}\left(4\frac{\partial G_{pjkl}}{\partial x^m} - \frac{\partial G_{mjkl}}{\partial x^p}\right)y^m y^j y^k = \gamma^i{}_{j000},$$

i.e., the coefficients of y^l from the equations of geodesics from (4), do not obey the specific component changes; hence they do *not* define a nonlinear connection.

3 Geodesics in the (h, v)-metric context

Let TM be endowed with: a nonlinear connection N, a metric structure

$$G = \underset{(0)}{g}{}_{ij}dx^i \otimes dx^j + \underset{(1)}{g}{}_{ij}\delta y^{(1)i} \otimes \delta y^{(1)j},$$

where the metrics $\underset{(0)}{g}$ and $\underset{(1)}{g}$ can be specified as in the previous sections. Consider as well a metrical normal linear d-connection D, $D\Gamma(N) = (L^i{}_{jk}, C^i{}_{jk})$ ([11]). Then N induces a local adapted basis $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$, and the dual adapted basis, $\{dx^i, \delta y^i\}$. We denote by $\langle\ ,\ \rangle$ the scalar product defined on TM by G, by $\underset{(\beta\alpha)}{\overset{(\gamma)}{T}}{}^i{}_{jk}$ the components of the torsion tensor $T(\delta_{\alpha k}, \delta_{\beta j}) = \underset{(\beta\alpha)}{\overset{(\gamma)}{T}}{}^i{}_{jk}\delta_{\gamma i}$, and by $\underset{(\alpha\beta\gamma)}{\overset{(\alpha)}{R}}{}^i{}_{jkl}$ the components of the curvature tensor $R(\delta_{\gamma l}, \delta_{\beta k})\delta_{\alpha j} = \underset{(\alpha\beta\gamma)}{\overset{(\alpha)}{R}}{}^i{}_{jkl}\delta_{\alpha i}$, where $\delta_{0i} = \frac{\delta}{\delta x^i}$, $\delta_{1i} = \frac{\partial}{\partial y^i}$.

For a curve $c : [0,1] \to TM$, $t \mapsto c(t) = (x^i(t), y^{(1)i}(t))$, we consider its velocity $V := V(t) = \dot{c} = V^{(\alpha)i}\delta_{\alpha i}$, where

$$V^{(0)i} = \frac{dx^i}{dt}, \quad V^{(1)i} = \frac{\delta y^{(1)i}}{dt}.$$

The energy of c is

$$E(c) = \int_0^1 \langle \dot{c}, \dot{c}\rangle dt = \int_0^1 \langle V, V\rangle dt = \int_0^1 \underset{(0)}{g}{}_{ij}V^{(0)i}V^{(0)j} + \underset{(1)}{g}{}_{ij}V^{(1)i}V^{(1)j}dt.$$

Theorem 1 (The first variation of energy). *If $c : [0,1] \to TM$, $\alpha : (-\varepsilon, \varepsilon) \times [0,1] \to TM$ is a variation of c by piecewise smooth curves with fixed ends, and $W = \frac{\partial\alpha}{\partial u}(0,t)$ is the associated deviation vector field, then the first variation of energy is given by*

$$\frac{1}{2}\left.\frac{dE(\bar{\alpha}(u))}{du}\right|_{u=0} = -\sum_{i=0}^{k-1}\langle W, \Delta_{t_i}V\rangle + \int_0^1 \langle T(W,V), V\rangle - \langle W, A\rangle dt,$$

where A is the acceleration vector field

$$A = D_{\dot{c}}V = \frac{DV}{dt} = A^{(0)i}\delta_{0i} + A^{(1)i}\delta_{1i}$$

and $\Delta_t X$ is the jump

$$\Delta_t X = X(t_+) - X(t_-), \quad t \in [0,1], \ X \in \mathcal{X}(TM).$$

We note that $\langle T(\,\cdot\,, V), V\rangle$ defines a 1-form. Hence there exists a vector field F on TM such that $\langle T(W,V), V\rangle = \langle F, W\rangle$. Then, denoting

$$V = V^{(\alpha)i}\delta_{\alpha i}, \quad W = W^{(\beta)j}\delta_{\beta j}, \quad F = \sum_{\alpha=0}^{1} F^{(\alpha)i}\delta_{\alpha i}$$

we have $\langle T(W,V), V\rangle = \sum_{\beta=0}^{1} \underset{(\beta)}{g}{}_{jh}F^{(\beta)h}W^{(\beta)j}$, and the components of the field F are given by

$$F^{(\alpha)i} = \sum_{\beta,\gamma=0}^{1} \underset{(\alpha)}{g}{}^{il}\underset{(\gamma)}{g}{}_{kh}\underset{(\beta\alpha)}{\overset{(\gamma)}{T}}{}^k{}_{jl}V^{(\beta)j}V^{(\gamma)h}, \quad \alpha = \overline{0,1}.$$

Remark. The vector field F does not depend on the chosen variation with fixed endpoints of c.

By replacing F into the expression of the first variation of energy, we get

$$\frac{1}{2}\left.\frac{dE\left(\bar{\alpha}(u)\right)}{du}\right|_{u=0} = -\sum_{i=0}^{k-1}\langle W, \Delta_{t_i} V\rangle + \int_0^1 \langle W, F - A\rangle dt.$$

For a smooth curve c on the whole $[0,1]$ the jumps in the sum cancel and we have

$$\frac{1}{2}\left.\frac{dE\left(\bar{\alpha}(u)\right)}{du}\right|_{u=0} = \int_0^1 \langle W, F - A\rangle dt,$$

which means that $u = 0$ is a critical point of E if and only if, along $c = \bar{a}(0)$, we have $F = A$. Consequently we state the following

Theorem 2. *Any geodesic* $c : [0,1] \to TM,\ t \to (x^i(t), y^{(1)i}(t))$ *of* (TM, G) *satisfies*

$$\frac{D}{dt}\frac{dc}{dt} = F.$$

Then, the smooth curve $c : [0,1] \to TM,\ t \to (x^i(t), y^{(1)i}(t))$ is a geodesic of TM iff

$$\frac{DV^{(0)i}}{dt} = F^{(0)i}, \quad \frac{DV^{(1)i}}{dt} = F^{(1)i}, \tag{3.1}$$

which rewrites explicitly as

$$\begin{aligned} \tfrac{dV^{(0)i}}{dt} + L^i{}_{jk}V^{(0)k}V^{(0)j} + C^i{}_{jk}V^{(1)k}V^{(0)j} &= \sum_{\beta,\gamma=0}^{1} \underset{(0)}{g}{}^{il} \underset{(\gamma)}{g}{}_{kh} \underset{(\beta 0)}{\overset{(\gamma)}{T}}{}^{k}_{jl} V^{(\beta)j}V^{(\gamma)h} \\ \tfrac{dV^{(1)i}}{dt} + L^i{}_{jk}V^{(0)k}V^{(1)j} + C^i{}_{jk}V^{(1)k}V^{(1)j} &= \sum_{\beta,\gamma=0}^{1} \underset{(1)}{g}{}^{il} \underset{(\gamma)}{g}{}_{kh} \underset{(\beta 1)}{\overset{(\gamma)}{T}}{}^{k}_{jl} V^{(\beta)j}V^{(\gamma)h}. \end{aligned} \tag{3.2}$$

Example. In particular, in a Finsler space (M, F), for $g_{ij} = \underset{(0)}{g}{}_{ij} = \underset{(1)}{g}{}_{ij} = \frac{1}{2}F^2_{,y^i y^j}$ considering the Cartan connection ([11]), we infer that (3.2) rewrite

$$\begin{cases} \dfrac{d^2x^i}{dt} + L^i{}_{jk}V^{(0)k}V^{(0)j} + C^i{}_{jk}V^{(1)k}V^{(0)j} = g^{il}g_{kh}\left(R^k_{jl}V^{(0)j}V^{(1)h} - P^k_{lj}V^{(1)h}V^{(1)j} - C^k_{lj}V^{(0)h}V^{(1)j}\right) \\ \dfrac{dV^{(1)i}}{dt} + L^i{}_{jk}V^{(0)k}V^{(1)j} + C^i{}_{jk}V^{(1)k}V^{(1)j} = g^{il}g_{kh}\left(P^k_{jl}V^{(0)j}V^{(1)h} + C^k_{jl}V^{(0)j}V^{(0)h}\right) \end{cases}$$

Remark. If we consider, instead of a normal linear d-connection $(L^i{}_{jk}, C^i{}_{jk})$, a (simple) d-connection $(L^i{}_{jk},\ L^a{}_{bk},\ C^i{}_{jc}\ , C^a{}_{bc})$, then the above equations become

$$\begin{cases} \frac{dV^{(0)i}}{dt} + L^i{}_{jk}V^{(0)k}V^{(0)j} + C^i{}_{jc}V^{(1)c}V^{(0)j} = F^{(0)i} \\ \frac{dV^{(1)a}}{dt} + L^a{}_{bk}V^{(0)k}V^{(1)b} + C^a{}_{bc}V^{(1)c}V^{(1)b} = F^{(1)a}. \end{cases}$$

4 The second variation of energy. Deviations of geodesics on TM

Consider as well TM endowed with a nonlinear connection N, a metric structure

$$G = \underset{(0)}{g}_{ij} dx^i \otimes dx^j + \underset{(1)}{g}_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j}$$

and a normal metrical linear d-connection D, $D\Gamma(N) = (L^i{}_{jk}, C^i{}_{jk})$.

Let $c : [0,1] \to TM$, $t \mapsto (x^i(t), y^i(t))$ be a geodesic, i.e., c is $\mathcal{C}^\infty$ on the whole $[0,1]$ and c is a critical point of the energy

$$E = \int_0^1 \langle \dot{c}, \dot{c} \rangle dt. \tag{4.1}$$

Let $\alpha : U \times [0,1] \to TM$ be a 2-parameter variation with fixed endpoints of c by smooth curves on $[0,1]$, U being a neightbourhood of $(0,0) \in \mathbb{R}^2$. We have $\alpha(0,0,t) = c(t)$, $\forall t \in [0,1]$. Let W_1, W_2 be the induced deviation vector fields

$$W_1(t) = \frac{\partial \alpha}{\partial u_1}(0,0,t), \quad W_2(t) = \frac{\partial \alpha}{\partial u_2}(0,0,t),$$

and let $\bar{\alpha}$ be the mapping defined on $\bar{U}$ by

$$\bar{\alpha}(u_{1,}, u_1)(t) = \alpha(u_1, u_2, t), \ (u_1, u_2, t) \in U \times [0,1].$$

The Hessian E_{**} of the energy (4.1) is

$$E_{**}(W_1, W_2) = \left. \frac{\partial^2 E(\bar{\alpha}(u_{1,}, u_2))}{\partial u_1 \partial u_2} \right|_{(0,0)}.$$

Let $\mathcal{F} = \mathcal{F}^{(\alpha)i} \delta_{\alpha i}$ be the vector field defined by

$$\left\langle T\left(\frac{\partial \alpha}{\partial u_2}, \frac{\partial \alpha}{\partial t}\right), \frac{\partial \alpha}{\partial t} \right\rangle = \left\langle \mathcal{F}, \frac{\partial \alpha}{\partial u_2} \right\rangle,$$

having the local coefficients

$$\mathcal{F}^{(\alpha)i} = \sum_{\beta,\gamma=0}^{1} \underset{(\alpha)}{g}^{il} \underset{(\gamma)}{g}_{kh} \underset{(\beta\alpha)}{\overset{(\gamma)}{T}}{}^{k}{}_{jl} \frac{\partial \alpha^{(\beta)j}}{\partial t} \frac{\partial \alpha^{(\gamma)h}}{\partial t} \rfloor_{(u_1,u_2,t)}, \ \alpha = \overline{0,1}. \tag{4.2}$$

Extending the results obtained in the Finslerian framework ([6], [7]) to the case of (h,v)-metrics (e.g., as in [8], [6]), we further state the following

Theorem 3 (The second variation of energy). *If $c : [0,1] \to TM$ is a geodesic and $\alpha : U \times [0,1] \to TM$ (where $\varepsilon > 0$) is a variation with fixed endpoints of c by piecewise smooth curves, then the Hessian E_{**} is given by:*

$$E_{**}(W_1, W_2) = -\sum_{i=1}^{k-1} \left\langle W_2, \Delta_{t_i}\left(T(W_1, V) + \tfrac{DW_1}{dt}\right)\right\rangle +$$

$$+ \int_0^1 \left\langle W_2, \left.\frac{D\mathcal{F}}{\partial u_1}\right|_{u_1=u_2=0} + R(V, W_1)V - \frac{D}{dt}T(W_1, V) - \frac{D^2 W_1}{dt^2}\right\rangle dt,$$

where $0 = t_0 < t_1 < \ldots < t_k = 1$ *is a division of* $[0,1]$ *such that* α *be smooth on each* $U \times (t_{i-1}, t_i)$, $i = \overline{1,k}$.

As consequence, if $c : [0,1] \to TM$ is a smooth geodesic and $\alpha : (-\varepsilon, \varepsilon) \times [0,1] \to TM$ $(\varepsilon > 0)$ is a variation of c through smooth geodesics, then the deviation vector fields - called also *generalized Jacobi fields*, $W = W^{(\alpha)i}\delta_{\alpha i}$ are given by

$$\frac{D^2 W^{(\alpha)i}}{dt^2} + \frac{D\overset{(\alpha)}{T}{}^i}{dt} = \left.\frac{D\mathcal{F}^{(\alpha)i}}{du}\right|_{u=0} + \overset{(\alpha)}{R}{}^i, \quad \alpha = \overline{0,1}, i = \overline{1,n},$$

where $\mathcal{F}^{(\alpha)i}$ are given by (4.2) and we denoted

$$\begin{cases} \overset{(\alpha)}{T}{}^i = \sum\limits_{\beta,\gamma=0}^{1} V^{(\beta)j} W^{(\gamma)k} \underset{(\beta\gamma)}{\overset{(\alpha)}{T}}{}^i{}_{jk} \\ \overset{(\alpha)}{R}{}^i := -\sum\limits_{\beta,\gamma=0}^{1} V^{(\alpha)h} V^{(\beta)j} W^{(\gamma)k} \underset{(\alpha\beta\gamma)}{\overset{(\alpha)}{R}}{}^i{}_{hjk}\,. \end{cases}$$

5 Projectability of horizontal geodesics of TM

Let N be an arbitrary nonlinear connection and let $(L^i{}_{jk}, C^i{}_{jk})$ be the coefficients of an arbitrary metrical normal linear d-connection. A curve $c : [0,1] \to TM$, $t \to (x^i(t), y^i(t))$ is a *horizontal geodesic* of TM iff

$$\begin{cases} V^{(1)i} \equiv \dfrac{dy^i}{dt} + N^i{}_j y^j = 0 \\ \dfrac{dV^{(0)i}}{dt} + L^i{}_{jk} V^{(0)j} V^{(0)k} = \underset{(0)}{g}{}^{il} \underset{(0)}{g}{}_{mh} \left(L^m{}_{jl} - L^m{}_{lj}\right) V^{(0)j} V^{(0)h} \\ \underset{(1)}{g}{}^{il} \underset{(0)}{g}{}_{mh} C^m_{jl} V^{(0)j} V^{(0)h} = 0. \end{cases} \tag{5.1}$$

The last two equations in (5.1) are obtained from (3.2), in which we have used the relations

$$\underset{(00)}{\overset{(0)}{T}}{}^m{}_{jl} = \left(L^m{}_{jl} - L^m{}_{lj}\right), \quad \underset{(01)}{\overset{(0)}{T}}{}^m{}_{jl} = C^m_{jl}.$$

We note that we take into account only curves $c : [0,1] \to TM$ with $y^i = \frac{dx^i}{dt} = V^{(0)i}$, i.e., *extensions* to TM of curves $t \mapsto x^i(t)$ on M, and we look for conditions for such horizontal geodesics to project to geodesics of M. For any curve on TM, we have $V^{(0)i} = \dfrac{dx^i}{dt}$, and hence from (5.1), we infer that the h-geodesics of TM which are extensions of curves of M are locally characterized by

$$\begin{cases} \dfrac{dy^i}{dt} + N^i{}_j y^j = 0 \\ \dfrac{dy^i}{dt} + L^i{}_{jk} y^j y^k = \underset{(0)}{g}{}^{il} \underset{(0)}{g}{}_{mh} \left(L^m{}_{jl} - L^m{}_{lj} \right) y^j y^h \\ \underset{(1)}{g}{}^{il} \underset{(0)}{g}{}_{mh} C^m_{jl} y^j y^h = 0. \end{cases} \tag{5.2}$$

We further obtain:

Proposition 4. *Let G^i be the coefficients of the Kern canonical semispray (2.2) of the Lagrangian $L = \underset{(0)}{g}{}_{ij} y^i y^j$. If one of the two following relations holds along any curve $t \to (x^i(t))$ of M:*

1. $2G^i\left(x, \dfrac{dx}{dt}\right) = \left(L^i{}_{jh} - \underset{(0)}{g}{}^{il} \underset{(0)}{g}{}_{mh} \left(L^m{}_{jl} - L^m{}_{lj} \right) \right) \dfrac{dx^j}{dt} \dfrac{dx^h}{dt}$;

2. $2G^i\left(x, \dfrac{dx}{dt}\right) = N^i{}_j y^j$;

then any horizontal geodesic of TM projects onto a geodesic of M.

Example. If F is a Finsler metric on M and N is the canonical (Cartan) nonlinear connection of F^2, given by $N^i{}_j = \dfrac{\partial G^i}{\partial y^j}$, then any horizontal curve (including the case of a horizontal geodesic) of TM is projected onto a geodesic of M.

In particular, for $\underset{(0)}{g} = \underset{(1)}{g}$ and $(N^i{}_j, L^i{}_{jk}, C^i{}_{jk})$ the Cartan connection, both the conditions 1) and 2) in the above Proposition are satisfied. Moreover, the third set of equations (5.2) is satisfied by any curve, and the first and the second one are both equivalent with the equations of geodesics of M. Then in this case, there holds:

Corollary 1 ([1]). *For the canonical Cartan connection and a given extension Γ on TM, we have:*

a) If Γ is a horizontal curve then Γ is a horizontal geodesic;

b) Γ is a horizontal curve iff Γ is projectable onto a geodesic of M.

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References

[1] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theories*, Dordrecht, Reidel, 1985.

[2] M. Anastasiei, I. Bucătaru, *A notable submersion in higher order geometry*, BJGA 1 (1996), 1, 1–9.

[3] M. Anastasiei, I. Bucătaru, *Jacobi fields in generalized Lagrange spaces*, Rev. Roum. Math. Pures Appl., 42 (1997), 9–10, 689–695.

[4] V. Balan, *On geodesics and deviations of geodesics in the fibered Finslerian approach*, Stud. Cerc. Mat., 46, 4 (1994), 415–422.

[5] V. Balan and N. Brinzei, *Einstein equations for the Berwald-Moore type Euclidean-locally Minkowski relativistic model*, BJGA 11, 2 (2006), 20–26.

[6] V. Balan and N. Voicu, *Distinguished geodesics and Jacobi fields on first order jet spaces*, CEJM (Central European Journal of Mathematics), electronic,http://www.cesj.com, 2 (2004), 2, 1–10.

[7] V. Balan and P. C. Stavrinos, *Deviations of geodesics in the fibered Finslerian approach*, in vol. of Kluwer Acad. Publ. FTPH no. 76, 1996, 65–74.

[8] J. Kern, *Lagrange geometry*, Arch. Math. 25 (1974), 438–443.

[9] S. V. Lebedev, *The Generalized Finslerian metric tensors*, Hypercomplex Numbers in Geometry and Physics 2, 4 (2005), 44–50.

[10] J. Milnor, *Morse Theory*, Princeton Univ. Press, Princeton, New Jersey, 1963.

[11] R. Miron and M. Anastasiei, *Vector Bundles. Lagrange Spaces. Applications in Relativity Theory*, (in Romanian), Ed. Acad. Bucharest, 1987.

[12] D. G. Pavlov, *Four-dimensional time, Hypercomplex numbers in Geometry and Physics*, Ed. Mozet, Russia, 1, 1 (2004), 31–39.

[13] N. Voicu, *Deviations of geodesics in the Geometry of second order* (in Romanian), Ph. D. Thesis, "Babes-Bolyiai" Univ., Cluj-Napoca, 2003.

Finsler Spaces with Polynomial Metric

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It is shown that Finsler spaces with polynomial metric allow metrical tensorial connections (i.e., which are linear for a given type of tensors). It is pointed out that many of these connections induce, in a natural way, metrical non-linear connections on the considered manifold.

MSC: 53B40, 53B15.

1 Introduction

Let M be a paracompact n-dimensional manifold and a an r-form, that is a covariant tensor $a \in \mathcal{T}_r^\circ$ of type $(0,r)$ on M with components $a_{i_1 \dots i_r}(x)$, $1 \leq i_1, \dots, i_r \leq n$ in a local coordinate system (x). Then $a_{i_1 \dots i_r}(x) y^{i_1} \dots y^{i_r}$, $y \in T_x M$ (summation over $1 \leq i_1, \dots, i_r \leq n$) is an r-th order homogeneous polynomial in $T_x M(y)$ $[y^{i_1} \dots y^{i_r} \equiv y^{i_1} \times \dots \times y^{i_r}]$. We suppose that

$$a_{i_i \dots i_r}(x) y^{i_1} \dots y^{i_r} = 1$$

is a star-shaped convex hypersurface in $T_x M(y)$. Then $F^n = (M, \mathcal{F})$ with the Finsler metric

$$\mathcal{F}^r(x,y) = a_{i_1 \dots i_r}(x) y^{i_1} \dots y^{i_r} \tag{1}$$

is a Finsler space with *polynomial metric*. Such F^n are generalizations of the Berwald-Moór metric (see [1] p. 53 or [20], [21], [14], [17], [18]). F^n with polynomial metric were recently investigated by several authors, such as V. Balan, N. Brinzei, S. Lebedev, D. G. Pavlov etc. in [2], [3], [8], [13], [19]. They considered these spaces endowed with linear metrical connections acting in the vector bundle

$$TM \times_M TM = \mathcal{V}TM = (VTM, \pi, \mathcal{V}^n)$$
$$\pi^{-1}(x,y) = \mathcal{V}^n = \{\xi(x,y)\},$$

where $\mathcal{V}^n$ is an n-dimensional real vector space, and (x,y) is a line-element. $\mathcal{V}TM$ is no tangent bundle, for $\dim TM = 2n \neq n = \dim \mathcal{V}^n$. In $\mathcal{V}TM$ there exist linear metrical connections (e.g. Cartan connection), which allow to develop a curvature theory, etc. in a way similar to that of Riemannian geometry. But using this bundle and line-elements (x,y) has some disadvantages too. The theory becomes more complicated, and the difference between the dimensions of the base space TM and the fiber $\mathcal{V}^n$ is sometimes inconvenient, especially in physics. A linear connection acting in the bundle $\tau_M = (TM,\, \pi,\, M,\, \mathcal{V}^n)$ is more simple, but in a Finsler space it cannot be metrical in general. Nevertheless there are many Finsler spaces which allow linear metrical connections in the tangent bundle. Such are the Riemannian spaces V^n, Minkowski spaces $\mathcal{M}^n$, locally Minkowski spaces $\ell\mathcal{M}^n$, and also the affine deformations $\mathcal{A}\ell\mathcal{M}^n$ of locally Minkowski spaces ([23], [24]), the Finsler spaces with 1-form metric ([15], [16]), the spaces modelled on a Minkowski space ([11], [12]). Some of these spaces may not exist on every paracompact manifold ([6], [24]). There are also Finsler spaces admitting metrical connections in τ_M which are only near to linear connections [25] or which are homogeneous only [13].

In this paper we want to show that Finsler spaces with polynomial metric allow metrical tensorial connections (linear for a given type of tensors). Many of them induce, in a natural way, metrical non-linear connections in τ_M.

2 Tensorial connection

Let us consider the tensors t of type $(r, 0)$, $t \in \mathcal{T}_0^r$ over the n-dimensional manifold M. $\mathcal{T}_0^r$ is a real vector space $\mathcal{V}^N$ of dimension $N = n^r$. Thus t^A, $A = 1, 2, \ldots, N$ can be considered as the components of t.

$$\hat{\mathcal{E}} = (\hat{E}, \pi, M, \mathcal{V}^N), \quad \pi : \hat{E} \to M, \quad \pi^{-1}(p) \approx \mathcal{V}^N, \qquad p \in M \tag{2}$$

is a tensor bundle, that is a vector bundle of rank N over M. A linear connection γ acting in $\hat{\mathcal{E}}$ is called *tensorial connection*. In a local coordinate system (x) it is given by the connection coefficients

$$\gamma_A{}^B{}_k(x), \qquad A, B = 1, 2, \ldots, N, \quad k = 1, 2, \ldots, n$$

and the parallel translated $\mathcal{P}^\gamma_{x(\tau)} t_0$ of $t_0 \in \pi^{-1}(x(\tau_0))$ along a curve $x(\tau)$ according to γ is defined by the solution $t(\tau)$ of the ODE system

$$\frac{Dt^B}{d\tau} \equiv \frac{dt^B}{d\tau} - \gamma_A{}^B{}_k(x(\tau)) t^A \frac{dx^k}{d\tau} \tag{3}$$

with initial value $t(0) = t_0$. With an appropriate γ one can realize any linear mapping between $\pi^{-1}(x(\tau_0)) \approx \mathcal{V}^N$ and $\pi^{-1}(x(\tau)) \approx \mathcal{V}^N$. – An affine connection Γ with coefficients $\Gamma_j{}^i{}_k(x)$ also induces certain (linear) mappings between the above fibers. These mappings are realized by special tensorial connections. In case of $r = 2$ the corresponding γ has the coefficients

$$\gamma_A{}^B{}_k(x) \equiv \gamma_{rs}{}^{ij}{}_k(x) = \Gamma_r{}^i{}_k(x)\delta_s^j + \delta_r^i \Gamma_s{}^j{}_k(x). \tag{4}$$

Clearly γ-s of this form are special ones, and they do not generate all linear mappings between $\pi^{-1}(x(\tau_0))$ and $\pi^{-1}(x(\tau))$. Also conversely, if a $\gamma_A{}^B{}_k$ can be represented in the form (4), then the tensorial connection γ reduces to the affine connection Γ.

The tensorial connection given by (3) is linear in $t \in \mathcal{T}_0^r$, and the operator $\frac{D}{d\tau}$ of (3) can be extended to the tensor algebra of tensors of type $(\lambda r, \mu r)$, where λ and μ can be arbitrary non-negative integers. Tensorial connection was introduced by E. Bompiani [9], and investigated by A. Cossu [10], L. Tamássy [22], M. Kucharzewski [14], and others.

Let $\overline{M}$ be an $N = n^r$ dimensional manifold with local coordinates $\overline{x}$, such that $M \subset \overline{M}$, and let $\overline{\gamma}(\overline{x})$ be a C° extension of γ. Then the restriction of $\overline{\gamma}$ to M yields $\gamma : \overline{\gamma}(\overline{x}) \restriction_M = \gamma(x)$. Then $(\overline{M}, \overline{\gamma})$ is an (ordinary) affine connection in the tangent bundle $\tau_{\overline{M}} = (T\overline{M}, \pi, \overline{M}, \mathcal{V}^N)$. So we obtain the

Proposition 1. *Any tensorial connection* $(M^n(x), \gamma(x))$ *is the restriction of an affinely connected space* $(\overline{M}^N(\overline{x}), \overline{\gamma}(\overline{x}))$ *in the form*

$$(M^n, \gamma(x)) = (\overline{M}^N, \overline{\gamma}(\overline{x})) \restriction M, \qquad N = n^r.$$

Here the restriction happens in the base manifold $\overline{M}$. This is in analogy to the fact that any Finsler space F^n can be considered as the restriction of a Riemannian space $V^{2n} = (TM, \mathcal{G})$, where $\mathcal{G}$ is the Sasakian type metric of F^n. Here the restriction happens in the fiber. The tangent space TTM of V^{2n} is restricted to the vertical bundle $\mathcal{V}TM$ of the Finsler space.

A tensorial connection γ has two curvature tensors $\mathcal{A}_E{}^C{}_i$; $\mathbb{R}_j{}^i{}_{k\ell}$, and a torsion tensor $\mathcal{S}_j{}^i{}_k$. Vanishing of $\mathcal{A}$ characterizes the reduction of γ to Γ. In this case also $\mathbb{R}$ and $\mathcal{S}$ reduce to the curvature R^Γ and the torsion $\mathcal{S}^\Gamma$ of Γ ([22]).

3 Tensorial connections in case of polynomial metric

The $a(x) \in \mathcal{T}_0^r$ appearing in (1) is *parallel* along $x(\tau)$ according to γ, if

$$\frac{da_A}{d\tau} = \gamma_A{}^B{}_k(x(\tau))a_B\frac{dx^k}{d\tau},$$

and $a(x)$ is an *absolute parallel* tensor field on M (or on a domain of it), if

$$\nabla_k a_A = 0, \tag{5}$$

that is

$$\frac{\partial a_A}{\partial x^k} = \gamma_A{}^B{}_k(x)a_B. \tag{5'}$$

The Finsler norm $\|y\|_F$ of a vector $y \in T_xM$ in our F^n with polynomial metric (1) is $\|y\|_F^r = \mathcal{F}^r(x,y) = a_Ab^A$, $A = i_1 \dots i_r$, $b^A = y^{i_1} \dots y^{i_r}$, and we define the Finsler norm $\|t\|_F$ of a tensor $t \in \mathcal{T}_0^r$ in our F^n by

$$\|t\|_F : a_A(x)t^A(x). \tag{6}$$

Thus

$$\|y\|_F^r = \mathcal{F}^r(x,y) = \|b\|_F. \tag{7}$$

The tensorial connection is called *metrical* if

$$\|\mathcal{P}^{\gamma}_{x(\tau)}t_0\|_F = \|t(\tau)\|_F = \text{const.}, \qquad \forall\, x(\tau) \subset M, \quad t_0 \in \mathcal{T}^r_0, \tag{8}$$

and thus

$$\frac{d}{d\tau}\|t(\tau)\|_F = \frac{D}{d\tau}\big(a_A(x(\tau))t^A(\tau)\big) = \left[(\nabla_k a_A)\frac{dx^k}{d\tau}\right] t^A + a_A\frac{Dt^A}{d\tau} = \frac{d}{d\tau}\text{const.} = 0 \tag{8'}$$

for any $t(\tau)$ parallel along any $x(\tau)$. Since for parallel $t(\tau)$ $\frac{Dt^A}{d\tau} = 0$, and for an appropriate $x(\tau)$ we can obtain every x_0 and $\dot{x}_0$, (8) is equivalent to (5) and to (5').

For given $a(x)$ (5') is a linear equation system at any point x_0 for the unknowns ${\gamma_A}^B{}_k(x_0)$. The equations of (5') are independent in the sense that each ${\gamma_A}^B{}_k(x_0)$ appears in a single equation only. Hence (5') is solvable for ${\gamma_A}^B{}_k(x)$. Thus we obtain

Theorem 1. *Any Finsler space with polynomial metric* (1) *has metrical tensorial connections.*

(5') consist of Nn equations, and in each of them (for fix A and k) appear N unknowns ${\gamma_A}^B{}_k$, of which $N-1$ can arbitrarily be chosen. Thus in the solution of (5') $Nn(N-1) = (N^2-N)n$ of the ${\gamma_A}^B{}_k$ remain arbitrary.

The upper script indices of a totally symmetric tensor $t^{i_1\ldots i_r} \in \mathcal{T}^r_0$ are the multiple combinations of order r from the elements $1, 2, \ldots, n$. These tensors form a linear subspace ${}^s\mathcal{T}^r_0$ of $\mathcal{T}^r_0$. The dimension of ${}^s\mathcal{T}^r_0$ is $C^m_{r,n} = \frac{(n-1+r)!}{(n-1)!r!} = C$, the number of the multiple combinations of order r from n elements $1, 2, \ldots, n$. The components of such a tensor will be denoted by t^α, $\alpha = 1, 2, \ldots, C$. Also $y^{i_1}\ldots y^{i_r} = b^{i_1\ldots i_r} = b^\alpha \in {}^s\mathcal{T}^r_0$. If in (1) we draw together those $a_{i_1\ldots i_r}$ in which the same $i_1, i_2, \ldots i_r$ appear (independently from the order), and denote their sum by g_α, then with respect to (6), (1) gets the form

$$\mathcal{F}^r(x,y) = g_\alpha b^\alpha = \|b\|_F, \qquad \alpha = 1, 2, \ldots, C. \tag{1'}$$

b is decomposable. It is an r-times tensor product of $y \in \mathcal{T}_x M$:

$$b = \overset{1}{y} \otimes \cdots \otimes \overset{r}{y}.$$

Thus

$$\phi := \{b\}$$

is a cone in ${}^s\mathcal{T}^r_0$. Its parameter representation is

$$b^\alpha = f^\alpha(y', \ldots, y^n) := y^{i_1}\ldots y^{i_r}, \quad \alpha = i_1\ldots i_r. \tag{9}$$

The correspondence between $(y^1, \ldots, y^n) \in \mathcal{V}^n(y)$ and $b \in \phi \subset {}^s\mathcal{T}^r_0$ is $1:1$. Thus $\dim\phi = n$. (9) is independent of $x \in M$. Thus ϕ has the same form in each fiber $\mathcal{V}^C \approx {}^s\mathcal{T}^r_0 \subset \mathcal{T}^r_0 \approx \pi^{-1}(x)$ of the bundle $\tilde{\mathcal{E}} = (\tilde{E}, \pi, M, \mathcal{V}^C)$.

One can see that

$$\mathcal{P}^{\gamma}_{x(\tau)} b_0 =: \tilde{b}(x(\tau)) \equiv \tilde{b}(\tau) \in \phi(x(\tau)), \quad \forall\, b_0,\ \forall\, x(\tau) \tag{10}$$

or in another form

$$\mathcal{P}^{\gamma}_{x(\tau)} \phi(x_0) = \phi(x(\tau)) \tag{10'}$$

does not hold in every tensorial connection γ. We want to obtain necessary and sufficient conditions for (10) to hold. We suppose that $\tilde{b}(x(\tau)) = \tilde{b}(\tau) \in \phi(x(\tau)) = \phi(x)$, where $\phi(x)$ is independent of x. Hence every $\tilde{b}(x(\tau)) = b(x)$ can be considered as a point of a single representative ϕ of the $\phi(x)$-s. Thus in case of (10) every $\frac{\partial b^\alpha}{\partial x^k}$ is a tangent of this ϕ:

$$\frac{\partial b^\alpha}{\partial x^k} \in T_b \phi. \tag{11}$$

But also conversely, if (10) is satisfied, then so is (10).

On the other hand, under the supposed conditions $\tilde{b}(\tau) = b(\tau)$ of (10) is a solution of $\frac{Db^\alpha}{d\tau} = 0$, that is (see (3))

$$\frac{db^\alpha}{d\tau} = \frac{\partial b^\alpha}{\partial x^k}\frac{dx^k}{d\tau} = \gamma_\beta{}^\alpha{}_k(x(\tau)) b^\beta \frac{dx^k}{d\tau}, \quad \alpha, \beta = 1, 2, \dots, C, \quad \forall\, x, \dot{x}.$$

Thus $\gamma_\beta{}^\alpha{}_k$ must satisfy the relation

$$\frac{\partial b^\alpha}{\partial x^k}(y) = \gamma_\beta{}^\alpha{}_k(x) b^\beta(y). \tag{12}$$

Furthermore any tangent of ϕ at y is a linear combination of $\frac{\partial f^\alpha}{\partial y^j} \equiv \frac{\partial b^\alpha}{\partial y^j}$. Thus the required necessary and sufficient condition (10) gets the form

$$c^j_k(y) \frac{\partial b^\alpha}{\partial y^j}(y) = \gamma_\beta{}^\alpha{}_k(x) b^\beta(y). \tag{13}$$

This must be satisfied identically in y.

(12) can be considered as a linear equation system for $\gamma_\beta{}^\alpha{}_k$ and c^j_k. We show that (12) has a solution, while many of the unknowns $\gamma_\beta{}^\alpha{}_k$ and c^j_k remain undetermined (free).

$b^\beta(y)$ is a homogeneous polynomial of order r in y. $\frac{\partial b^\alpha}{\partial y^j}$ is also a homogeneous polynomial of order $r-1$. Thus c^j_k must be a homogeneous polynomial of order $1 : c^j_k(y) = {}_s c^j_k y^s$. So (12) gets the form

$${}_s c^j_k y^s \frac{\partial b^\alpha}{\partial y^j}(y) = \gamma_\beta{}^\alpha{}_k(x) b^\beta(y). \tag{13'}$$

This is a special, very simple equation system. For any fixed k_0 we obtain a subsystem

$${}_s c^j y^s \frac{\partial b^\alpha}{\partial y^j}(y) = \gamma_\beta{}^\alpha(x) b^\beta(y), \quad {}_s c^j = {}_s c^j_{k_0},\ \gamma_\beta{}^\alpha = \gamma_\beta{}^\alpha{}_{k_0}. \tag{14}$$

The unknowns ${}_sc^j_{k_0}$ and ${\gamma_\beta}^\alpha{}_{k_0}$ appear in one single subsystem only. Since every subsystem has the same structure, our statement for one of them is true for all of them. Let us fix $\alpha = \alpha_0$. Then on both sides of (13) there is a homogeneous polynomial of order r in y, and (13) must hold identically in y. Thus the coefficients of $y^{i_1} \dots y^{i_r}$ consisting of the different ${}_sc^j$ and ${\gamma_\beta}^\alpha$ must be equal on the two sides. These yield homogeneous linear equations, C in number, for ${}_sc^j$ and ${\gamma_\beta}^\alpha$. The number of the unknowns ${}_sc^j$ and ${\gamma_\beta}^\alpha$ is $n^2 + C^2$. For the different α-s (13) consists of C equations. So the number of the equations for ${}_sc^j$ and ${\gamma_\beta}^\alpha$ stemming from (13) is C^2, and the number of the unknowns remains $n^2 + C^2$. (13') consists of n subsystems for the different k_0 with new unknows in each. Thus (13') yields, as identities in y^s , C^2n equations with $n^3 + C^2n$ unknowns. So we obtain

Proposition 2. *There are many tensorial connections γ taking by parallel translation any decomposable tensor $b = \overset{i}{y} \otimes \cdots \otimes \overset{r}{y}$ into a similar one: $\mathcal{P}^\gamma_{x(\tau)} b_0 = b(\tau)$.*

4 Induced non-linear connection in τ_M

A tensorial connection γ for which $\mathcal{P}^\gamma_{x(\tau)} b_0 \overset{(10)}{=} b(x(\tau))$, or in another form $\mathcal{P}^\gamma_{x(\tau)} \phi(x_0) \overset{(10')}{=} \phi(x(\tau)) \approx \phi$ holds, induces a non-linear connection in τ_M. Namely, as also the diagram

$$\begin{tikzcd} b_0 \in \phi(x_0) \arrow[r, "\mathcal{P}^\gamma_{x(\tau)}"] & b(\tau) \in \phi(x(\tau)) \arrow[d, "f^{-1}"] \\ y_0 \in T_{x_0}M \arrow[u, "f"] \arrow[r, dashed, "\mathcal{N}"] & y(\tau) \in T_{x(\tau)}M \end{tikzcd}$$

shows (f^α from (9))

$$\mathcal{N} := (f^\alpha)^{-1} \circ \mathcal{P}^\gamma_{x(\tau)} \circ f^\alpha \tag{15}$$

takes any $y_0 \in T_{x_0}M$ into a $y(\tau) \in T_{x(\tau)}M$. Each mapping on the right side of (14) is homogeneous in y. Thus

$$\mathcal{P}^\mathcal{N}_{x(\tau)} y_0 = y(\tau). \tag{16}$$

$\mathcal{N}$ is non-linear in y, for $\mathcal{P}^\gamma$ is so in b. Thus we obtain

Theorem 2. *Any tensorial connection, which takes tensors $b = \overset{1}{y} \otimes \cdots \otimes \overset{r}{y}$ into similar ones determines in τ_M among the vectors $y \in T_xM$ a non-linear connection $\mathcal{N}$ in a natural way.*

We want to investigate *metrical* tensorial connections γ of a Finsler space with polynomial metric, which induce non-linear connections $\mathcal{N}$ in τ_M. Then γ satisfies (13'), and it is metrical. A tensorial connection is metrical, if (5') or, in view of the symmetry of a_A,

$$\frac{\partial g_\alpha}{\partial x^k} = {\gamma_\alpha}^\beta{}_k(x) g_\beta \tag{17}$$

holds. At a given (x) (16) means Cn linear equations for the unknowns $\gamma_\alpha{}^\beta{}_k$. So (13') (respectively the equations stemming from the fact that the equations of (13') must be identities in y^s) combined with (16) consists of $Cn + C^2n$ (simple) linear equations, and the number of the unknowns ${}_sc^j_k$ and $\gamma_\beta{}^\alpha{}_k$ remains $C^2n + n^3$. The rank of the combined system is maximal. If the number of the unknowns is not less than the number of the equations, that is if $C^2n + n^3 \geq C^2n + Cn$, or

$$n^2 \geq C = C^m_{n,r}, \tag{18}$$

then the combined system is solvable. Since (in consequence of (16)) γ is metrical, in this case we have

$$\|\mathcal{P}^\gamma_{\alpha(\tau)} b_0\|_F \overset{(8)}{=} \|b(\tau)\|_F = \text{const.}$$

$$\overset{(7)}{=} \mathcal{F}(x, y(\tau)) = \|y(\tau)\|^r_F = \|\mathcal{P}^{\mathcal{N}}_{x(\tau)} y_0\|^r_F.$$

Thus $\|\mathcal{P}^{\mathcal{N}}_{x(\tau)} y_0\|_F = \text{const.}$. This yields

Theorem 3. *If γ is metrical (satisfies* (16)*), and takes every $b = \overset{1}{y}\otimes, \dots, \overset{r}{y}$ into a similar tensor (which satisfies* (13')*), then also the induced non-linear connection $\mathcal{N}$ is metrical with respect to the F^n with polynomial metric.*

The condition of the solvability of the combined system is (17). For which n and r will it be satisfied? It is clear from the notion of multiple combination that $C^m_{n,r}$ is monotone increasing in r for every fix n, and also in n for every fix r. Therefore there exists a maximal r for every n for which $n^2 \geq C^m_{n,r}$. We denote this r by r_n. Then we obtain

Proposition 3. (17) *holds iff $r \leq r_n$. In this case the combined system* (13') *and* (16) *is solvable, and the induced non-linear connection $\mathcal{N}$ is metrical.*

In case of $r = 2$ we have $C^m_{n,r} = \frac{n(n+1)}{2} < n^2$. Thus (17) holds for $\forall\, n$, and so we have tensorial connections γ inducing metrical non-linear connections $\mathcal{N}$ in τ_M. In this case $\mathcal{F}^2(x,y) = a_\alpha(x) b^a = a_{ij}(x) y^i y^j$. This means that for $r = 2$ the Finsler space with polynomial metric is a Riemann space: $F^n = V^n$. Then $\gamma_\alpha{}^\beta{}_k(x) = \gamma_{ij}{}^{rs}{}_k(x) = \Gamma_i{}^r{}_k(x)\delta^s_j + \delta^r_i \Gamma_j{}^s{}_k(x)$. This γ is constructed from the symmetric (torsion free) or non-symmetric Christoffel symbols $\Gamma_j{}^i{}_k$ of V^n. This γ yields a metrical tensorial connection, and the metrical connection $\mathcal{N}$ in τ_M becomes linear with coefficients $\Gamma_j{}^i{}_k(x)$.

In case of $r = 3$ (17) reads as $C^m_{n,3} = \frac{n(n+1)(n+2)}{6} \leq n^2$ or equivalently $n^2 + 1 \leq 3n$. This holds for $n = 2$, but for $r = 3$ and $n = 3$ (17) is not yet true. For $n_0 \geq 3$, $r \geq 3$ we have $n_0^2 < C^m_{n_0,2} < C^m_{n_0,r}$, since $C^m_{n_0,r}$ is increasing in r. Thus for $n \geq 3$, $r \geq 3$ (17) does not hold. For $n = 2$ $C^m_{2,r} = r+1$. Thus (17) holds for $n = 2$, $r = 3$: $2^2 = C^m_{2,3}$ (as we have already seen), but $C^m_{2,3} < C^m_{2,r}$, $r > 3$, since $C^m_{2,r}$ is increasing in r. So we have $n^2 = 4 = C^m_{2,3} < C^m_{2,r}$, that is (17) holds neither for $n = 2$, $r > 3$.

But there may exist special $g_\alpha(x)$ for which the number K of the independent equations of (16) is smaller than Cn, and thus the combined system (13') and (16) still has a solution, for example if $\frac{\partial g_{\alpha_1}}{\partial \alpha^k} + \frac{\partial g_{\alpha_2}}{\partial x^k} = \frac{\partial g_{\alpha_3}}{\partial x^k}$ for certain (or several) k.

Theorem 4. *If γ is metrical (satisfies* (16)*), and takes every $b = \overset{1}{y} \otimes \cdots \otimes \overset{r}{y}$ into a similar tensor (satisfies* (13')*), then the induced non-linear connection $\mathcal{N}$ is also metrical with respect to the F^n with polynomial metric. The condition for this is $n^2 \geq C^m_{n,r}$, or $n^3 \geq K$ if* (16) *has only K independent equations..*

Such γ exists for any Finsler space with polynomial metric only if $r = 2$ (in this case the Finsler space is a Riemannian space) or in case of $r = n = 3$. Such γ exists also for arbitrary r and n, but not for every polynomial metric.

Finally we make two remarks:

Remark 1. $a_A(x)$ of (1) may have the form

$$a_{ijk\ell}(x) = g_{ij}(x) h_{m\ell}(x), \tag{19}$$

where $g_{ij}(x)$ and $h_{m\ell}(x)$ are metric tensors of two Riemannian spaces V_1^n and V_2^n on M. Then

$$\mathcal{F}^4(x,y) = \|y\|_F^4 = \|y\|^2_{V_1} \|y\|^2_{V_2}.$$

This may have a mathematical interest. $\|y\|_{V_1}$ and $\|y\|_{V_2}$ can also mean two different impacts of a physical phenomenon. In this case (18) has a physical interest.

Remark 2. A Randers space $R^n = (M, \mathbb{R}(x,y))$ is a special Finsler space ([7], [14]), where

$$\mathbb{R}(x,y) = (g_{ij}(x) y^i y^j)^{1/2} + b_i(x) y^i$$

in place of $\mathcal{F}(x,y)$ means the Randers metric. In a degenerate case we may have $\mathbb{R}(x,y) = b_i(x) y^i$. If we endow in the vector bundle $\hat{\mathcal{E}}$ (see (2)) of rank N each fiber $\pi^{-1}(x) \approx \mathcal{V}^N$ with the metric $\mathbb{R}(x,y) = a_A(x) b^A$, then we obtain a degenerate Randers vector bundle R^n_N. Thus any Finsler space with polynomial metric (1) can be considered as a degenerate Randers vector bundle. – It could have some interest to consider a Finsler space with polynomial metric as a degenerate Randers vector bundle.

References

[1] G. S. Asanov, *Finsler Geometry and Gauge Theories,* Reidel, 1985.

[2] V. Balan and N. Brinzei, *Einstein equations for (h,v)-Bervald-Moór relativistic models,* Balkan J. Geom. Appl. **11** (2006), 20–27.

[3] V. Balan, N. Brinzei and S. Lebedev, *Geodesics, connections and Jacobi fields for Berwald-Moór quartic metrics,* Hypercomplex Numbers in Geometry and Physics, **2 (6)**, Vol 3, 2006, 113–122.

[4] D. Bao and S. S. Chern, *A note on the Gauss-Bonnet theorem for Finsler spaces,* Ann. Math. **143** (1996), 233–252.

[5] D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry,* Springer, New York, 2000.

[6] N. Brinzei, *Projective relations for Shimada spaces,* to appear.

[7] E. Bompiani, *Le connessioni tensoriali,* Atti Acad. Naz. Lincei **1** (1946), 478–482.

[8] A. Cossu, *Nozioni generali sulle connessioni di specie qualunque,* Rend. Mat. ed Appl. **21** (1962), 162–218.

[9] Y. Ichijyo, *Finsler manifold modeled on Minkowski spaces,* J. Math. Kyoto Univ. **16** (1976), 639–652.

[10] Y. Ichijyo, *Finsler manifold with a linear connection,* J. Math. Tokushima Univ. **10** (1976), 1–11.

[11] L. Kozma and V. Balan, *On metrical homogeneous connections of a Finsler point space,* Publ. Math. Debrecen **49** (1996), 59–68.

[12] M. Kucharzewski, *Über die Tensorübertragung,* Annali di Math. Pura ed Appl. **54** (1961), 64–83.

[13] S. Lebedev, *The generalized Finslerian metric tensors,* Hypercomplex Numbers in Geometry and Physics, **2 (4)**, Vol 2, 2005, 44–50, and in this book.

[14] M. Matsumoto, *Foundation of Finsler Geometry and Special Finsler Spaces,* Kaisheisha, Kyoto, 1986.

[15] M. Matsumoto and H. Shimada, *On Finsler spaces with 1-form metric,* Tensor N. S. **32** (1978), 161–169.

[16] M. Matsumoto and H. Shimada, *On Finsler spaces with 1-form metric, II,* Tensor N. S. **32** (1978), 275–278.

[17] M. Matsumoto and S. Numata, *On Finsler spaces with cubic metric,* Tensor N. S. **33** (1979), 153–162.

[18] M. Matsumoto and K. Okubo, *Theory of Finsler spaces with m-th root metric,* Tensor N. S. **56** (1995), 9–104.

[19] D. G. Pavlov, *Generalization of scalar product axioms,* Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1, 2004, 5–18.

[20] H. Shimada, *On Finsler spaces with 1-form metric II, Berwald-Moór metric* $L = \sqrt[n]{y^1 y^2 \dots y^n}$, Tensor N. S. **32** (1978), 375–378.

[21] H. Shimada, *On Finsler spaces with metric* $L = \sqrt[m]{a_{i_1 \dots i_m}(x) y^{i_1} \dots y^{i_m}}$, Tensor N. S. **33** (1979), 365–372.

[22] L. Tamássy, *Über den Affinzusammenhang von, zu Tangenzialräumen gehörenden Procdukträumen,* Acta Math. Acad. Sci. Hungar **11** (1960), 65–82.

[23] L. Tamássy, *Point Finsler spaces with metrical linear connections,* Publ. Math. Debrecen **56** (2000), 643–655.

[24] L. Tamássy, *Finsler Geometry in the tangent bundle,* Advanced Studies in Pure Math. to appear.

[25] L. Tamássy, *Metrical almost linear connections in TM for Randers spaces,* to appear.

Pairs of Metrical Finsler Structures and Finsler Connections Compatible to Them

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We consider a pair of metrical Finsler structure $g_{ij}(x,y)$, $s_{ij}(x,y)$, $(x,y) \in TM$, $i,j=\overline{1,n}$, $\dim M = n$ and we investigate the cases in which is possible to find Finsler connections compatible to them: $rank \left\| g_{ij}\left(x,y\right)\right\| = n$, $rank \left\| s_{ij}\left(x,y\right)\right\| = n-k$, $k \in \{0,1,...,n-1\}$, $\forall \left(x,y\right) \in TM \setminus \{0\}$.

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1 Metrical Finsler structures and metrical Finsler connections ([7])

Let M be an $n-$dimensional differentiable manifold and $x=(x^i)$ and $y=(y^i)$ denote a point of M and a supporting element respectively. We put $\partial_i = \partial/\partial x^i$, $\dot{\partial}_i = \partial/\partial y^i, (i=1,2,...,n)$.

Let $g_{ij}\left(x,y\right) = \left(\dot{\partial}_i \dot{\partial}_j F^2\right)/2$ be a Finsler metric and $N\left(N^i{}_j\right)$ a nonlinear connection, which us given the adapted basis $\left\{\delta_i,\ \dot{\partial}_i\right\}$ of the tangent bundle $TM =: HM \oplus VM$:

$$\delta_i = \frac{\delta}{\delta x^i} = \partial_i - N^j{}_i \dot{\partial}_j. \tag{1.1}$$

We denote $\{dx^i, dy^i\}$ the dual basis of adapted basis, where

$$\delta y^i = dy^i + N^i{}_j dx^j. \tag{1.2}$$

We shall express a Finsler connection $F\Gamma$ in terms of its coefficients as $F\Gamma = \left(N^j{}_k, F^i{}_{jk}, C^i{}_{jk}\right)$, (cf. with M. Matsumoto [6], R. Miron [7] and E. Stoica [13]). A Finsler connection having a fixed nonlinear connection N is also denoted by $F\Gamma\left(N\right) = \left(F^i{}_{jk}, C^i{}_{jk}\right)$. And the respective h- and v-covariant derivatives are denoted by short and long bars, *e.g.*, $g_{ij\shortmid k}, g_{ij|k}$ (with respect to $F\Gamma$), $g_{ij\overset{\circ}{\shortmid} k}, g_{ij\overset{\circ}{|}k}$ (with respect to $\overset{\circ}{F\Gamma}$), etc.

Given a Finsler metric g_{ij}, a Finsler connection $F\Gamma$ is called **metrical**, if it satisfies

$$g_{ij\shortmid k} = 0, \quad g_{ij|k} = 0. \tag{1.3}$$

For a Finsler metric g_{ij}, we have so-called Obata's operators, [10]:

$$\underset{1}{\Lambda}{}^{pq}_{ij} = \frac{1}{2}\left(\delta^p_i \delta^q_j - g_{ij} g^{pq}\right), \quad \underset{2}{\Lambda}{}^{qp}_{ij} = \frac{1}{2}\left(\delta^p_i \delta^q_j + g_{ij} g^{pq}\right), \tag{1.4}$$

where $(g^{ij}) = (g_{ij})^{-1}$. Then we have

Theorem 1.1 *Let* $F\overset{\circ}{\Gamma}(N) = \left(\overset{\circ}{\Gamma}{}^i{}_{jk}, \overset{\circ}{C}{}^i{}_{jk}\right)$ *be a fixed Finsler connection. For a Finsler metric* g_{ij}, *we define tensor fields* $U^i{}_{jk}, \tilde{U}^i{}_{jk}$ *by*

$$U^i{}_{jk} = -\frac{1}{2} g^{ir} g_{\underset{\circ}{rj|k}}, \quad \tilde{U}^i{}_{jk} = -\frac{1}{2} g^{ir} g_{\overset{\circ}{rj|k}}. \tag{1.5}$$

Then a Finsler connection $F\Gamma(N) = (F^i{}_{jk}, C^i{}_{jk})$ *is metrical, if and only if the difference tensor fields* $B^i{}_{jk}, D^i{}_{jk}$ *given by*

$$F^i{}_{jk} = \overset{\circ}{F}{}^i{}_{jk} - B^i{}_{jk}, \quad C^i{}_{jk} = \overset{\circ}{C}{}^i{}_{jk} - D^i{}_{jk}, \tag{1.6}$$

are solutions of the equations

$$\underset{2}{\Lambda}{}^{ip}_{qj} B^q{}_{pk} = U^i{}_{jk}, \quad \underset{2}{\Lambda}{}^{ip}_{qj} D^q{}_{pk} = \tilde{U}^i{}_{jk}. \tag{1.7}$$

Conform with Obata's theory, [10], the above equations have solutions and their general forms are given by

Theorem 1.2 (R. Miron, [7]) Let $F\overset{\circ}{\Gamma}(N) = \left(\overset{\circ}{F}{}^i{}_{jk}, \overset{\circ}{C}{}^i{}_{jk}\right)$ be a fixed Finsler connection. For a Finsler metric g_{ij}, there exists a metrical Finsler connection $F\Gamma(N) = (F^i{}_{jk}, C^i{}_{jk})$ and the set of all such connections is given by

$$\begin{aligned} F^i{}_{jk} &= \overset{\circ}{F}{}^i{}_{jk} + \frac{1}{2} g^{ir} g_{\underset{\circ}{rj|k}} + \underset{1}{\Lambda}{}^{ip}_{qj} X^q{}_{pk}, \\ C^i{}_{jk} &= \overset{\circ}{C}{}^i{}_{jk} + \frac{1}{2} g^{ir} g_{\overset{\circ}{rj|k}} + \underset{1}{\Lambda}{}^{ip}_{qj} Y^q{}_{pk}, \end{aligned} \tag{1.8}$$

where $X^i{}_{jk}, Y^i{}_{jk}$ are arbitrary Finsler tensor fields.

2 Finsler connections compatible with a pair of Finsler metrics

Let g_{ij} and s_{ij} be two given Finsler metrics. A Finsler connection is called **compatible** with the pair (g_{ij}, s_{ij}), if it is metrical with respect to both g_{ij} and s_{ij}:

$$g_{ij|k} = 0, \quad g_{ij|k} = 0, \quad s_{ij|k} = 0, \quad s_{ij|k} = 0. \tag{2.1}$$

We define Obata's operators by (1.4) and

$$\underset{1}{O}{}^{pq}_{ij} = \frac{1}{2}\left(\delta^p_i \delta^q_j - s_{ij} s^{pq}\right), \quad \underset{2}{O}{}^{pq}_{ij} = \frac{1}{2}\left(\delta^p_i \delta^q_j + s_{ij} s^{pq}\right), \tag{2.2}$$

where $(s^{ij}) = (s_{ij})^{-1}$. Then we have

Theorem 2.1 *Let* $F\overset{\circ}{\Gamma}(N) = \left(\overset{\circ}{F^i}_{jk}, \overset{\circ}{C^i}_{jk}\right)$ *be a fixed Finsler connection. For a pair of Finsler metrics* (g_{ij}, s_{ij}) *we define Finsler tensor fields* $U^i{}_{jk}, \tilde{U}^i{}_{jk}, V^i{}_{jk}, \tilde{V}^i{}_{jk}$ *by* (1.5) *and*

$$V^i{}_{jk} = -\frac{1}{2}s^{ir}s_{rj\overset{\circ}{\shortmid}k}, \quad \tilde{V}^i{}_{jk} = -\frac{1}{2}s^{ir}s_{rj\overset{\circ}{|}k}. \tag{2.3}$$

Then a Finsler connection $F\Gamma(N) = (F^i{}_{jk}, C^i{}_{jk})$ *is compatible with the pair* (g_{ij}, s_{ij}), *if and only if the difference tensor fields* $B^i{}_{jk}, D^i{}_{jk}$ *given by* (1.6) *are solutions of the equations* (1.7) *and following equations*

$$\underset{2}{O}{}^{ip}_{qj}B^q{}_{pk} = V^i{}_{jk}, \quad \underset{2}{O}{}^{ip}_{qj}D^q{}_{pk} = \tilde{V}^i{}_{jk}. \tag{2.4}$$

It is complicated to solve the above equations.

We shall show the case when the equations have solutions.

A pair of two Finsler metrics g_{ij}, s_{ij} is called **natural**, if there exists a nonvanishing Finsler function $\mu(x, y)$ such that

$$g_{ip}g_{jq}s^{pq} = \mu s_{ij}, \tag{2.5}$$

or equivalently, if the commutativities

$$\underset{\alpha}{\Lambda}{}^{ip}_{qj}\underset{\beta}{O}{}^{qr}_{tp} = \underset{\beta}{O}{}^{ip}_{qj}\underset{\alpha}{\Lambda}{}^{qr}_{tp}, \quad (\alpha, \beta = 1, 2), \tag{2.6}$$

hold. Then, we have

Proposition 2.1 *All the commutativities* (2.6) *hold if any one of them holds.*

Proposition 2.2 *Let* (g_{ij}, s_{ij}) *be a natural pair of Finsler metrics. If there exists a Finsler connection compatible with the pair, the function* μ *in* (2.5) *is constant.*

Proof. The equations (2.1) are equivalent by the following equations:

$$g^{ij}{}_{\shortmid k} = 0, \quad g^{ij}{}_{|k} = 0, \quad s^{ij}{}_{\shortmid k} = 0, \quad s^{ij}{}_{|k} = 0. \tag{2.7}$$

By (2.1) and (2.1′) we have $\mu_{\shortmid k}s_{ij} = 0$, $\mu\,|_k\, s_{ij} = 0$, which are reduced to $\mu_{\shortmid k} = 0$, $\mu\,|_k = 0$ because $s_{ij}s^{ij} = n \neq 0$. Hence the nonvanishing function μ is constant.

Proposition 2.3 *Let* g_{ij} *be a Finsler metric. There exists a Finsler metric* s_{ij} *such that the pair* (g_{ij}, s_{ij}) *is natural by a constant* $\mu = \varepsilon c^2\,(\varepsilon = \pm 1, c > 0)$, *if and only if there exists a Finsler tensor field* $t^i{}_j$ *of type* $(1, 1)$ *satisfying*

$$\varepsilon t^i{}_r t^r{}_j = \delta^i_j, \quad \varepsilon g_{pq}t^p{}_i t^q{}_j = g_{ij}. \tag{2.8}$$

The correspondence between $t^i{}_j$ and s_{ij} in Proposition 2.3 is given by

$$t^i{}_j = cg^{ir}s_{rj}, \quad s_{ij} = \frac{1}{c}g_{ir}t^r{}_j. \tag{2.9}$$

Remark 2.1 If $\varepsilon = -1$, then $\mu = -c^2$ and $t^i{}_j$ is an almost complex Finsler structure $f^i{}_j : f^2 = -I,\ (n = 2m)$. In this case, the natural pair (g_{ij}, s_{ij}) is called of **elliptical type,** or a **(g, f, −1)−structure** (cf. with Gh. Atanasiu [1], Gh. Atanasiu, M. Hashiguchi, R. Miron [3]), or an **anti-Hermitian structure**:

$$f^i{}_r f^r{}_j = -\delta^i_j, \quad g_{pq}f^p{}_i f^q{}_j = -g_{ij}. \tag{2.10}$$

Remark 2.2 If $\varepsilon = +1$, then $\mu = c^2$ and $t^i{}_j$ is an almost product Finsler structure $p^i{}_j : p^2 = I$. In this case, the natural pair (g_{ij}, s_{ij}) is called of **hyperbolical type,** or a **(g, p, +1)−structure** (see [1], [3]:

$$p^i{}_r p^r{}_j = \delta^i_j, \quad g_{rt}p^r{}_i p^t{}_j = g_{ij}. \tag{2.11}$$

Using Proposition 2.3 we can show that for a natural pair (elliptic or hyperbolic) with a constant $\mu \neq 0$ the equations (1.7) and (2.4) have solutions and their general forms are given by

Theorem 2.2 *Let $\overset{\circ}{F\Gamma}(N) = \left(\overset{\circ}{F}{}^i{}_{jk}, \overset{\circ}{C}{}^i{}_{jk}\right)$ be a fixed Finsler connection. For a natural pair with a constant $\mu \neq 0$ of Finsler metric g_{ij}, s_{ij}, there exists a Finsler connection $F\Gamma(N) = (F^i{}_{jk}, C^i{}_{jk})$ compatible with the pair and the set of all such connections is given by*

$$\begin{aligned} F^i{}_{jk} &= \overset{\circ}{F}{}^i{}_{jk} + \frac{1}{2}\left(g^{ir}g_{rj\overset{\circ}{|}k} + \underset{1}{\Lambda}{}^{ip}_{qj}s^{qt}s_{tp\overset{\circ}{|}k} \right) + \underset{1}{\Lambda}{}^{ip}_{qj}\underset{1}{O}{}^{qr}_{tp}X^t{}_{rk}, \\ C^i{}_{jk} &= \overset{\circ}{C}{}^i{}_{jk} + \frac{1}{2}\left(g^{ir}g_{rj}\overset{\circ}{|}_k + \underset{1}{\Lambda}{}^{ip}_{qj}s^{qt}s_{tp}\overset{\circ}{|}_k \right) + \underset{1}{\Lambda}{}^{ip}_{qj}\underset{1}{O}{}^{qr}_{tp}Y^t{}_{rk}, \end{aligned} \tag{2.12}$$

where $X^i{}_{jk}, Y^i{}_{jk}$ are arbitrary Finsler tensor fields.

3 The case of Finsler metric with an additional structure

The previous results for a pair of Finsler metrics $g_{ij}(x,y), s_{ij}(x,y)$ are generalized to the case $s_{ij}(x,y)$ is degenerate.

Let a Finsler space (M, g_{ij}) admit a symmetric and degenerate Finsler tensor field $s_{ij}(x,y)$:

$$s_{ij} = s_{ji} \tag{3.1}$$

$$rank\,(s_{ij}) = n - k, \tag{3.2}$$

where k is an integer and $0 < k < n$. Then (M, g_{ij}) is called to have an **additional structure of index k**. The case of a Finsler metric $s_{ij}(x,y)$ is contained in the following duscussions as the exceptional case $k = 0$.

The matrix (g_{ij}) has the inverse $\left(g^{jk}\right)$, but the matrix (s_{ij}) is not regular. So we shall construct some matrix $\left(s^{jk}\right)$ which plays the role similar to the inverse matrix. (see, V. Oproiu [11], [12]). Because (g_{ij}) is positive-definite, then on each local chart there are exaktly k independent Finsler vector fields $\xi^i_a\,(a = 1, ..., k)$ with the properties

$$s_{ij}\xi^i_a = 0, \quad g_{ij}\xi^i_a\xi^j_b = \delta_{ab} \quad (a, b = 1, ..., k)\,. \tag{3.3}$$

Then we define local Finsler covector fields $\eta^a_i\,(a = 1, ..., k)$ by

$$\eta^a_i = g_{ij}\xi^j_a. \tag{3.4}$$

If we define local Finsler tensor fields $l^i{}_j$ and $m^i{}_j$ by

$$l^i{}_j = \sum_a \xi^i_a\eta^a_i, \quad m^i{}_j = \delta^i_j - l^i{}_j, \tag{3.5}$$

then $l^i{}_j$ and $m^i{}_j$ are independent on the choice of ξ^i_a and globally defined as the respective projectors on the kernel $\mathbf{K}$ of the mapping $s_{ij}:\ \xi^j_a \longrightarrow s_{ij}\xi^j_a$ and the orthogonal $\mathbf{H}$ to $\mathbf{K}$ with respect to g_{ij}. Then a global Finsler tensor field s^{jk} is uniquely determined from (g_{ij}, s_{ij}) by

$$s_{ij}s^{jk} = m^k{}_i, \quad l^i{}_js^{jk} = 0. \tag{3.6}$$

A Finsler connection of a Finsler space (M, g_{ij}) with an additional structure s_{ij} is called **compatible** with the pair (g_{ij}, s_{ij}), if it satisfies (2.1). Then the condition that a Finsler connection $F\Gamma$ is compatible with the pair (g_{ij}, s_{ij}) is given by Theorem 2.1, if we define $V^i{}_{jk}, \tilde{V}^i{}_{jk}$ by

$$\begin{aligned} V^i{}_{jk} &= -\frac{1}{2}\left(s^{ir}s_{rj\overset{\circ}{|}k} + 3l^i{}_tl^t{}_{j\overset{\circ}{|}k} - l^i{}_{j\overset{\circ}{|}k}\right), \\ \tilde{V}^i{}_{jk} &= -\frac{1}{2}\left(s^{ir}s_{rj}\overset{\circ}{|}_k + 3l^i{}_tl^t{}_j\overset{\circ}{|}_k - l^i{}_j\overset{\circ}{|}_k\right), \end{aligned} \tag{3.7}$$

and Obata's operators $\underset{\alpha}{O}{}^{pq}_{ij}\ (\alpha = 1, 2)$ by

$$\begin{aligned} \underset{1}{O}{}^{pq}_{ij} &= \frac{1}{2}\left(\delta^p_i\delta^q_j - \delta^p_il^q_j - l^p_i\delta^q_j + 3l^p_il^q_j - s_{ij}s^{pq}\right), \\ \underset{2}{O}{}^{pq}_{ij} &= \frac{1}{2}\left(\delta^p_i\delta^q_j + \delta^p_il^q_j + l^p_i\delta^q_j - 3l^p_il^q_j + s_{ij}s^{pq}\right), \end{aligned} \tag{3.8}$$

and impose on the $B^i{}_{jk}$ and $D^i{}_{jk}$ the additional conditions:

$$\begin{aligned} l^r{}_is_{tj}B^t{}_{rk} &= -l^r{}_is_{rj\overset{\circ}{|}k}, \quad & l^r{}_is_{tj}B^t{}_{rk} &= -l^r{}_is_{rj}\overset{\circ}{|}_k, \\ l^i{}_tm^r{}_jB^t{}_{rk} &= -l^i{}_tl^t{}_{j\overset{\circ}{|}k}, \quad & l^i{}_tm^r{}_jD^t{}_{rk} &= -l^itl^t{}_j\overset{\circ}{|}_k. \end{aligned} \tag{3.9}$$

If we define the naturality of a pair (g_{ij}, s_{ij}) by (2.5), or equivalently (2.6) where $\underset{\alpha}{O}{}^{pq}_{ij}$ are defined by (3.8), then Propositions 2.1 and 2.2 still hold. Corresponding to Proposition 2.3, the condition that a Finsler space (M, g_{ij}) admits an additional structure s_{ij} of index k such that the pair (g_{ij}, s_{ij}) is natural by a constant $\mu = \varepsilon c^2$ $(\varepsilon = \pm 1, c > 0)$ is given by the existence of a Finsler tensor field $t^i{}_j$ of type $(1,1)$, k Finsler vector fields ξ^i_a $(a = 1, ..., k)$ and k Finsler covector fields η^a_i $(i = 1, ..., k)$ satisfying

$$\begin{gathered}\varepsilon t^i{}_r t^r{}_j = \delta^i_j - \xi^i_a \eta^a_j, \quad \varepsilon g_{pq} t^p{}_i t^q{}_j = g_{ij} - \sum_a \eta^a_i \eta^a_j, \\ \eta^a_i t^i{}_j = 0, \quad t^i{}_j \xi^j_a = 0, \ \eta^a_i \quad \xi^i_b = \delta^a_b.\end{gathered} \tag{3.10}$$

Remark 3.1 If $\varepsilon = -1$, then $\mu = -c^2$ and $t^i{}_j$ is an degenerate almost complex Finsler structure $f^i{}_j(x, y)$:

$$\begin{gathered}f^i{}_r f^r{}_j = -\delta^i_j + \xi^i_a \eta^a_j, \quad g_{pq} f^p{}_i f^q{}_j = -g_{ij} + \sum_a \eta^a_i \eta^a_j, \\ \eta^a_i f^i{}_j = 0, \quad f^i{}_j \xi^j_a = 0, \quad \eta^a_i \xi^i_b = \delta^a_b.\end{gathered} \tag{3.11}$$

In this case we have a $(g, f, \xi, \eta, -1)$ −structure, [5], [9].

Remark 3.2 If $\varepsilon = +1$, then $\mu = c^2$ and $t^i{}_j$ is an degenerate almost product Finsler structure $p^i{}_j(x, y)$:

$$\begin{gathered}p^i{}_r p^r{}_j = \delta^i_j - \xi^i_a \eta^a_j, \quad g_{rt} p^r{}_i p^t{}_j = g_{ij} - \sum_a \eta^a_i \eta^a_j, \\ \eta^a_i p^i{}_j = 0, \quad p^i{}_j \xi^j_a = 0, \quad \eta^a_i \xi^i_b = \delta^a_b.\end{gathered} \tag{3.12}$$

and we have a $(g, p, \xi, \eta, +1)$ −structure, [3], [9].

The existence and arbitrariness of Finsler connections compatible with a pair $(g_{ij},\ s_{ij})$ with a constant $\mu \neq 0$, is given by

Theorem 3.1 Let $F\Gamma\left(\overset{\circ}{N}\right) = \left(\overset{\circ}{F}{}^i{}_{jk}, \overset{\circ}{C}{}^i{}_{jk}\right)$ be a fixed Finsler connection. There exists a Finsler connection $F\Gamma(N) = \left(F^i{}_{jk}, C^i{}_{jk}\right)$ compatible with the pair and the set of all such connections is given by

$$\begin{aligned}F^i{}_{jk} &= \overset{\circ}{F}{}^i{}_{jk} + \frac{1}{2}\left[g^{ir} g_{rj\overset{\circ}{|}k} + \underset{1}{\Lambda}{}^{ip}_{qj}\left(s^{qt} s_{tp\overset{\circ}{|}k} + 3 l^q{}_t l^t{}_{p\overset{\circ}{|}k} - l^q{}_{p\overset{\circ}{|}k}\right)\right] + \underset{1}{\Lambda}{}^{ip}_{qj} \underset{1}{O}{}^{qr}_{tp} X^t{}_{rk}\,, \\ C^i{}_{jk} &= \overset{\circ}{C}{}^i{}_{jk} + \frac{1}{2}\left[g^{ir} g_{rj}\overset{\circ}{|}_k + \underset{1}{\Lambda}{}^{ip}_{qj}\left(s^{qt} s_{tp}\overset{\circ}{|}_k + 3 l^q{}_t l^t{}_p\overset{\circ}{|}_k - l^q{}_p\overset{\circ}{|}_k\right)\right] + \underset{1}{\Lambda}{}^{ip}_{qj} \underset{1}{O}{}^{qr}_{tp} Y^t{}_{rk}.\end{aligned} \tag{3.13}$$

where $\underset{1}{O}{}^{pq}_{ij}$ is given by (3.8) and $X^i{}_{jk}, Y^i{}_{jk}$ are arbitrary Finsler tensor fields.

Lastly, it is noted whether the naturality is necessary in order that the system of equations (1.7), (2.4), (3.9) with unknowns $B^i{}_{jk}, D^i{}_{jk}$ has a solution is an open problem.

References

[1] Gh. Atanasiu, *Variétés différentiables douées de couples de structures Finsler,* Proc. of the Nat. Sem. on Finsler Spaces, Univ. of Brasov, România, vol. II, 1982, 35–67.

[2] Gh. Atanasiu, *Structures Finsler presque horsimplectiques,* An.Ştiinţ. Univ. "Al. I. Cuza" Iaşi, Secţ. I-a Mat, 30-4, 1984, 15–18.

[3] Gh. Atanasiu, M. Hashiguchi, R. Miron, *Supergeneralized Finsler spaces,* Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.), 18, 1985, 19–34.

[4] Gh. Atanasiu, M. Hashiguchi, R. Miron, *Lagrange connections compatible with a pair of generalized Lagrange metrics,* Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.), 19, 1986, 1–6.

[5] S. I. Goldberg, K. Yano, *Globally framed f-manifolds,* Ilinois J. Math., 15, 1971, 456–474.

[6] M. Matsumoto, *The theory of Finsler connections,* Publ. Study Group Geom., 5, Okayama Univ., 1970.

[7] R. Miron, *Metrical Finsler structures and metrical Finsler connections,* J. Math. Kyoto Univ., 23, 1983, 219–224.

[8] R. Miron, Gh. Atanasiu, *Existence et arbitraire des connexions compatibles aux structure Riemann généralisées du type Hermitien,* Tensor, N. S., 38, 1982, 8–12.

[9] R. Miron, Gh. Atanasiu, *Existence et arbitrariété des connexions compatible à une structure Riemann généralisée de type presque k-horsymplectique métrique,* Kodai Math. J., 6, 1983, 228–237.

[10] M. Obata, *Affine connections on manifold with almost complex, quaternion or Hermitian structure,* Jap. J. Math., 26, 1957, 43–77.

[11] V. Oproiu, *Degenerate Riemannian and degenerate conformal connections,* An. Univ. "Al. I. Cuza" Iaşi, 16, 1970, 357–376.

[12] V. Oproiu, *Degenerate almost symplectic structure and degenerate almost symplectic connections,* Bull. Math. t. 14 (62), nr. 2, 1970, 197–207

[13] E. Stoica, *A geometrical characterization of normal Finsler connections,* An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi, Secţ. I-a Math., 30, 1984, 3.

[14] K. Yano, S. Ishihara, *Tangent and cotangent bundle,* M. Dekker, New-York, 1973,

CMC and Minimal Surfaces in Berwald-Moor Spaces

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For Randers and Kropina Finsler spaces are described the extended equations of minimal and CMC hypersurfaces. For the Berwald-Moor type Finsler metric are then considered different types of symmetric polynomials generating the fundamental function and classes of CMC surfaces are evidentiated. Maple 9.5 representations of indicatrices point out structural differences among Berwald-Moor fundamental functions of different order, leading to different CMC approaches.

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1 Introduction

Recently Z. Shen ([16]), and further M. Souza and K. Tenenblat ([17]) have investigated minimal surfaces immersed in Finsler spaces from differential geometric point of view. Still, earlier rigorous attempts using functional analysis exist in the works of G. Bellettini and M. Paolini (after 1995, e.g., [7, 8, 9]). In 1998, based on the notion of Hausdorff measure, Z. Shen ([16]) has introduced the notion of mean curvature on submanifolds of Finsler spaces as follows.

If $(\tilde{M}, \tilde{F})$ is a Finsler structure, and $\varphi : (M, F) \to (\tilde{M}, \tilde{F})$ is an isometric immersion (hence F is induced by $\tilde{F}$), then the mean curvature of M is given by ([16, (57), p. 563])

$$H_\varphi(X) = \frac{1}{G}\left(G_{;x^i} - G_{;z^i_a z^j_b}\varphi^j_{;u^a u^b} - G_{;x^j z^i_a}\varphi^j_{;u^a}\right) X^i,$$

where lower indices stand for corresponding partial derivatives and:

- $(u^a, v^b)_{a,b\in\overline{1,n}}$ are local coordinates in TM $(\dim M = n)$;
- $(x^i, y^j)_{i,j\in\overline{1,m}}$ are local coordinates in $T\tilde{M}$ $(\dim \tilde{M} = m)$;
- z^i_a are the entries of the Jacobian matrix $[J(\varphi)] = (\partial\varphi^i/\partial u^a)_{a=\overline{1,n},\, i=\overline{1,m}}$;
- $\varphi_t : M \to \tilde{M}$, $t \in (-\varepsilon, \varepsilon)$, $\varphi_0 = \varphi$, is the variation of the surface;
- X is the vector field $X_x = \frac{\partial\varphi_t}{\partial t}\,|_{t=0}\,(x)$ induced along φ attached to the variation;
- G is the Finsler induced volume form

$$G_{\tilde{e}}(z) = \frac{vol[B^n]}{vol\{(v^a) \in \mathbb{R}^n \mid \tilde{F}(v^a z^i_a \tilde{e}_i) \leq 1\}}, \tag{1.1}$$

where $z = (z^i_a)_{a=\overline{1,n},\, i=\overline{1,m}} \in GL_{m\times n}(\mathbb{R})$, $\tilde{e} = \{\tilde{e}_i\}_{i=\overline{1,m}}$ is an arbitrary basis in $\mathbb{R}^m$ and $B^n \subset \mathbb{R}^n$ is the standard Euclidean ball.

It was proved that the variation of the volume in M reaches a minimum for $H_\varphi = 0$ ([16]). Recent advances in constructing minimal surfaces ($n = 2$) based on (1.1) were provided in ([17]), by characterizing the minimal surfaces of revolution in Randers spaces $(\tilde{M} = R^3, \tilde{F})$ with the Finsler (α, β)-fundamental function

$$\tilde{F}(x,y) = \alpha(x,y) + \beta(x,y), \quad \alpha(x,y) = \sqrt{a_{ij}(x)^i y^j}, \;\; \beta(x,y) = b_i(x) y^i,$$

for the particular case when $a_{ij} = \delta_{ij}$ (the Euclidean metric) and $\beta = b \cdot \mathrm{d}x^3$, with $b \in [0,1)$.

We further consider a real smooth manifold $\tilde{M}$ of dimension $n+1$ endowed with a positive 1-homogeneous locally Minkowski Finsler fundamental function $F : T\tilde{M} \to \mathbb{R}$ ([13]).

2 Generalized Randers-Kropina hypersurfaces ([4])

Let $H = Im\ \varphi$, $\varphi :\ D \subset \mathbb{R}^n \to \tilde{M} = \mathbb{R}^{n+1}$ be a simple hypersurface. We denote $z^i_\alpha = \frac{\partial \varphi^i}{\partial u^\alpha}$, $u = (u^1, \ldots, u^n) \in D$. We shall further determine the volume of the body $Q \subset T_{\varphi(u)}H$ bounded by the induced on $T_{\varphi(u)}H$ indicatrix from $\tilde{M}$

$$\Sigma_* = T_{\varphi(u)}H \cap \{y \in T_{\varphi(u)}\mathbb{R}^{n+1} | F(y) = 1\}.$$

If $v = v^\alpha \frac{\partial}{\partial u^\alpha} \in T_u D$, then $\varphi_{*,u}(v) = z^i_\alpha v^\alpha \left.\frac{\partial}{\partial y^i}\right|_{\varphi(u)} \in T_{\varphi(u)}H$ and hence at some fixed point $u \in D$, Q is given by

$$Q = \{v \in T_u D \mid F(\varphi(u), \varphi_{*,u}(v)) \leq 1\}.$$

We have the following:

Theorem 1. *If the body Q is given by*

$$Q : \;\sum_{i=1}^{n} (z^i_\alpha v^\alpha)^2 + \mu (z^{n+1}_\alpha v^\alpha)^2 + 2\nu z^{n+1}_\alpha v^\alpha + \rho \leq 0, \tag{2.1}$$

where $\mu, \nu, \rho \in \mathbb{R}$, then

$$Vol(Q) = \begin{cases} \dfrac{Vol(B_n)}{\sqrt{\delta}\cdot(1+\tau)^{(n+1)/2}} \cdot \left(\dfrac{\nu^2 \tau}{\mu - 1} - \rho(1+\tau)\right)^{n/2}, & for\ \mu \neq 1 \\ \dfrac{Vol(B_n)\cdot(-\rho + \nu^2 z^{n+1}_a z^{n+1}_b h^{ab})^{n/2}}{\sqrt{\delta}}, & for\ \mu = 1, \end{cases} \tag{2.2}$$

where τ *and* δ *are given by*

$$\tau = (\mu - 1) z_a^{n+1} z_b^{n+1} h^{ab}, \quad \delta = \det(h_{ab})_{a,b=\overline{1,n}},$$

and $B_n \subset \mathbb{R}^n$ *is the standard* n*-dimensional ball and* h^{ab} *is the dual of* h_{ab} *(*$h^{as} h_{sb} = \delta_b^a$*).*

In particular, we obtain the following result:

Corollary 1. *a) In the Randers case*

$$F(x, y) = \sqrt{\sum_{i=1}^{n+1} (y^i)^2} + b y^{n+1}, \quad b \in [0, 1), \tag{2.3}$$

we obtain the known result ([17, (5), p. 627]),

$$Vol(Q_R) = \frac{Vol(B_n)}{\sqrt{\delta}(1 - b^2 z_a^{n+1} z_b^{n+1} h^{ab})^{(n+1)/2}}.$$

b) In the Kropina case

$$F(x, y) = (b y^{n+1})^{-1} \cdot \sum_{i=1}^{n+1} (y^i)^2, \quad b \in [0, 1), \tag{2.4}$$

we have

$$Vol(Q_K) = \frac{Vol(B_n) \left(\frac{b^2}{4} z_a^{n+1} z_b^{n+1} h^{ab} \right)^{n/2}}{\sqrt{\delta}}.$$

Remarks. In the Kropina case, the function G in (1.1) has the expression

$$G = \frac{Vol(B_n)}{Vol(Q_K)} = \frac{\sqrt{\delta}}{(z_a^{n+1} z_b^{n+1} h^{ab} \cdot b^2/4)^{n/2}} = 2^n \cdot C B^{-n/2},$$

where we have used the notations from [17], $B = b^2 z_a^{n+1} z_b^{n+1} h^{ab}$, $C = \sqrt{\delta}$. Then the mean curvature vector field has the components

$$\bar{H}_i = \frac{1}{G} \left(\frac{\partial^2 G}{\partial z_\varepsilon^i z_\eta^j} \cdot \frac{\partial^2 \varphi^j}{\partial u^\varepsilon \partial u^\eta} \right), \quad i = \overline{1, n+1},$$

and the volume form of the hypersurface H is

$$dV_F = \frac{\sqrt{\delta}}{\left(\frac{b^2}{4} z_a^{n+1} z_b^{n+1} h^{ab} \right)^{n/2}} du^1 \wedge \cdots \wedge du^n.$$

Theorem 2. *The mean curvature vector field of the hypersurface M in the Kropina space $\tilde{M} = \mathbb{R}^{n+1}$ with the fundamental function (2.4) has the following expression in terms of B and C*

$$H_i = 2^n B^{-(n+4)/2}\left[\frac{\partial^2 C}{\partial z^i_\varepsilon \partial z^j_\eta} B^2 + \frac{n(n+2)}{4} C \frac{\partial B}{\partial z^i_\varepsilon}\frac{\partial B}{\partial z^j_\eta} - \right.$$
$$\left. - \frac{nB}{2}\left(\frac{\partial C}{\partial z^i_\varepsilon}\frac{\partial B}{\partial z^j_\eta} + \frac{\partial C}{\partial z^j_\eta}\frac{\partial B}{\partial z^i_\varepsilon} + C\frac{\partial^2 B}{\partial z^i_\varepsilon \partial z^j_\eta}\right)\right]\frac{\partial^2 \varphi^j}{\partial u^\varepsilon \partial u^\eta}, \quad i = \overline{1, n+1}.$$

Corollary 2. *The mean curvature vector field of the surface M in the Kropina space $\tilde{M} = \mathbb{R}^{n+1}$ with the fundamental function (2.4) has the following expression*

$$\begin{aligned} H_i = \frac{4C}{E^3}\Bigg[& 6E^2 \frac{\partial C}{\partial z^i_\varepsilon}\frac{\partial C}{\partial z^j_\eta} + 2C^2 \frac{\partial E}{\partial z^i_\varepsilon}\frac{\partial E}{\partial z^j_\eta} - C^2 E \frac{\partial^2 E}{\partial z^i_\varepsilon \partial z^j_\eta} - \\ & -3CE\left(\frac{\partial E}{\partial z^i_\varepsilon}\frac{\partial C}{\partial z^j_\eta} + \frac{\partial E}{\partial z^j_\eta}\frac{\partial C}{\partial z^i_\varepsilon}\right) + 3E^2 C \frac{\partial^2 C}{\partial z^i_\varepsilon \partial z^j_\eta}\Bigg]\frac{\partial^2 \varphi^j}{\partial u^\varepsilon \partial u^\eta}, \quad i = \overline{1, n+1}, \end{aligned} \tag{2.5}$$

where

$$E = b^2 \sum_{k=1}^{3}\sum_{\alpha,\beta=1}^{2} (-1)^{\alpha+\beta} z^k_{\tilde{\alpha}} z^k_{\tilde{\beta}} z^3_\alpha z^3_\beta, \quad \tilde{\alpha} = 3 - \alpha.$$

Corollary 3. *The mean curvature of the surface M in the Kropina space (2.4) is $H_* = H_i X^i$, where H_i are given by (2.5),*

$$X \in Ker(G_* Z^1) \cap Ker(G_* Z^2) \cap \{y \in T_{\varphi(u)}\tilde{M} \mid F(y) = 1\},$$

$Z^1 = (z^1_1, z^2_1, z^3_1)$, $Z^2 = (z^1_2, z^2_2, z^3_2)$, and $G_* v$ is defined by the equality $(G_* v)(v') = \langle v, v'\rangle_F = \dfrac{1}{2}\dfrac{\partial F^2}{\partial y^i \partial y^j} v^i v'^j$.

Corollary 4. *Let $M = \Sigma = Im\, \varphi$ be a surface of revolution described by*

$$\varphi(t, \theta) = (f(t)\cos\theta, f(t)\sin\theta, t), \quad (t, \theta) \in D = \mathbb{R} \times [0, 2\pi).$$

Then M is minimal iff the function f satisfies the ODE

$$1 + f'^2 = 3ff''(1 + 2f'^2).$$

3 The Berwald-Moor Finsler case

We shall further point out the obstructions present in the case of a Berwald-Moor Finsler metric and evidentiate the means of construction of spatial and temporal CMC and minimal surfaces. The substantial difference between the Randers-Kropina framework and the Berwald-Moor Finsler metric relies in the fact that the indicatrix $\Sigma : F(x,y) = 1, x \in M$ is in general noncompact for all values of x. Hence, one may not talk about the volume contained inside this hypersurface Σ, which in the latter case extends to infinity and the volume is provided by a divergent integral.

However, specializing to certain temporal or spatial slices, one may define within them CMC or minimal submanifolds of codimension 1, in particular surfaces.First we note that in the case of Minkowski Finsler metrics of Berwald-Moor type

$$F(x,y) = \sqrt[k]{P_k(y^1,\dots,y^n)}, \quad (\dim M = n \geq 3),$$

provided by appropriate order square roots of homogeneous polynomials P_k, even in the case when the indicatrix Σ is compact and strongly convex, e.g.,

$$F(x,y) = \sqrt[2k]{(y^1)^{2k} + \cdots + (y^n)^{2k}}, \quad (k \geq 2), \tag{3.1}$$

the task of computing the volume bounded by Σ becomes difficult for higher orders k (see Appendix I). This points out once more that from technical point of view choosing an appropriate submanifold which would decrease the dimension, is a desirable attempt.

We shall discuss further several cases of nonpositive signature of the Finsler metric tensor field given by the halved y-Hessian of F^2.

1. The $H(4) \sim \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ - type Berwald-Moor Minkowski - Finsler metrics [11]:

$$\begin{aligned} F_2(y) &= \sqrt{|(a+b)(c+d) + ab + cd|}, \\ F_3(y) &= \sqrt[3]{|ab(c+d) + cd(a+b)|}, \\ F_4(y) &= \sqrt[4]{|abcd|}, \end{aligned} \tag{3.2}$$

where $n = 4$, $y = (y^1, y^2, y^3, y^4) = (a,b,c,d) \in T_p(\mathbb{R}^4)$. After performing the Hadamard change of basis of matrix $C = \frac{1}{4}\begin{pmatrix} A & A \\ A & -A \end{pmatrix}$, with $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ given by $y^T = C\hat{y}^T$, $\hat{y} = (t,x,y,z)$, the functions (3.2) transform into

$$\begin{aligned} \hat{F}_2(\hat{y}) &= \sqrt{|6t^2 - 2(x^2+y^2+z^2)|}, \\ \hat{F}_3(\hat{y}) &= \sqrt[3]{|8xyz + 4t(t^2 - x^2 - y^2 - z^2)|}, \\ \hat{F}_4(\hat{y}) &= \sqrt[4]{|x^4 + y^4 + z^4 + t^4 + 8txyz - 2[(x^2+y^2)(z^2+t^2) + x^2y^2 + z^2t^2]|}. \end{aligned}$$

Hence for $\hat{F}_2$ one might consider the slice submanifold $\hat{y}^1 \equiv t = const$ where the CMC imbedded surfaces are the Euclidean ones.

For $\tilde{F}_4$, besides considering the spatial slices $v^i = const$, $(i \in \overline{1,4})$ one might look for subclasses of CMC surfaces which satisfy additional PDEs, by reformulating the energy-minimizing problem using Lagrange multipliers imposed, e.g., by

$$(v^1)^4 + \cdots + (v^n)^4 \equiv \tilde{F}_4(\hat{y})|_{\hat{y}^i=(C^{-1})^i_j z^j_\alpha v^\alpha,\ i=\overline{1,4}},$$

or, for the initial basis,

$$(v^1)^4 + \cdots + (v^n)^4 \equiv (z^1_\alpha v^\alpha)(z^2_\beta v^\beta)(z^3_\gamma v^\gamma)(z^4_\delta v^\delta),$$

where the Greek indices run through $\overline{1,n}$, with $n \geq 1$.

2. In general, for $m \geq 3$ and Berwald-Moor metrics of type (3.1), valid additional PDEs which impose the change of energy to provide surface-like CMC surfaces are

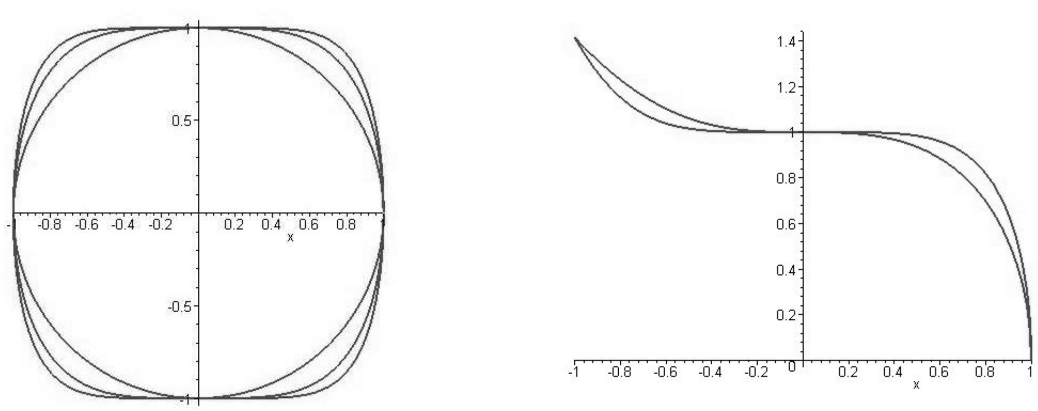
$$(v^1)^{2k} + \cdots + (v^n)^{2k} \equiv F(y)|_{y^i=z^i_\alpha v^\alpha,\ i=\overline{1,m}},$$

with the same conventions as above.

3. A notable difference exhibited by Berwald-Moor Finsler fundamental functions

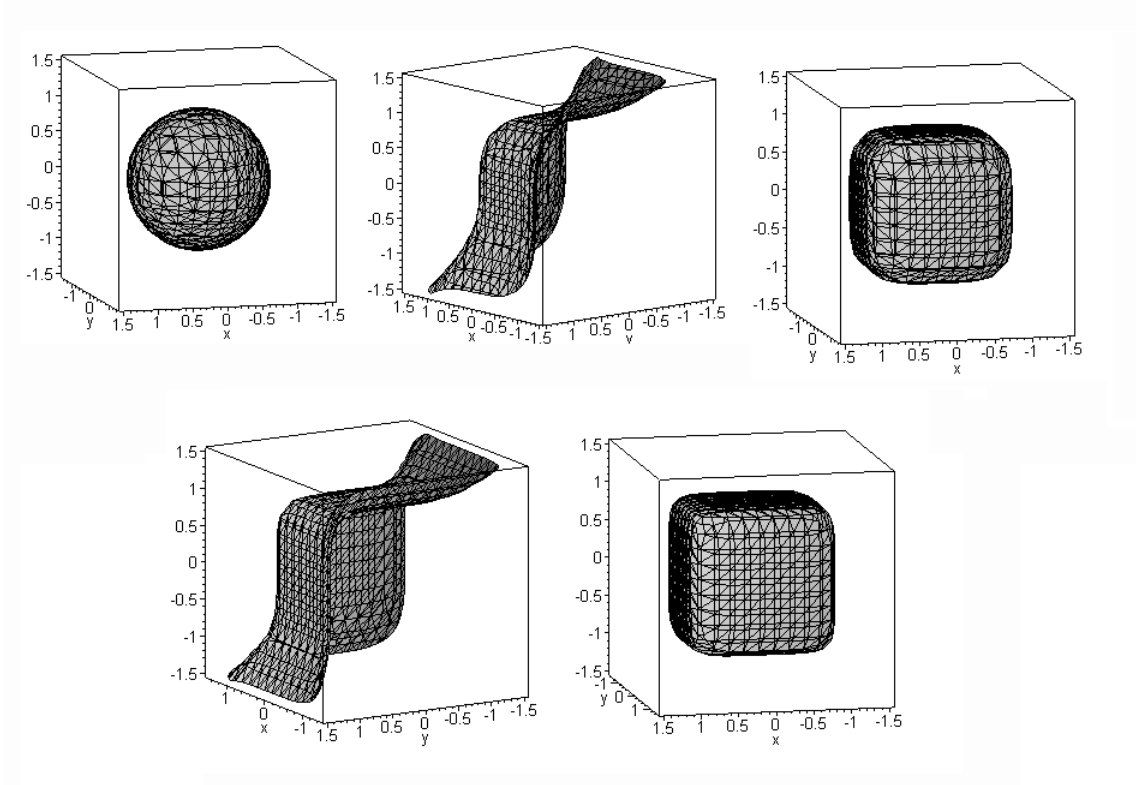
$$F(v) = \sqrt[k]{(v^1)^k + \ldots + (v^n)^k}, \quad n \in \{2,3\} \tag{3.3}$$

and hence, by their indicatrices, is the dependence of the topologic properties on the index k. For k even, the indicatrices are compact and have a strictly convex interior set, while for k odd, the indicatrices are unbounded and define no finite volume. This is illustrated for $m = 2$ by the following Maple plots of indicatrices $F(v) = 1$ with F provided by (3.3):



Berwald-Moor indicatrices ($m = 2$; even ($k \in \{2,4,6\}$) and odd ($k \in \{3,5\}$) root index).

In higher dimensions (e.g. for $m = 3$) the topology strongly differs as well:



Berwald-Moor indicatrices for $k \in \{2, \ldots, 6\}$ $(m = 3)$

Moreover, even for small even values of k, to compute the encompassed volume inside a bounded indicatrix implies the usage of special functions. Though the case $k = 2$ is calssical, providing volumes of (hyper)-spheres ($Vol(Q)_{m=2,k=2} = \pi$, $Vol(Q)_{m=2,k=2} = 4\pi/3$, etc), for larger values of k we get results as:

$$Vol(Q)_{m=2,k=4} = \frac{1}{4} B\left(\frac{1}{4}, \frac{3}{2}\right), \quad Vol(Q)_{m=2,k=6} = \frac{1}{6} B\left(\frac{1}{6}, \frac{3}{2}\right),$$

where $B(\,\cdot\,,\,\cdot\,)$ is the Bessel function.

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References

[1] J. C. Álvarez Paiva, A. C. Thompson, *Volumes on normed and Finsler spaces*, In: Riemann-Finsler Geometry, MSRI Publications, 49 (2004), 1–46.

[2] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Geometry*, D. Reidel, Dordrecht, 1985.

[3] V. Balan, *DPW method for CMC surfaces in Randers spaces*, in: "Modern Trends in Geometry and Topology" (Eds: D. Andrica, P. A. Blaga, S. Moroianu), Proc. of

The Seventh International Workshop on Differential Geometry and Its Applications, 5–11 September 2005, Deva, Romania, Cluj University Press 2006, 79–86.

[4] V. Balan, *Minimal surfaces in special Finsler spaces*, Proc. of The National Seminar of Finsler, Lagrange and Hamilton Spaces, 22–23 September **2006**, Braşov, Romania, to appear.

[5] D. Bao, S.-S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer-Verlag, 2000.

[6] A. Bejancu, *Finsler Geometry and Applications*, E. Horwood, 1990.

[7] G. Belletini, M. Paolini, *Anisotropic motion by mean curvature in the context of Finsler geometry*, Hokkaido Math. Jour. 25 (1995), 537–566.

[8] G. Belletini, M. Paolini, *Some aspects of mean curvature flow in presence of nonsmooth anisotropies*, Progress in Mathematics, Birkhäuser 202 (2001), 245–253.

[9] G. Belletini, M. Paolini, S. Venturini, *Some results on surface measures in claculus of variations*, Ann. di Mat. Pura ed Appl., IV vol. CLXX (1996), 329–259.

[10] G. Breck, *Minimalité des sous-variétés totalement géodésiques en géométrie Finslerienne*, arXiv:math.DG/0409320, v1, 18 Sept. 2004.

[11] V. M. Chernov, *On defining equations for the elements of associative and commutative algebras and on associated metric forms*, Hypercomplex Numbers in Geometry and Physics, **2 (4)**, Vol 2, 2005, 57–74, and in this book.

[12] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry* I, II, Interscience Publishers, New York, (1963), (1969).

[13] R. Miron, M. Anastasiei, *The Geometry of Vector Bundles. Theory and Applications*, Kluwer Acad. Publishers, FTPH, no. 59, 1994.

[14] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, 1959.

[15] Z. Shen, *Differental Geometry of Sprays and Finsler Spaces*, Kluwer Acad. Publishers, 2001.

[16] Z. Shen, *On Finsler geometry of submanifolds*, Mathematische Annalen, Springer-Verlag 311 (1998), 549–576.

[17] M. Souza, K. Tenenblat, *Minimal surface of rotation in Finsler space with a Randers metric*, Math. Ann. 325 (2003), 625–642.

The Definition of a Simultaneity in Finsler Space-Time

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Starting with A. Poincare's studies about the definition of simultaneity of events in an inertial system of a reference, in this paper a new definition of simultaneous events using the signal method in Finsler Space-Time, is investigated. General transformations which preserve the metric function of the considered projective space are obtained. Using the Hamiltonian formalism, the relations for energy and impulse of a particle, and their transformations are discussed.

MSC2000n: 53B40, 83D05, 70S05.

1 Introduction

In well-known work [1] A. Poincare are reduced for the first time with definition of simultaneity distant events in an inertial system of a reference. The geometrical form in three-dimensional space is sphere (or a so-called surface of a stationary value of a phase) as an orb possessing a central symmetry, and geometrical object is the distance. Using methods of metric geometry, A. Poincare for the first time has considered a formalism of a four-dimensional space-time and has found all invariants of a Lorenz group [2]. Finally, G. Minkowski [3] used a A. Poincare's formalism, and has suggested local isotropic four-dimensional pseudoeuclidean space-time in Galilean coordinates which is base physical relativistic theories (for a case of a special relativity theory). Metric function

$$F = ds = \left[(cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2\right]^{1/2}, \tag{1.1}$$

equal to distance between events in space-time, has at $F = 0$ two characteristic for a signal.

Studies by French scientist have great significance for relativistic mechanics and they are un-deservedly belittled at present. In 1904 the Kazan Society of Physics and Mathematics awards A. Poincare of a gold medal of Fund of a name of N. I. Lobachevsky, and the prize of a name of N. I. Lobachevsky is adjudged to D. Gilbert. N. I. Lobachevsky of 19 years (1827–1846) was as the rector of the Kazan Imperial University founded on November, 17 (5), 1804 by emperor Alexander I, and for the first time discovered non-Euclidean geometry [4] in 1826. In particular, this geometry is realized in three-dimensional Fock velocities-space

[5]. A. Poincare and N. I. Lobachevsky stated, that different physical phenomena would be described in terms of various geometries.

There are some models of expansion of pseudoeuclidean geometry. One of perspective approaches is the study local Finsler space-time which reference property is presence of an anisotropy. Recently in works [6–9] the new model of a four-dimensional space-time as local anisotropic Finsler geometry with Berwald-Moor metric function

$$F = \big[(cdt + dx + dy + dz) \cdot (cdt - dx + dy - dz) \cdot \\ \cdot (cdt + dx - dy - dz) \cdot (cdt - dx - dy + dz)\big]^{1/4} \qquad (1.2)$$

and its generalizations

$$F = \big[(cdt - dx - dy - dz)^{1+r_1+r_2+r_3} (cdt + dx - dy + dz)^{1-r_1+r_2-r_3} \times \\ \times (cdt - dx + dy + dz)^{1+r_1-r_2-r_3} (cdt + dx + dy - dz)^{1-r_1-r_2+r_3}\big]^{1/4}, \qquad (1.3)$$

are investigated. This metrical function has four characteristics for a signal. At $r_1 = r_2 = r_3 = 0$ and replacement $x \to -x$, $y \to -y$, $z \to -z$ metric function (1.3) coincides with (1.2).

The geometrical form in three-dimensional space is the specially oriented coordinate tetrahedron, which does not possess a central symmetry. Geometrical object is the volume. In such model it is natural to apply methods of a projective geometry with the relevant theory of invariants and measures among which, as is known, there is a definition of distance of the metric form, however there is a measure of an angle, etc. Transformations of the projective homogeneous coordinates t, x, y and z at transitions between moving inertial systems of references which maintain a form-invariance of metric functions (1.2) and (1.3) have been obtained. Also we have obtained the transformations of the projective nonhomogeneous coordinates u_x, u_y, u_z (components of a three-dimensional velocity). Transformations of impulse and energy of a particle to Finsler geometry with (1.3) are given in [10].

Metrics functions (1.2) and (1.3) refer to a class of Riemann functions [11]

$$F = \left[\left(a_i dx^i\right)^{1+a} \left(b_i dx^i\right)^{1+b} \left(c_i dx^i\right)^{1+c} \left(e_i dx^i\right)^{1+e}\right]^{1/4}, \quad a + b + c + e = 0. \quad (1.4)$$

The aim of the present work is the study of the definition of simultaneity distant events in a general view for such Finsler structure of a four-dimensional projective geometry and to find new transformations of the projective coordinates, energies and impulse of a particle.

2 The definition of simultaneity distant events

Let's consider distant events in four points of three-dimensional space of an inertial system of the reference (K). Let from a point O in a time T signals in four points A and A^n ($n = 1, 2, 3$) are sent. Signals reach these points in a time t ($t > T$). After reflection from points A^n signals are returned in a point O in a time T^n ($T^n > t$). Observable magnitudes are T and T^n in a point O.

Definition 1. There is a unique time (or the definition of a simultaneity of events is given) for points O, A and A^n at realization of a equality

$$(t - T) = \left(T^1 - t\right) + \left(T^2 - t\right) + \left(T^3 - t\right). \tag{2.1}$$

The given definition means equality of an interval of time at propagation of a signal from a point O up to a point A to the sum total of intervals of times at propagation from three points A^n in a point O, noted as $t_{OA} = t_{A^1O} + t_{A^2O} + t_{A^3O}$ ($t_{OA} = -t_{AO} > 0$, $t_{A^nO} = -t_{OA^n} > 0$). From (2.1) we obtain a value of a time coordinate t

$$t = T + \frac{1}{4}\sum_{n}^{3}\left(T^n - T\right) = \frac{1}{4}\Big(T + \sum_{n}^{3} T^n\Big), \tag{2.2}$$

depending on the times T and T^n in a point O. The value t is arithmetic mean. According to (2.2) clocks in points O, A also A^n are synchronized.

Definition 2. The value

$$\overline{OA} = \overline{A^1O} + \overline{A^2O} + \overline{A^3O} \tag{2.3}$$

is a length of a segment paths from a point O up to a point A and is equaled to the sum total of lengths of paths from three points up to a point O.

Definition 3. The value

$$c = \frac{\overline{OA} + \overline{A^1O} + \overline{A^2O} + \overline{A^3O}}{t_{OA} + t_{A^1O} + t_{A^2O} + t_{A^3O}} \tag{2.4}$$

is the universal constant and defines a physical velocity of a signal in various inertial systems of references.

According to (2.1) and (2.3), from (2.4) we have a relation $t - T = \overline{OA}\big/c$ from which follows $\overline{OA} = -\overline{AO} > 0$. Similar definitions are fulfilled for points A^1, A^2 and A^3. For example, using definitions for a point A^1,, we find

$$\begin{gathered} t_{OA^1} = t_{AO} + t_{A^2O} + t_{A^3O}, \quad \overline{OA^1} = \overline{AO} + \overline{A^2O} + \overline{A^3O}, \\ c = \frac{\overline{OA^1} + \overline{AO} + \overline{A^2O} + \overline{A^3O}}{t_{OA^1} + t_{AO} + t_{A^2O} + t_{A^3O}} \end{gathered} \tag{2.5}$$

and we have $T^1 - t = \overline{A^1O}\big/c$.

Thus, for three points we shall note the following equalities

$$T^n - t = \frac{\overline{A^nO}}{c}, \quad \overline{A^nO} = -\overline{OA^n} > 0. \tag{2.6}$$

Let the point O is an origin of coordinates of three-dimensional space. From (2.3) and (2.6) follow, in view of inequalities for lengths of segments of paths, the linear forms for coordinates

$$\overline{OA} = (\boldsymbol{\varepsilon x}), \quad \overline{A^n O} = (\boldsymbol{\varepsilon}^n \boldsymbol{x}), \tag{2.7}$$

The expressions (2.7) are products of constant vectors $\boldsymbol{\varepsilon} = \{\varepsilon_x, \varepsilon_y, \varepsilon_z\}$, $\boldsymbol{\varepsilon}^n = \left\{\varepsilon_x^n, \varepsilon_y^n, \varepsilon_z^n\right\}$ and a vector $\boldsymbol{x} = \{x, y, z\}$. The length of segments of paths will consist of lengths of segments, directional along axes and parallel it direct lines. We have four preferred directions as vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^n$ and four equalities for characteristics

$$T = t - \frac{(\boldsymbol{\varepsilon x})}{c}, \quad T^n = t + \frac{(\boldsymbol{\varepsilon}^n \boldsymbol{x})}{c}, \tag{2.8}$$

The given equalities are characteristic for a projective geometry where t, x, y, z are the projective homogeneous coordinates. In a metric space-time geometry of Minkowski is considered with (1.1) two equalities $T = t - |\boldsymbol{x}|/c$, $T^1 = t + |\boldsymbol{x}|/c$ for the characteristics implying from definition of simultaneity of Poincare, definition of distance between points and definitions of the universal constant value c.

From (2.8) follows, that the signal represents a simple plane wave. For such plane wave the surface of a constant value of a phase is a plane, which goes with a phase velocity not dependent on frequency. The four planes moves in directions of four vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^n$. Such amount of vectors is minimum for formation in three-dimensional space of the made surface as a tetrahedron.

Equality 1. The equality is fulfilled

$$\frac{1}{4}\left(-\varepsilon_i + \varepsilon_i^1 + \varepsilon_i^2 + \varepsilon_i^3\right) = 0, \tag{2.9}$$

which is a result of relations (2.2), (2.8) and is a linear relation of vectors

$$\boldsymbol{\varepsilon} = \sum_n^3 \boldsymbol{\varepsilon}^n. \tag{2.10}$$

Further we shall consider two inertial systems of references (K) and (K') which coincide. Then we use equalities $T = T'$ and $T^n = T'^n$ and we have

$$t - \frac{(\boldsymbol{\varepsilon x})}{c} = t' - \frac{(\boldsymbol{\varepsilon x}')}{c}, \quad t + \frac{(\boldsymbol{\varepsilon}^n \boldsymbol{x})}{c} = t' + \frac{(\boldsymbol{\varepsilon}^n \boldsymbol{x}')}{c}. \tag{2.11}$$

Sum equalities (2.11), we find $t' = t$. For coordinates we have condition

$$\boldsymbol{\varepsilon}(\boldsymbol{\varepsilon x}) + \sum_n^3 \boldsymbol{\varepsilon}^n (\boldsymbol{\varepsilon}^n \boldsymbol{x}) = \boldsymbol{\varepsilon}(\boldsymbol{\varepsilon x}') + \sum_n^3 \boldsymbol{\varepsilon}^n (\boldsymbol{\varepsilon}^n \boldsymbol{x}'). \tag{2.12}$$

As $\boldsymbol{x} = \boldsymbol{x}'$, the following new equality is necessary.

Equality 2. The equality is fulfilled

$$\frac{1}{4}\left(\varepsilon_i\varepsilon_j+\varepsilon_i^1\varepsilon_j^1+\varepsilon_i^2\varepsilon_j^2+\varepsilon_i^3\varepsilon_j^3\right)=\delta_{ij}, \tag{2.13}$$

where δ_{ij} is a Kronecker symbol (or a unit three-dimensional matrix).

Using (2.13), we shall reduce relations for characteristics in various systems of references

$$t^2+\boldsymbol{x}^2/c^2=\frac{1}{4}\left(T^2+\sum_n^3\left(T^n\right)^2\right),\quad t'^2+\boldsymbol{x}'^2/c^2=\frac{1}{4}\left(T'^2+\sum_n^3\left(T'^n\right)^2\right) \tag{2.14}$$

Four equalities (2.8) is defined by $\mathbf{T}=\mathbf{HX}$, where

$$\mathbf{H}=\begin{pmatrix}1 & -\varepsilon_x & -\varepsilon_y & -\varepsilon_z\\ 1 & \varepsilon_x^1 & \varepsilon_y^1 & \varepsilon_z^1\\ 1 & \varepsilon_x^2 & \varepsilon_y^2 & \varepsilon_z^2\\ 1 & \varepsilon_x^3 & \varepsilon_y^3 & \varepsilon_z^3\end{pmatrix},\quad \mathbf{T}=\begin{pmatrix}T\\ T^1\\ T^2\\ T^3\end{pmatrix},\quad \mathbf{X}=\begin{pmatrix}t\\ x/c\\ y/c\\ z/c\end{pmatrix}. \tag{2.15}$$

It is consider of a matrix product $\mathbf{HH}^T$ (where $\mathbf{H}^T$ – a transpose of a matrix). Relations (2.14) are valid in only case when the following conditions satisfied

$$1-(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^n)=0,\quad 1+(\boldsymbol{\varepsilon}^n\boldsymbol{\varepsilon}^r)=0\quad(n\neq r),\quad 1+\boldsymbol{\varepsilon}^2=1+(\boldsymbol{\varepsilon}^n)^2=4. \tag{2.16}$$

The module of vectors is equal $|\boldsymbol{\varepsilon}|=|\boldsymbol{\varepsilon}^n|=\sqrt{3}$, and $\mathbf{HH}^T=4\mathbf{I}$ ($\mathbf{I}$ – a unit four-dimensional matrix). Using the equalities (2.16) we obtain arbitrarily oriented coordinate tetrahedron. The volume of such coordinate tetrahedron with vertexes on the points of four vectors $(-\boldsymbol{\varepsilon})$ also $\boldsymbol{\varepsilon}^n$ accepts a value $V_{tetr}=\det\mathbf{H}/6=\left(\boldsymbol{\varepsilon}^1\left[\boldsymbol{\varepsilon}^2\boldsymbol{\varepsilon}^3\right]\right)/3$ equal to third of volume of a parallelepiped, constructed on noncoplanar vectors $\boldsymbol{\varepsilon}^1$, $\boldsymbol{\varepsilon}^2$, $\boldsymbol{\varepsilon}^3$. The surface of a constant value of a phase represents other coordinate tetrahedron with four edges, perpendicular to vectors $(-\boldsymbol{\varepsilon})$ and $\boldsymbol{\varepsilon}^n$.

Further we have, according to (2.2), (2.8), (2.9), (2.13) and (2.14), the following relations

$$\begin{gathered}t^2+\boldsymbol{x}^2/c^2=\frac{1}{4}\sum_m^4\left(T^m\right)^2,\\ \boldsymbol{x}/c=\frac{1}{4}\sum_m^4\boldsymbol{\varepsilon}^mT^m,\quad t=\frac{1}{4}\sum_m^4T^m,\quad \frac{1}{4}\sum_m^4\varepsilon_i^m=0,\quad \frac{1}{4}\sum_m^4\varepsilon_i^m\varepsilon_j^m=\delta_{ij},\\ 3t^2-\boldsymbol{x}^2/c^2=\frac{1}{2}\left(T_1T_2+T_1T_3+T_1T_4+T_2T_3+T_2T_4+T_3T_4\right),\\ \boldsymbol{u}=c\frac{\sum_m^4\boldsymbol{\varepsilon}^mT^m}{\sum_m^4T^m},\quad |\boldsymbol{u}|=\sqrt{\boldsymbol{u}^2}=c\left[\frac{\frac{1}{4}\sum_m^4\left(T^m\right)^2}{\left(\frac{1}{4}\sum_m^4T^m\right)^2}-1\right]^{1/2},\end{gathered} \tag{2.17}$$

expressed through times $T^m = t + \boldsymbol{\varepsilon}^m \boldsymbol{x}$ $(m = 1, 2, 3, 4)$ and $\boldsymbol{\varepsilon}^m \to (-\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^n)$. As $T^m \geq 0$ the requirement $1 + (\boldsymbol{\varepsilon}^m \boldsymbol{u}) / c \geq 0$ with an equal sign, hence, is valid at driving on characteristics. In (2.17) we have the square-law form from the projective homogeneous coordinates, values of components of a vector, coordinate time, a coordinate velocity of a signal and its module. At equality to unity the square-law form defines a hypersurface of the second order in the space-time, intersecting all four characteristics (2.8) for a signal. Any two rows of matrixes $\mathbf{H}$ are orthogonal in a four-dimensional Euclidean space with Galilean coordinates $\{ct, \boldsymbol{x}\}$. Thus, magnitudes T^m of the linear vector function of the first sort give four axes of a considered hypersurface. The considered coordinate tetrahedron is a boundary tetrahedron of some exact body in a four-dimensional Euclidean space. It is known, that such bodies, restricted three-dimensional tetrahedrons is present three.

Definition 4. Proper time in a point $\{x, y, z\}$ is defined by metric function (1.3) in the generalized expression

$$T_0 = \frac{F}{c} = \prod_m^4 (T^m)^{p^m} = \left[(T)^{1+(\boldsymbol{\varepsilon r})} \left(T^1\right)^{1-\left(\boldsymbol{\varepsilon}^1 \boldsymbol{r}\right)} \left(T^2\right)^{1-\left(\boldsymbol{\varepsilon}^2 \boldsymbol{r}\right)} \left(T^3\right)^{1-\left(\boldsymbol{\varepsilon}^3 \boldsymbol{r}\right)} \right]^{1/4} =$$

$$= \left\{ \left[t - \frac{(\boldsymbol{\varepsilon x})}{c} \right]^{1+(\boldsymbol{\varepsilon r})} \left[t + \frac{(\boldsymbol{\varepsilon}^1 \boldsymbol{x})}{c} \right]^{1-\left(\boldsymbol{\varepsilon}^1 \boldsymbol{r}\right)} \left[t + \frac{(\boldsymbol{\varepsilon}^2 \boldsymbol{x})}{c} \right]^{1-\left(\boldsymbol{\varepsilon}^2 \boldsymbol{r}\right)} \left[t + \frac{(\boldsymbol{\varepsilon}^3 \boldsymbol{x})}{c} \right]^{1-\left(\boldsymbol{\varepsilon}^3 \boldsymbol{r}\right)} \right\}^{1/4}. \quad (2.18)$$

Here the vector-parameter $\boldsymbol{r} = \{r_1, r_2, r_3\}$ has a constant value to magnitudes $p^m = (1/4) \left[1 - (\boldsymbol{\varepsilon}^m \boldsymbol{r})\right]$ for which equalities are fulfilled

$$\frac{1}{4} \left(1 + \boldsymbol{r}^2\right) = \sum_m^4 (p^m)^2, \quad -\boldsymbol{r} = \sum_m^4 \boldsymbol{\varepsilon}^m p^m, \quad 1 = \sum_m^4 p^m. \quad (2.19)$$

At a value $F = 0$ the equation (2.18) represents the equation of a hypersurface with four characteristics in space-time. It means presence of four real roots for time t.

Different relativistic techniques in Finsler space-time were considered in [12–16]. Unidirection physical or coordinate speed of signal has nonisotropic quantity. Using general nonstandard clock synchronization [17, 18] in definition of simultaneity.

3 Transformations of the projective homogeneous coordinate

Let's consider transformations of the projective homogeneous coordinates at transition between moving inertial systems of references (K) and (K') with the relative velocities $\boldsymbol{v} = \{v_x, v_y, v_z\}$ and $\boldsymbol{v}' = \left\{v'_x, v'_y, v'_z\right\}$, accordingly. Velocities $\boldsymbol{v}$ and $\boldsymbol{v}'$ with $c = 1$ express in scale unities of systems of references, according to a principle of a relativity. Transformations leave form-invariant metric function (2.18) in a global geometry and volume of a coordinate tetrahedron. At transition to local Finsler geometry the projective homogeneous coordinates are substituted with their differentials. The vectors $\boldsymbol{\varepsilon}^m$ are stationary values and we have

$$\left\{ \prod_m^4 \left[dt + (\boldsymbol{\varepsilon}^m d\boldsymbol{x})\right]^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})} \right\}^{1/4} = \left\{ \prod_m^4 \left[dt' + (\boldsymbol{\varepsilon}^m d\boldsymbol{x}')\right]^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})} \right\}^{1/4}. \quad (3.1)$$

Extending the method of coefficient "k" earlier used for a case of one spatial coordinate [12–16], we shall note relations $T^m = k^m(\boldsymbol{v},\boldsymbol{r})\,T'^m$ and $T'^m = k^m(\boldsymbol{v}',\boldsymbol{r})\,T^m$. We have

$$[t + (\boldsymbol{\varepsilon}^m \boldsymbol{x})] = k^m(\boldsymbol{v},\boldsymbol{r})\,[t' + (\boldsymbol{\varepsilon}^m \boldsymbol{x}')]\,,\qquad [t' + (\boldsymbol{\varepsilon}^m \boldsymbol{x}')] = k^m(\boldsymbol{v}',\boldsymbol{r})\,[t + (\boldsymbol{\varepsilon}^m \boldsymbol{x})]\,, \tag{3.2}$$

where coefficients $k^m(\boldsymbol{v},\boldsymbol{r})$ also $k^m(\boldsymbol{v}',\boldsymbol{r})$ characterize Doppler effects of a frequencies ω^m and ω'^m a plane wave on four preferred directions

$$\omega^m k^m(\boldsymbol{v},\boldsymbol{r}) = \omega'^m, \quad \omega'^m k^m(\boldsymbol{v}',\boldsymbol{r}) = \omega^m. \tag{3.3}$$

From (3.1) one gets the relations

$$\prod_m^4 [k^m(\boldsymbol{v},\boldsymbol{r})]^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})} = \prod_m^4 [k^m(\boldsymbol{v}',\boldsymbol{r})]^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})} = 1, \quad k^m(\boldsymbol{v},\boldsymbol{r})\,k^m(\boldsymbol{v}',\boldsymbol{r}) = 1.$$

At $\boldsymbol{x}' = 0$ we have $\boldsymbol{x} = \boldsymbol{v}t$, and at $\boldsymbol{x} = 0$ we have $\boldsymbol{x}' = \boldsymbol{v}'t'$. Then from (2.18) and (3.2) we obtain the following equalities

$$t = t'N(\boldsymbol{v}',\boldsymbol{r})\,, \quad k^m(\boldsymbol{v}',\boldsymbol{r})\,t = [1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')]\,t',\qquad t' = tN(\boldsymbol{v},\boldsymbol{r})\,, \quad k^m(\boldsymbol{v},\boldsymbol{r})\,t' = [1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})]\,t. \tag{3.4}$$

where expressions with three-dimensional velocities

$$\begin{gathered}
\frac{T_0}{t} = N(\boldsymbol{v},\boldsymbol{r}) = \frac{N(\boldsymbol{v})}{A(\boldsymbol{v},\boldsymbol{r})} = \left\{\prod_m^4 [1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})]^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})}\right\}^{1/4},\\
\frac{T_0}{t'} = N(\boldsymbol{v}',\boldsymbol{r}) = \frac{N(\boldsymbol{v}')}{A(\boldsymbol{v}',\boldsymbol{r})} = \left\{\prod_m^4 [1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')]^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})}\right\}^{1/4},\\
N(\boldsymbol{v}) = \left\{\prod_m^4 [1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})]\right\}^{1/4}, \quad N(\boldsymbol{v}') = \left\{\prod_m^4 [1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')]\right\}^{1/4},\\
A(\boldsymbol{v},\boldsymbol{r}) = \prod_m^4 \left[\frac{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})}{N(\boldsymbol{v})}\right]^{(\boldsymbol{\varepsilon}^m \boldsymbol{r})/4}, \quad A(\boldsymbol{v}',\boldsymbol{r}) = \left\{\prod_m^4 \left[\frac{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')}{N(\boldsymbol{v}')}\right]^{(\boldsymbol{\varepsilon}^m \boldsymbol{r})/4}\right\},
\end{gathered} \tag{3.5}$$

The functions $A(\boldsymbol{v},\boldsymbol{r})$, $A(\boldsymbol{v}',\boldsymbol{r})$ depend on a vector $\boldsymbol{r}$ and $N(\boldsymbol{v}) = N(\boldsymbol{v},0)$.

Finally from (3.4) we obtain values of coefficients and the following identities

$$\begin{gathered}
k^m(\boldsymbol{v},\boldsymbol{r}) = \frac{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})}{N(\boldsymbol{v},\boldsymbol{r})}, \quad k^m(\boldsymbol{v}',\boldsymbol{r}) = \frac{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')}{N(\boldsymbol{v}',\boldsymbol{r})}, \quad \frac{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})}{N(\boldsymbol{v},\boldsymbol{r})} \cdot \frac{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')}{N(\boldsymbol{v}',\boldsymbol{r})} = 1,\\
\left\{\frac{1}{4}\sum_m^4 [k^m(\boldsymbol{v},\boldsymbol{r})]^2\right\}^{1/2} = \frac{\sqrt{1 + \boldsymbol{v}^2}}{N(\boldsymbol{v},\boldsymbol{r})}, \quad \left\{\frac{1}{4}\sum_m^4 [k^m(\boldsymbol{v}',\boldsymbol{r})]^2\right\}^{1/2} = \frac{\sqrt{1 + \boldsymbol{v}'^2}}{N(\boldsymbol{v}',\boldsymbol{r})},\\
\frac{1}{4}\sum_m^4 k^m(\boldsymbol{v},\boldsymbol{r}) = \frac{1}{N(\boldsymbol{v},\boldsymbol{r})}, \quad \frac{1}{4}\sum_m^4 k^m(\boldsymbol{v}',\boldsymbol{r}) = \frac{1}{N(\boldsymbol{v}',\boldsymbol{r})},
\end{gathered} \tag{3.6}$$

where (K) and (K') – relative velocities of reference systems.

Direct and inverse transformations of characteristics (3.2) will become

$$[t + (\boldsymbol{\varepsilon}^m \boldsymbol{x})] = \frac{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})}{N(\boldsymbol{v}, \boldsymbol{r})} [t' + (\boldsymbol{\varepsilon}^m \boldsymbol{x}')] = \frac{N(\boldsymbol{v}', \boldsymbol{r})}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')} [t' + (\boldsymbol{\varepsilon}^m \boldsymbol{x}')],$$

$$[t' + (\boldsymbol{\varepsilon}^m \boldsymbol{x}')] = \frac{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')}{N(\boldsymbol{v}', \boldsymbol{r})} [t + (\boldsymbol{\varepsilon}^m \boldsymbol{x})] = \frac{N(\boldsymbol{v}, \boldsymbol{r})}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})} [t + (\boldsymbol{\varepsilon}^m \boldsymbol{x})], \tag{3.7}$$

From (3.3) we have formulas for Doppler effect

$$\omega'^m = \frac{A(\boldsymbol{v}, \boldsymbol{r}) [1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})]}{N(\boldsymbol{v})} \omega^m, \quad \omega^m = \frac{A(\boldsymbol{v}', \boldsymbol{r}) [1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')]}{N(\boldsymbol{v}')} \omega'^m, \tag{3.8}$$

with the relative velocities of moving of a radiant and the receiver of signals according to a principle of a relativity.

According to (3.7), we find equality

$$\frac{t + (\boldsymbol{\varepsilon}^m \boldsymbol{x})}{t + (\boldsymbol{\varepsilon}^k \boldsymbol{x})} = \frac{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})}{1 + (\boldsymbol{\varepsilon}^k \boldsymbol{v})} \cdot \frac{t' + (\boldsymbol{\varepsilon}^m \boldsymbol{x}')}{t' + (\boldsymbol{\varepsilon}^k \boldsymbol{x}')}, \tag{3.9}$$

Using of a linear relation of vectors (2.10) we obtain the following relations

$$\frac{1}{N(\boldsymbol{v}, \boldsymbol{r})\, N(\boldsymbol{v}', \boldsymbol{r})} = \frac{1}{4} \sum_m^4 \frac{1}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})} = \frac{1}{4} \sum_m^4 \frac{1}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')} = \frac{1}{1 + (\boldsymbol{v}\boldsymbol{v}')},$$

$$1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}') = \frac{1}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})} \left[\frac{1}{4} \sum_m^4 \frac{1}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})} \right]^{-1}, \tag{3.10}$$

$$1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}) = \frac{1}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')} \left[\frac{1}{4} \sum_m^4 \frac{1}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')} \right]^{-1},$$

where $1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}) \neq 0$ and $1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}') \neq 0$. From (3.10) it follows formulas

$$\boldsymbol{v}' = \left[\frac{1}{4} \sum_m^4 \frac{\boldsymbol{\varepsilon}^m}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})} \right] \left[\frac{1}{4} \sum_m^4 \frac{1}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})} \right]^{-1} = \\ = \left[-\frac{1}{4} \sum_m^4 \frac{\boldsymbol{\varepsilon}^m (\boldsymbol{\varepsilon}^m \boldsymbol{v})}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})} \right] \left[\frac{1}{4} \sum_m^4 \frac{1}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})} \right]^{-1},$$

$$\boldsymbol{v} = \left[\frac{1}{4} \sum_m^4 \frac{\boldsymbol{\varepsilon}^m}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')} \right] \left[\frac{1}{4} \sum_m^4 \frac{1}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')} \right]^{-1} = \\ = \left[-\frac{1}{4} \sum_m^4 \frac{\boldsymbol{\varepsilon}^m (\boldsymbol{\varepsilon}^m \boldsymbol{v}')}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')} \right] \left[\frac{1}{4} \sum_m^4 \frac{1}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')} \right]^{-1}, \tag{3.11}$$

$$A(\boldsymbol{v}, \boldsymbol{r})\, A(\boldsymbol{v}', \boldsymbol{r}) = 1, \quad N(\boldsymbol{v}, \boldsymbol{r})\, N(\boldsymbol{v}', \boldsymbol{r}) = N(\boldsymbol{v})\, N(\boldsymbol{v}').$$

Taking into account equalities (2.10) and (2.13) we shall receive direct transformations with $c \neq 1$ in vectorial form I

$$
\begin{gathered}
\boldsymbol{x} = \frac{A(\boldsymbol{v}/c, \boldsymbol{r})}{N(\boldsymbol{v}/c)} \left[\boldsymbol{x}' + \boldsymbol{v}t' + \frac{1}{4c} \sum_{m}^{4} \boldsymbol{\varepsilon}^m (\boldsymbol{\varepsilon}^m \boldsymbol{v}) (\boldsymbol{\varepsilon}^m \boldsymbol{x}') \right] = \\
= \frac{N(\boldsymbol{v}'/c)}{4A(\boldsymbol{v}'/c, \boldsymbol{r})} \sum_{m}^{4} \boldsymbol{\varepsilon}^m \left[\frac{\boldsymbol{\varepsilon}^m (\boldsymbol{x}' - \boldsymbol{v}'t')}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')/c} \right], \\
t = \frac{A(\boldsymbol{v}/c, \boldsymbol{r})}{N(\boldsymbol{v}/c)} \left[t' + \frac{1}{c^2} (\boldsymbol{v}\boldsymbol{x}') \right] = \frac{N(\boldsymbol{v}'/c)}{4A(\boldsymbol{v}'/c, \boldsymbol{r})} \sum_{m}^{4} \frac{t' + (\boldsymbol{\varepsilon}^m \boldsymbol{x}')/c}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')/c}
\end{gathered}
\tag{3.12}
$$

and inverse between systems of references (K) and (K')

$$
\begin{gathered}
\boldsymbol{x}' = \frac{A(\boldsymbol{v}'/c, \boldsymbol{r})}{N(\boldsymbol{v}'/c)} \left[\boldsymbol{x} + \boldsymbol{v}'t + \frac{1}{4c} \sum_{m}^{4} \boldsymbol{\varepsilon}^m (\boldsymbol{\varepsilon}^m \boldsymbol{v}') (\boldsymbol{\varepsilon}^m \boldsymbol{x}) \right] = \\
= \frac{N(\boldsymbol{v}/c)}{4A(\boldsymbol{v}/c, \boldsymbol{r})} \sum_{m}^{4} \boldsymbol{\varepsilon}^m \left[\frac{\boldsymbol{\varepsilon}^m (\boldsymbol{x} - \boldsymbol{v}t)}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})/c} \right], \\
t' = \frac{A(\boldsymbol{v}'/c, \boldsymbol{r})}{N(\boldsymbol{v}'/c)} \left[t + \frac{1}{c^2} (\boldsymbol{v}'\boldsymbol{x}) \right] = \frac{N(\boldsymbol{v}/c)}{4A(\boldsymbol{v}/c, \boldsymbol{r})} \sum_{m}^{4} \frac{t + (\boldsymbol{\varepsilon}^m \boldsymbol{x})/c}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v})/c}
\end{gathered}
\tag{3.13}
$$

At $\boldsymbol{v} = \{v_x, 0, 0\}$ we have from (3.13) direct transformation

$$
\begin{gathered}
x = \frac{A(v_x/c, \boldsymbol{r})}{N(v_x/c)} \left[x' + v_x t' + \frac{v_x}{4c} \sum_{m}^{4} (\varepsilon_x^m)^2 (\boldsymbol{\varepsilon}^m \boldsymbol{x}') \right] = \\
= \frac{N(\boldsymbol{v}'/c)}{4A(\boldsymbol{v}'/c, \boldsymbol{r})} \sum_{m}^{4} \boldsymbol{\varepsilon}_x^m \left[\frac{\boldsymbol{\varepsilon}^m (\boldsymbol{x}' - \boldsymbol{v}'t')}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')/c} \right], \\
y = \frac{A(v_x/c, \boldsymbol{r})}{N(v_x/c)} \left[y' + \frac{v_x}{4c} \sum_{m}^{4} (\varepsilon_y^m \varepsilon_x^m) (\boldsymbol{\varepsilon}^m \boldsymbol{x}') \right] = \\
= \frac{N(\boldsymbol{v}'/c)}{4A(\boldsymbol{v}'/c, \boldsymbol{r})} \sum_{m}^{4} \boldsymbol{\varepsilon}_y^m \left[\frac{\boldsymbol{\varepsilon}^m (\boldsymbol{x}' - \boldsymbol{v}'t')}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')/c} \right], \\
z = \frac{A(v_x/c, \boldsymbol{r})}{N(v_x/c)} \left[z' + \frac{v_x}{4c} \sum_{m}^{4} (\varepsilon_z^m \varepsilon_x^m) (\boldsymbol{\varepsilon}^m \boldsymbol{x}') \right] = \\
= \frac{N(\boldsymbol{v}'/c)}{4A(\boldsymbol{v}'/c, \boldsymbol{r})} \sum_{m}^{4} \boldsymbol{\varepsilon}_z^m \left[\frac{\boldsymbol{\varepsilon}^m (\boldsymbol{x}' - \boldsymbol{v}'t')}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')/c} \right], \\
t = \frac{A(v_x/c, \boldsymbol{r})}{N(v_x/c)} \left[t' + \frac{v_x x'}{c^2} \right] = \frac{N(\boldsymbol{v}'/c)}{4A(\boldsymbol{v}'/c, \boldsymbol{r})} \sum_{m}^{4} \frac{t' + (\boldsymbol{\varepsilon}^m \boldsymbol{x}')/c}{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{v}')/c},
\end{gathered}
\tag{3.14}
$$

where functions

$$N(v_x/c) = \left[\prod_m^4 (1+\varepsilon_x^m v_x/c)\right]^{1/4}, \quad A(v_x/c, \boldsymbol{r}) = \prod_m^4 \left[\frac{1+\varepsilon_x^m v_x/c}{N(v_x/c)}\right]^{(\boldsymbol{\varepsilon}^m \boldsymbol{r})/4}. \tag{3.15}$$

The velocity of a system of reference (K) is relative equaled (K')

$$\begin{aligned} \boldsymbol{v}' = \left[\frac{1}{4}\sum_m^4 \frac{\boldsymbol{\varepsilon}^m}{1+\varepsilon_x^m v_x/c}\right]\left[\frac{1}{4}\sum_m^4 \frac{1}{1+\varepsilon_x^m v_x/c}\right]^{-1} = \\ = \left[-\frac{1}{4}\sum_m^4 \frac{\boldsymbol{\varepsilon}^m \varepsilon_x^m v_x}{1+\varepsilon_x^m v_x/c}\right]\left[\frac{1}{4}\sum_m^4 \frac{1}{1+\varepsilon_x^m v_x/c}\right]^{-1}. \end{aligned} \tag{3.16}$$

4 The composition law of elements of coordinate velocities group and its properties

Let's consider the law of a composition of elements of group of coordinate three-dimensional velocities with $c = 1$. From (2.18) we obtain equality

$$N(\boldsymbol{u}', \boldsymbol{r})\, t' = N(\boldsymbol{u}, \boldsymbol{r})\, t, \quad N(0, \boldsymbol{r}) = 1. \tag{4.1}$$

Taking into account (3.12) and (3.13), we shall receive relations

$$\begin{gathered} N(\boldsymbol{u}', \boldsymbol{r}) = \frac{N(\boldsymbol{v}', \boldsymbol{r})\, N(\boldsymbol{u}, \boldsymbol{r})}{1+(\boldsymbol{u}\boldsymbol{v}')}, \quad N(\boldsymbol{u}, \boldsymbol{r}) = \frac{N(\boldsymbol{v}, \boldsymbol{r})\, N(\boldsymbol{u}', \boldsymbol{r})}{1+(\boldsymbol{u}'\boldsymbol{v})}, \\ N(\boldsymbol{v})\, N(\boldsymbol{v}') = [1+(\boldsymbol{u}'\boldsymbol{v})]\,[1+(\boldsymbol{u}\boldsymbol{v}')]\,, \end{gathered} \tag{4.2}$$

where $\boldsymbol{u} = \boldsymbol{x}/t$ also $\boldsymbol{u}' = \boldsymbol{x}'/t'$ are coordinate velocities in systems (K) and (K').

The law of a composition of elements of group in a representation of a group as function $k^m(\boldsymbol{u}, \boldsymbol{r})$ will be noted, according to (3.7) and (4.1), as follows

$$\begin{gathered} \frac{1+(\boldsymbol{\varepsilon}^m \boldsymbol{u})}{N(\boldsymbol{u})} = \frac{1+(\boldsymbol{\varepsilon}^m \boldsymbol{v})}{N(\boldsymbol{v})} \cdot \frac{1+(\boldsymbol{\varepsilon}^m \boldsymbol{u}')}{N(\boldsymbol{u}')}, \\ A(\boldsymbol{u}, \boldsymbol{r}) = A(\boldsymbol{v}, \boldsymbol{r})\, A(\boldsymbol{u}', \boldsymbol{r})\,, \quad A(\boldsymbol{u}', \boldsymbol{r}) = A(\boldsymbol{v}', \boldsymbol{r})\, A(\boldsymbol{u}, \boldsymbol{r}). \end{gathered} \tag{4.3}$$

for the function depending only from a velocity, and function $A(\boldsymbol{u}, \boldsymbol{r})$. For both functions binary operation of the law of a composition as usual operation of multiplication is fulfilled. Let $\boldsymbol{w}'$ be a velocity of moving of the third inertial system of a reference (K'') concerning second (K'), and $\boldsymbol{z}''$ concerning first (K). Then we have operations of multiplication of functions $k^m(\boldsymbol{u}, \boldsymbol{r})$ and, according to (3.6), equalities

$$k^m(\boldsymbol{u},\boldsymbol{r}) = k^m(\boldsymbol{v},\boldsymbol{r})\,k^m(\boldsymbol{u}',\boldsymbol{r}), \quad k^m(\boldsymbol{u}',\boldsymbol{r}) = k^m(\boldsymbol{w}',\boldsymbol{r})\,k^m(\boldsymbol{u}'',\boldsymbol{r}),$$

$$k^m(\boldsymbol{u},\boldsymbol{r}) = k^m(\boldsymbol{z}'',\boldsymbol{r})\,k^m(\boldsymbol{u}'',\boldsymbol{r}), \quad k^m(\boldsymbol{v},\boldsymbol{r})\,k^m(\boldsymbol{w}',\boldsymbol{r}) = k^m(\boldsymbol{z}'',\boldsymbol{r}),$$

$$\begin{gathered}
\frac{\dfrac{1}{4}\displaystyle\sum_m^4 k^m(\boldsymbol{u},\boldsymbol{r})}{\left\{\dfrac{1}{4}\displaystyle\sum_m^4 \left[k^m(\boldsymbol{u},\boldsymbol{r})\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{1+\boldsymbol{u}^2}}, \\
\frac{\dfrac{1}{4}\displaystyle\sum_m^4 k^m(\boldsymbol{v},\boldsymbol{r})\,k^m(\boldsymbol{u}',\boldsymbol{r})}{\left\{\dfrac{1}{4}\displaystyle\sum_m^4 \left[k^m(\boldsymbol{v},\boldsymbol{r})\right]^2\right\}^{1/2}\left\{\dfrac{1}{4}\displaystyle\sum_m^4 \left[k^m(\boldsymbol{u}',\boldsymbol{r})\right]^2\right\}^{1/2}} = \frac{1+(\boldsymbol{v}\boldsymbol{u}')}{\sqrt{1+\boldsymbol{v}^2}\sqrt{1+\boldsymbol{u}'^2}},
\end{gathered} \tag{4.4}$$

where the law of a composition

$$\boldsymbol{u} = \boldsymbol{v}\circ\boldsymbol{u}', \quad \boldsymbol{u}' = \boldsymbol{w}'\circ\boldsymbol{u}'', \quad \boldsymbol{u} = \boldsymbol{z}''\circ\boldsymbol{u}'', \quad \boldsymbol{v}\circ\boldsymbol{w}' = \boldsymbol{z}''. \tag{4.5}$$

Direct and inverse transformations of the dimensionless coordinate three-dimensional velocities

$$\begin{gathered}
\boldsymbol{u} = \frac{\boldsymbol{u}'+\boldsymbol{v}+\dfrac{1}{4}\displaystyle\sum_m^4 \boldsymbol{\varepsilon}^m(\boldsymbol{\varepsilon}^m\boldsymbol{u}')(\boldsymbol{\varepsilon}^m\boldsymbol{v})}{1+(\boldsymbol{u}'\boldsymbol{v})} = \frac{\sum\limits_m^4 \boldsymbol{\varepsilon}^m\left[\dfrac{\boldsymbol{\varepsilon}^m(\boldsymbol{u}'-\boldsymbol{v}')}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{v}')}\right]}{\sum\limits_m^4 \dfrac{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u}')}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{v}')}}, \\
\boldsymbol{u}' = \frac{\boldsymbol{u}+\boldsymbol{v}'+\dfrac{1}{4}\displaystyle\sum_m^4 \boldsymbol{\varepsilon}^m(\boldsymbol{\varepsilon}^m\boldsymbol{u})(\boldsymbol{\varepsilon}^m\boldsymbol{v}')}{1+(\boldsymbol{u}\boldsymbol{v}')} = \frac{\sum\limits_m^4 \boldsymbol{\varepsilon}^m\left[\dfrac{\boldsymbol{\varepsilon}^m(\boldsymbol{u}-\boldsymbol{v})}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{v})}\right]}{\sum\limits_m^4 \dfrac{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u})}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{v})}}
\end{gathered} \tag{4.6}$$

in vectorial form I do not depend on vector-parameter $\boldsymbol{r}$.

Thus, we have nonadditive group of elements for coordinate three-dimensional velocities, inhering *only to different systems of references* with the law of a composition (4.6) containing square-law nonlinearity.

Formulas (4.6) are linear-fractional functions of velocities $\boldsymbol{u}$, $\boldsymbol{u}'$ and represent direct and inverse projective (collinear) transformations of the nonhomogeneous coordinates to a projective geometry.

Let's consider the basic properties of the law of a composition $\boldsymbol{u} = \boldsymbol{u}_1\circ\boldsymbol{u}_2$, not distinguishing a velocity in different systems and the relative velocities between them. The law of a composition

$$\boldsymbol{u}_1 \circ \boldsymbol{u}_2 = \boldsymbol{u}_2 \circ \boldsymbol{u}_1 = \frac{\boldsymbol{u}_1 + \boldsymbol{u}_2 + \dfrac{1}{4}\sum\limits_m^4 \boldsymbol{\varepsilon}^m\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_1\right)\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_2\right)}{1 + \left(\boldsymbol{u}_1 \boldsymbol{u}_2\right)} \tag{4.7}$$

has property of a commutability, that is the group is Abelian.

Group postulates are fulfilled.

1. An associativity:

$$\begin{gathered}\left(\boldsymbol{u}_1 \circ \boldsymbol{u}_2\right) \circ \boldsymbol{u}_3 = \boldsymbol{u}_1 \circ \left(\boldsymbol{u}_2 \circ \boldsymbol{u}_3\right) = \Big\{\boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3 + \frac{1}{4}\sum_m^4 \big[\boldsymbol{\varepsilon}^m\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_1\right)\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_2\right) + \\ + \boldsymbol{\varepsilon}^m\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_2\right)\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_3\right) + \boldsymbol{\varepsilon}^m\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_3\right)\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_1\right)\big] + \frac{1}{4}\sum_m^4 \boldsymbol{\varepsilon}^m\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_1\right)\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_2\right)\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_3\right)\Big\} \times \\ \times \Big\{1 + \left(\boldsymbol{u}_1 \boldsymbol{u}_2\right) + \left(\boldsymbol{u}_2 \boldsymbol{u}_3\right) + \left(\boldsymbol{u}_3 \boldsymbol{u}_1\right) + \frac{1}{4}\sum_m^4 \left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_1\right)\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_2\right)\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}_3\right)\Big\}^{-1}.\end{gathered} \tag{4.8}$$

2. A unity element: $\boldsymbol{u} \circ \mathbf{E} = \boldsymbol{u}$. The unity element corresponds to a zero value of a velocity.

3. An inverse device: $\boldsymbol{u} \circ \boldsymbol{u}^{-1} = \mathbf{E}$. Expression of an inverse element is equaled

$$\begin{aligned}\boldsymbol{u}^{-1} &= \left[\frac{1}{4}\sum_m^4 \frac{\boldsymbol{\varepsilon}^m}{1 + \left(\boldsymbol{\varepsilon}^m \boldsymbol{u}\right)}\right]\left[\frac{1}{4}\sum_m^4 \frac{1}{1 + \left(\boldsymbol{\varepsilon}^m \boldsymbol{u}\right)}\right]^{-1} = \\ &= \left[-\frac{1}{4}\sum_m^4 \frac{\boldsymbol{\varepsilon}^m\left(\boldsymbol{\varepsilon}^m \boldsymbol{u}\right)}{1 + \left(\boldsymbol{\varepsilon}^m \boldsymbol{u}\right)}\right]\left[\frac{1}{4}\sum_m^4 \frac{1}{1 + \left(\boldsymbol{\varepsilon}^m \boldsymbol{u}\right)}\right]^{-1}\end{aligned} \tag{4.9}$$

at side conditions $1 + (\boldsymbol{\varepsilon}^m \boldsymbol{u}) \neq 0$ and $1 + (\boldsymbol{u}\boldsymbol{u}^{-1}) \neq 0$. According to (4.9) follows, that the relative velocity $\boldsymbol{v}' = \boldsymbol{v}^{-1}$ is an inverse element of group for the relative velocity $\boldsymbol{v}$, not equal to an opposite element $(-\boldsymbol{v})$. Then the law of a composition $\boldsymbol{u} = \boldsymbol{v} \circ \boldsymbol{u}'$ will be noted as $\boldsymbol{u} = (\boldsymbol{v}')^{-1} \circ \boldsymbol{u}'$. It means property of a noncommutability the law of a composition of three-dimensional velocities, *inhering only to one frame of reference* at replacement $\boldsymbol{v}'$ by $\boldsymbol{u}'$. The inverse element (4.9) is equal to the sum total of the vectors $\boldsymbol{\varepsilon}^m$ increased on coefficients, depending from a velocity $\boldsymbol{u}$.

Using the law of a composition and the reduced properties, we have the following equalities

$$1+\left(\boldsymbol{u}\boldsymbol{u}^{-1}\right)=\left[\frac{1}{4}\sum_{m}^{4}\frac{1}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u})}\right]^{-1}=\left[\frac{1}{4}\sum_{m}^{4}\frac{1}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u}^{-1})}\right]^{-1}=N\left(\boldsymbol{u}\right)N\left(\boldsymbol{u}^{-1}\right),$$

$$\frac{\boldsymbol{u}+\boldsymbol{u}^{-1}}{1+(\boldsymbol{u}\boldsymbol{u}^{-1})}=\frac{1}{4}\sum_{m}^{4}\frac{\boldsymbol{\varepsilon}^m+\boldsymbol{u}}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u})}=\frac{1}{4}\sum_{m}^{4}\frac{\boldsymbol{\varepsilon}^m+\boldsymbol{u}^{-1}}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u}^{-1})},$$

$$\boldsymbol{u}=\left(\boldsymbol{u}^{-1}\right)^{-1}=\left[\frac{1}{4}\sum_{m}^{4}\frac{\boldsymbol{\varepsilon}^m}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u}^{-1})}\right]\left[\frac{1}{4}\sum_{m}^{4}\frac{1}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u}^{-1})}\right]^{-1},$$

$$N\left(\boldsymbol{u}\right)=\frac{N\left(\boldsymbol{u}_1\right)N\left(\boldsymbol{u}_2\right)}{1+(\boldsymbol{u}_1\boldsymbol{u}_2)},$$

$$\frac{\boldsymbol{u}}{N\left(\boldsymbol{u}\right)}=\frac{\boldsymbol{u}_1+\boldsymbol{u}_2+\frac{1}{4}\sum_{m}^{4}\boldsymbol{\varepsilon}^m\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}_1\right)\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}_2\right)}{N\left(\boldsymbol{u}_1\right)N\left(\boldsymbol{u}_2\right)},$$

$$\left[1+(\boldsymbol{\varepsilon}^m\boldsymbol{u})\right]\left[1+(\boldsymbol{u}_1\boldsymbol{u}_2)\right]=\left[1+(\boldsymbol{\varepsilon}^m\boldsymbol{u}_1)\right]\left[1+(\boldsymbol{\varepsilon}^m\boldsymbol{u}_2)\right], \tag{4.10}$$

$$\boldsymbol{u}_1=\boldsymbol{u}\circ\boldsymbol{u}_2^{-1}=\left\{\frac{1}{4}\sum_{m}^{4}\frac{\boldsymbol{\varepsilon}^m+\boldsymbol{u}}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u}_2)}+\frac{1}{4}\sum_{k}^{4}\sum_{m}^{4}\frac{\boldsymbol{\varepsilon}^m\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}\right)\left(\boldsymbol{\varepsilon}^k\boldsymbol{u}_2\right)}{1+(\boldsymbol{\varepsilon}^k\boldsymbol{u}_2)}\right\}\left\{\frac{1}{4}\sum_{m}^{4}\frac{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u})}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u}_2)}\right\}^{-1}$$

$$\boldsymbol{u}_2=\boldsymbol{u}_1^{-1}\circ\boldsymbol{u}=\left\{\frac{1}{4}\sum_{m}^{4}\frac{\boldsymbol{\varepsilon}^m+u}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u}_1)}+\frac{1}{4}\sum_{k}^{4}\sum_{m}^{4}\frac{\boldsymbol{\varepsilon}^m\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}\right)\left(\boldsymbol{\varepsilon}^k\boldsymbol{u}_1\right)}{1+(\boldsymbol{\varepsilon}^k\boldsymbol{u}_1)}\right\}\left\{\frac{1}{4}\sum_{m}^{4}\frac{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u})}{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u}_1)}\right\}^{-1}$$

$$\frac{1}{4}\sum_{m}^{4}\boldsymbol{\varepsilon}^m\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}\right)=\boldsymbol{u},\quad \frac{1}{4}\sum_{m}^{4}\boldsymbol{\varepsilon}^m\left(\boldsymbol{\varepsilon}^m\boldsymbol{\varepsilon}^k\right)\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}\right)=\boldsymbol{\varepsilon}^k\left(\boldsymbol{\varepsilon}^k\boldsymbol{u}\right)-\boldsymbol{u},$$

$$\frac{1}{4}\sum_{m}^{4}\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}_1\right)\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}_2\right)=\frac{1}{16}\sum_{k}^{4}\sum_{m}^{4}\left(\boldsymbol{\varepsilon}^k\boldsymbol{\varepsilon}^m\right)\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}_1\right)\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}_2\right)=\left(\boldsymbol{u}_1\boldsymbol{u}_2\right),$$

$$\frac{1}{4}\sum_{m}^{4}\left(\boldsymbol{\varepsilon}^k\boldsymbol{\varepsilon}^m\right)\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}_1\right)\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}_2\right)=\left(\boldsymbol{\varepsilon}^k\boldsymbol{u}_1\right)\left(\boldsymbol{\varepsilon}^k\boldsymbol{u}_2\right)-\left(\boldsymbol{u}_1\boldsymbol{u}_2\right),$$

$$|\boldsymbol{u}|=\sqrt{\boldsymbol{u}^2}=\sqrt{\frac{1}{4}\sum_{m}^{4}\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}\right)\left(\boldsymbol{\varepsilon}^m\boldsymbol{u}\right)}.$$

Vectors of the preferred directions are not elements of group of velocities. They have no inverse elements. Therefore we have *only a formal equality*

$$\boldsymbol{\varepsilon}^m\circ\boldsymbol{u}=\boldsymbol{u}\circ\boldsymbol{\varepsilon}^m=\boldsymbol{\varepsilon}^m. \tag{4.11}$$

Finally, we shall consider a special case when the law of a composition depends only on values of the nonhomogeneous projective coordinates. The following new equality to *Equalities 1 and 2* is necessary, for example.

Equality 3. The equality is fulfilled

$$\frac{1}{4}\sum_{m}^{4}\varepsilon_i^m\varepsilon_j^m\varepsilon_r^m = \varepsilon_{ijr}, \tag{4.12}$$

where ε_{ijr} – the symmetric symbol with properties $\varepsilon_{ijr} = 1$ at $i \neq j \neq r$, and remaining values are zero. Then, the law of a composition of vectors in coordinate representation is

$$u_i = \left[u_{1i} + u_{pi} + \sum_{j,k}^{3}\varepsilon_{ijk}u_{1j}u_{2k}\right] \times \left[1 + \sum_{i,j}^{3}\delta_{ij}u_{1i}u_{2j}\right]^{-1}. \tag{4.13}$$

According to *Equalities 1–3* we have following relations

$$t^3 + 3t\boldsymbol{x}^2/c^2 + 6xyz/c^3 = \tfrac{1}{4}\sum_{m}^{4}\left(T^m\right)^3,$$
$$4t\left[t^2 - \boldsymbol{x}^2/c^2\right] + 8xyz/c^3 = T^1T^2T^3 + T^1T^2T^4 + T^1T^3T^4 + T^2T^3T^4,$$
$$\tfrac{1}{4}\sum_{m}^{4}\left(T^m - t\right) = 0, \quad \tfrac{1}{4}\sum_{m}^{4}\left(T^m - t\right)^2 = \boldsymbol{x}^2/c^2, \quad \tfrac{1}{4}\sum_{m}^{4}\left(T^m - t\right)^3 = 6xyz/c^3.$$

5 An angular measure

Definition 5. Expression of the additive angular measure is equaled

$$\alpha^m(\boldsymbol{u}) = \ln\frac{1 + (\boldsymbol{\varepsilon}^m\boldsymbol{u})}{N(\boldsymbol{u})}, \quad \alpha^m(0) = 0. \tag{5.1}$$

According to (3.6) and (5.1), we obtain relations

$$\alpha^m(\boldsymbol{u}) = \alpha^m(\boldsymbol{u}_1) + \alpha^m(\boldsymbol{u}_2), \quad \sum_{m}^{4}\alpha^m(\boldsymbol{u}) = 0,$$
$$\frac{1}{4}\sum_{m}^{4}\exp\left[\alpha^m(\boldsymbol{u})\right] = \frac{1}{N(\boldsymbol{u})}, \quad \frac{1}{4}\sum_{m}^{4}\exp\left[-\alpha^m(\boldsymbol{u})\right] = \frac{1}{N(\boldsymbol{u}^{-1})},$$
$$\alpha^m(\boldsymbol{u}^{-1}) = -\alpha^m(\boldsymbol{u}) = \ln\frac{1 + (\boldsymbol{\varepsilon}^m\boldsymbol{u}^{-1})}{N(\boldsymbol{u}^{-1})}, \tag{5.2}$$
$$\boldsymbol{u} = \frac{\sum_{m}^{4}\boldsymbol{\varepsilon}^m e^{\alpha^m(\boldsymbol{u})}}{\sum_{m}^{4}e^{\alpha^m(\boldsymbol{u})}}, \quad \boldsymbol{u}^{-1} = \frac{\sum_{m}^{4}\boldsymbol{\varepsilon}^m e^{-\alpha^m(\boldsymbol{u})}}{\sum_{m}^{4}e^{-\alpha^m(\boldsymbol{u})}}.$$

Definition 6. The vector-parameter $\boldsymbol{\beta} = \{\beta_1, \beta_2, \beta_3\}$ of an angular measure is defined so

$$\boldsymbol{\beta} = \frac{1}{4}\sum_{m}^{4}\boldsymbol{\varepsilon}^m\alpha^m(\boldsymbol{u}). \tag{5.3}$$

Taking into account (2.15), (3.5) and (5.2), we have equalities for an angular measure and relations

$$
\begin{gathered}
\alpha^m(\boldsymbol{u}) = (\boldsymbol{\varepsilon}^m \boldsymbol{\beta}), \quad (\boldsymbol{r}\boldsymbol{\beta}) = \ln A(\boldsymbol{u}, \boldsymbol{r}) = \ln \prod_m^4 \left[\frac{1 + (\boldsymbol{\varepsilon}^m \boldsymbol{u})}{N(\boldsymbol{u})}\right]^{(\boldsymbol{\varepsilon}^m \boldsymbol{r})/4}, \\
\boldsymbol{\beta}(\boldsymbol{u}) = \boldsymbol{\beta}(\boldsymbol{v}) + \boldsymbol{\beta}(\boldsymbol{u}'), \quad t - (\boldsymbol{r}\boldsymbol{x})/c = \sum_m^4 T^m p^m, \\
\boldsymbol{x} - \boldsymbol{r}ct - \frac{1}{4}\sum_m^4 \boldsymbol{\varepsilon}^m (\boldsymbol{\varepsilon}^m \boldsymbol{x})(\boldsymbol{\varepsilon}^m \boldsymbol{r}) = c\sum_m^4 \boldsymbol{\varepsilon}^m T^m p^m, \\
(\boldsymbol{u}/c) \circ (-\boldsymbol{r}) = \frac{\sum_m^4 \boldsymbol{\varepsilon}^m T^m p^m}{\sum_m^4 T^m p^m}.
\end{gathered}
\tag{5.4}
$$

From transformations of characteristics (3.7) we have

$$
\begin{aligned}
t + (\boldsymbol{\varepsilon}^m \boldsymbol{x}) &= e^{(\boldsymbol{r}\boldsymbol{\beta})} e^{(\boldsymbol{\varepsilon}^m \boldsymbol{\beta})} \left[t' + (\boldsymbol{\varepsilon}^m \boldsymbol{x}')\right], \\
t' + (\boldsymbol{\varepsilon}^m \boldsymbol{x}') &= e^{(\boldsymbol{r}\boldsymbol{\beta}')} e^{(\boldsymbol{\varepsilon}^m \boldsymbol{\beta}')} \left[t + (\boldsymbol{\varepsilon}^m \boldsymbol{x})\right].
\end{aligned}
\tag{5.5}
$$

In (5.5) the factor depending on vectors-parameters $\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{v})$, $\boldsymbol{\beta}' = \boldsymbol{\beta}(\boldsymbol{v}') = -\boldsymbol{\beta}$ and $\boldsymbol{r}$ is conformal multiplier.

Taking (2.17) and (5.1) into account we can obtain

$$
\begin{gathered}
\frac{T^m}{\left(\frac{1}{4}\sum_m^4 T^m\right)} = e^{\beta + (\boldsymbol{\varepsilon}^m \boldsymbol{\beta})}, \quad \frac{cT^m}{F} = e^{(\boldsymbol{r}\boldsymbol{\beta}) + (\boldsymbol{\varepsilon}^m \boldsymbol{\beta})}, \\
\frac{F}{\left(\frac{1}{4}\sum_m^4 cT^m\right)} = e^{\beta - (\boldsymbol{r}\boldsymbol{\beta})}, \quad \beta = \ln N(\boldsymbol{u}).
\end{gathered}
\tag{5.6}
$$

Let's enter values of angles $\alpha_4 = -\alpha(\boldsymbol{v})$, $\alpha_n = \alpha^n(\boldsymbol{v})$ $(n = 1, 2, 3)$, and we have, according to (5.5), direct transformation of coordinates and time with $c \neq 1$ in the vectorial form II

$$
\begin{gathered}
\boldsymbol{x} = e^{(\boldsymbol{r}\boldsymbol{\beta})}\left\{\frac{c}{4}\sum_n^3 \left[e^{(\boldsymbol{\varepsilon}^n \boldsymbol{\beta})} + e^{-(\boldsymbol{\varepsilon}\boldsymbol{\beta})}\right]\boldsymbol{\varepsilon}^n t + \frac{1}{4}\sum_n^3 \left[e^{(\boldsymbol{\varepsilon}^n \boldsymbol{\beta})}\boldsymbol{\varepsilon}^n + e^{-(\boldsymbol{\varepsilon}\boldsymbol{\beta})}\sum_k^3 \boldsymbol{\varepsilon}^k\right](\boldsymbol{\varepsilon}^n \boldsymbol{x}')\right\}, \\
t = e^{(\boldsymbol{r}\boldsymbol{\beta})}\left\{\frac{1}{4}\left[\sum_n^3 e^{(\boldsymbol{\varepsilon}^n \boldsymbol{\beta})} + e^{-(\boldsymbol{\varepsilon}\boldsymbol{\beta})}\right]t' + \frac{1}{4c}\sum_n^3 \left[e^{(\boldsymbol{\varepsilon}^n \boldsymbol{\beta})} - e^{-(\boldsymbol{\varepsilon}\boldsymbol{\beta})}\right](\boldsymbol{\varepsilon}^n \boldsymbol{x}')\right\}.
\end{gathered}
\tag{5.7}
$$

The replacement $\boldsymbol{x} \to \boldsymbol{x}'$, $t \to t'$ and $\boldsymbol{\beta} \to -\boldsymbol{\beta}$ leads to the inverse transformations.

From (5.2) and (5.3) one finds expression of a vector of a three-dimensional velocity $\boldsymbol{v}$ and vector-parameter

$$\begin{gathered}\boldsymbol{v}=c\frac{\sum\limits_{n}^{3}\boldsymbol{\varepsilon}^{n}\left[e^{(\boldsymbol{\varepsilon}^{n}\boldsymbol{\beta})}-e^{-(\boldsymbol{\varepsilon}\boldsymbol{\beta})}\right]}{\left[\sum\limits_{n}^{3}e^{(\boldsymbol{\varepsilon}^{n}\beta)}+e^{-(\boldsymbol{\varepsilon}\beta)}\right]}\,,\\ \boldsymbol{\beta}=\frac{1}{4}\sum_{n}^{3}\boldsymbol{\varepsilon}^{n}\left[\alpha^{n}\left(\boldsymbol{u}\right)-\alpha\left(\boldsymbol{u}\right)\right],\quad(\boldsymbol{\varepsilon}\boldsymbol{\beta})=\sum_{n}^{3}\left(\boldsymbol{\varepsilon}^{n}\boldsymbol{\beta}\right),\end{gathered}\tag{5.8}$$

depending on considered angles.

6 An energy and impulse of a particle

It is consider the particle, which moves relative to inertial system of a reference (K) . The Lagrangian with $c\neq 1$ is

$$\begin{gathered}L=-m_{0}c^{2}F\left(d\boldsymbol{x}/\,dt,\boldsymbol{r}\right)=-m_{0}c^{2}N\left(\boldsymbol{u}/c,r\right),\\ N\left(\boldsymbol{u}/c,\boldsymbol{r}\right)=\left\{\prod_{m}^{4}\left[1+\left(\boldsymbol{\varepsilon}^{m}\boldsymbol{u}\right)/\,c\right]^{1-(\boldsymbol{\varepsilon}^{m}\boldsymbol{r})}\right\}^{1/4}.\end{gathered}\tag{6.1}$$

Here the requirement $\boldsymbol{r}\neq-\boldsymbol{\varepsilon}^{k}$ satisfies at $m_0\neq 0$ as at equality $\boldsymbol{r}$ with a vector of one $\boldsymbol{\varepsilon}^{k}$ from (6.1) the Lagrangian $L=-m_0c^2\left[1-(\boldsymbol{r}\boldsymbol{u})/\,c\right]$, which linearly depends on a velocity implies.

Using Hamilton formalism, we have impulse and an energy of a particle in vectorial forms

$$\begin{gathered}\partial=\frac{\partial L}{\partial\boldsymbol{u}}=m_{0}cN\left(\boldsymbol{u}/c,\boldsymbol{r}\right)\left[\frac{1}{4}\sum_{m}^{4}\boldsymbol{\varepsilon}^{m}\frac{\boldsymbol{\varepsilon}^{m}\left(\boldsymbol{u}/c+\boldsymbol{r}\right)}{1+\left(\boldsymbol{\varepsilon}^{m}\boldsymbol{u}\right)/\,c}\right]=\\ =\frac{m_{0}c}{N\left(\boldsymbol{u}^{-1}/c,\boldsymbol{r}\right)}\left[-\boldsymbol{u}^{-1}/\,c+\boldsymbol{r}+\frac{1}{4c}\sum_{m}^{4}\boldsymbol{\varepsilon}^{m}\left(\boldsymbol{\varepsilon}^{m}\boldsymbol{u}^{-1}\right)\left(\boldsymbol{\varepsilon}^{m}\boldsymbol{r}\right)\right],\\ E=\left(\partial\boldsymbol{u}\right)-L=m_{0}c^{2}N\left(\boldsymbol{u}/c,\boldsymbol{r}\right)\left[\frac{1}{4}\sum_{m}^{4}\frac{1-\left(\boldsymbol{\varepsilon}^{m}\boldsymbol{r}\right)}{1+\left(\boldsymbol{\varepsilon}^{m}\boldsymbol{u}\right)/\,c}\right]=\\ =\frac{m_{0}c^{2}}{N\left(\boldsymbol{u}^{-1}/c,\boldsymbol{r}\right)}\left[1-\left(\boldsymbol{u}^{-1}\boldsymbol{r}\right)/\,c\right],\end{gathered}\tag{6.2}$$

Let us consider three special cases. In the first case at $\partial=0$ one gets $E=m_0c^2N\left(-\boldsymbol{r},\boldsymbol{r}\right)$ for a moving particle with a velocity $\boldsymbol{u}=-c\boldsymbol{r}$. The second case at $\boldsymbol{r}=0$ corresponds to the generalized Berwald-Moor metric function. For such space-time we have values $\partial=-m_0\boldsymbol{u}^{-1}/\,N\left(\boldsymbol{u}^{-1}/c\right)$ and $E=m_0c^2/\,N\left(\boldsymbol{u}^{-1}/c\right)$. If $\boldsymbol{u}=0$, in the third case we shall receive values of an energy $E_0=m_0c^2$, impulse

$\partial_0 = m_0 c\boldsymbol{r}$ and vector-parameter $c\partial_0/E_0 = \boldsymbol{r}$ of the rest particle, discussed in [10]. We shall mark also presence of a moment $\mathbf{M}_0 = m_0 c\,[\boldsymbol{x}\boldsymbol{r}]$ of a rest particle, which varies at transformations (3.12). Generally expressions (6.2) with the account (4.10) give a relation as

$$\begin{aligned} E - c\left(\boldsymbol{\varepsilon}^m \partial\right) &= \frac{1 + \left(\boldsymbol{\varepsilon}^m \boldsymbol{u}^{-1}\right)/c}{N\left(\boldsymbol{u}^{-1}/c, \boldsymbol{r}\right)} \left[E_0 - c\left(\boldsymbol{\varepsilon}^m \partial_0\right)\right], \\ E_0 - c\left(\boldsymbol{\varepsilon}^m \partial_0\right) &= \frac{1 + \left(\boldsymbol{\varepsilon}^m \boldsymbol{u}\right)/c}{N\left(\boldsymbol{u}/c, \boldsymbol{r}\right)} \left[E - c\left(\boldsymbol{\varepsilon}^m \partial\right)\right], \end{aligned} \tag{6.3}$$

from which the energy, impulse of a rest particle and vector-parameter imply

$$\begin{aligned} E_0 &= \frac{E - (\partial \boldsymbol{u})}{N\left(\boldsymbol{u}/c, \boldsymbol{r}\right)}, \\ \partial_0 &= \frac{1}{N\left(\boldsymbol{u}/c, \boldsymbol{r}\right)} \left[\partial - E\boldsymbol{u}/c^2 + \frac{1}{4c}\sum_m^4 \boldsymbol{\varepsilon}^m \left(\boldsymbol{\varepsilon}^m \boldsymbol{u}\right)\left(\boldsymbol{\varepsilon}^m \partial\right)\right], \\ \boldsymbol{r} &= \frac{c\partial_0}{E_0} = \frac{\left[\partial - E\boldsymbol{u}/c^2 + \dfrac{1}{4c}\displaystyle\sum_m^4 \boldsymbol{\varepsilon}^m \left(\boldsymbol{\varepsilon}^m \boldsymbol{u}\right)\left(\boldsymbol{\varepsilon}^m \partial\right)\right]}{E - (\partial \boldsymbol{u})}. \end{aligned} \tag{6.4}$$

From (6.3) we have the following formula correlations of an energy and impulse

$$\left\{\prod_m^4 \left[E - c\left(\boldsymbol{\varepsilon}^m \partial\right)\right]^{1 - \left(\boldsymbol{\varepsilon}^m \boldsymbol{r}\right)}\right\}^{1/4} = m_0 c^2 \left\{\prod_m^4 \left[1 - \left(\boldsymbol{\varepsilon}^m \boldsymbol{r}\right)\right]^{1 - \left(\boldsymbol{\varepsilon}^m \boldsymbol{r}\right)}\right\}^{1/4} \tag{6.5}$$

and also value of velocities

$$\begin{aligned} \boldsymbol{u} &= \frac{\partial E}{\partial \partial} = c\sum_m^4 \frac{\boldsymbol{\varepsilon}^m \left(1 - \boldsymbol{\varepsilon}^m \boldsymbol{r}\right)}{E - c\left(\boldsymbol{\varepsilon}^m \partial\right)} \left[\sum_m^4 \frac{1 - \left(\boldsymbol{\varepsilon}^m \boldsymbol{r}\right)}{E - c\left(\boldsymbol{\varepsilon}^m \partial\right)}\right]^{-1}, \\ \frac{c^2 \partial}{E} &= \left[\frac{1}{4}\sum_m^4 \boldsymbol{\varepsilon}^m \frac{\boldsymbol{\varepsilon}^m \left(\boldsymbol{u} + c\boldsymbol{r}\right)}{1 + \left(\boldsymbol{\varepsilon}^m \boldsymbol{u}\right)/c}\right] \left[\sum_m^4 \frac{1 - \left(\boldsymbol{\varepsilon}^m \boldsymbol{r}\right)}{1 + \left(\boldsymbol{\varepsilon}^m \boldsymbol{u}\right)/c}\right]^{-1}. \end{aligned} \tag{6.6}$$

Let's present (6.3) in a general view

$$\begin{aligned} E - c\left(\boldsymbol{\varepsilon}^m \partial\right) &= \frac{1 + \left(\boldsymbol{\varepsilon}^m \boldsymbol{v}'\right)/c}{N\left(\boldsymbol{v}'/c, \boldsymbol{r}\right)} \left[E' - c\left(\boldsymbol{\varepsilon}^m \partial'\right)\right] = \frac{N\left(\boldsymbol{v}/c, \boldsymbol{r}\right)}{1 + \left(\boldsymbol{\varepsilon}^m \boldsymbol{v}\right)/c} \left[E' - c\left(\boldsymbol{\varepsilon}^m \partial'\right)\right], \\ E' - c\left(\boldsymbol{\varepsilon}^m \partial'\right) &= \frac{1 + \left(\boldsymbol{\varepsilon}^m \boldsymbol{v}\right)/c}{N\left(\boldsymbol{v}/c, \boldsymbol{r}\right)} \left[E - c\left(\boldsymbol{\varepsilon}^m \partial\right)\right] = \frac{N\left(\boldsymbol{v}'/c, \boldsymbol{r}\right)}{1 + \left(\boldsymbol{\varepsilon}^m \boldsymbol{v}'\right)/c} \left[E - c\left(\boldsymbol{\varepsilon}^m \partial\right)\right], \end{aligned} \tag{6.7}$$

where coefficients $k^m\left(\boldsymbol{v}/c, \boldsymbol{r}\right)$ and $k^m\left(\boldsymbol{v}'/c, \boldsymbol{r}\right)$ are used. Then from (6.7) we shall receive direct transformations of an energy and impulse between inertial systems of references (K) and (K') in vectorial form I

$$\partial = \frac{A\left(\boldsymbol{v}'/c,\boldsymbol{r}\right)}{N\left(\boldsymbol{v}'/c\right)}\left[\partial' - \frac{E'\boldsymbol{v}'}{c^2} + \frac{1}{4c}\sum_m^4 \boldsymbol{\varepsilon}^m\left(\boldsymbol{\varepsilon}^m v'\right)\left(\boldsymbol{\varepsilon}^m\partial'\right)\right] =$$
$$= \frac{N\left(\boldsymbol{v}/c\right)}{4A\left(\boldsymbol{v}/c,\boldsymbol{r}\right)}\sum_m^4 \boldsymbol{\varepsilon}^m\left[\frac{\boldsymbol{\varepsilon}^m\left(\partial' + E'\boldsymbol{v}/c^2\right)}{1+\left(\boldsymbol{\varepsilon}^m\boldsymbol{v}\right)/c}\right], \qquad (6.8)$$
$$E = \frac{A\left(\boldsymbol{v}'/c,\boldsymbol{r}\right)}{N\left(\boldsymbol{v}'/c\right)}\left[E' - \left(\boldsymbol{v}'\partial'\right)\right] = \frac{N\left(\boldsymbol{v}/c\right)}{4A\left(\boldsymbol{v}/c,\boldsymbol{r}\right)}\sum_m^4 \frac{E' - c\left(\boldsymbol{\varepsilon}^m\partial'\right)}{1+\left(\boldsymbol{\varepsilon}^m\boldsymbol{v}\right)/c},$$

which leave form-invariant a relation (6.5). The replacement $\partial \to \partial'$, $E \to E'$ and $\boldsymbol{v} \to \boldsymbol{v}'$ lead to the inverse transformations.

Let's enter in four-dimensional space with coordinates $\{E, c\partial\}$ four characteristics $E^m = = E - c\left(\boldsymbol{\varepsilon}^m\partial\right)$ for which relations are valid

$$E^2 + c^2\partial^2 = \frac{1}{4}\sum_m^4 \left(E^m\right)^2, \quad E = \frac{1}{4}\sum_m^4 E^m,$$
$$-\partial = \frac{1}{4c}\sum_m^4 \boldsymbol{\varepsilon}^m E^m, \quad -\frac{c\partial}{E} = \frac{\sum_m^4 \boldsymbol{\varepsilon}^m E^m}{\sum_m^4 E^m}. \qquad (6.9)$$

In (6.9) we have the square-law form, which at equality to a value $\left(m_0c^2\right)^2$ defines the hypersurface of the second order intersecting all four characteristics. Expressions E^m of the linear vector function of the first sort give four axes of a considered hypersurface. In (6.9) we have the square-law form, which at equality to a value defines the hypersurface of the second order intersecting all four characteristics. Expressions of the linear vector function of the first sort give four axes of a considered hypersurface.

Transformations (3.12) and (6.8) also imply from an invariance of a relation $E^mT^m = E'^mT'^m$, noted in the form

$$\left[E - c\left(\boldsymbol{\varepsilon}^m\partial\right)\right]\left[t + \left(\boldsymbol{\varepsilon}^m\boldsymbol{x}\right)/c\right] = \left[E' - c\left(\boldsymbol{\varepsilon}^m\partial'\right)\right]\left[t' + \left(\boldsymbol{\varepsilon}^m\boldsymbol{x}'\right)/c\right]. \qquad (6.10)$$

According to (5.5) from (6.10) we have transformations with an angular measure

$$E^m = e^{(\boldsymbol{r}\boldsymbol{\beta}')}e^{(\boldsymbol{\varepsilon}^m\boldsymbol{\beta}')}E'^m, \quad E'^m = e^{(\boldsymbol{r}\boldsymbol{\beta})}e^{(\boldsymbol{\varepsilon}^m\boldsymbol{\beta})}E^m. \qquad (6.11)$$

From transformations (6.2), (6.4) and (6.8) we find laws of a composition of elements of group with $c \neq 1$

$$\left(-c\partial/E\right) = \left(\boldsymbol{u}^{-1}/c\right)\circ\left(-\boldsymbol{r}\right), \quad \left(-\boldsymbol{r}\right)^{-1} = \left(-c\partial/E\right)^{-1}\circ\left(\boldsymbol{u}^{-1}/c\right),$$
$$\left(-c\partial/E\right) = \left(\boldsymbol{v}'/c\right)\circ\left(-c\partial'/E'\right), \quad \left(-c\partial'/E'\right) = \left(\boldsymbol{v}/c\right)\circ\left(-c\partial/E\right), \qquad (6.12)$$
$$\left(-\boldsymbol{r}\right) = \left(\boldsymbol{u}/c\right)\circ\left(-c\partial/E\right) = \left(\boldsymbol{u}'/c\right)\circ\left(-c\partial'/E'\right).$$

Hence, the dimensionless velocities $\boldsymbol{u}/c$, $(-c\partial/E)$, and vector-parameter $(-\boldsymbol{r})$ are equivalent elements of group of three-dimensional velocities and the following conditions $1-(\boldsymbol{\varepsilon}^m\partial)\,c/E \geq 0,\ 1-(\boldsymbol{\varepsilon}^m\boldsymbol{r}) > 0$ are valid. Last equality in (6.12) means an invariance of vector-parameter $\boldsymbol{r}$, as it was necessary to expect.

For a particle with $m_0 = 0$ from (6.3) we obtain the equality $E = c\,(\boldsymbol{\varepsilon}^m\partial)$. According to (4.11) and (6.12) we given by the expressions

$$\begin{gathered}
-\boldsymbol{u}^{-1} = \boldsymbol{r} = -\boldsymbol{\varepsilon}^k, \quad \partial = mc\boldsymbol{r}\left(1+\boldsymbol{r}^2\right), \quad E = mc^2\left(1+\boldsymbol{r}^2\right), \\
m = \lim_{\substack{m_0\to 0\\ \boldsymbol{u}\to c\boldsymbol{\varepsilon}^k}} \frac{m_0}{N\left(\boldsymbol{u}/c,\boldsymbol{r}\right)}, \quad \partial = \frac{E\boldsymbol{r}}{c}, \quad E = mc^2\left(1+\frac{c^2\partial^2}{E^2}\right).
\end{gathered} \tag{6.13}$$

Here $\boldsymbol{r}$ coincides with the fixed value of a vector of the preferred direction and m there is a mass of "photon" in Finsler space-time.

According to (4.10) we shall write out some relations

$$\begin{gathered}
EN\left(-c\partial/E, r\right) = m_0c^2N\left(-\boldsymbol{r},\boldsymbol{r}\right), \quad N\left(0,\boldsymbol{r}\right) = 1, \\
N\left(-c\partial/E\right) = \left\{\prod_m^4 \left[1-(\boldsymbol{\varepsilon}^m\partial)\,c/E\right]^{1-(\boldsymbol{\varepsilon}^m\boldsymbol{r})}\right\}^{1/4}, \\
N\left(-\boldsymbol{r},\boldsymbol{r}\right) = \left\{\prod_m^4 \left[1-(\boldsymbol{\varepsilon}^m\boldsymbol{r})\right]^{1-(\boldsymbol{\varepsilon}^m\boldsymbol{r})}\right\}^{1/4}, \\
1-(\boldsymbol{\varepsilon}^m\boldsymbol{r}) = \frac{\left[1+(\boldsymbol{\varepsilon}^m\boldsymbol{u})/c\right]\left[E-c\,(\boldsymbol{\varepsilon}^m\partial)\right]}{E-(\boldsymbol{u}\partial)}, \\
\frac{1-(\boldsymbol{\varepsilon}^m\boldsymbol{r})}{N\left(-\boldsymbol{r}\right)} = \frac{1+(\boldsymbol{\varepsilon}^m\boldsymbol{u})/c}{N\left(\boldsymbol{u}/c\right)}\cdot\frac{1-(\boldsymbol{\varepsilon}^m\partial)\,c/E}{N\left(-\partial c/E\right)}, \\
N\left(-\boldsymbol{r},\boldsymbol{r}\right) = \frac{N\left(\boldsymbol{u}/c,\boldsymbol{r}\right)N\left(-c\partial/E,\boldsymbol{r}\right)}{1-(\boldsymbol{u}\partial)/E}.
\end{gathered} \tag{6.14}$$

The inverse element of vector-parameter is

$$\begin{gathered}
\left(-\boldsymbol{r}\right)^{-1} = \left[\frac{1}{4}\sum_m^4 \frac{\boldsymbol{\varepsilon}^m}{1-(\boldsymbol{\varepsilon}^m\boldsymbol{r})}\right]\left[\frac{1}{4}\sum_m^4 \frac{1}{1-(\boldsymbol{\varepsilon}^m\boldsymbol{r})}\right]^{-1} = \\
= \left[\frac{1}{4}\sum_m^4 \frac{\boldsymbol{\varepsilon}^m(\boldsymbol{\varepsilon}^m\boldsymbol{r})}{1-(\boldsymbol{\varepsilon}^m\boldsymbol{r})}\right]\left[\frac{1}{4}\sum_m^4 \frac{1}{1-(\boldsymbol{\varepsilon}^m\boldsymbol{r})}\right]^{-1}
\end{gathered} \tag{6.15}$$

Also the following equalities we get

$$
\begin{gathered}
\ln T_0 = \sum_m^4 p^m \ln T^m, \quad \ln N\left(-\boldsymbol{r}, \boldsymbol{r}\right) = \sum_m^4 p^m \ln p^m, \\
T_0 = \lim_{q\to 0} N_q\left(T\right), \quad N\left(-\boldsymbol{r}, \boldsymbol{r}\right) = \lim_{q\to 0} N_q\left(p\right), \\
N_q\left(T\right) = \left\{\sum_m^4 \left(T^m\right)^q p^m\right\}^{1/q}, \quad N_q\left(p\right) = \left\{\sum_m^4 \left(p^m\right)^{q+1}\right\}^{1/q}, \\
\frac{1-\left(\boldsymbol{\varepsilon}^m \boldsymbol{r}\right)}{N\left(-\boldsymbol{r}\right)} \cdot \frac{1+\left[\boldsymbol{\varepsilon}^m \left(-\boldsymbol{r}\right)^{-1}\right]}{N\left[\left(-\boldsymbol{r}\right)^{-1}\right]} = 1, \\
\prod_m^4 \left[t + \left(\boldsymbol{\varepsilon}^m \boldsymbol{x}\right)/c\right]^{\frac{1-\left(\boldsymbol{\varepsilon}^m \boldsymbol{r}\right)}{N\left(-\boldsymbol{r}\right)}} = \prod_m^4 \left[t + \left(\boldsymbol{\varepsilon}^m \boldsymbol{x}\right)/c\right]^{\frac{N\left[\left(-\boldsymbol{r}\right)^{-1}\right]}{1+\left[\boldsymbol{\varepsilon}^m \left(-\boldsymbol{r}\right)^{-1}\right]}}.
\end{gathered}
\tag{6.16}
$$

At probability concept [14–16] of value p^m are interpreted as a probability distribution, and T^m – as the random observable variables describing of geometry. Then function $N_q\left(T\right)$ for a value $1 \leq q < \infty$ is expressions of a half-norm [19]. For a half-norm it is admissible $N_q\left(T\right) = 0$ at $T \neq 0$. This property the half-norm differs from norm $N_2\left(T\right) = \left\{\sum_m^4 \left(T^m\right)^2 p^m\right\}^{1/2}$ at $q = 2$. If $\boldsymbol{r} = 0$, we have equality probability distribution $p^m = 1/4$.

Finally, we shall present a signal for an establishment of the definition of a simultaneity (2.2) as a de Broglie plane wave in a normal form

$$
\psi\left(\boldsymbol{x}, t\right) = A \exp i[Et - \left(\partial \boldsymbol{x}\right)]/\hbar = A \exp i\omega\left[t - \left(\boldsymbol{k}\boldsymbol{x}\right)/\omega\right], \tag{6.17}
$$

where A – amplitude, $\boldsymbol{k}$ – a wave vector, $E = \hbar\omega$ and $\partial = \hbar\boldsymbol{k}$. According to (6.10) magnitude

$$
\varphi = \frac{[Et - \left(\partial \boldsymbol{x}\right)]}{\hbar} = \frac{1}{4\hbar}\sum_m^4 E^m T^m = \frac{1}{4\hbar}\sum_m^4 E'^m T'^m \tag{6.18}
$$

is a form-invariant phase of a wave.

For a particle with $\partial = 0$ also $\boldsymbol{u} = 0$ we have, accordingly, waves in forms

$$
\begin{gathered}
\psi\left(\boldsymbol{x}, t\right) = A \exp i[Et]/\hbar = A \exp\left[i\omega_0 t N\left(-\boldsymbol{r}, \boldsymbol{r}\right)\right], \quad \omega_0 = m_0 c^2/\hbar, \\
\psi\left(\boldsymbol{x}, t\right) = A \exp i[E_0 t - \left(\partial_0 \boldsymbol{x}\right)]/\hbar = A \exp i\omega_0\left[t - \left(\boldsymbol{r}\boldsymbol{x}\right)/c\right].
\end{gathered}
\tag{6.19}
$$

The wave function in a generalized Berwald-Moor space-time with $\boldsymbol{r} = 0$ satisfies, according to (6.5), to the following functional wave equation

$$
\left\{\prod_m^4 \left[\frac{\partial}{\partial t} + \boldsymbol{\varepsilon}^m \frac{\partial}{\partial \boldsymbol{x}}\right]\right\}\psi\left(\boldsymbol{x}, t\right) = \left(\frac{m_0 c^2}{\hbar}\right)^4 \psi\left(\boldsymbol{x}, t\right). \tag{6.20}
$$

7 Discussion

In work *Definitions* on the basis of which the theory of the anisotropic Finsler space-time is under construction are reduced. We shall consider some deductions implying from obtained results.

We use values of vectors $\boldsymbol{\varepsilon} = (-1,-1,-1)$, $\boldsymbol{\varepsilon}^1 = (-1,1,-1)$, $\boldsymbol{\varepsilon}^2 = (1,-1,-1)$, $\boldsymbol{\varepsilon}^3 = (-1,-1,1)$ for specially oriented coordinate tetrahedron [8, 9] and satisfying to *Equalities 1–3*. In outcome the symmetrical matrix H in (2.15) is a Hadamard matrix of order 4 with elements equal to numbers ± 1

$$\mathbf{H}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{pmatrix}, \quad \mathbf{H}_2 = \begin{pmatrix} \mathbf{H}_1 & \mathbf{H}_1 \\ \mathbf{H}_1 & -\mathbf{H}_1 \end{pmatrix}, \quad \mathbf{H}_1 = 1. \tag{7.1}$$

The Hadamard matrix can be constructed by the Sylvester method the recursion evaluation from matrixes $\mathbf{H}_1$ and $\mathbf{H}_2$ and is widely used in an information theory. As the first line and the first column will consist of numbers $+1$ we have the normalized Hadamard matrix. The elements of row of matrixes are discrete values of Walsh orthogonal functions.

In works [6, 7] values of vectors $\boldsymbol{\varepsilon} = (1,1,1)$, $\boldsymbol{\varepsilon}^1 = (1,-1,1)$, $\boldsymbol{\varepsilon}^2 = (-1,1,1)$, $\boldsymbol{\varepsilon}^3 = (1,1,-1)$ are used. Then nonnormalized matrix has properties

$$\begin{gathered} \bar{H}_4 = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} \bar{H}_2 & -H_2 \\ \bar{H}_2 & H_2 \end{pmatrix}, \quad \bar{H}_2 = \begin{pmatrix} \bar{H}_1 & -H_1 \\ \bar{H}_1 & H_1 \end{pmatrix}, \quad \bar{H}_1 = 1, \\ \bar{H}_4 \bar{H}_4^T = 4I, \quad H_4 \bar{H}_4 = 4G_4, \quad G_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{gathered} \tag{7.2}$$

From (3.12), (3.13) and (4.6) we shall receive known direct and inverse transformations of the projective homogeneous and nonhomogeneous coordinates for metric functions (1.2) and (1.3), de-pending only from components of the relative velocities. From (6.2) and (6.5) known relations for an energy and impulse [10] imply. For example, from (3.14) we have the following direct transformations

$$\begin{aligned} x &= \left(\frac{1+v_x/c}{1-v_x/c}\right)^{r_1/2} \frac{x' + v_x t'}{\sqrt{1-v_x^2/c^2}}, & t &= \left(\frac{1+v_x/c}{1-v_x/c}\right)^{r_1/2} \frac{t' + \left(v_x/c^2\right) x'}{\sqrt{1-v_x^2/c^2}}, \\ y &= \left(\frac{1+v_x/c}{1-v_x/c}\right)^{r_1/2} \frac{y' + (v_x/c)\, z'}{\sqrt{1-v_x^2/c^2}}, & z &= \left(\frac{1+v_x/c}{1-v_x/c}\right)^{r_1/2} \frac{z' + (v_x/c)\, y'}{\sqrt{1-v_x^2/c^2}}, \end{aligned} \tag{7.3}$$

which at $r_1 = 0$ coincide with known [9]. Components of vector-parameter of an angular measure $\boldsymbol{\beta}$ are equaled to values of group parameters and the arguments entered in [7, 8]. Expression β are equaled to the value relevant to it from [8]. From (6.8) we shall receive known transformations of impulse and the energies [10] noted in other form.

At the formal limit $c \to \infty$ from (3.12), (3.13) both (4.6) we obtain direct and inverse Galilei transformations

$$\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{v}t', \quad \boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t, \quad t' = t \tag{7.4}$$

and the law of a composition in an Abelian group of three-dimensional velocities

$$\boldsymbol{u} = \boldsymbol{u}' + \boldsymbol{v}. \tag{7.5}$$

Here we have the relative velocity $\boldsymbol{v}' = \boldsymbol{v}^{-1} = -\boldsymbol{v}$ implying from (3.11). For (7.5) group postulates are fulfilled.

Let us consider the kinematics effects of a course of time. We shall note transformation of time (3.13) so

$$t' = \frac{A\left(\boldsymbol{v}'/c, \boldsymbol{r}\right)}{N\left(\boldsymbol{v}'/c\right)}\left[t + \frac{1}{c^2}\left(\boldsymbol{v}'\boldsymbol{x}\right)\right] = \frac{N\left(\boldsymbol{v}/c\right)}{A\left(\boldsymbol{v}/c, \boldsymbol{r}\right)}\left[t + \frac{1}{4}\sum_{m}^{4}\frac{\boldsymbol{\varepsilon}^m\left(\boldsymbol{x} - \boldsymbol{v}t\right)}{c + \left(\boldsymbol{\varepsilon}^m\boldsymbol{v}\right)}\right]. \tag{7.6}$$

At $\boldsymbol{x} = \boldsymbol{v}t$ we have from (7.6), according to (3.10), the formula for time-dilation in origin $\boldsymbol{x}' = 0$ of a frame of reference (K')

$$t' = \frac{N\left(\boldsymbol{v}/c\right)}{A\left(\boldsymbol{v}/c, \boldsymbol{r}\right)}\, t \tag{7.7}$$

Expression (7.7) is equaled (3.4).

Let in a system of reference (K') simultaneous two events in origin $\boldsymbol{x}' = 0$ and in point $\boldsymbol{x}'$ that is $\Delta t' = t'\left(\boldsymbol{x}'\right) - t'\left(0\right) = 0$. Then with a system of reference (K) these events are non-simultaneous and from (7.6) the formula for effect of a relative of simultaneity distant events in points $\boldsymbol{x}_1 = \boldsymbol{v}t$ and $\boldsymbol{x}_2 = \boldsymbol{x}$ we obtain

$$\Delta t = t\left(\boldsymbol{x}\right) - t\left(\boldsymbol{v}t\right) = -\frac{1}{4}\sum_{m}^{4}\frac{\boldsymbol{\varepsilon}^m\left(\Delta\boldsymbol{x}\right)}{c + \left(\boldsymbol{\varepsilon}^m\boldsymbol{v}\right)} \tag{7.8}$$

where $\Delta\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$.

The effect of length-contraction of segments of paths is

$$\boldsymbol{x}' = \frac{N\left(\boldsymbol{v}/c\right)c}{4A\left(\boldsymbol{v}/c, \boldsymbol{r}\right)}\sum_{m}^{4}\boldsymbol{\varepsilon}^m\frac{\left(\boldsymbol{\varepsilon}^m\Delta\boldsymbol{x}\right)}{c + \left(\boldsymbol{\varepsilon}^m\boldsymbol{v}\right)}. \tag{7.9}$$

Let's consider strict kinematics reviewing Finsler space-time with transformations (7.4) and (7.5) classical physics with absolute simultaneity distant events.

Definition 7. There is a unique time (or the definition of a simultaneity of events is given) for points O, A and A^n at realization of a equality

$$\sum_m^4 c^m \left(T^m - t\right) = 0, \tag{7.10}$$

where c^m is a velocity of a signal in directions of the preferred vectors $\boldsymbol{\varepsilon}^m$ $\left(\sum\limits_m^4 \boldsymbol{\varepsilon}^m = 0\right)$ of an inertial system of a reference (K).

Lengths of segments of the paths moving by a signal, are values $(\boldsymbol{\varepsilon}^m x)$ and for characteristics we have the following equalities

$$T^m = t + \frac{(\boldsymbol{\varepsilon}^m \boldsymbol{x})}{c^m}. \tag{7.11}$$

Definition 8. Expression

$$c = \frac{1}{4}\sum_m^4 c^m \tag{7.12}$$

is the universal constant and defines a "average" physical velocity of a signal in various inertial sys-tems of references.

According to (7.12), from (7.10) we obtain the following relations

$$t = \frac{\sum\limits_m^4 c^m T^m}{\sum\limits_m^4 c^m} = \frac{1}{4c}\sum_m^4 c^m T^m, \quad \boldsymbol{x}/c = \frac{1}{4}\sum_m^4 \boldsymbol{\varepsilon}^m c^m T^m, \quad \boldsymbol{u} = c\,\frac{\sum\limits_m^4 \boldsymbol{\varepsilon}^m c^m T^m}{\sum\limits_m^4 c^m T^m}. \tag{7.13}$$

Definition 9. Form-invariant metric function in local Finsler geometry is defined so

$$F = \left\{\prod_m^4 \left[c^m dt + \boldsymbol{\varepsilon}^m d\boldsymbol{x}\right]^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})}\right\}^{1/4} = \left\{\prod_m^4 \left[c'^m dt' + \boldsymbol{\varepsilon}^m d\boldsymbol{x}'\right]^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})}\right\}^{1/4} \tag{7.14}$$

Metric function (7.14) refers to a class of functions (1.4).

We use a method of coefficient "k" and we shall note relations in vectorial form

$$t + \frac{(\boldsymbol{\varepsilon}^m \boldsymbol{x})}{c^m} = k^m\left(\boldsymbol{v}, \boldsymbol{r}\right)\left[t' + \frac{(\boldsymbol{\varepsilon}^m \boldsymbol{x}')}{c'^m}\right], \quad t' + \frac{(\boldsymbol{\varepsilon}^m \boldsymbol{x}')}{c'^m} = k^m\left(\boldsymbol{v}', \boldsymbol{r}\right)\left[t + \frac{(\boldsymbol{\varepsilon}^m \boldsymbol{x})}{c^m}\right] \tag{7.15}$$

where $k^m\left(\boldsymbol{v}, \boldsymbol{r}\right) k^m\left(\boldsymbol{v}', \boldsymbol{r}\right) = 1$. We shall substitute (7.15) in (7.14) and we have

$$\begin{aligned} &\prod_m^4 \left(c^m\right)^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})} \prod_m^4 \left[k^m\left(\boldsymbol{v}, \boldsymbol{r}\right)\right]^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})} = \prod_m^4 \left(c'^m\right)^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})}, \\ &\prod_m^4 \left(c'^m\right)^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})} \prod_m^4 \left[k^m\left(\boldsymbol{v}', \boldsymbol{r}\right)\right]^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})} = \prod_m^4 \left(c^m\right)^{1-(\boldsymbol{\varepsilon}^m \boldsymbol{r})}. \end{aligned} \tag{7.16}$$

Further we obtain the equalities implying from (7.14) and (7.15), under known conditions $\boldsymbol{x}' = 0$ at $\boldsymbol{x} = \boldsymbol{v}t$ and $\boldsymbol{x} = 0$ at $\boldsymbol{x}' = \boldsymbol{v}'t'$, and also the new definition an anisotropy of a velocity of a signal in a moving system of reference.

Definition 10. The linear vector function of the first sort is a velocity of a signal in a system (K')

$$c'^m = c^m + (\boldsymbol{\varepsilon}^m \boldsymbol{v}) \tag{7.17}$$

depending from the relative velocity..

In a result we shall receive values of coefficients

$$k^m(\boldsymbol{v}, \boldsymbol{r}) = 1 + \frac{(\boldsymbol{\varepsilon}^m \boldsymbol{v})}{c^m}, \quad k^m(\boldsymbol{v}', \boldsymbol{r}) = 1 + \frac{(\boldsymbol{\varepsilon}^m \boldsymbol{v}')}{c'^m} \tag{7.18}$$

with $\boldsymbol{v}' = -\boldsymbol{v}$ and equality $c'^m T'^m = c^m T^m$. According to (7.15), we obtain Galilei transformations (7.4) and formulas for Doppler effect on four preferred directions

$$\omega'^m = \left[1 + \frac{(\boldsymbol{\varepsilon}^m \boldsymbol{v})}{c^m}\right] \omega^m, \quad \omega^m = \left[1 + \frac{(\boldsymbol{\varepsilon}^m \boldsymbol{v}')}{c'^m}\right] \omega'^m. \tag{7.19}$$

At $c^m = c$ and use of a dynamic substantiation of three effects in formulas (7.7)–(7.9), we have equality $c'^m = c$ and transformations (3.12) for metric function (3.1). Such interpretation of transformations (3.12) is compounded with idea of not carried away ether of classical physics with Galilei transformations with a velocity of a signal (7.17) and their further transformation.

References

[1] H. Poincare, *La mesure du temps,* Rev. Metaphys. Mordle, 1898. V. 6, 1–13.

[2] H. Poincare, *Sur la dynamique de l'ectron,* Rend. Circolo Mat. Palermo. 1906. V. 21, 129–176.

[3] H. I. Minkowski, *Raum und Zeit,* address delivered at the 80th Assembly of German Scientists and Physicians (Koln, 1908), Phys. Ztscht. 1909. Bd. 10, 104–134.

[4] N. I. Lobachevsky, *Complete Collected Works,* V. II. Gostechisd, M.-L., 1949 (in Russian).

[5] V. A. Fock, *Theory of Space, Time and Gravitation.* Moscow: Gostechisd, 1955 (in Russian).

[6] G. Yu. Bogoslovsky, H. F. Goenner, *On the possibility of the phase transitions in the geometric structure of space-time,* Phys. Lett. A. 1998. V. 244, 222–226.

[7] G. Yu. Bogoslovskiy, H. F. Goenner, *Finslerian spaces possessing local relativistic symmetry,* Gen. Relativ. Grav. 1999. V. 31. No. 10, 1565–1603.

[8] D. G. Pavlov, *Hypercomplex numbers, associated metric spaces, and extension of relativistic hyperboloid,* arXiv: gr-gc/0206004. 2002. V. 1, 1–16.

[9] D. G. Pavlov, G. I. Garas'ko, *The notions of distance and velocity modulus in the linear Finsler spaces,* Hypercomplex Numbers in Geometry and Physics, 2005, 1 (3), 1–15 (in Russian).

[10] G. Yu. Bogoslovskiy, *4-momentum of a particle and the equation of a mass surface in completely anisotropic space-time,* Hypercomplex numbers in geometry and physics. 2005, 2 (4), 27–43 (in Russian).

[11] B. Riemann, *Abhandlungen der Konigl.* Gesellshaft d. Wissenshaften. Gottingen, 1867. V. 13, 14.

[12] R. G. Zaripov, *Definition of a simultaneity and Finsler structure of flat anisotropic space-time,* Gravitation and a relativity theory. Kazan: Kazan Univ. Press, 1992. N 29, 64–71 (in Russian).

[13] R. G. Zaripov, *Clock synchronization and Finsler structure of a flat anisotropic Space-Time,* Proceedings International Scientific Meeting PIRT-2003 "Physical Interpretations of Relativity Theory", Moscow, Liverpool, Sunderland, 2003, 241–248.

[14] R. G. Zaripov, *The Law of a composition of physical velocities in locally anisotropic Finsler Space-Time*, Proceedings International Scientific Meeting PIRT-2005 "Physical Interpretations of Relativity Theory", Moscow, Liverpool, Sunderland, 2005, 148–158.

[15] R. G. Zaripov, *Binary a system of numbers and Finsler geometry of local anisotropic space-time,* Hypercomplex Numbers in Geometry and Physics, 2005, 1 (3), 47–60.

[16] R. G. Zaripov, *Synchronization of clocks and Finsler geometry of local anisotropic space-time, The new problems of a field theory,* ed. A. V. Aminova. Kazan: Kazan Univ. Press, 2006. V. 5, 99–114 (in Russian).

[17] R. G. Zaripov, *To definition of a simultaneity in a Special Relativity Theory, Gravitation and relativity theory.* Kazan: Kazan Univ. Press, 1978, N 14–15, 60–69 (in Russian).

[18] R. G. Zaripov, *About physical concept of the definition of a simultaneity, Gravitation and a Relativity Theory.* Kazan: Kazan Univ. Press, 1980. N 17, 47–51 (in Russian).

[19] R. G. Zaripov, *New of a Measure and Methods in Information Theory.* Kazan: Kazan Tech. University Press, 2005 (in Russian); online book at http://www.imm.knc.ru/zaripov-measures.ru.html.

On the World Function and the Relation between Geometries

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It is shown that the World function can be regarded as a link between the qualitatively different geometries with a certain congruence of world lines (geodesics). If the space in which the World function is defined is a polynumber space, then the hypothesis of the analyticity of the vector field of generalized velocities of the world lines leads to the strict limitations on the structure of the World function. The main result states that the Minkowskian space and the polynumber space correspond to the same physical World.

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1 Introduction

The idea that everything happening in the physical world is governed by a single scalar function has originated long ago and can hardly be attributed to any scientist or even to a group of scientists. It is this function that will be called the World function here. For example, H. Weyl [1] uses the term "World function" when discussing the Mie theory. It is not definitely clear what to choose as a World function. G. Mie in his theory (a field theory) suggested to choose the Lagrangian of the field, i.e. to take the density of the Lagrange function as a World function. In this paper the field equations and the field theories will not be discussed.

For the observer using the classical mechanics and Finsler geometry, it is sufficient to know how all the material points move, in other words it is sufficient to know the congruence of the world lines in the space-time. In Finsler geometry such a congruence is a normal congruence of the geodesics [2], i.e. there exists such a scalar function, S, the level surfaces of which are transversal to the given congruence of geodesics. In classical mechanics such function is usually called 'action as a function of coordinates'. In this paper it is this finction, S, that will be considered the World function.

So, let us adopt that in the coordinate space $x^1, x^2, ..., x^n$ the scalar function $S(x)$ corresponding to the notion of *action as a fuction of coordinates* $x^1, x^2, ..., x^n$ in classical mechanics plays a role of the World function. Taken as it is, the scalar function, S, can not define the field of velocities and, therefore, can not define a congruence of geodesics each of which corresponds to an observer or to a material particle. One needs an additional procedure, $\tilde{\varphi}$, providing the possibility to pass from the covariant 'vectors' to the contravariant 'vectors'. In any Finsler geometry, Φ_n, there is such a procedure. Thus, the pair $\{S; \tilde{\varphi}\}$ as well as the pair $\{S; \Phi_n\}$

defines the congruence of the world lines for all the points of the space, that is defines the evolution of this space.

Let x^0, x^1, x^2, x^3 be the Minkowskian space with the length element defined as

$$ds = mc\sqrt{(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2} \equiv mc\sqrt{\overset{o}{g}_{ij}\, x^i x^j}\,, \tag{1}$$

where the factor mc – optional from the geometric point of view – provides a better possibility to give a physical interpretation of the geometrical objects; m and c are the rest mass of the particle and the light velocity in vacuum. The tangent equation of the indicatrix in such a space can be written as follows

$$(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = (mc)^2\,. \tag{2}$$

Then, $S(x^0, x^1, x^2, x^3)$, the action as a function of coordinates in the Minkowski space must suffice the Hamilton-Jacoby equation

$$\left(\frac{dS}{dx^0}\right)^2 - \left(\frac{dS}{dx^1}\right)^2 - \left(\frac{dS}{dx^2}\right)^2 - \left(\frac{dS}{dx^3}\right)^2 = (mc)^2\,. \tag{3}$$

Let us now take an arbitrary function $\tilde{S}$ which suffice

$$\left(\frac{d\tilde{S}}{dx^0}\right)^2 - \left(\frac{d\tilde{S}}{dx^1}\right)^2 - \left(\frac{d\tilde{S}}{dx^2}\right)^2 - \left(\frac{d\tilde{S}}{dx^3}\right)^2 > 0\,. \tag{4}$$

and substitute it into (3). The result is that the function $\tilde{S}$ is a solution of the Hamilton-Jacoby equation which corresponds to the Finsler geometry with the length element

$$d\tilde{s} = \kappa(x) \cdot mc\sqrt{(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2} \tag{5}$$

and the tangent equation of the indicatrix

$$(p_0)^2 - (p_1)^2 - (p_2)^2 - (p_3)^2 = \kappa(x)^2 \cdot (mc)^2\,, \tag{6}$$

where

$$\kappa(x) \equiv \frac{1}{mc}\sqrt{\left(\frac{d\tilde{S}}{dx^0}\right)^2 - \left(\frac{d\tilde{S}}{dx^1}\right)^2 - \left(\frac{d\tilde{S}}{dx^2}\right)^2 - \left(\frac{d\tilde{S}}{dx^3}\right)^2}\,. \tag{7}$$

Notice, that if the length elements of two geometries, ds, $d\tilde{s}$, defined in the same coordinate space $x^1, x^2, ..., x^n$ are related as

$$d\tilde{s} = \kappa(x)ds\,, \tag{8}$$

where $k(x) > 0$ is an arbitrary function of a point, then these two geometries are called conformly connected [2]. The geometry $d\tilde{s}$ differs from the geometry

ds in such a way that in every infinitely small vicinity of every point of space, $x^1, x^2, ..., x^n$, there is a scale transformation while the extension-contraction coefficient, $\kappa(x)$, depends on the point.

Thus, we see that if we know the arbitrary scalar function, $\tilde{S}$, sufficing (4) in the flat Minkowski space (1), then we know the World function in the space given by (5) which is conformly connected to the Minkowski space. Therefore, the world lines equations in space (5) can be written as:

$$\dot{x}^i = \overset{o}{g}{}^{ij} \frac{d\tilde{S}}{dx^j} \lambda(x, y) \,, \tag{9}$$

where $\dot{x}^i \equiv \frac{dx^i}{d\tau}$ is a derivative over τ (evolution parameter) along the world line, and $\lambda(x, y) > 0$ is an arbitrary function.

All the above said is true (with regard to the obvious changes of notation in formulas) for the Euclidean or for pseudo Euclidean geometry of the arbitrary dimension n, but only for $n = 2$ one could correlate a system of the associative commutative non-degenerate numbers (correspondingly, complex numbers, C_2, and hyperbolic numbers, H_2), to the Euclidean or to pseudo Euclidean space.

In this approach the form of the World function is not limited by anything but (4). To make the form of the World function concrete for the polynumber space, P_n, one could use the analyticity condition - the condition giving a relation between the World function and the analytical functions of the polynumber variable, P_n. In this paper this is done in the form of Hypotheses I, II. The other realizations are also possible.

1.1 Complex plane

Hypothesis I_{C_2}: Components of the vector field that produces the world lines corresponding to the given World function, are the components of the analytical function of the complex variable.

According to this Hypothesis

$$\lambda(x, y) \cdot \frac{\partial \tilde{S}}{\partial x} = u \,, \qquad \lambda(x, y) \cdot \frac{\partial \tilde{S}}{\partial y} = v \,, \tag{10}$$

where $F(z) = u(x, y) + iv(x, y)$ is an analytical function of the complex variable $z = x + iy$. Then the Cauchi-Riemann relations give the following partial differential equations for the World function $\tilde{S}$:

$$\frac{\partial}{\partial x} \lambda(x, y) \frac{\partial \tilde{S}}{\partial x} = \frac{\partial}{\partial y} \lambda(x, y) \frac{\partial \tilde{S}}{\partial y} \,, \qquad \frac{\partial}{\partial y} \lambda(x, y) \frac{\partial \tilde{S}}{\partial x} = -\frac{\partial}{\partial x} \lambda(x, y) \frac{\partial \tilde{S}}{\partial y} \,. \tag{11}$$

If $\lambda(x, y) \equiv 1$, then the equations (11) simplify:

$$\frac{\partial^2 \tilde{S}}{\partial x^2} - \frac{\partial^2 \tilde{S}}{\partial y^2} = 0 \,, \qquad \frac{\partial^2 \tilde{S}}{\partial x \partial y} = 0 \,. \tag{12}$$

The general solution of this system of equations is

$$\tilde{S} = \frac{A}{2}(x^2 + y^2) + a_1 x + a_2 y + b\,, \tag{13}$$

where A, a_1, a_2, b are real numbers. Notice, that if $A \neq 0$, then function $\tilde{S}$ is not a component of the analytical function of complex variable.

Hypothesis II_{C_2}: *The components of the vector field that produces the world lines corresponding to the given World function, are the components of the function of complex variable conjugate to the analytical function of complex variable.*

Then according to this hypothesis

$$\lambda(x,y)\cdot\frac{\partial\tilde{S}}{\partial x} = u\,, \qquad \lambda(x,y)\cdot\frac{\partial\tilde{S}}{\partial y} = -v\,, \tag{14}$$

where $F(z) = u(x,y) + iv(x,y)$ is an analytical function of complex variable $z = x + iy$. The Cauchi-Riemann relations give the following partial differential equations for the World function $\tilde{S}$:

$$\frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde{S}}{\partial x} = -\frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde{S}}{\partial y}\,, \qquad \frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde{S}}{\partial x} = \frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde{S}}{\partial y}\,. \tag{15}$$

If $\lambda(x,y) \equiv 1$, then the equations (15) simplify and give a single partial differential equation:

$$\frac{\partial^2\tilde{S}}{\partial x^2} + \frac{\partial^2\tilde{S}}{\partial y^2} = 0\,. \tag{16}$$

Thus, provided the Hypothesis II_{C_2} is true and $\lambda(x,y) \equiv 1$, the function $\tilde{S}$ is a component of the analytical function of the complex variable, and the corresponding geometry which is conformly connected to the Euclidean plane can be obtained with the help of the conformal transformation of the Euclidean plane.

1.2 Hyperbolic plane

The metric tensor for the hyperbolic plane has the form

$$\overset{o}{g}_{ij} = diag(1, -1)\,, \tag{17}$$

and the Cauchi-Riemann relations for the analytical functions $F(z) = u(x,y) + jv(x,y)$ of the variable $H_2 \ni z = x + jy$, $j^2 = 1$ can be written as:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}\,. \tag{18}$$

Hypothesis I_{H_2}: *Components of the vector field that produces the world lines corresponding to the given World function, are the components of the analytical function of the variable* H_2.

According to this hypothesis and in analogy to (9) for $n = 2$, one gets

$$\lambda(x,y)\cdot\frac{\partial\tilde{S}}{\partial x} = u\,, \qquad \lambda(x,y)\cdot\frac{\partial\tilde{S}}{\partial y} = -v\,, \tag{19}$$

where $F(z) = u(x,y) + iv(x,y)$ is an analytical function of the variable $H_2 \ni z = x + jy$. Then the Cauchi-Riemann relations give the following partial differential equations for the World function, $\tilde{S}$:

$$\frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde{S}}{\partial x} = -\frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde{S}}{\partial y}\,, \qquad \frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde{S}}{\partial x} = -\frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde{S}}{\partial y}\,. \tag{20}$$

If $\lambda(x,y) \equiv 1$, then the equations (20) simplify:

$$\frac{\partial^2\tilde{S}}{\partial x^2} + \frac{\partial^2\tilde{S}}{\partial y^2} = 0\,, \qquad \frac{\partial^2\tilde{S}}{\partial x\partial y} = 0\,. \tag{21}$$

The general solution of this system of equations is

$$\tilde{S} = \frac{A}{2}(x^2 - y^2) + a_1 x + a_2 y + b\,, \tag{22}$$

where A, a_1, a_2, b are real numbers. Notice, that if $A \neq 0$, function $\tilde{S}$ is not a component of the analytical function of variable H_2.

Hypothesis II_{H_2}: The components of the vector field that produces the world lines corresponding to the given World function, are the components of the function of variable H_2 conjugate to the analytical function of the variable H_2.

According to this hypothesis

$$\lambda(x,y)\cdot\frac{\partial\tilde{S}}{\partial x} = u\,, \qquad \lambda(x,y)\cdot\frac{\partial\tilde{S}}{\partial y} = v\,, \tag{23}$$

where $F(z) = u(x,y) + jv(x,y)$ is an analytical function of the variable $H_2 \ni z = x + iy$. Then the Cauchi-Riemann relations give the following partial differential equations for the World function $\tilde{S}$:

$$\frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde{S}}{\partial x} = \frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde{S}}{\partial y}\,, \qquad \frac{\partial}{\partial y}\lambda(x,y)\frac{\partial\tilde{S}}{\partial x} = \frac{\partial}{\partial x}\lambda(x,y)\frac{\partial\tilde{S}}{\partial y}\,. \tag{24}$$

If $\lambda(x,y) \equiv 1$, then the equations (24) simplify and give a single partial differential equation:

$$\frac{\partial^2\tilde{S}}{\partial x^2} - \frac{\partial^2\tilde{S}}{\partial y^2} = 0\,. \tag{25}$$

Thus, provided the Hypothesis II_{H_2} is true and $\lambda(x,y) \equiv 1$, the function $\tilde{S}$ is a component of the analytical function of the variable H_2, and the corresponding geometry which is conformly connected to the hyperbolic plane can be obtained with the help of the conformal transformation of the hyperbolic plane.

2 Polynumbers P_n

Let us regard a system of the non-degenerate n-numbers P_n. The corresponding coordinate space, $x^1, x^2, ..., x^n$, is a Finsler metric space with the length element

$$ds = mc \sqrt[n]{\overset{o}{g}_{i_1 i_2 \ldots i_n}\, dx^{i_1} dx^{i_2} \ldots dx^{i_n}}\,, \tag{26}$$

$\overset{o}{g}_{i_1 i_2 \ldots i_n}$ is a metric tensor that does not depend on point. The Finsler spaces of this kind have been studied in mathematical literature for a long time (see, for example, [3] - [6]), but the fact that all the polynumber spaces are just the Finsler spaces of this type was discovered not long ago in [7], [8] and the subsequent papers of the same authors.

The components of the generalized momentum in the geometry (26) can be calculated according to the formulas:

$$p_i = mc \frac{\overset{o}{g}_{i j_2 \ldots j_n}\, dx^{j_2} \ldots dx^{j_n}}{\left(\overset{o}{g}_{i_1 i_2 \ldots i_n}\, dx^{i_1} dx^{i_2} \ldots dx^{i_n}\right)^{\frac{n-1}{n}}}\,. \tag{27}$$

Finsler geometry with the length element (26) will be called resolvable if the tangent equation for the indicatrix can be written as

$$\overset{o}{g}{}^{i_1 i_2 \ldots i_n}\, p_{i_1} p_{i_2} \ldots p_{i_n} = \mu^n (mc)^n\,, \tag{28}$$

where $\mu > 0$ is a constant. For the Riemannian or pseudo Riemannian geometry the re solvability means that the determinant of the metric tensor is not equal to zero. It seems that the Finsler geometry in the space of the non-degenerate polynumbers is always resolvable, but this statement demands the strict proof.

As it can be seen from expressions (26) - (28), tensors $\overset{o}{g}_{i_1 i_2 \ldots i_n}$, $\overset{o}{g}{}^{i_1 i_2 \ldots i_n}$ must suffice the following relation of the resolvable Finsler geometry

$$\begin{aligned} &\overset{o}{g}{}^{j_1 j_2 \ldots j_n} \times \\ &\times \overset{o}{g}_{j_1 i_2 \ldots i_n}\, dx^{i_2} \ldots dx^{i_n}\, \overset{o}{g}_{j_2 k_2 \ldots k_n}\, dx^{k_2} \ldots dx^{k_n} \ldots \overset{o}{g}_{j_n m_2 \ldots m_n}\, dx^{m_2} \ldots dx^{m_n} = \\ &= \mu^n \left(\overset{o}{g}_{i_1 i_2 \ldots i_n}\, dx^{i_1} dx^{i_2} \ldots dx^{i_n}\right)^{n-1}\,. \end{aligned} \tag{29}$$

Action as a function of coordinates in geometry (26) suffices the Hamilton-Jacoby equation:

$$\overset{o}{g}{}^{j_1 j_2 \ldots j_n} \frac{\partial \tilde{S}}{\partial x^{j_1}} \frac{\partial \tilde{S}}{\partial x^{j_2}} \ldots \frac{\partial \tilde{S}}{\partial x^{j_n}} = \mu^n (mc)^n\,. \tag{30}$$

Let us regard an arbitrary World function, $\tilde{S}(x^1, x^2, ..., x^n)$, with the only condition

$$\overset{o}{g}{}^{j_1 j_2 \ldots j_n} \frac{\partial \tilde{S}}{\partial x^{j_1}} \frac{\partial \tilde{S}}{\partial x^{j_2}} \ldots \frac{\partial \tilde{S}}{\partial x^{j_n}} > 0\,, \tag{31}$$

Then function $\tilde{S}(x)$ is the action for the geometry conformly connected to geometry (26) with the length element

$$d\tilde{s} = \kappa(x) \cdot mc \sqrt[n]{\overset{o}{g}_{i_1 i_2 ... i_n} \, x^{i_1} x^{i_2} ... x^{i_n}}\,, \tag{32}$$

where $\kappa(x) > 0$ is the extension–contraction coefficient which varies from point to point of the coordinate space

$$\kappa(x) = \frac{1}{\mu \cdot mc} \sqrt[n]{\overset{o}{g}{}^{j_1 j_2 ... j_n} \, \frac{\partial \tilde{S}}{\partial x^{j_1}} \frac{\partial \tilde{S}}{\partial x^{j_2}} ... \frac{\partial \tilde{S}}{\partial x^{j_n}}}\,, \tag{33}$$

and the World function, $\tilde{S}$, is the solution of the Hamilton-Jacoby equation of the following form:

$$\overset{o}{g}{}^{j_1 j_2 ... j_n} \, \frac{\partial \tilde{S}}{\partial x^{j_1}} \frac{\partial \tilde{S}}{\partial x^{j_2}} ... \frac{\partial \tilde{S}}{\partial x^{j_n}} = \kappa(x)^n \cdot \mu^n (mc)^n \,. \tag{34}$$

The field of velocities that defines the congruence of the world lines can be expressed by the World function, $\tilde{S}$, by the formula

$$\dot{x}^i = \overset{o}{g}{}^{i j_2 ... j_n} \, \frac{\partial \tilde{S}}{\partial x^{j_2}} ... \frac{\partial \tilde{S}}{\partial x^{j_n}} \cdot \lambda(x)^{n-1}\,, \tag{35}$$

where $\lambda(x) > 0$ is an arbitrary scalar function.

The algebra of polynumbers $P_n \ni X = x^1 e_1 + x^2 e_2 + ... + x^n e_n$ is completely defined by the multiplication rule for the basis elements:

$$e_i e_j = p^k_{ij} e_k \tag{36}$$

that is by the number tensor, p^k_{ij}. Notice, that the polynumbers, P_n, are called non-degenerate if

$$det(q_{ij}) \neq 0\,, \qquad q_{ij} \equiv p^k_{im} p^m_{kj}\,. \tag{37}$$

In this case one can construct tensor q^{ij}. If ϵ^i are the coefficients of the expansion of the unity $1 \in P_n$ in the basis e_i, then the Cauchi-Riemann relation for the analytical function $F(X) = f(x)^i e_i$ of the variable P_n can be written in the following form:

$$\frac{\partial f^i}{\partial x^k} - p^k_{ij} \epsilon^m \frac{\partial f^j}{\partial x^m} = 0\,. \tag{38}$$

Hypothesis I_{P_n}: Components of the vector field that produces the world lines corresponding to the given World function, are the components of the analytical function of the variable P_n.

If $F(X) = f(x)^i e_i$ is an analytical function of the variable P_n, then this hypothesis leads to the expression:

$$f^i(x^1, x^2, ..., x^n) = \overset{o}{g}{}^{i j_2 ... j_n} \, \frac{\partial \tilde{S}}{\partial x^{j_2}} ... \frac{\partial \tilde{S}}{\partial x^{j_n}} \cdot \lambda(x)^{n-1}\,. \tag{39}$$

Substituting these components of the analytical function expressed by the World function into the Cauchi-Riemann relations, we get such a system of partial differential equations that if Hypothesis I_{P_n} is fulfilled, then the World function suffices this system.

Hypothesis II_{P_2}: *Components of the vector field that produces the world lines corresponding to the given World function, are the components of the function of the variable* P_n *conjugate to the analytical function of the same variable with the help of a special unary operation.*

Let us define the unary operation $\bar{X} = Y$ acting on the set $P_n \ni X, Y$ in the following way:

$$y^i = \overset{o}{g}{}^{ij_2 \dots j_n} q_{j_2 m_2} \dots q_{j_n m_n} x^{m_2} \dots x^{m_n} \,. \tag{40}$$

For complex numbers C_2 and hyperbolic numbers H_2 such a unary operation is a regular conjugation, while on the polynumber set H_4 (and H_n) this operation coincides with the operation of normal conjugation [9] within the accuracy of a number factor. The unary operation (40) can be generalized for $(n-1)$ arguments, it will remain symmetrical due to its definition. To distinguish such a unary operation and a corresponding $(n-1)$-ary operation from other conjugations in the polynumber algebras, let us call such an operation *symmetrical conjugation*.

Comparing formulas (35) and (40) and changing x^i to f^i, we see that the realization of the *Hypothesis* II_{P_2} leads to the relations

$$q_{ij} f^j = \frac{\partial \tilde{S}}{\partial x^i} \lambda(x) \,, \tag{41}$$

or

$$f^i = q^{ij} \frac{\partial \tilde{S}}{\partial x^j} \lambda(x) \,, \tag{42}$$

that is the quantities $q^{ij} \frac{\partial \tilde{S}}{\partial x^j} \lambda(x)$ are the components of the analytical function of the variable P_n.

Let us show that one and the same pair {World function; congruence of the world lines} can be realized in various Finsler geometries.

We introduce the notation

$$g^{ij}(x) = \left[\frac{1}{\kappa(x) \cdot \mu \cdot cm} \right]^{n-2} \overset{o}{g}{}^{ijj_3 \dots j_n} \frac{\partial \tilde{S}}{\partial x^{j_3}} \dots \frac{\partial \tilde{S}}{\partial x^{j_n}} \,. \tag{43}$$

Let $det(g^{ij}(x)) \neq 0$, then we can construct the twice covariant tensor $g_{ij}(x)$. Let us regard the pseudo Riemannian geometry with the length element

$$ds' = \kappa(x) \cdot \mu \cdot cm \sqrt{g_{ij} dx^i dx^j} \,. \tag{44}$$

The tangent equation of the indicatrix in such a geometry is

$$g^{ij} p_i p_j = \kappa(x)^2 \cdot \mu^2 \cdot (cm)^2 \,, \tag{45}$$

and the Hamilton-Jacoby equation for the action $S'(x)$ is

$$g^{ij}\frac{\partial S'}{\partial x^i}\frac{\partial S'}{\partial x^j} = \kappa(x)^2 \cdot \mu^2 \cdot (cm)^2 \,. \tag{46}$$

Substitute the expression (43) into this equation and get

$$\overset{o}{g}{}^{j_1 j_2 j_3 \ldots j_n}\ \frac{\partial S'}{\partial x^{j_1}}\frac{\partial S'}{\partial x^{j_2}}\frac{\partial \tilde{S}}{\partial x^{j_3}}\ldots\frac{\partial \tilde{S}}{\partial x^{j_n}} = \kappa(x)^n \cdot \mu^n (mc)^n \,. \tag{47}$$

Thus, we see that the function $S' = \tilde{S}$ is the solution of the equation (46), that is function $\tilde{S}$ remains the World function in the geometry (44).

The field of velocities in the geometry (44) is defined by the formula

$$\dot{x}^i = g^{ij}\frac{\partial \tilde{S}}{\partial x^j}\cdot \lambda'(x)\,, \tag{48}$$

where $\lambda'(x) > 0$ is a scalar function. Substitute the expression (43) into this equation

$$\lambda'(x) = \kappa(x)^{n-2} \cdot \mu^{n-2} \cdot (cm)^{n-2} \cdot \lambda(x)^{n-1} \tag{49}$$

and get the formula

$$\dot{x}^i = \overset{o}{g}{}^{i j_2 \ldots j_n}\ \frac{\partial \tilde{S}}{\partial x^{j_2}}\ldots\frac{\partial \tilde{S}}{\partial x^{j_n}} \cdot \lambda(x)^{n-1}, \tag{50}$$

which coincides with the formula (35).

So, one and the same pair {World function; congruence of the world lines} can be realized in the qualitatively different geometries.

One can use metric tensor $g_{i_1 i_2 \ldots i_m}(x^{i_1} x^{i_2} \ldots x^{i_n})$ to obtain metric tensor with less number of indices, $r < m$. To do this one should contract some indices with vector or tensor contravariant fields (see, for example, [3] – [6]). The speculations given above show that the best method of contraction for polynumber spaces P_n is the following:

$$g_{i_1 i_2 \ldots i_r}(x^{i_1} x^{i_2} \ldots x^{i_n}) = a(x) \cdot g_{i_1 i_2 \ldots i_m}(x^{i_1} x^{i_2} \ldots x^{i_n}) f_{(1)}^{i_{r+1}} f_{(2)}^{i_{r+2}} \ldots f_{(m-r)}^{i_m},$$

where $a(x)$ is some scalar function and $f^i_{(A)}$ are the components of the analytical functions of variable P_n or the components of some conjugated functions to the analytical functions of the same variable.

3 Hypercomplex numbers H_4

Notice, that the system of hypercomplex numbers H_4 is isomorphic to the algebra of real square diagonal matrices 4×4. The corresponding coordinate space is the metric Finsler space with Berwald-Moor metric. In H_4 there is a special basis, e_1, e_2, e_3, e_4, with the following multiplication rule

$$e_i e_j = p_{ij}^k e_k \,, \qquad p_{ij}^k = \begin{cases} 1\,, & i = j = k\,, \\ 0\,, & \end{cases} \tag{51}$$

The components of tensors q_{ij} (37), q^{ij} in this basis give a unity matrix:

$$(q_{ij}) = (q_{ij}) = diag(1,1,1,1)\,. \tag{52}$$

The length element in the H_4 space in the special basis (51) is

$$ds = mc\sqrt[4]{dx^1 dx^2 dx^3 dx^4} \equiv mc\sqrt[4]{\overset{o}{g}_{ijkm}\, dx^i dx^j dx^k dx^m}\,, \tag{53}$$

where

$$\overset{o}{g}_{ijkm} = \begin{cases} \frac{1}{24}\,, & \text{for all different} \quad i,j,k,m \\ 0\,, & \text{else.} \end{cases} \tag{54}$$

The components of the generalized momentum are defined by the formula

$$p_i = \frac{mc}{4} \cdot \frac{\sqrt[4]{dx^1 dx^2 dx^3 dx^4}}{dx^i}\,, \tag{55}$$

and the tangent equation of the indicatrix is

$$p_1 p_2 p_3 p_4 = \left(\frac{mc}{4}\right)^4\,, \tag{56}$$

or in the covariant form

$$\overset{o}{g}{}^{ijkm}\, p_i p_j p_k p_m = \left(\frac{mc}{4}\right)^4\,, \tag{57}$$

For the special basis used above, we have

$$\left(\overset{o}{g}{}^{ijkm}\right) = \left(\overset{o}{g}_{ijkm}\right). \tag{58}$$

Action as a function of coordinates in the H_4 space suffices the equation

$$\overset{o}{g}{}^{ijkm}\, \frac{\partial\tilde{S}}{\partial x^i}\frac{\partial\tilde{S}}{\partial x^j}\frac{\partial\tilde{S}}{\partial x^k}\frac{\partial\tilde{S}}{\partial x^m} = \left(\frac{mc}{4}\right)^4\,, \tag{59}$$

or

$$\frac{\partial\tilde{S}}{\partial x^1}\frac{\partial\tilde{S}}{\partial x^2}\frac{\partial\tilde{S}}{\partial x^3}\frac{\partial\tilde{S}}{\partial x^4} = \left(\frac{mc}{4}\right)^4\,. \tag{60}$$

Substitute into (60) some World function, $\tilde{S}(x)$, that suffices the only condition

$$\frac{\partial\tilde{S}}{\partial x^1}\frac{\partial\tilde{S}}{\partial x^2}\frac{\partial\tilde{S}}{\partial x^3}\frac{\partial\tilde{S}}{\partial x^4} > 0\,, \tag{61}$$

and get

$$\frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^3}\frac{\partial \tilde{S}}{\partial x^4} = \kappa(x)^4 \cdot \left(\frac{mc}{4}\right)^4 , \tag{62}$$

This means that the function, $\tilde{S}(x)$, is a World function in the geometry which is conformly connected to the Berwald-Moor geometry (53), which is a geometry with the length element

$$ds = \kappa(x) \cdot mc\sqrt[4]{dx^1 dx^2 dx^3 dx^4} , \tag{63}$$

The extension-contraction coefficient is given by

$$\kappa(x) = \frac{4}{mc}\sqrt[4]{\frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^3}\frac{\partial \tilde{S}}{\partial x^4}} . \tag{64}$$

In this geometry the field of velocities defining the congruence of the world lines is

$$\dot{x}^i = \frac{\frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^3}\frac{\partial \tilde{S}}{\partial x^4}}{\frac{\partial \tilde{S}}{\partial x^i}} \cdot \lambda(x)^3 , \tag{65}$$

where $\lambda(x) > 0$ is a scalar function.

Hypothesis I_{H_4}: Components of the vector field that produces the world lines corresponding to the given World function, are the components of the analytical function of the variable H_4.

In the special basis in question an arbitrary analytical function of the variable H_4 has the form

$$F(X) = f^1(x^1)e_1 + f^2(x^2)e_2 + f^3(x^3)e_3 + f^4(x^4)e_4 , \tag{66}$$

where f^i are the arbitrary functions of a single real variable. That is why the Hypothesis I_{H_4} leads to the demand

$$f^i(x^{i-}) = \frac{\frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^3}\frac{\partial \tilde{S}}{\partial x^4}}{\frac{\partial \tilde{S}}{\partial x^i}} \cdot \lambda(x)^3 . \tag{67}$$

Multipliing the expressions (67) with different indices and performing some transformations, one gets

$$\frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^3}\frac{\partial \tilde{S}}{\partial x^4} = \frac{\sqrt[3]{f^1 f^2 f^3 f^4}}{\lambda^4} \tag{68}$$

and

$$\frac{\partial \tilde{S}}{\partial x^i} = \frac{\sqrt[3]{f^1 f^2 f^3 f^4}}{\lambda f^i} . \tag{69}$$

Using the commutativity of the partial derivatives

$$\frac{\partial}{\partial x^j}\frac{\partial \tilde{S}}{\partial x^i} = \frac{\partial}{\partial x^i}\frac{\partial \tilde{S}}{\partial x^j} \tag{70}$$

we get the system of six differential equations for $\lambda(x)$. Writing down one of them for $i = 1$, $j = 2$, one gets:

$$3(f^1)^2\frac{\partial\lambda}{\partial x^1} - 3(f^2)^2\frac{\partial\lambda}{\partial x^2} = \lambda(f^1 - f^2)\,. \tag{71}$$

If $\lambda = const$, then $f^i = f^j = const$, which means that $\tilde{S}$ is a following linear function of coordinates:

$$\tilde{S} = a\left(x^1 + x^2 + x^3 + x^4\right) + b\,, \tag{72}$$

where a, b are constants.

If $\lambda \neq const$ and $f^i \neq 0$, then we introduce the following notation for the indefinite integrals

$$I^i = \int^{x^i} \frac{dx^{i-}}{(f^{i-})^2}\,, \qquad J^i = \int^{x^i} \frac{dx^{i-}}{3f^{i-}}\,, \tag{73}$$

and the equation (71) and its analogues give

$$\lambda(x^1,x^2,x^3,x^4) = \exp\left(W(I^1+I^2+I^3+I^4) + J^1+J^2+J^3+J^4\right)\,, \tag{74}$$

where W is an arbitrary function of a single real variable. The World function, $\tilde{S}$, can be obtained with the help of a line integral of the second kind for an arbitrary path in the H_4 space. This path connects the fixed point with the point $M(x^1,x^2,x^3,x^4)$.

The expressions (69), (73) and (74) mean that the derivatives $\frac{\partial\tilde{S}}{\partial x^i}$ are not the components of the analytical function of the variables H_4 or their linear combinations. The only exception takes place when all these derivarives are equal and equal to a constant a (72). The same can be stated for the function, $\tilde{S}$, if we exclude the linear dependence (72). But for every analytical function, $F(X)$, with $f^i \neq 0$, there is a corresponding World function, $\tilde{S}$, that can be expressed with the help of the squares of the components of $F(X)$, while the corresponding field of velocities defining the world lines is an analytical function, $F(X)$, of variables H_4.

Hypothesis II_{H_4}: *Components of the vector field that produces the world lines corresponding to the given World function, are the components of the function of the variable* H_4 *symmetrically conjugate to the analytical function of the same variable.*

According to (52), (54), (58) the symmetrical conjugation (40) in the H_4 space coincide with the normal conjugation [9], and in the mentioned special basis the expression (40) becomes

$$y^i = \frac{x^1x^2x^3x^4}{x^i}\,. \tag{75}$$

Taking into acount this formula and the expression (65) as a consequence of the Hypothesis II, one gets

$$\frac{f^1 f^2 f^3 f^4}{f^i} = \frac{\frac{\partial\tilde{S}}{\partial x^1}\frac{\partial\tilde{S}}{\partial x^2}\frac{\partial\tilde{S}}{\partial x^3}\frac{\partial\tilde{S}}{\partial x^4}}{\frac{\partial\tilde{S}}{\partial x^i}} \cdot \lambda(x)^3\,, \tag{76}$$

or

$$f^i(x^{i-}) = \frac{\partial\tilde{S}}{\partial x^i} \cdot \lambda(x)\,. \tag{77}$$

If $\lambda = const$, then

$$\tilde{S} = \frac{1}{4}\left(\tilde{f}^1(x^{1-}) + \tilde{f}^2(x^{2-}) + \tilde{f}^3(x^{3-}) + \tilde{f}^4(x^{4-})\right)\,, \tag{78}$$

where $\tilde{f}^i(x^{i-})$ is a function of a single real variable x^{i-}. Within the accuracy of a number factor, these functions are the primitives of the components $f^i(x^{i-})$ of the initial analytical functions $F(X)$. The properties of the polynumbers H_4 provide the formal coincidence of the scalar function, $\tilde{S}$, (78), with the component of the analytical function

$$\tilde{F}(X) = \tilde{f}^i(x^{i-})e_i \tag{79}$$

for the unity element in the basis $1, j, k, jk$; $j^2 = k^2 = (jk)^2 = 1$:

$$\left.\begin{aligned} 1 &= e_1 + e_2 + e_3 + e_4\,, & j &= e_1 + e_2 - e_3 - e_4\,,\\ k &= e_1 - e_2 + e_3 - e_4\,, & jk &= e_1 - e_2 - e_3 + e_4\,. \end{aligned}\right\} \tag{80}$$

Let $\lambda \neq const$, then the expression (77) gives the system of six equations to define function $\lambda(x)$:

$$f^i\frac{\partial\lambda}{\partial x^j} = f^j\frac{\partial\lambda}{\partial x^i} \tag{81}$$

The general solution of this system is

$$\lambda(x) = \Lambda\left(\tilde{f}^1(x^{1-}) + \tilde{f}^2(x^{2-}) + \tilde{f}^3(x^{3-}) + \tilde{f}^4(x^{4-})\right)\,, \tag{82}$$

where Λ is a function of a single real variable, and $\tilde{f}^i(x^{i-})$ are the primitives of the components $f^i(x^{i-})$ of the initial analytical function $F(X)$.

The World function $\tilde{S}$ can be obtained with the help of a line integral of the second kind for an arbitrary path in the H_4 space. This path connects the fixed point with the point $M(x^1, x^2, x^3, x^4)$.

In general case, the derivatives $\frac{\partial\tilde{S}}{\partial x^i}$ are not the components of the analytical function of the variable H_4 or their linear combinations. The same can be stated for function $\tilde{S}$. But for every analytical function $F(X)$ there is a corresponding World function, $\tilde{S}$, that can be expressed with the help of the squares of the components of $F(X)$, while the corresponding field of velocities defining the world lines is symmetrically conjugate to the analytical function $F(X)$ of variables H_4.

Let us suggest that we know the World function in the space (63) which is conformly connected to the Berwald-Moor space. Let us regard tensor

$$g^{ij}(x) = \frac{1}{\kappa(x)^2 \cdot \mu^2 \cdot (mc)^2} \overset{o}{g}{}^{ijkm} \frac{\partial \tilde{S}}{\partial x^k} \frac{\partial \tilde{S}}{\partial x^m}, \tag{83}$$

in which $\mu = 1/4$ according to (57). Let $det(g^{ij}(x)) \neq 0$, then in the same coordinate space, x^1, x^2, x^3, x^4, one can define the pseudo Riemannian geometry with the length element

$$ds' = \kappa(x) \cdot \mu \cdot mc\sqrt{g_{ij}dx^i dx^j} \tag{84}$$

and the tangent equation for the indicatrix

$$g^{ij}p'_i p'_j = \kappa(x)^2 \cdot \mu^2 \cdot (mc)^2 . \tag{85}$$

The Hamilton-Jacoby equation for the action, S', is

$$g^{ij}\frac{\partial S'}{\partial x^i}\frac{\partial S'}{\partial x^j} = \kappa(x)^2 \cdot \mu^2 \cdot (mc)^2 , \tag{86}$$

and the field of velocities defining the congruence of the world lines has the form

$$\dot{x}^i = g^{ij}\frac{\partial S'}{\partial x^j}\lambda'(x) , \tag{87}$$

where $\lambda'(x)$ is a scalar function. Substituting the expression for g^{ij} (83) into the last two formulas, one can see that the solution of the equation (86) is the World function $S' = \tilde{S}$, and the congruences of the world lines in the spaces (63) and (84) coincide.

Let us regard tensor

$$G^{ij}(x) = \overset{o}{g}{}^{ijkm} \frac{\partial \tilde{S}}{\partial x^k} \frac{\partial \tilde{S}}{\partial x^m}, \tag{88}$$

which coincides with tensor g^{ij} (83) within the accuracy of a number factor. In the matrix form

$$\left(G^{ij}(x)\right) = \frac{1}{12}\begin{pmatrix} 0 & \frac{\partial \tilde{S}}{\partial x^3}\frac{\partial \tilde{S}}{\partial x^4} & \frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^4} & \frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^3} \\ \frac{\partial \tilde{S}}{\partial x^3}\frac{\partial \tilde{S}}{\partial x^4} & 0 & \frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^4} & \frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^3} \\ \frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^4} & \frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^4} & 0 & \frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^2} \\ \frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^3} & \frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^3} & \frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^2} & 0 \end{pmatrix} . \tag{89}$$

Since

$$det\left(G^{ij}\right) = -\frac{3}{12^4}\left(\frac{\partial \tilde{S}}{\partial x^1}\frac{\partial \tilde{S}}{\partial x^2}\frac{\partial \tilde{S}}{\partial x^3}\frac{\partial \tilde{S}}{\partial x^4}\right)^2 \neq 0 , \tag{90}$$

due to the inequality (61), one can construct tensor G_{ij}, and, thus, construct tensor g_{ij}.

The basis, e_1, e_2, e_3, e_4, used in this Section is not the physical basis commonly used. So, let us pass to the basis (80), though not for the general case but for the simplest World function

$$\tilde{S} = \frac{1}{4}\left(x^1 + x^2 + x^3 + x^4\right) + const\,, \tag{91}$$

which in the basis (80) has the form

$$\tilde{S} = x^0 + const\,, \tag{92}$$

where x^0 is the coordinate of the unity element in the basis (80). In this case matrix (G^{ij}) takes the form

$$\left(G^{ij}(x)\right) = \frac{1}{12\cdot 4^2}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{93}$$

To obtain matrix (G^{ij}) of tensor G^{ij} in the new basis (80), that is matrix $\left(G^{i'j'}\right)$, one should multiply the matrix (G^{ij}) (with regard to the fact that the transition matrix is symmetrical) from the left and from the right by the matrix reverse to the transition matrix. The result is

$$\left(G^{i'j'}(x)\right) = \frac{1}{4^4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{94}$$

Thus, the World function (91) in space H_4 corresponds to the pseudo Euclidean geometry with the signature $(1, -1, -1, -1)$.

Conclusion

All the above said means that the relation between the World function, $\tilde{S}$, defined in a polynumber space P_n, and the analytical functions of the variable P_n can be postulated in various forms.

The most strong limitations on the form of the World function, $\tilde{S}$, are given by *Hypothesis I*: *Components of the vector field that produces the world lines corresponding to the given World function, are the components of the analytical function of the variable P_n.*

Less strong though strong enough limitations on the form of the World function, $\tilde{S}$, are given by *Hypothesis II*: *Components of the vector field that produces the* *world lines corresponding to the given World function, are the components of the function of the variable P_n symetrically conjugate to the analytical function of the same variable.*

It seems that *Hypothesis II* is more closely linked to Physics.

Although the approach used to describe the World with the help of a World function demands some operation of the "index rising" for the covariant tensors (and this operation can be always realized for a fixed geometry), the all-sufficient pair {World function; congruence of the world lines} can correspond to qualitatively different geometries.

In this paper it is shown that Finsler space H_4 with the Berwald-Moor metric corresponds to the Minkowski space.

Finally, regarding the physical World as the congruence of the world lines in the four dimensional space-time, we conclude that the geometry is not a fixed notion. One can pass from one geometry to another depending on the problems of interest, and with this not only the congruence of the world lines, i.e. World itself, will be conserved, but the World function also.

Thus Minkowskian space and polynumber space H_4 correspond to the same physical World.

References

[1] H. Weyl, *Space, Time, Matter,* Dover, 1922.

[2] P. K. Rashevsky, *Riemann Geometry and Tensor Analysis,* Science, M., 1967 (rus).

[3] M. Matsumoto, S. Numata, *On Finsler spaces with 1-form metric,* Tensor, N. S., Vol. 32 (1978), 161.

[4] M. Matsumoto, S. Numata, *On Finsler spaces with 1-form metric II. Berwald-Moor's metric.,* Tensor, N. S., Vol. 32 (1978), 275.

[5] M. Matsumoto, S. Numata, *On Finsler spaces with a cubic metric,* Tensor, N. S., Vol. 33 (1979), 153.

[6] H. Shimada, *On Finsler spaces with the metric L of m-th root metric,* Tensor, N. S., Vol. 33 (1979), 365–372.

[7] D. G. Pavlov, *Generalization of scalar product axioms,* Hypercomplex Numbers in Geometry and Physics, Vol. 1 (2004), 5–19.

[8] G. I. Garas'ko. *Generalized analytical functions of the polynumber variable,* Hypercomplex Numbers in Geometry and Physics, Vol. 1 (2004), 75–88.

[9] D. G. Pavlov, G. I. Garas'ko, *Normal conjugation on the polynumber manifold,* Hypercomplex Numbers in Geometry and Physics, Vol. 2 (2004), 6–14.

Construction of the Pseudo-Riemannian geometry on the Base of the Berwald-Moor Geometry

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A way to construct the metric tensor of a 4-dimensional pseudo-Riemannian space (Space-Time) is suggested, emerging from the 4-contravariant tensor of the tangent indicatrix equation of the Berwald-Moor space and the World function. The Berwald-Moor space appears to be closely related to the Minkowski space. The break of the analyticity of the World function leads to the non-trivial curving of the 4-dimensional Space-Time and, particularly, to the Newtonian potential in the nonrelativistic limit. So, one remarks that the algebra of commutative and associative hypercomplex numbers, denoted by H_4, and the corresponding Finsler geometry can be used as a mathematical model of the real Space-Time. This model is conjectured to be more productive than the pseudo-Riemannian constructions prevailing in Physics now.

MSC: 53B40, 30G35, 83D05.

1 Introduction

The fascinating beauty of the theory of the functions of complex variable reveals itself, for example, in the harmony of the algebraic fractals on the Euclidian plane. It makes many researches look for the analogous number systems, the elements of which could be correlated not to the points on the plane but to the points of the 4-dimensional space-time. In case of the success of such a search, we could really trust the famous Pythagoras saying 'all the existing is number'. On this way, the interesting results were obtained for quaternions [1], biquaternions [2-4], octaves [5] and so forth. Nevertheless, none of these number system theories can be compared even to the theory of the relatively simple 2-component complex numbers. The main reason for this seems to be the lack of the commutativity (and sometimes even of the associativity) of the multiplication in these algebras. Although the authors of this paper realize the conceptual bases of all the variety of algebras, the commutativity of the multiplication is the integral property of all the principal number systems that contain natural, integer, rational, real and complex numbers. Finally, the commutativity and the associativity of the multiplication are among the axioms of arithmetic which presents the foundation of mathematics, and it would be strange if the algebraic system which is the most natural for our real world does not correspond to the rules of regular counting.

One of the systems free from this drawback is the algebra of the commutative and associative hyper complex numbers, related to the direct sum of the four real algebras, which will be denoted as H_4. The algebra of these numbers is isomorphic to the algebra of the 4-dimensional square real diagonal matrices, and the corresponding space is a linear Finsler space with the Berwald-Moor metric (the last fact was proved by the authors in [6]). It should be mentioned that Finsler space with the Berwald-Moor metric has been known and partially investigated for a long time [7–8].

One of the main properties of this space is the existence of such a range of the parameters that the 3-dimensional distances (from the point of view of the observer who uses the radar method to measure them [9]) correspond to the positively defined metric function the limit of which is the quadratic form [10]. In other words, the 3-dimensional world observed by an "H_4 inhabitant" is Euclidian within certain accuracy. Moreover, when one passes to the relativistic velocities, the 4-dimensional intervals between the H_4 events present the Minkowski space correlations [11]. All this makes possible to suggest that the H_4 space and the corresponding Finsler geometry can be used as a mathematical model of the real space-time, and maybe this model would be even more productive than the pseudo Riemannian constructions prevailing in Physics now.

Any hypercomplex algebra is completely defined by the multiplication rule for the elements of a certain fixed basis. In the H_4 number system there is a special – isotropic – basis e_1, e_2, e_3, e_4, such that

$$e_i e_j = p_{ij}^k e_k \qquad p_{ij}^k = \begin{cases} 1\,, & \text{if } \ i = j = k\,, \\ 0\,, & \text{else}\,. \end{cases} \tag{1}$$

Any analytical function in this basis can be given as

$$F(X) = f^1(\xi^1)e_1 + f^2(\xi^2)e_2 + f^3(\xi^3)e_3 + f^4(\xi^4)e_4\,, \tag{2}$$

where

$$H_4 \ni X = \xi^1 e_1 + \xi^2 e_2 + \xi^3 e_3 + \xi^4 e_4\,, \tag{3}$$

and f^i are four arbitrary smooth functions of a single real variable.

In H_4 there is one more – orthogonal – selected basis 1, j, k, jk, which is related to the isotropic basis by the following formulas

$$\left.\begin{aligned} 1 &= e_1 + e_2 + e_3 + e_4\,, \\ j &= e_1 + e_2 - e_3 - e_4\,, \\ k &= e_1 - e_2 + e_3 - e_4\,, \\ jk &= e_1 - e_2 - e_3 + e_4\,, \end{aligned}\right\} \tag{4}$$

where *1* is the unity of algebra, and the corresponding component of the analytical function of the H_4 variable is defined by the formula

$$u = \frac{1}{4}\left[f^1(\xi^1) + f^2(\xi^2) + f^3(\xi^3) + f^4(\xi^4)\right] . \tag{5}$$

If X is a radius vector, then the coordinate space ξ^1, ξ^2, ξ^3, ξ^4 is a Berwald-Moor space with the length element

$$ds = \sqrt[4]{d\xi^1 d\xi^2 d\xi^3 d\xi^4} \equiv \sqrt[4]{g_{ijkl} d\xi^i d\xi^j d\xi^k d\xi^l}\,, \tag{6}$$

where

$$g_{ijkl} = \begin{cases} \frac{1}{4!}\,, & (i \neq j \neq k \neq l), \\ 0\,, & (else). \end{cases} \tag{7}$$

For this geometry the tangent indicatrix equation is

$$g^{ijkl} p_i p_j p_k p_l - 1 = 0\,, \tag{8}$$

where

$$g^{ijkl} = \begin{cases} \dfrac{4^4}{4!}\,, & (i \neq j \neq k \neq l), \\ 0\,, & (\text{else}), \end{cases} \tag{9}$$

$$p_i = \frac{g_{ijkl} d\xi^j d\xi^k d\xi^l}{(g_{mrst} d\xi^m d\xi^r d\xi^s d\xi^t)^{3/4}} \tag{10}$$

are the components of the generalized momentum or generalized momenta.

If we have tensors p^k_{ij}, g_{ijkl}, g^{ijkl} and vector fields of the analytical functions $F_{(A)}(X)$ of the H_4 variables, we could construct the metric tensors in the 4-dimensional space-time in many ways. For example,

$$g_{ij}(\xi) = g_{ijkl} f^k_{(1)} f^l_{(2)}\,, \tag{11}$$

Now one can investigate the obtained Riemannian geometry. The main drawback of this approach is the variety of the ways to construct it.

It is known [12] that if the tangent indicatrix equation is defined as

$$\Phi(p;\xi) = 0\,, \tag{12}$$

then the geodesics will be the solutions of the canonical system of differential equations

$$\dot{\xi}^i = \frac{\partial \Phi}{\partial p_i} \cdot \lambda(p;\xi)\,, \qquad \dot{p}_i = -\frac{\partial \Phi}{\partial \xi^i} \cdot \lambda(p;\xi)\,, \tag{13}$$

$\lambda(p;\xi) \neq 0$ is an arbitrary smooth function, and a dot above ξ^i and p_i means the derivation by the evolution parameter, τ.

2 Construction of the metric function of the pseudo-Riemannian space

Let us regard a space which is conformally connected to the H_4 space, that is to the space with the length element

$$ds' = \kappa(\xi) \cdot \sqrt[4]{g_{ijkl} d\xi^i d\xi^j d\xi^k d\xi^l}\,, \tag{14}$$

where $\kappa(\xi) > 0$ is a scalar function which is a contraction-extension coefficient depending on the point.

Let there be a normal congruence of geodesics (world lines). Then there is a scalar function $S(\xi)$ (see, e.g. [12]) such that its level hyper surfaces are transversal to this normal congruence of the world lines and this function is a solution of the equation

$$g^{ijkl} \frac{\partial S}{\partial \xi^i} \frac{\partial S}{\partial \xi^j} \frac{\partial S}{\partial \xi^k} \frac{\partial S}{\partial \xi^l} = \kappa(\xi)^4\,, \tag{15}$$

while the generalized momenta along this congruence of the world lines are related to $S(\xi)$ by

$$p_i = \frac{\partial S}{\partial \xi^i}\,, \tag{16}$$

The equations for the world lines obtain the form

$$\dot{\xi}^i = g^{ijkl} \frac{\partial S}{\partial \xi^j} \frac{\partial S}{\partial \xi^k} \frac{\partial S}{\partial \xi^l} \cdot \lambda(\xi)\,, \tag{17}$$

were $\lambda(\xi) \neq 0$.

In Physics the function $S(\xi)$ is called "action as a function of coordinates" and (15) is known as the Hamilton-Jacoby equation. In [10] the function $S(\xi)$ was called the *World function.*

If there is a congruence of the world lines, then the evolution of every point in space is known, particularly, the velocity field is known, but the energy characteristics of the material objects (observers) corresponding to a given world line are not known. The knowledge of the World function $S(\xi)$ makes it possible to calculate the generalized momenta p_i, corresponding to the energy characteristics, and the invariant energy characteristic, $\kappa(\xi)$, which has also the meaning of the local contraction-extension coefficient of the plane H_4 space.

So, if our world view is the classical mechanics, then any pair out of the three: World function, congruence of the world lines, Finsler geometry – gives us the complete knowledge of the World.

Let us construct a twice contravariant tensor *gij(?)* in the following way:

$$g^{ij}(\xi) = \frac{1}{\kappa(\xi)^4} \cdot g^{ijkl} \frac{\partial S}{\partial \xi^k} \frac{\partial S}{\partial \xi^l}\,. \tag{18}$$

Since

$$det(g^{ij}(\xi)) = -\frac{4^4}{3^3\kappa(\xi)^8} \neq 0\,, \tag{19}$$

then everywhere where the geometry (14) is defined, one can construct a tensor $g_{ij}(\xi)$ such that

$$g^{ik}(\xi)g_{kj}(\xi) = \delta^i_j\,, \tag{20}$$

$$g_{ij}(\xi) = 4\cdot\begin{pmatrix} -2\left(\frac{\partial S}{\partial\xi^1}\right)^2 & \frac{\partial S}{\partial\xi^1}\frac{\partial S}{\partial\xi^2} & \frac{\partial S}{\partial\xi^1}\frac{\partial S}{\partial\xi^3} & \frac{\partial S}{\partial\xi^1}\frac{\partial S}{\partial\xi^4} \\ \frac{\partial S}{\partial\xi^1}\frac{\partial S}{\partial\xi^2} & -2\left(\frac{\partial S}{\partial\xi^2}\right)^2 & \frac{\partial S}{\partial\xi^2}\frac{\partial S}{\partial\xi^3} & \frac{\partial S}{\partial\xi^2}\frac{\partial S}{\partial\xi^4} \\ \frac{\partial S}{\partial\xi^1}\frac{\partial S}{\partial\xi^3} & \frac{\partial S}{\partial\xi^2}\frac{\partial S}{\partial\xi^3} & -2\left(\frac{\partial S}{\partial\xi^3}\right)^2 & \frac{\partial S}{\partial\xi^3}\frac{\partial S}{\partial\xi^4} \\ \frac{\partial S}{\partial\xi^1}\frac{\partial S}{\partial\xi^4} & \frac{\partial S}{\partial\xi^2}\frac{\partial S}{\partial\xi^4} & \frac{\partial S}{\partial\xi^3}\frac{\partial S}{\partial\xi^4} & -2\left(\frac{\partial S}{\partial\xi^4}\right)^2 \end{pmatrix}. \tag{21}$$

No doubt that in the same coordinate space $\xi^1, \xi^2, \xi^3, \xi^4$ such tensor $g_{ij}(\xi)$ defines a Riemannian or pseudo Riemannian geometry with the length element

$$ds'' = \sqrt{g_{ij}(\xi)d\xi^i d\xi^j}\,. \tag{22}$$

The construction of tensor $g_{ij}(\xi)$ leads directly to the conclusion: the change of geometry (14) to the geometry (22) does not lead to the change of the initial congruence of the world lines and corresponding World function $S(\xi)$.

Therefore, in our concept one and the same World, i.e. the pair {World function; congruence of the world lines}, corresponds to a whole class of related but qualitatively different Finsler geometries.

3 Analyticity condition and the Minkowski space

Let the World function $S(\xi)$ be the (unity) component of an analytical function of the H_4 variable in the orthogonal basis (4), that is

$$S(\xi) = \frac{1}{4}\left[f^1(\xi^1) + f^2(\xi^2) + f^3(\xi^3) + f^4(\xi^4)\right]\,. \tag{23}$$

Then

$$g^{ijkl}\frac{\partial S}{\partial\xi^i}\frac{\partial S}{\partial\xi^j}\frac{\partial S}{\partial\xi^k}\frac{\partial S}{\partial\xi^l} = \frac{\partial f^1(\xi^1)}{\partial\xi^1}\frac{\partial f^2(\xi^2)}{\partial\xi^2}\frac{\partial f^3(\xi^3)}{\partial\xi^3}\frac{\partial f^4(\xi^4)}{\partial\xi^4} = \kappa(\xi)^4 > 0\,, \tag{24}$$

and this leads to the limitation on the functions, *fi*:

$$\frac{\partial f^1(\xi^1)}{\partial\xi^1}\frac{\partial f^2(\xi^2)}{\partial\xi^2}\frac{\partial f^3(\xi^3)}{\partial\xi^3}\frac{\partial f^4(\xi^4)}{\partial\xi^4} > 0\,. \tag{25}$$

It follows from (24) that the space with the length element (14) can be obtained from the space with the length element (6) with the help of the conformal transformation, which means that the condition of the analyticity of the World function can be treated in a sense as the condition of the conformal symmetry.

Let us construct tensor $g_{ij}(\xi)$ following the algorithm developed in the previous section. It turns out that in a region where functions f^i have no singularities there will always be such a coordinate system x^0, x^1, x^2, x^3 in which the length element ds'' has a form

$$ds'' = \sqrt{(x^0)^2 - (x^1)^2 - (x^3)^2 - (x^3)^2}\,. \tag{26}$$

Let us express the coordinates x^0, x^1, x^2, x^3 in terms of the initial coordinates $\xi^1, \xi^2, \xi^3, \xi^4$:

$$\left.\begin{aligned}
x^0 &= \frac{1}{4}\left(f^1(\xi^1) + f^2(\xi^2) + f^3(\xi^3) + f^4(\xi^4)\right)\,,\\
x^1 &= \frac{\sqrt{3}}{4}\left(f^1(\xi^1) + f^2(\xi^2) - f^3(\xi^3) - f^4(\xi^4)\right)\,,\\
x^2 &= \frac{\sqrt{3}}{4}\left(f^1(\xi^1) - f^2(\xi^2) + f^3(\xi^3) - f^4(\xi^4)\right)\,,\\
x^3 &= \frac{\sqrt{3}}{4}\left(f^1(\xi^1) - f^2(\xi^2) - f^3(\xi^3) + f^4(\xi^4)\right)\,.
\end{aligned}\right\} \tag{27}$$

Therefore, to obtain the non-trivial curving of the space-time one should use the World functions with the broken conformal symmetry.

4 Newtonian potential

Let us show that there are World functions that lead to the non-trivial pseudo Riemannian 4-dimensional spaces. Let us regard a function

$$S(\xi) = \frac{1}{4}\left(\xi^1 + \xi^2 + \xi^3 + \xi^4\right) + \alpha \cdot \psi(\varrho)\,, \tag{28}$$

where α is the parameter characterizing the break of the analyticity of the World function (the break of the conformal symmetry in the H_4 space), ψ is an arbitrary function of a single argument

$$\varrho = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}\,, \tag{29}$$

and y^0, y^1, y^2, y^3 are the coordinates in the orthogonal basis $1, j, k, jk$:

$$\left.\begin{aligned}
y^0 &= \frac{1}{4}(\xi^1 + \xi^2 + \xi^3 + \xi^4)\,,\\
y^1 &= \frac{1}{4}(\xi^1 + \xi^2 - \xi^3 - \xi^4)\,,\\
y^2 &= \frac{1}{4}(\xi^1 - \xi^2 + \xi^3 - \xi^4)\,,\\
y^3 &= \frac{1}{4}(\xi^1 - \xi^2 - \xi^3 + \xi^4)\,.
\end{aligned}\right\} \tag{30}$$

Then the derivatives of the World functions over the coordinates ξ^i can be expressed in the following way:

$$\left.\begin{aligned} \frac{\partial S}{\partial \xi^1} &= \frac{1}{4}\left[1+\frac{\alpha}{\varrho}\frac{d\psi}{d\varrho}\left(y^1+y^2+y^3\right)\right],\\ \frac{\partial S}{\partial \xi^2} &= \frac{1}{4}\left[1+\frac{\alpha}{\varrho}\frac{d\psi}{d\varrho}\left(y^1-y^2-y^3\right)\right],\\ \frac{\partial S}{\partial \xi^3} &= \frac{1}{4}\left[1+\frac{\alpha}{\varrho}\frac{d\psi}{d\varrho}\left(-y^1+y^2-y^3\right)\right],\\ \frac{\partial S}{\partial \xi^4} &= \frac{1}{4}\left[1+\frac{\alpha}{\varrho}\frac{d\psi}{d\varrho}\left(-y^1-y^2+y^3\right)\right]. \end{aligned}\right\} \tag{31}$$

Let us calculate the components of the metric tensor in coordinates y^0, y^1, y^2, y^3 using the invariance of the square of the length element

$$g_{ij}(\xi)d\xi^i d\xi^j = \tilde{g}_{ij}(y)dy^i dy^j \tag{32}$$

Grouping the terms, one gets

$$\tilde{g}_{00} = 1-3\alpha^2\left(\frac{d\psi}{d\varrho}\right)^2, \quad \tilde{g}_{\beta\beta_-} = -3\left\{1+\alpha^2\left(\frac{d\psi}{d\varrho}\right)^2\left[1-\frac{4(y^\alpha)^2}{3\rho^2}\right]\right\}, \tag{33}$$

$$2\tilde{g_{0\beta}} = -4\left[\alpha\frac{d\psi}{d\varrho}\frac{y^\beta}{\varrho}+3\alpha^2\left(\frac{d\psi}{d\varrho}\right)^2\cdot\frac{y^1y^2y^3}{y^\beta\varrho^2}\right], \tag{34}$$

$$2\tilde{g}_{\beta\gamma} = -4\left[3\alpha\frac{d\psi}{d\varrho}\frac{y^\delta}{\varrho}+\alpha^2\left(\frac{d\psi}{d\varrho}\right)^2\cdot\frac{y^\beta y^\gamma}{\varrho^2}\right], \tag{35}$$

where β, γ, δ, $= 1,2,3$; $\beta \equiv \beta_-$ but no summation is performed here; in the last formula all the indices β, γ, δ are different.

If $\alpha = 0$, then

$$(\tilde{g}_{ij}) = diag(1,-3,-3,-3)\,. \tag{36}$$

This means that the real physical coordinates x^0, x^1, x^2, x^3 of the space-time are expressed by the coordinates y^0, y^1, y^2, y^3 in the following way

$$x^0 = y^0\,, \qquad x^\beta = \sqrt{3}\cdot y^\beta\,. \tag{37}$$

Let us pass to the physical coordinates x^0, x^1, x^2, x^3:

$$\tilde{g}_{ij}(y)dy^i dy^j = \bar{g}_{ij}(x)dx^i dx^j\,, \tag{38}$$

where

$$\bar{g}_{00} = \tilde{g}_{00}\,, \qquad \bar{g}_{0\beta} = \frac{1}{\sqrt{3}}\cdot\tilde{g}_{0\beta}\,, \qquad \bar{g}_{\beta\gamma} = \frac{1}{3}\cdot\tilde{g}_{\beta\gamma}\,. \tag{39}$$

Let us denote

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \equiv \sqrt{3} \cdot \varrho \,, \tag{40}$$

Then

$$\bar{g}_{00} = 1 - 9\alpha^2 \left(\frac{d\psi}{dr}\right)^2 , \quad \bar{g}_{\beta\beta_-} = - \left\{ 1 + 3\alpha^2 \left(\frac{d\psi}{dr}\right)^2 \left[1 - \frac{4(x^\alpha)^2}{3r^2} \right] \right\} , \tag{41}$$

$$2\bar{g_{0\beta}} = -4 \left[\alpha \frac{d\psi}{dr}\frac{x^\beta}{r} + 3\sqrt{3}\alpha^2 \left(\frac{d\psi}{dr}\right)^2 \cdot \frac{x^1 x^2 x^3}{x^\beta r^2} \right] , \tag{42}$$

$$2\bar{g}_{\beta\gamma} = -4 \left[\sqrt{3}\alpha \frac{d\psi}{dr}\frac{x^\delta}{r} + \alpha^2 \left(\frac{d\psi}{dr}\right)^2 \cdot \frac{x^\beta x^\gamma}{r^2} \right] . \tag{43}$$

The metric tensor $\bar{g}_{ij}(x) = \bar{g}_{ij}(x^1, x^2, x^3)$ depends only on the space coordinates x^1, x^2, x^3, and this corresponds to the stationary gravitational field, stationary Universe. The probe particle of mass m moves along the geodesic of the pseudo Riemannian space with metric tensor $\bar{g}_{ij}(x^1, x^2, x^3)$.

Let a particle move in a fixed frame and have velocity much less than the light velocity, c:

$$\frac{dx^\beta}{dt} = v^\beta \,, \qquad |v^\beta| \ll c \,, \tag{44}$$

The gravitational fields are weak, that is the condition $|v^\beta| << 1$ remains valid for all the time of the particle motion. Let us obtain the Lagrange function, L, to describe such non-relativistic motion of the probe particle in the weak gravity field. To do this, develop the right hand side of the expression

$$L = -mc \cdot \frac{\sqrt{\bar{g}_{ij}(x^1, x^2, x^3) dx^i dx^j}}{dt} \tag{45}$$

Within the accuracy of $\left(\frac{v}{c}\right)^2$

$$L = -mc^2 \sqrt{\bar{g}_{00}} \cdot \sqrt{1 + \frac{1}{\bar{g}_{00}} \left(2\bar{g}_{0\beta}\frac{v^\beta}{c} + \bar{g}_{\beta\gamma}\frac{v^\beta v^\gamma}{c^2} \right)} \,, \tag{46}$$

$$L \simeq -mc^2 \sqrt{\bar{g}_{00}} \cdot \left\{ 1 + \frac{1}{2\bar{g}_{00}} \left(2\bar{g}_{0\beta}\frac{v^\beta}{c} + \bar{g}_{\beta\gamma}\frac{v^\beta v^\gamma}{c^2} \right) - \frac{1}{8\bar{g}_{00}^2} \left(2\bar{g}_{0\beta}\frac{v^\beta}{c} \right)^2 \right\} . \tag{47}$$

Opening the brackets in the right hand side, we get an additive term which is the full time derivative of a certain function *f(r)*, it depends linearly on the velocity components and, thus, it can be omitted. Leaving the same designation for the Lagrange function, we get

$$L \simeq -mc^2 \sqrt{\bar{g}_{00}} \cdot \left\{ 1 + \frac{1}{2\bar{g}_{00}} \cdot \bar{g}_{\beta\gamma}\frac{v^\beta v^\gamma}{c^2} - \frac{1}{8\bar{g}_{00}^2} \left(2\bar{g}_{0\beta}\frac{v^\beta}{c} \right)^2 \right\} . \tag{48}$$

Our goal is the Lagrange function of the form

$$L = \frac{m\vec{v}^2}{2} - U(\vec{x})\,, \tag{49}$$

where $U(\vec{x})$ is the potential energy of the probe particle, $\vec{x} \equiv (x^1, x^2, x^3)$, $\vec{v} \equiv (v^1, v^2, v^3)$, $r^2 = \vec{x}^2$, $\vec{v}^2 = (v^1)^2 + (v^2)^2 + (v^3)^2 \equiv v^2$. To reach it we have to make some assumptions about the correlation between the parameter, α and light velocity:

$$\alpha = \frac{\nu}{c}\,, \quad \text{when} \quad c \to \infty \quad \alpha \to 0\,. \tag{50}$$

Besides, let α be of the same order (or smaller) with the relation $\left|\frac{v}{c}\right|$. Then leaving only the terms that don't disappear at $c \to \infty$ in the (48), one gets

$$L \simeq -mc^2 + mc^2 \frac{9}{2}\frac{\nu^2}{c^2}\left(\frac{d\psi}{dr}\right)^2 + m \cdot \frac{v^1v^1 + v^2v^2 + v^3v^3}{2}\,. \tag{51}$$

Since $(-mc^2)$ is a full time derivative of function $(-mc^2 \cdot t)$, we omit it and get

$$L \simeq \frac{m\vec{v}^2}{2} + \frac{9m\nu^2}{2}\left(\frac{d\psi}{dr}\right)^2\,. \tag{52}$$

Let a mass M be motionless in the frame origin, and then the potential energy of the probe particle with mass m located at x^1, x^2, x^3 is equal to

$$U(r) = -\gamma\frac{mM}{r}\,, \tag{53}$$

where γ is the gravitational constant. Comparing (49) and (52), we get the equation for $\psi(r)$:

$$\frac{9m\nu^2}{2}\left(\frac{d\psi}{dr}\right)^2 = \gamma\frac{mM}{r} \quad \Rightarrow \quad \frac{d\psi}{dr} = \pm\frac{\sqrt{2\gamma M}}{3\nu}\frac{1}{r^{1/2}}\,. \tag{54}$$

Therefore,

$$\psi(r) = \pm\frac{2\sqrt{2\gamma M}}{3\nu} \cdot r^{1/2} + \psi_0 \qquad (\psi_0 = const). \tag{55}$$

Finally, the World function is equal to

$$S = x^0 \pm \frac{2\sqrt{2\gamma M}}{3c} \cdot r^{1/2} + C_0 \qquad (C_0 = const), \tag{56}$$

When it performs a conformal transformation of the length element of the plane Berwald-Moor space, it induces a pseudo Riemannian geometry in the Minkowski space. For a non-relativistic probe particle of mass m, this geometry gives the motion equations for the Kepler problem for the point mass M located in the origin of the space frame.

The more complicated World function, maybe also leading to the stationary Universe, has the form

$$S(\xi) = \frac{1}{4}\left(\xi^1 + \xi^2 + \xi^3 + \xi^4\right)\left[1 + \alpha_1 \cdot \psi_1(\varrho)\right] + \alpha_2 \cdot \psi_2(\varrho)\,, \tag{57}$$

where α_A are the parameters of the analyticity break of the World function (parameters of the conformal symmetry break in the H_4 space), ψ_A are the arbitrary functions of single argument ϱ (29), (30).

Conclusion

The results obtained in this paper point at the deep correlation between the Einstein geometries and Finsler spaces with Berwald-Moor metric. We managed to find the concrete Finsler space with the Berwald-Moor metric which in the limit appeared to be related to the curved pseudo Riemannian space with the Newtonian gravitational potential. This fact points at the principal possibility to built more interesting constructions, particularly, such Finsler spaces whose limit cases would be the known relativistic solutions.

References

[1] A. V. Berezin, Yu. A. Kurochkin, E. A. Tolkachev, *Quaternions in Relativity Physics,* Science and Technic, Minsk, 1989 (rus).

[2] A. P. Efremov, *Quaternions: algebra and physical theories,* Hypercomplex Numbers in Geometry and Physics, 1, 2004.

[3] V. V. Kassandrov, *Algebrodinamics: primodial light, particles-caustics and flow of Time,* Hypercomplex Numbers in Geometry and Physics, 1, 2004.

[4] G. Kazanova, *From Clifford Algebra to Hydrogen Atom.* M.: Platon, 1997

[5] J. C. Baez, *The octonions,* Bull. Amer. Mathem. Soc. **39: 2**, 145–205 (2002), arXiv: math.RA/0105155 v4.

[6] D. G. Pavlov, *Hypercomplex Numbers, Associated Metric Spaces, and Extension of Relativistic Hyperboloid,* arXiv: gr-qc/0206004.

[7] G. Yu. Bogoslovsky, H. F. Goenner: Phys. Lett. A 244, N 4, (1998), 222.

[8] G. Yu. Bogoslovsky, H. F. Goenner: Gen. Relativ. Gravit. 31, N 10, (1999) 1565.

[9] R. I. Pimenov, *Construction of Time in Science: on the way to understanding the time phenomenon. Part 1. Interdisciplinary Research.* 153–199, MSU Publ., 1996.

[10] G. I. Garas'ko, D. G. Pavlov, *Three-dimensional distance and velocity modulus in the linear Finsler spaces,* Hypercomplex Numbers in Geometry and Physics, 3, 2005.

[11] D. G. Pavlov, *Number and the geometry of Space-Time.* In: *Metaphysics, XXI A.D.,* ed. by Yu. S. Vladimirov. M.: Binom, 2006.

[12] P. K. Rashevsky, *The Geometrical Theory of Partial Differential Equations*, 2-nd ed., Editorial URSS, M., 2003 (rus).

On Field Theory and Finsler Spaces

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The paper introduces the approach to construction of the Lagrangian of the field (fields). This approach is based solely on the metric function of the Finsler space: the Lagrangian is constructed as the unit divided by the volume swept by the unit vector running through all the points of the indicatrix in the tangent space under the assumption of the tangent space being Euclidean. For the space, which is conformally connected to the Minkowski space, under the assumption of the exponential time dependence and spherically symmetrical coordinates dependence the cosmological equation is written, which yields Hubble law for distances from the origin which are much less than the size of the universe. The cosmological equation is written for the field describing the universe with the geometry conformally connected to the geometry of polynumbers $H(4)$ with the Berwald-Moore metrics.

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1 Introduction

Both in classical theory [1] and in the theory of quantized fields [2] the most 'convenient' method of field equations construction deals with such concepts as Lagrangian, action and the principle of the least action (Hamilton's principle). According to this approach, the relation is defined unambiguously [3] between continuous transformations (with respect to which the action is invariant) and physical laws of conservation, that can be verified empirically.

If x^0, x^1, x^2, x^3 are coordinates, $f(x) \equiv f(x^0, x^1, x^2, x^3)$ is a scalar field in Minkowski space, and $\mathfrak{L}$, given by

$$\mathfrak{L} \equiv \mathfrak{L}\left(f(x); \frac{\partial f}{\partial x^0}, \frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3} \right) , \tag{1}$$

is the Lagrangian, then the integral of the Lagrangian over certain 4-dimensional volume V in space-time,

$$I[f] = \int\limits_V^4 \mathfrak{L} \; dx^0 dx^1 dx^2 dx^3 \tag{2}$$

is said to be 'action'. Under the assumption, that variations of the field function δf are equal to zero on the boundary of integration domain, and taking into consideration the requirement of stationarity of action,

$$\delta I[f] = 0 \,, \tag{3}$$

applying the well-known method, we get Euler-Lagrange equation, the field equation:

$$\frac{\partial}{\partial x^i}\frac{\partial \mathfrak{L}}{\partial\left(\frac{\partial f}{\partial x^i}\right)} - \frac{\partial \mathfrak{L}}{\partial f} = 0\,. \tag{4}$$

Usually Lagrangian is selected with the purpose to obtain finally the given field equations, or constructed with the purpose to ensure the desired symmetry and to meet certain auxiliary requirements: e.g. when selecting the Lagrangian we may try to obtain the linear partial differential equations of second order. Construction of the essentially new Lagrangians describing non-linear physical processes reperesents is, in certain sense, a kind of 'art'.

The functional (2) may be interpreted from a purely geometrical standpoint: i.e. not as the integral (of the Lagrangian $\mathfrak{L}$ as the integrand) in the Minkowski space, but as the volume in the space (more complex space) with the volume element given by:

$$dV = \mathfrak{L}\; dx^0 dx^1 dx^2 dx^3\,. \tag{5}$$

Consider the Finsler space $x^1, x^2, ..., x^n$ [4] with the metric function

$$L(dx; x) \equiv L(dx^1, dx^2, ..., dx^n; x^1, x^2, ..., x^n)\,. \tag{6}$$

In this space, let the length element ds be defined by

$$ds = L(dx^1, dx^2, ..., dx^n; x^1, x^2, ..., x^n)\,. \tag{7}$$

The metric properties of Finsler space may be more evidently described in terms of the concept of indicatrix. In every point $M(x^1, x^2, ..., x^n)$ of the main space the indicatrix is defined in the corresponding tangent centroaffine space $\xi^1, \xi^2, ..., \xi^n$ as a hyperspace made up from the 'endpoints' of unit radius-vectors $\xi_{(1)}$. Points of this hypersurface satisfy the equality:

$$L(\xi^1, \xi^2, ..., \xi^n; x^1, x^2, ..., x^n) = 1\,. \tag{8}$$

If the system of indicatrices is defined in every point of the main space, or (what is the same) the sets of unit vectors are defined, the Finsler geometry is defined completely. To calculate the length of the vector $(dx^1, dx^2, ..., dx^n)$, one has to find a unit vector $\xi_{(1)}$ co-directional with the vector dx, then the scalar coefficient ds in the relation

$$dx^i = ds\cdot \xi^i_{(1)} \tag{9}$$

will be the length of the vector dx. From the last relation it follows that, the length element

$$ds = \frac{|dx|_{\mathrm{eu}}}{|\xi_{(1)}|_{\mathrm{eu}}}\,, \tag{10}$$

where $|dx|_{\mathrm{eu}}$, $|\xi_{(1)}|_{\mathrm{eu}}$ are lengths of vectors $(dx^1, dx^2, ..., dx^n)$ and $(\xi^1, \xi^2, ..., \xi^n)$ respectively, calculated as if the spaces $dx^1, dx^2, ..., dx^n$ and $\xi^1, \xi^2, ..., \xi^n$ were Euclidean, and coordinate systems employed were Cartesian.

If under these assumptions, this is possible to calculate the volume of the indicatrix, i.e. the n-dimensional volume, swept by the unit vector $\xi_{(1)}$ in the tangent space $\xi^1, \xi^2, ..., \xi^n$, running through all the points of the indicatrix, then in the Finsler space it is possible (similar to (10)) to define the volume element dV by

$$dV = const \cdot \frac{dx^1 dx^2 ... dx^n}{(V_{ind})_{eu}}\,, \tag{11}$$

where $(V_{ind})_{eu}$ is the volume of the indicatrix, calculated under the assumption, that the tangent space is Euclidean and the coordinates are Cartesian. This is quite evident, that volume element, defined in this way is invariant with respect to coordinate transformations.

Consider n-dimensional Riemannian space. In this case the metric function is given by

$$L(dx;x) = \sqrt{g_{ij} dx^i dx^j}\,, \tag{12}$$

and the equation of the indicatrix is given by

$$g_{ij}\xi^i\xi^j = 1\,. \tag{13}$$

This equation defines the hypersurface of order 2, namely the ellipsoid. If the space $\xi^1, \xi^2, ..., \xi^n$ is Euclidean, then the volume of this ellipsoid is equal to

$$(V_{ind})_{eu} = \frac{const'}{\sqrt{det(g_{ij})}}\,. \tag{14}$$

Substituting the last relation into (11), we obtain the formula for the volume element in an arbitrary Riemannian space:

$$dV = const \cdot \sqrt{det(g_{ij})} \quad dx^1 dx^2 ... dx^n\,, \tag{15}$$

This relation is a conventional definition of invariant volume element in the Riemannian space.

For pseudo-Riemannian spaces, when there are no additional constraints on the indicatrix, we get

$$(V_{ind})_{eu} = \infty \qquad \Rightarrow \qquad dV = 0 \cdot dx^1 dx^2 ... dx^n\,. \tag{16}$$

But this is possible to provide the line of reasoning which allows one to propose for pseudo-Riemannian space the definition of the invariant volume element in the form, similar to(15). The same reasoning should be provided to obtain the invariant volume element in Finsler spaces, where the problem (16) takes place. As a start, we should consider some flat space, close to the space, where the volume element should be defined.

We will provide this reasoning for a particular example: for pseudo-Riemannian space with the signature $(+,-,-,-)$. In this case we will start with Minkowski space x^0, x^1, x^2, x^3, with the metric function of the form

$$L(dx) = \sqrt{(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2} \equiv \sqrt{\overset{o}{g}_{ij}\, dx^i dx^j}\,, \tag{17}$$

and with the tangential equation of the indicatrix in the form

$$(\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 = 1. \tag{18}$$

This equation is of the second order and it defines the hypersurface, which is a hyperboloid of two sheets; thus, the problem of calculating the volume of the indicatrix does take place. As both the metric function and indicatrix equation are the same for all the point in the space, then regardless of how the corresponding integral is regularized, we will obtain a real number, the same for all the points in the space . Let us denote this number by $(V_{ind})_{eu}$. In order to obtain the invariant volume element in Minkowski space using (11), the quantity $(V_{ind})_{eu}$ should be written in the form

$$(V_{ind})_{eu} = \frac{const'}{\sqrt{-det\left(\overset{o}{g}_{ij}\right)}}\,. \tag{19}$$

Now we change the coordinates x^0, x^1, x^2, x^3 to curvilinear coordinates $x^{0'}, x^{1'}, x^{2'}, x^{3'}$. As a result, $\overset{o}{g}_{ij}$ will be substituted by $g(x')_{i'j'}$, and the volume element in Minkowski space in the curvilinear coordinates $x^{0'}$, $x^{1'}$, $x^{2'}$, $x^{3'}$ will be given by

$$dV = const \cdot \sqrt{-det\left(g(x')_{i'j'}\right)}\quad dx^{0'}dx^{1'}dx^{2'}dx^{3'}\,, \tag{20}$$

but this is still the same Minkowski space.

Consider the pseudo-Riemannian space which is conformally connected [4] with Minkowski space

$$ds = \kappa(x)\cdot\sqrt{\overset{o}{g}_{ij}\, dx^i dx^j}\,, \tag{21}$$

where $\kappa(x) > 0$. This space cannot be converted to the Minkowski space by any coordinate transform. The indicatrix equation for this pseudo-Riemannian space may be written in the form:

$$(\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 = \frac{1}{\kappa^2(x)}\,. \tag{22}$$

Comparing (22) with the equation (18), one can notice, that the hypersurface, given by (22), can be obtained from the hypersurface given by (18), via scaling coordinate transform with the coefficient $\frac{1}{\kappa(x)}$. Thus, if we assign volume (19) to the indicatrix (18), then to the indicatrix (22) we should assign the volume by

$$(V_{ind})_{eu} = \frac{const'}{\kappa^4(x)\sqrt{-det\left(\overset{o}{g}_{ij}\right)}} = \frac{const'}{\sqrt{-det\left(g(x)_{ij}\right)}}\,, \tag{23}$$

where
$$g(x)_{ij} \equiv \kappa^2(x) \overset{o}{g}_{ij} \ . \tag{24}$$

From the reasoning provided above, it follows that in the pseudo-Riemannian space with the metric tensor $g(x)_{ij}$ and the signature $(+,-,-,-)$ this is possible to define the volume element by

$$dV = const \cdot \sqrt{-det\,(g(x)_{ij})} \quad dx^0 dx^1 dx^2 dx^3\,, \tag{25}$$

and this corresponds to the approach conventional for GRT[1] .

The problem (16) in pseudo-Riemannian spaces may be handled more rigorously (however, this is outside the scope of this paper), but we will have to deal with the spaces that are more general than pseudo-Riemannian spaces. This may be explained on the example of Minkowski space. If instead of Minkowski space with the metric function (17) we consider the Finsler space with the metric function

$$L(dx) = \sqrt{(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2} + q_0 dx^0 \tag{26}$$

and the constraint $dx^0 \geq 0$, where $q_0 > 0$, then for this space the volume of the indicatrix $(V_{ind})_{eu}$ will be a finite real number, depending on the parameter q_0, such that $(V_{ind})_{eu}$tends to ∞, as the parameter q_0 vanishes.

Thus, we will assume that in any Finsler space, where the problem (16) takes place, this problem is solvable. Then, this is possible to claim that if the metric function of this space contains certain fields, the geometry of Finsler space yields automatically the Lagrangian
$$\mathfrak{L} = \frac{const}{(V_{ind})_{eu}}\,, \tag{27}$$
and from this Lagrangian one can obtain the field equations.

Remark. Henceforth the constants which appear in the relations (11), (14),..., (27) will be omitted, as these constants are not involved in the the field equations.

2 The spaces, conformally connected to Euclidean spaces

In the space conformally connected to n-dimensional Euclidean space, the length element is given by

$$ds = \kappa(x) \cdot \sqrt{(x^1)^2 + (x^2)^2 + ... + (x^n)^2}\,, \tag{28}$$

where $\kappa(x) > 0$. As in this case the following relation holds,

$$\sqrt{det(g_{ij})} = \kappa^n(x)\,, \tag{29}$$

the Lagrangian takes the form
$$\mathfrak{L} = \kappa^n(x)\,. \tag{30}$$

To construct the field equation with the help of this Lagrangian, it is necessary to represent the scalar field $\kappa(x)$ in terms of another field so that the lagrangian will involve the derivatives of the new field. A method to achieve this goal is proposed in [5], [6].

The generalized momenta in the space (28) are given by

$$p_i = \kappa(x) \frac{dx^i}{\sqrt{(x^1)^2 + (x^2)^2 + ... + (x^n)^2}}\,, \tag{31}$$

and the tangential equation of the indicatrix may be written in the form:

$$p_1^2 + p_2^2 + ... + p_n^2 - \kappa^2(x) = 0\,. \tag{32}$$

Consider scalar function $S(x)$, which in the space $x^1, x^2, ..., x^n$. Let this function define the normal congruence of geodesics: in classical mechanics this function is called the 'action as a function of coordinates', and in the paper [5] the function $S(x)$ is called the World function. This function must satisfy the Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial x^1}\right)^2 + \left(\frac{\partial S}{\partial x^2}\right)^2 + ... + \left(\frac{\partial S}{\partial x^n}\right)^2 = \kappa^2(x)\,. \tag{33}$$

Thus,

$$\mathfrak{L} = \left[\left(\frac{\partial S}{\partial x^1}\right)^2 + \left(\frac{\partial S}{\partial x^2}\right)^2 + ... + \left(\frac{\partial S}{\partial x^n}\right)^2\right]^{\frac{n}{2}}\,, \tag{34}$$

and the field equation (4) takes the form:

$$\frac{\partial}{\partial x^i}\left\{\frac{\partial S}{\partial x^i}\left[\left(\frac{\partial S}{\partial x^1}\right)^2 + \left(\frac{\partial S}{\partial x^2}\right)^2 + ... + \left(\frac{\partial S}{\partial x^n}\right)^2\right]^{\frac{n}{2}-1}\right\} = 0\,. \tag{35}$$

Note, that for $n > 2$ this equation is a non-linear partial differential equation of second order.

For the space, conformally connected to the 2-dimensional Euclidean plain (x, y), the equation (35) may be written in the form

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = 0\,, \tag{36}$$

i.e. the function $S(x, y)$ satisfies the Laplace equation; therefore this function is a component of the analytical function of complex variable. Thus,

$$\kappa(x, y) = \sqrt{\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2} \tag{37}$$

is the factor of the conformal transformation of the length element in the Euclidean space

$$ds' = \sqrt{(x'^2 + (y'^2} = \kappa(x,y)\sqrt{x^2+y^2} \tag{38}$$

for the conformal transformation

$$x' = u(x,y)\,, \qquad y' = \pm v(x,y)\,, \tag{39}$$

where the function S is one of the components of the analytical function $u + iv$ of complex variable $x + iy$.

Now, we will solve the equation (35) under the assumption that function S is a function of radius only

$$r = \sqrt{(x^1)^2 + (x^2)^2 + \ldots + (x^n)^2}\,. \tag{40}$$

To find the solution, this will be more convenient, if volume element is represented as a function of spherical coordinates. Then, after integration on all the angles, we obtain

$$dV_r = r^{n-1}\left|\frac{dS}{dr}\right|^n dr \qquad \Rightarrow \qquad \mathfrak{L}_r = r^{n-1}\left|\frac{dS}{dr}\right|^n . \tag{41}$$

Then the field equation will take the form :

$$\frac{d}{dr}\left[r^{n-1}\left|\frac{dS}{dr}\right|^{n-1}\right] = 0\,. \tag{42}$$

Via integration of the last relation, we get

$$\frac{dS}{dr} = \frac{C}{r}\,, \qquad S = C\ln\frac{r}{r_0}\,, \tag{43}$$

where $C \neq 0$, $r_0 > 0$ are real. Thus,

$$\kappa(x) = \left|\frac{dS}{dr}\right| = \frac{|C|}{r}\,. \tag{44}$$

In this space the geodesics are given by the relations

$$\dot{x}^i = \frac{dS}{dx^i}\cdot\lambda(x)\,, \tag{45}$$

where $\lambda(x) \neq 0$ is a function, $\dot{x}^i$ is the parameter derivative of x^i along the geodesic τ. Set $\lambda(x) = r$, then the relation (45) yields

$$\dot{x}^i = x^i\,. \tag{46}$$

Let $j > 1$, then

$$\frac{dx^j}{dx^1} = \frac{x^j}{x^1} \qquad \Rightarrow \qquad x^j = C^j x^1\,, \tag{47}$$

thus, the geodesics in this space are straight lines, going through the origin with the directing vector $(1, C^2, C^3, \ldots, C^n)$.

3 The spaces, conformally connected to pseudo-Euclidean spaces with the signature $(+, -, -, ..., -)$

In the space, which is conformally connected to the n-dimensional pseudo-Euclidean space with the signature $(+, -, -, ..., -)$ the length element is given by

$$ds = \kappa(x) \cdot \sqrt{(x^0)^2 - (x^1)^2 - ... - (x^{n-1})^2}\,, \tag{48}$$

where $\kappa(x) > 0$. As in this case the following relation holds

$$\sqrt{(-1)^{n-1} det\,(g_{ij})} = \kappa^n(x)\,, \tag{49}$$

the Lagrangian can be represented in the form

$$\mathfrak{L} = \kappa^n(x)\,. \tag{50}$$

In order to construct the field equation from this Lagrangian, it is required to express the scalar field $\kappa(x)$ via another field so that, the Lagrangian will contain the derivatives of the new field [5], [6].

The generalized momenta in the space (48) are given by:

$$p_0 = \frac{\kappa(x)\; dx^0}{\sqrt{(x^0)^2 - (x^1)^2 - ... - (x^{n-1})^2}},\; p_\mu = -\frac{\kappa(x)\; dx^\mu}{\sqrt{(x^0)^2 - (x^1)^2 - ... - (x^{n-1})^2}}, \tag{51}$$

where $\mu = 1, 2, ..., (n-1)$, and tangential equation of the indicatrix may be represented in the form:

$$p_0^2 - p_1^2 - ... - p_{n-1}^2 - \kappa^2(x) = 0\,. \tag{52}$$

The scalar function $S(x)$, which in the space $x^0, x^1, ..., x^{n-1}$ defines the normal congruence of geodesics, must satisfy Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial x^0}\right)^2 - \left(\frac{\partial S}{\partial x^1}\right)^2 - ... - \left(\frac{\partial S}{\partial x^{n-1}}\right)^2 = \kappa^2(x)\,. \tag{53}$$

Thus,

$$\mathfrak{L} = \left[\left(\frac{\partial S}{\partial x^0}\right)^2 - \left(\frac{\partial S}{\partial x^1}\right)^2 - ... - \left(\frac{\partial S}{\partial x^{n-1}}\right)^2\right]^{\frac{n}{2}}\,, \tag{54}$$

and the field equation (4) takes the form:

$$\begin{aligned}&\frac{\partial}{\partial x^0}\left\{\frac{\partial S}{\partial x^0}\left[\left(\frac{\partial S}{\partial x^0}\right)^2 - \left(\frac{\partial S}{\partial x^1}\right)^2 - ... - \left(\frac{\partial S}{\partial x^{n-1}}\right)^2\right]^{\frac{n}{2}-1}\right\} - \\ &-\frac{\partial}{\partial x^\mu}\left\{\frac{\partial S}{\partial x^\mu}\left[\left(\frac{\partial S}{\partial x^0}\right)^1 - \left(\frac{\partial S}{\partial x^2}\right)^2 - ... - \left(\frac{\partial S}{\partial x^{n-1}}\right)^2\right]^{\frac{n}{2}-1}\right\} = 0\,.\end{aligned} \tag{55}$$

Interestingly, that for $n > 2$ this equation is a non-linear partial differential equations of second order and this equation is satisfied if the function S satisfies the eikonal equation

$$\left(\frac{\partial S}{\partial x^0}\right)^1 - \left(\frac{\partial S}{\partial x^2}\right)^2 - \ldots - \left(\frac{\partial S}{\partial x^{n-1}}\right)^2 = 0\,.$$

For the field equation (55) to be the wave equation, the function S must simultaneously satisfy one more condition:

$$\left(\frac{\partial S}{\partial x^0}\right)^1 - \left(\frac{\partial S}{\partial x^2}\right)^2 - \ldots - \left(\frac{\partial S}{\partial x^{n-1}}\right)^2 = const\,.$$

For the space conformally connected with the 2-dimensional pseudo-Euclidean plain (x, y), the relation (55) takes the form

$$\frac{\partial^2 S}{\partial x^2} - \frac{\partial^2 S}{\partial y^2} = 0\,, \tag{56}$$

that is for the two-dimensional case the field equation (55) is a wave equation.

Now we will solve the equation (55) under the assumption that the function S depends only on the interval

$$s = \sqrt{(x^0)^2 - (x^1)^2 - \ldots - (x^{n-1})^2}\,. \tag{57}$$

For this we will consider the volume element, choosing as one of the variables the interval s. In the process of integration on hyperbolic angles certain difficulties may take place, which are similar to (16) and which can be resolved in the similar way, thus

$$dV_s = s^{n-1}\left|\frac{dS}{ds}\right|^n ds \qquad \Rightarrow \qquad \mathfrak{L}_s = s^{n-1}\left|\frac{dS}{ds}\right|^n, \tag{58}$$

and the field equation takes the form:

$$\frac{d}{ds}\left[s^{n-1}\left|\frac{dS}{ds}\right|^{n-1}\right] = 0\,. \tag{59}$$

Integrating the last equality, we obtain

$$\frac{dS}{ds} = \frac{C}{s}\,, \qquad S = C\ln\frac{s}{s_0}\,, \tag{60}$$

where $C \neq 0$, $s_0 > 0$ are real. Thus,

$$\kappa(x) = \left|\frac{dS}{ds}\right| = \frac{|C|}{s}\,. \tag{61}$$

The geodesics in this space are given by

$$\dot{x}^0 = \frac{dS}{dx^0} \cdot \lambda(x) \,, \qquad \dot{x}^\mu = -\frac{dS}{dx^\mu} \cdot \lambda(x) \,, \tag{62}$$

where $\lambda(x) \neq 0$ is a function , $\dot{x}^i$ is a derivative of x^i with respect to the evolution parameter τ, $\mu = 1, 2, ..., n-1$. Set $\lambda(x) = \dfrac{s^2}{|C|}$, then from (62) it follows that

$$\dot{x}^i = x^i \,, \tag{63}$$

or

$$\frac{dx^\mu}{dx^0} = \frac{x^\mu}{x^0} \qquad \Rightarrow \qquad x^\mu = C^\mu x^0 \,, \tag{64}$$

that is the geodesics (extremals) in this space are straight lines, 'going' through the origin with the directing vector $(1, C^2, C^3, ..., C^n)$. The interval will also change linearly with respect to the coordinate x^0,

$$s = \sqrt{1 - (C^1)^2 - ... - (C^{n-1})^2} \cdot x^0 \,, \qquad x^0 > 0. \tag{65}$$

As we will further use the space, which is conformally connected to the Minkowski space, for construction of the cosmological equation, we will provide certain formulae of this section for $n = 4$, using the metric tensor of Minkowski space $\overset{o}{g}_{ij}$:

Relation between the function $S(x)$ and the factor $\kappa(x)$:

$$\overset{o}{g}{}^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} = \kappa^2(x) \,, \tag{66}$$

Lagrangian:

$$\mathfrak{L} = \left(\overset{o}{g}{}^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} \right)^2 \,, \tag{67}$$

Field equation:

$$\overset{o}{g}{}^{kl} \frac{\partial}{\partial x^k} \left[\frac{\partial S}{\partial x^l} \left(\overset{o}{g}{}^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} \right) \right] = 0 \,. \tag{68}$$

4 Model cosmological equation in the space, conformally connected to the Minkowski space

We will write the equation (68) under the assumption that the function S is of the form

$$S(x^0, r) = S_0 e^{-\gamma x^0} \psi(r) \,, \tag{69}$$

where $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, and γ , S_0 are constant. This is much simpler to obtain the field equation of this form, if in relations for volume element instead of spatial coordinates x^1, x^2, x^3 the spherical coordinate system is used. After integration on spherical angles (omitting the constant), we will obtain the following relation for the Lagrangian

$$\mathfrak{L} = r^2 \left[\left(\frac{\partial S}{\partial x^0} \right)^2 - \left(\frac{\partial S}{\partial r} \right)^2 \right]^2 , \tag{70}$$

and the field equation will take the form:

$$r^2 \frac{\partial}{\partial x^0} \left\{ \frac{\partial S}{\partial x^0} \left[\left(\frac{\partial S}{\partial x^0} \right)^2 - \left(\frac{\partial S}{\partial r} \right)^2 \right] \right\} - \frac{\partial}{\partial r} \left\{ r^2 \frac{\partial S}{\partial r} \left[\left(\frac{\partial S}{\partial x^0} \right)^2 - \left(\frac{\partial S}{\partial r} \right)^2 \right] \right\} = 0 . \tag{71}$$

Substituting into the last relation the function $S(x^0, r)$ (69), we obtain

$$3\gamma^2 r^2 \psi \left[\gamma^2 \psi^2 - \left(\frac{d\psi}{dr} \right)^2 \right] - \frac{d}{dr} \left\{ r^2 \frac{d\psi}{dr} \left[\gamma^2 \psi^2 - \left(\frac{d\psi}{dr} \right)^2 \right] \right\} = 0 . \tag{72}$$

Let define the dimensionless variable $\xi \equiv \gamma r$, then the last equation may be rewritten in the from:

$$3\xi^2 \psi \left[\psi^2 - \left(\frac{d\psi}{d\xi} \right)^2 \right] - \frac{d}{d\xi} \left\{ \xi^2 \frac{d\psi}{d\xi} \left[\psi^2 - \left(\frac{d\psi}{d\xi} \right)^2 \right] \right\} = 0 . \tag{73}$$

As this equation is homogeneous with respect to the unknown function$\psi(\xi)$, we will suppose the solution to be of the form

$$\psi(\xi) = \psi_0 \exp \left(\int_0^\xi \varphi(\xi) d\xi \right) , \tag{74}$$

where ψ_0 is a constant. This constant for construction of the function S is multiplied by the constant S_0, thus we will set $\psi_0 = 1$. Substituting (74) into (73), we obtain

$$\frac{d}{d\xi} \left[\xi^2 \varphi (1 - \varphi^2) \right] - 3\xi^2 (1 - \varphi^2)^2 = 0 , \tag{75}$$

or

$$\xi (1 - 3\varphi^2) \frac{d\varphi}{d\xi} + 2\varphi (1 - \varphi^2) - 3\xi (1 - \varphi^2)^2 = 0 . \tag{76}$$

There were no success in finding the analytical solution of the last equation.

For the domain $\xi \ll 1$ we will find a solution in a form of power series

$$\varphi \simeq A\xi + B\xi^2 + C\xi^3 + O(\xi^4) . \tag{77}$$

Substituting this expansion into equation (76), after grouping the terms, we will obtain

$$\varphi \simeq \xi - \frac{1}{5}\xi^3 + O(\xi^4)\,. \tag{78}$$

The sample bodies (stars) move along the geodesics (extremals) of the space with the length element

$$ds = \kappa(x^0, r)\sqrt{(dx^0)^2 - (dr)^2} \tag{79}$$

and the tangential equation of indicatrix

$$p_0^2 - p_r^2 = \kappa^2(x^0, r)\,. \tag{80}$$

For the field S (69), (74) the scaling factor of conformal transformation may be calculated as

$$\kappa(x^0, r) = \sqrt{\left(\frac{\partial S}{\partial x^0}\right)^2 - \left(\frac{\partial S}{\partial r}\right)^2} = \gamma \cdot \sqrt{1 - \varphi^2} \cdot S(x^0, r)\,. \tag{81}$$

From the last relation, it follows that $|\varphi| < 1$. The motion equations in this case will be of the form

$$\dot{x}^0 = \frac{\partial S}{\partial x^0}\lambda = -\gamma S \lambda\,, \qquad \dot{r} = -\frac{\partial S}{\partial r}\lambda = -\gamma S \varphi(\gamma r)\lambda\,, \tag{82}$$

where the dot represents the total derivative with respect to certain evolution parameter τ, and an arbitrary function $\lambda \neq 0$. Then

$$\frac{dr}{dx^0} = \varphi(\gamma r) \qquad \Rightarrow \qquad \frac{dr}{dt} = c\varphi(\gamma r)\,. \tag{83}$$

As $|\varphi| < 1$, then

$$\left|\frac{dr}{dx^0}\right| < 1 \qquad \text{and} \qquad \left|\frac{dr}{dt}\right| < c\,.$$

Let consider, the behavior of the velocity of the sample body in the domain $\xi \ll 1$, for this we substitute (78) into the obtained relation:

$$\frac{dr}{dt} = c\gamma\left(1 - \frac{1}{5}\gamma^2 r^2\right) \cdot r\,. \tag{84}$$

If we denote the Hubble's constant by H_0, then according to the obtained relation the Hubble law holds when $\gamma r < \frac{1}{10}$, and $H_0 = c\gamma$, and the tendence, how the 'Hubble constant' $H(r)$ evolves initially as the distance from the center grows is of the form:

$$\frac{dr}{dt} = H(r) \cdot r\,, \qquad H(r) = H_0 \cdot \left[1 - \frac{1}{5}\left(\frac{H_0}{c}\right)^2 \cdot r^2\right]. \tag{85}$$

I.e. in the domain $\xi \ll 1$ this constant $H(r)$ decreases as the distance from the origin grows.

To provide any ideas about the size of the universe and the dependence $H(r)$ for all possible values of the variable r, this is required to analyze the solution $\varphi(\xi)$ of the equation (76), the solution which (as $\xi \to 0$) takes the form (78). Neither analytically, nor numerically we didnot succeed in this analysis, as approaching the value $\varphi = \frac{1}{\sqrt{3}}$ the behavior of the solution becomes quite complicated (unstable). If we suppose the the solution of the equation (76) can be obtained and analyzed, then general form of the quantity $H(r)$ may be written in the following way:

$$H(r) = H_0 \cdot \left[\frac{\varphi\left(\frac{H_0}{c} r\right)}{\frac{H_0}{c} r} \right] . \tag{86}$$

If we consider motion trajectories in the space x^0, x^1, x^2, x^3 with the World function S (69), these trajectories will be given by the equations

$$\frac{dx^\mu}{dx^0} = \varphi(\gamma r)\, \frac{x^\mu}{r} \,,$$

that is the motion will be along the rays from the origin, and this means that the sample particle move rectilinearly, but certainly the motion will be still non-uniform.

As the space with the length element (79) is a pseudo-Riemannian space with the metric tensor

$$g_{ij}(x^0, r) = \kappa^2(x^0, r) \cdot \overset{o}{g}_{ij} \,, \tag{87}$$

where $\overset{o}{g}_{ij}$ is the metric tensor in the Minkowski space and

$$\kappa(x^0, r) = \gamma S \sqrt{1 - \varphi^2} \,, \tag{88}$$

then for this space this is possible to calculate the curvature tensor and its contractions, and directly from the Einstein equations one may obtain the matter energy-momentum tensor T_{km}, which is involved in the Einstein equations and which corresponds to the space with the metric tensor (87). Interestingly, that the equations for the gravitational field, certainly, for this energy-momentum tensor will hold automatically, but with the tensor T_{km} this is not possible, in general, to associate the laws of conservation of energy and momentum.

Let us introduce a new quantity, which can be employed quite usefully

$$a = \ln(\kappa^2/const) \,. \tag{89}$$

Then, using the well-known classical formulae, we obtain the expressions for the connectivity object:

$$\Gamma^i_{kl} = \frac{1}{2} \left(\frac{\partial a}{\partial x^l} \delta^i_k + \frac{\partial a}{\partial x^k} \delta^i_l - \overset{o}{g}{}^{is}\, \frac{\partial a}{\partial x^s}\, \overset{o}{g}_{kl} \right) , \tag{90}$$

curvature tensor:

$$
\begin{aligned}
&R^i_{klm} = \\
&= \frac{1}{2}\left(\frac{\partial^2 a}{\partial x^l \partial x^k}\,\delta^i_m - \frac{\partial^2 a}{\partial x^k \partial x^m}\,\delta^i_l - \overset{o}{g}{}^{is}\,\frac{\partial^2 a}{\partial x^l \partial x^s}\,\overset{o}{g}_{km} + \overset{o}{g}{}^{is}\,\frac{\partial^2 a}{\partial x^m \partial x^s}\,\overset{o}{g}_{kl}\right) + \\
&\frac{1}{4}\left(\frac{\partial a}{\partial x^m}\frac{\partial a}{\partial x^k}\,\delta^i_l - \frac{\partial a}{\partial x^l}\frac{\partial a}{\partial x^k}\,\delta^i_m - \overset{o}{g}{}^{ns}\,\frac{\partial a}{\partial x^n}\frac{\partial a}{\partial x^s}\,\delta^i_l\,\overset{o}{g}_{km} + \right. \\
&\left. + \frac{\partial a}{\partial x^l}\,\overset{o}{g}_{km}\overset{o}{g}{}^{is}\,\frac{\partial a}{\partial x^s} + \overset{o}{g}{}^{ns}\,\frac{\partial a}{\partial x^n}\frac{\partial a}{\partial x^s}\,\delta^i_m\,\overset{o}{g}_{kl} - \frac{\partial a}{\partial x^m}\,\overset{o}{g}_{kl}\overset{o}{g}{}^{is}\,\frac{\partial a}{\partial x^s}\right),
\end{aligned}
\tag{91}
$$

Ricci tensor:

$$
\begin{aligned}
&R_{km} \equiv R^l_{klm} = \\
&= \frac{1}{2}\left(-2\frac{\partial^2 a}{\partial x^k \partial x^m} - \overset{o}{g}{}^{ns}\,\frac{\partial^2 a}{\partial x^n \partial x^s}\,\overset{o}{g}_{km} + \frac{\partial a}{\partial x^k}\frac{\partial a}{\partial x^m} - \overset{o}{g}{}^{ns}\,\frac{\partial a}{\partial x^n}\frac{\partial a}{\partial x^s}\,\overset{o}{g}_{km}\right),
\end{aligned}
\tag{92}
$$

scalar curvature of the space:

$$
R \equiv g^{km} R_{km} = \frac{1}{\kappa^2}\,\overset{o}{g}{}^{km}\,R_{km} = -\frac{3}{\kappa^2}\left(2\,\overset{o}{g}{}^{km}\,\frac{\partial^2 a}{\partial x^k \partial x^m} + \overset{o}{g}{}^{km}\,\frac{\partial a}{\partial x^k}\frac{\partial a}{\partial x^m}\right), \tag{93}
$$

matter energy-momentum tensor:

$$
T_{km} = \frac{c^4}{8\pi k}\left(R_{km} - \frac{1}{2}\kappa^2\,\overset{o}{g}_{km}\,R\right), \tag{94}
$$

where k is the gravitation constant. Hence,

$$
T \equiv g^{km} T_{km} = \frac{1}{\kappa^2}\,\overset{o}{g}{}^{km}\,T_{km} = -\frac{c^4}{8\pi k} R\,. \tag{95}
$$

But using the 'independence' on the Einstain gravitation field equations, we can calculate the full energy-momentum tensor $\hat{T}_{km}$. For the Lagrangian of the field $\mathfrak{L}$ (67) we obtain

$$
\hat{T}^k_m = \frac{\partial S}{\partial x^m}\frac{\partial \mathfrak{L}}{\partial \frac{\partial S}{\partial x^k}} - \delta^k_m \mathfrak{L} = 4\,\overset{o}{g}{}^{ks}\,\frac{\partial S}{\partial x^s}\frac{\partial S}{\partial x^m}\left(\overset{o}{g}{}^{rs}\,\frac{\partial S}{\partial x^r}\frac{\partial S}{\partial x^s}\right) - \delta^k_m\left(\overset{o}{g}{}^{rs}\,\frac{\partial S}{\partial x^r}\frac{\partial S}{\partial x^s}\right)^2, \tag{96}
$$

after contraction on 2 used indices, we get

$$
\hat{T}^k_k \equiv 0\,. \tag{97}
$$

Finally, one may note that the tensors T_{km} and $\hat{T}_{km}$ are essentially different.

5 The space, conformally connected to 4-dimensional Berwald-Moore space

The length element in this space (in special isotropic basis) will have the form

$$ds = \kappa(\xi^1, \xi^2, \xi^3, \xi^4)\sqrt[4]{d\xi^1 d\xi^2 d\xi^3 d\xi^4}\,. \tag{98}$$

The generalized momenta will satisfy the relations

$$p_i = \frac{1}{4}\kappa(\xi)\frac{\sqrt[4]{d\xi^1 d\xi^2 d\xi^3 d\xi^4}}{d\xi^i}\,. \tag{99}$$

If $\eta^1, \eta^2, \eta^3, \eta^4$ are coordinates of tangent centroaffine space in the point $M(\xi^1, \xi^2, \xi^3, \xi^4)$ of the main space, then the indicatrix equation will have the form

$$\eta^1\eta^2\eta^3\eta^4 = \frac{1}{\kappa^4(\xi)}\,, \tag{100}$$

and the tangential equation of indicatrix will have e.g. the form,

$$p_1 p_2 p_3 p_4 = \frac{\kappa^4(\xi)}{4^4}\,. \tag{101}$$

Then the function S, defines normal congruence of geodesics, and satisfies the following non-linear partial differential equation

$$\frac{\partial S}{\partial \xi^1}\frac{\partial S}{\partial \xi^2}\frac{\partial S}{\partial \xi^3}\frac{\partial S}{\partial \xi^4} = \frac{\kappa^4(\xi)}{4^4}\,. \tag{102}$$

From the relation (100) we obtain that

$$(V_{ind})_{eu} = const \cdot \frac{1}{\kappa^4}\,. \tag{103}$$

Thus, the Lagrangian of the scalar field S will have the form:

$$\mathfrak{L} = \frac{\partial S}{\partial \xi^1}\frac{\partial S}{\partial \xi^2}\frac{\partial S}{\partial \xi^3}\frac{\partial S}{\partial \xi^4}\,. \tag{104}$$

Correspondingly, the field equation will take the form

$$\frac{\partial}{\partial \xi^1}\left(\frac{\partial S}{\partial \xi^2}\frac{\partial S}{\partial \xi^3}\frac{\partial S}{\partial \xi^4}\right)+\frac{\partial}{\partial \xi^2}\left(\frac{\partial S}{\partial \xi^1}\frac{\partial S}{\partial \xi^3}\frac{\partial S}{\partial \xi^4}\right)+\frac{\partial}{\partial \xi^3}\left(\frac{\partial S}{\partial \xi^1}\frac{\partial S}{\partial \xi^2}\frac{\partial S}{\partial \xi^4}\right)+\frac{\partial}{\partial \xi^4}\left(\frac{\partial S}{\partial \xi^1}\frac{\partial S}{\partial \xi^2}\frac{\partial S}{\partial \xi^3}\right) = 0\,. \tag{105}$$

Any function S, which depends on not all the coordinates $\xi^1, \xi^2, \xi^3, \xi^4$ satisfies this equation.

Let the field S depend on only one variable

$$s = \sqrt[4]{\xi^1\xi^2\xi^3\xi^4}\,, \tag{106}$$

Substituting $S(s)$ into the field equation (105) and using the formula

$$\frac{\partial s}{\partial \xi^i} = \frac{1}{4}\frac{s}{\xi^i}\,, \tag{107}$$

we obtain

$$\frac{d}{ds}\left(s\frac{dS}{ds}\right) = 0\,. \tag{108}$$

The same equation may be obtained easier, if the volume element

$$dV = \mathfrak{L}\, d\xi^1 d\xi^2 d\xi^3 d\xi^4\,, \tag{109}$$

is written, as a function of variable s and three angular variables. After integration of this element over the angles we obtain

$$dV_s = s^3\left(\frac{dS}{ds}\right)^4 ds\,. \tag{110}$$

Via integration of the equation (108), we get

$$S(s) = S_0 \ln\frac{s}{s_0}\,, \tag{111}$$

where S_0, s_0 are constants of integration, and also the relation for the factor κ,

$$\kappa = \frac{|A|}{s}\,. \tag{112}$$

This is quite interesting to compare the last two relations with the relations (43), (44) and (60), (61).

Now we will find the trajectories of the motion of sample particles in the four-dimensional Berwald-Moore space, if the function S, defining the congruence of geodesics, has the form (111), i.e. the factor satisfies the relation (112). The motion equations in this case will have the form

$$\dot{\xi}^i = \frac{\dfrac{\partial S}{\partial \xi^1}\dfrac{\partial S}{\partial \xi^2}\dfrac{\partial S}{\partial \xi^3}\dfrac{\partial S}{\partial \xi^4}}{\dfrac{\partial S}{\partial \xi^i}}\lambda(\xi)\,, \tag{113}$$

where $\lambda(\xi) \neq 0$ is a certain scalar function. Taking into consideration the relation (107) and via appropriate selection of $\lambda(\xi)$, motion equations may take a more simple form

$$\dot{\xi}^i = \xi^i\,. \tag{114}$$

Set the variable

$$x^0 = \xi^1 + \xi^2 + \xi^3 + \xi^4\,, \tag{115}$$

which in the four-dimensional Berwald-Moore plays the same role as the coordinate x^0 in the Minkowski space, then

$$\frac{d\xi^i}{dx^0} = \frac{\xi^i}{x^0} \qquad \Rightarrow \qquad \xi^i = \xi^i_0 \cdot x^0 \,, \tag{116}$$

where ξ^i_0 are constant. Thus, all the motion trajectories are straight lines, passing through the origin, and the motion of sample bodies will be uniform and rectilinear, with respect to the time variable x^0.

Conclusion

The proposed new approach of the non-ambiguous construction of the field Lagrangians basing on the metric function of the Finsler space requires that the fields which are involved in the Lagrangian without their partial derivatives with respect to coordinates, are expressed via other fields so that these partial derivatives over coordinates are involved in the Lagrangian, otherwise, this is not possible to obtain the field equations as partial differential equations. Thus, the 'art' of Lagrangian construction is replaced with the 'art' of representation of physical fields using other fields.

For n-dimensional Riemannian or pseudo-Riemannian spaces with the metric tensor $g_{ij}(x)$, the Lagrangian is given by

$$\mathfrak{L} = \sqrt{|det(g_{ij}(x))|} \,.$$

The metric tensor $g_{ij}(x)$ may be represented, for example, in the following form:

$$g_{ij}(x) = \sum_{a=1}^{N} \varepsilon_{(a)} \frac{\partial f_{(a)}}{\partial x^i} \frac{\partial f_{(a)}}{\partial x^j} \,,$$

here $\varepsilon_{(a)} = \pm 1$ are independent sign multiplicands, $f_{(a)}(x)$ are scalar functions, and $N \geq n$. If $N < n$, then $det\,(g_{ij}(x)) = 0$.

References

[1] Landau, Livshitz, *Field Theory,* 'Nauka', M., 1967 (in Russian).

[2] N. N. Bogolyubov, D. V. Shirkov, *Introduction to Theory of Quantized Fields,* 'Nauka', M., 1973 (in Russian).

[3] Neter E., *Invariant variational problems,* In: *Variation Principles in Mechanics.* eds. L.S. Polak, 'GIPML', M., 1959, 611–630.

[4] P. K. Rashevsky, *The Geometrical Theory of Partial Differential Equations*, 2-nd ed., Editorial URSS, M., 2003 (in Russian).

[5] Garas'ko G. I., *On the World function and relation between geometries,* Hypercomplex Numbers in Geometry and Physics, **1 (5)**, Vol 3, 2006, 3–17

[6] Garas'ko G. I., Pavlov D. G., *Construction of pseudo-Riemannian geometry on the basis of Berwald-Moore geometry,* Ibidem, 18–26

Quaternions: Algebra, Geometry and Physical Theories

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A review of modern study of algebraic, geometric and differential properties of quaternionic (Q) numbers with their applications. Traditional and "tensor" formulation of Q-units with their possible representations are discussed; groups of Q-units transformations leaving Q-multiplication rule form-invariant are determined. A series of mathematical and physical applications is offered, among them use of Q-triads as a moveable frame, analysis of Q-spaces families, Q-formulation of Newtonian mechanics in arbitrary rotating frames, realization of a Q-Relativity model comprising all effects of Special Relativity and admitting description of kinematics of non-inertial motion. A list of "Quaternionic Coincidences" is presented revealing surprising interconnection between basic relations of some physical theories and Q-numbers mathematics.

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Introduction

The discovery of quaternionic (Q) numbers dated by 1843 is usually attributed to Hamilton [1, 2], but in the previous century Euler and Gauss made a contribution to mathematics of Q-type objects; moreover Rodriguez offered multiplication rule for elements of similar algebra [3–5]. Active opposition of Gibbs and Heaviside to Hamilton's disciples gave a start to the modern vector algebra, and later to vector analysis, and quaternions practically ceased to be a tool of mathematical physics, despite of exclusive nature of their algebra confirmed by Frobenius theorem. At the beginning of 20 century last bastion of Q-numbers amateurs, "Association for the Promotion of the Study of Quaternions", was ruined. The only reminiscence of once famous hypercomplex numbers was the set of Pauli matrices. Later on quaternions appeared incidentally as a mathematical mean for alternative description of already known physical models [6, 7] or due to surprising simplicity and beauty they were used to solve rigid body cinematic problems [8]. An interest to quaternionic numbers essentially increased in last two decades when a new generation of theoreticians started feeling in quaternions deep potential yet undiscovered (e. g. [9 – 11]).

This work is an attempt to give more systematic overview of contemporary state of Q-number mathematics, its applications to physical theories and possible perspectives in this area. In the context some quite specific even surprising physical models, but worth to pay attention to, are shortly discussed.

The review arranged as follows. In section 1 general relations of the quaternionic algebra are briefly described in the traditional hamiltonian formulation as well as in tensor-like format. Section 2 is devoted to description of structure of three "imaginary" quaternionic units. In section 3 the elements of differential Q-geometry are given with examples of their mathematical application. Section 4 comprises Q-formulation of Newtonian mechanics in the rotating frames of reference. Quaternionic Relativity Theory with a number of cinematic relativistic effects is found in section 5. Section 6 contains the list of "Great Quaternion Coincidences" and final discussion.

1. Algebra of quaternions

Traditional approach

According to Hamilton, a quaternion is a mathematical object of the form

$$Q \equiv a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},$$

where a, b, c, d are real numbers, a is a coefficient at real unit "1", and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ – three imaginary quaternion units. The multiplication rule for these units given by Hamilton and often used in literature is

$$1\mathbf{i} = \mathbf{i}1 \equiv \mathbf{i}, \qquad 1\mathbf{j} = \mathbf{j}1 \equiv \mathbf{j}, \qquad 1\mathbf{k} = \mathbf{k}1 \equiv \mathbf{k},$$
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$
$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \qquad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

These very cumbersome equations mean, that Q-multiplication loses a commutativity.

$$Q_1Q_2 \neq Q_2Q_1,$$

so that a notion of the right and the left multiplication appears, but it remains associative.

$$(Q_1Q_2)Q_3 = Q_1(Q_2Q_3).$$

Two rather different algebraic parts are separated naturally in a quaternion, these once could be denoted as scalar and vector:

$$scal\ Q = a, \qquad vect\ Q = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}.$$

Addition (subtraction) of quaternions is performed by components, scalar and vector parts are added (subtracted) separately. With respect to addition the Q-algebra is commutative and associative.

Further step is quaternion conjugation introduced similarly to that of the complex numbers

$$\bar{Q} \equiv scal\ Q - vect\ Q = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k},$$

modulus of a Q-number is defined as

$$|Q| \equiv \sqrt{Q\bar{Q}} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

This permit to formulate a quaternionic division being as multiplication "right" and "left"

$$Q_L = \frac{Q_1 \bar{Q}_2}{|Q_2|^2}, \qquad Q_R = \frac{\bar{Q}_2 Q_1}{|Q_2|^2}.$$

Definition of Q-modulus enhances the famous four squares identity

$$|Q_1 Q_2|^2 = |Q_1|^2 \, |Q_2|^2 \, .$$

Due to the properties mentioned above the Q-numbers form the algebra, which belongs to the elite group of four the so-called exclusive – "very good" – algebras: of real, complex, quaternionic numbers and the octonions (Frobenious and Horwits theorems of 1878–1898 [12]).

Special attention should be paid to Q-units representations. In terms of Hamilton real unit is simply 1 while three imaginary units similarly to complex numbers algebra are denoted as **i, j, k**. Later a simple 2×2 matrices representation of these units was revealed

$$\mathbf{i} = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{j} = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \mathbf{k} = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This representation of course is not unique. Here is a simple example. If in the above expressions imaginary unit i of complex numbers is represented as 2×2 with real elements

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then three vector Q-units turn out to be represented by real 4×4 matrices. The procedure of the matrix rank duplication can obviously be continued further.

"Tensor" form and representations

If each Q-unit is endowed with its proper number (as components of a tensor)

$$(\mathbf{i}, \mathbf{j}, \mathbf{k}) \to (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \mathbf{q}, \qquad k, j, k, l, m, n, \ldots = 1, 2, 3,$$

then quaternionic multiplication rule acquires compact form

$$1\mathbf{q}_k = \mathbf{q}_k 1 = \mathbf{q}_k, \qquad \mathbf{q}_j \mathbf{q}_k = -\delta_{jk} + \varepsilon_{jkn} \mathbf{q}_n,$$

where δ_{kn} and ε_{knj} – 3-dimension (3D) symbols Kronecker and Levi-Chivita.

It is easy to show that a number of the Q-units representations even only by 2×2 matrices is infinite. Indeed for any 2×2 matrices with properties

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \qquad B = \begin{pmatrix} d & e \\ f & -d \end{pmatrix}, \qquad TrA = TrB = 0,$$

the first two Q-units can be constructed as follows

$$\mathbf{q}_1 = \frac{A}{\sqrt{\det A}}, \qquad \mathbf{q}_2 = \frac{B}{\sqrt{\det B}},$$

while the third one is

$$\mathbf{q}_3 \equiv \mathbf{q}_1\mathbf{q}_2 = \frac{AB}{\sqrt{\det A \det B}} \qquad \text{provided that} \;\; Tr(AB) = 0.$$

The scalar unit $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is always invariant.

Transformations of Q-units and invariancy of the multiplication rule

a. Spinor-type transformations

If U is an operator changing at once all the units, and there is an inverse operator $U^{-1}: \;\; UU^{-1} = E$, then transformations

$$\mathbf{q}_{k'} \equiv U\mathbf{q}_k U^{-1} \quad \text{and} \quad 1' \equiv U1U^{-1} = E1 = 1$$

retain the multiplication rule

$$1\mathbf{q}_k = \mathbf{q}_k 1 = \mathbf{q}_k, \quad \mathbf{q}_j\mathbf{q}_k = -\delta_{jk} + \varepsilon_{jkn}\mathbf{q}_n$$

form-invariant

$$\mathbf{q}_{k'}\mathbf{q}_{n'} = U\mathbf{q}_k U^{-1} U\mathbf{q}_n U = U\delta_{kn}U^{-1} + \varepsilon_{knj}U\mathbf{q}_j U^{-1} = \delta_{kn} + \varepsilon_{knj}\mathbf{q}_{j'}.$$

Such operator can be represented for example by 2×2 matrix

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det U = 1,$$

or unimodular quaternion,

$$U = \frac{a+d}{2} + \sqrt{1 - \left(\frac{a+d}{2}\right)^2}\,\mathbf{q},$$

where

$$\mathbf{q} \equiv \left(\sqrt{1 - \left(\frac{a+d}{2}\right)^2}\right)^{-1} \begin{pmatrix} \frac{a-d}{2} & b \\ c & -\frac{a-d}{2} \end{pmatrix}.$$

In general this transformation contains 3 independent complex parameter (or 6 real ones), then $U \in SL(2, C)$. In special case of only three real parameters, then $U \in SU(2)$.

b. Vector type transformations

Vector Q-units can be transformed by 3×3 matrix $O_{k'n}$

$$\mathbf{q}_{k'} = O_{k'n}\mathbf{q}_n.$$

The requirement of Q-multiplication form-invariance forces the transformation matrix to be orthogonal and unimodular

$$O_{k'n}O_{j'n} = \delta_{kn} \Rightarrow O^{-1}_{nk'} = O_{k'n}, \quad \det O = 1.$$

This transformation in general has 6 independent real parameters, then $O \in SO(3, C)$. In the special case of three parameters $O \in SO(3, R)$. Below a variant of representation of the transformation matrix O is given with x, y, z being arbitrary real or complex functions

$$O = \begin{pmatrix} \sqrt{1-x^2-z^2} & -\frac{x\sqrt{1-y^2-z^2}+yz\sqrt{1-x^2-z^2}}{1-z^2} & \frac{xy-z\sqrt{1-x^2-z^2}\sqrt{1-y^2-z^2}}{1-z^2} \\ x & \frac{\sqrt{1-x^2-z^2}\sqrt{1-y^2-z^2}-xyz}{1-z^2} & \frac{-y\sqrt{1-x^2-z^2}-xz\sqrt{1-y^2-z^2}}{1-z^2} \\ z & y & \sqrt{1-y^2-z^2} \end{pmatrix}.$$

This matrix can be represented as a product of three irreducible multipliers

$$O = \begin{pmatrix} \sqrt{\frac{1-x^2-z^2}{1-z^2}} & -\frac{x}{\sqrt{1-z^2}} & 0 \\ \frac{x}{\sqrt{1-z^2}} & \sqrt{\frac{1-x^2-z^2}{1-z^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1-z^2} & 0 & -z \\ 0 & 1 & 0 \\ z & 0 & \sqrt{1-z^2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1-y^2-z^2}{1-z^2}} & -\frac{y}{\sqrt{1-z^2}} \\ 0 & \frac{y}{\sqrt{1-z^2}} & \sqrt{\frac{1-y^2-z^2}{1-z^2}} \end{pmatrix}.$$

after substitutions $z \equiv \sin \mathrm{B}$, $x \equiv -\sin \mathrm{A} \cos \mathrm{B}$, $y \equiv -\sin \Gamma \cos \mathrm{B}$, where A, B, Γ – are complex "angles", it takes the form

$$O = \begin{pmatrix} \cos \mathrm{A} & \sin \mathrm{A} & 0 \\ -\sin \mathrm{A} & \cos \mathrm{A} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \mathrm{B} & 0 & -\sin \mathrm{B} \\ 0 & 1 & 0 \\ \sin \mathrm{B} & 0 & \cos \mathrm{B} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Gamma & \sin \Gamma \\ 0 & -\sin \Gamma & \cos \Gamma \end{pmatrix} = O_3^{\mathrm{A}} O_2^{\mathrm{B}} O_1^{\Gamma}.$$

If the angles are real: $\mathrm{A} = \alpha$, $\mathrm{B} = \beta$, $\Gamma = \gamma$, then this transformation is an ordinary vector rotation consisting of three simple rotations around numbered orthogonal axes: $O \Rightarrow R, R = R_3^{\alpha} R_2^{\beta} R_1^{\gamma}$. Correlation between related "spinor" and "vector" transformations is easily determined:

$$O_{k'n} = -\frac{1}{2} Tr(U\mathbf{q}_k U^{-1}\mathbf{q}_n), \qquad U = \frac{1 - O_{k'n}\mathbf{q}_k\mathbf{q}_n}{2\sqrt{1 + O_{mm'}}}.$$

Q-geometry in three dimensional space

Hamilton was the first to note that triad of Q-units behaves as three strictly tied unit vectors (with length i) initiating Cartesian coordinate system, somewhat exotic because of its "imaginarity". Due to the fact the Q-triad in 3D-space $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ will be called 'quaternionic basis' (Q-basis). Now Q-units transformations have apparent geometrical sense of various rotations of the Q-basis. An example: a simple rotation by real angle α around axis # 3

$$\mathbf{q}' = R_3^\alpha \mathbf{q}.$$

Notion of Q-basis helps to introduce 3D quaternionic vectors (Q-vectors), defined as

$$\mathbf{a} = a_k \mathbf{q}_k,$$

here all its components a_k are real. The most important property of Q-vector – is its invariancy with respect to vector transformations from the group SO(3,R)

$$\mathbf{a}' = a_{k'} \mathbf{q}_{k'} = a_{k'} R_{k'j} \mathbf{q}_j = a_j \mathbf{q}_j = \mathbf{a}.$$

The projection of Q-vector onto arbitrary coordinate axis (represented by any different Q-unit) can be found in two ways. First, if at least one set of projections of Q-vector and rotation matrices $R_{nk'}$ are known then projections of this vector on rotated axis are immediately found

$$a_{k'} = a_n R_{nk'}.$$

The second approach is related to existence of internal structure of the Q-units; a brief analysis of it is given in the next section.

2. Structure of quaternionic "imaginary" units

Eigenfunctions of Q-units [13]

Each vector Q-unit can be thought of as operator, so eigenfunctions and eigenvalues problem can be formulated for it

$$\mathbf{q}\psi = \lambda\psi, \qquad \varphi\mathbf{q} = \mu\varphi.$$

The solution of this problem are the eigenvalues ("imaginary length" of Q-unit with division by parity)

$$\lambda = \mu = \pm i,$$

and two sets of eigenfunctions(one for each parity), possible given by columns $\psi^\pm$ and rows $\varphi^\pm$, being the functions of components $\mathbf{q}$.

Here is an example explicit form of eigenfunction: for the Q-unit represented by matrix

$$\mathbf{q} = -\frac{i}{T}\begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

where $T \equiv a^2 + bc \neq 0, b \neq 0, c \neq 0,$ its eigenfunctions are defined as

$$\varphi^{\pm} = x \begin{pmatrix} 1 & \pm\frac{b}{T \pm a} \end{pmatrix}, \qquad \psi^{\pm} = y \begin{pmatrix} 1 \\ \mp\frac{c}{T \pm a} \end{pmatrix},$$

where x, y are arbitrary complex factors.

The freedom of components, arising in the calculations is reduced by convenient normalization condition

$$\varphi^{\pm}\psi^{\pm} = 1,$$

while the eigenfunctions orthogonality (by parity) is an inherited property

$$\varphi^{\mp}\psi^{\pm} = 0.$$

One can construct tensor products of eigenfunctions and obtain 2×2 matrices

$$C^{\pm} \equiv \psi^{\pm}\varphi^{\pm},$$

possessing a properties reciprocal with respect to the ones of vector $\mathbf{q}$:

$$\det C = 0, \quad Tr\, C = 1,$$

whereas

$$\det \mathbf{q} = 1, \quad Tr\, \mathbf{q} = 0.$$

Matrix C is idempotent

$$C^n = C,$$

and can be expressed throw their own unit Q-vector

$$C^{\pm} = \frac{1 \pm i\mathbf{q}}{2}.$$

When inversed this expression gives information about Q-unit internal structure

$$\mathbf{q} = \pm i(2C^{\pm} - 1) = \pm i(2\psi^{\pm}\varphi^{\pm} - 1),$$

which turns out to consist of a combination of its eigenfunctions and scalar units.

Since each Q-unit has its own eigenfunctions the Q-triad as a whole possesses unique set of eigenfunctions $\{\varphi^{\pm}_{(k)}, \psi^{\pm}_{(k)}\}$. There is an interesting algebraic observation concerning this set. Three Q-units are interrelated by obviously nonlinear combination – multiplication e. g.

$$\mathbf{q}_3 = \mathbf{q}_1\mathbf{q}_2,$$

but it is easy to show that corresponding eigenfunctions depend on each other linearly:

$$\varphi^{\pm}_{(3)} = \sqrt{\mp i}\varphi^{\pm}_{(1)} \pm \sqrt{i}\varphi^{\pm}_{(2)}, \quad \psi^{\pm}_{(3)} = \sqrt{\pm i}\psi^{\pm}_{(1)} \pm \sqrt{-i}\psi^{\pm}_{(2)}.$$

Q-eigenfunctions help to represent a spinor-type transformation of Q-units retaining Q-multiplication invariant in the familiar form

$$\psi^{\pm}_{(k')} = U\psi^{\pm}_{(k)}, \quad \varphi^{\pm}_{(k')} = \varphi^{\pm}_{(k)} U^{-1},$$

so that the eigenfunctions can be regarded as a set of specific spinor functions, allowing in subject in general to $SL(2C)$ transformations. Yet another mathematical observation should be noted: from pairs of eigenfunctions, belonging to different Q-units of one triad and having one parity, one can construct 24 scalar invariants $SL(2C)$ group; these invariants are real or complex numbers, e. g.:

$$\sigma^{+}_{12} \equiv \varphi^{+}_{(1)}\psi^{+}_{(2)} = \sqrt{-\frac{i}{2}} = \frac{1-i}{2}.$$

Quaternionic eigenfunctions as projectors

Eigenfunctions act on their own Q-basis as following

$$\varphi^{\pm}_{(1)}\mathbf{q}_1\psi^{\pm}_{(1)} = \pm i, \quad \varphi^{\pm}_{(1)}\mathbf{q}_2\psi^{\pm}_{(1)} = 0, \quad \varphi^{\pm}_{(1)}\mathbf{q}_3\psi^{\pm}_{(1)} = 0,$$

or in general

$$\varphi^{\pm}_{(k)}\mathbf{q}_n\psi^{\pm}_{(k)} = \pm i\delta_{kn} \qquad \text{(no summation by k)}.$$

It looks like that eigenfunctions select a projection of the unit Q-vector, generating them. This idea is confirmed by an example of an action of eigenfunctions of one Q-basis onto the vectors of the rotated Q-basis

$$\varphi^{\pm}_{(k)}\mathbf{q}_{n'}\psi^{\pm}_{(k)} = \varphi^{\pm}_{(k)}R_{n'm}\mathbf{q}_m\psi^{\pm}_{(k)} = \pm iR_{n'k} = \pm i\cos\angle(\mathbf{q}_{n'}, \mathbf{q}_k) \quad \text{(no summation by k)},$$

the result of the action is 'nearly' projection of Q-basis $\mathbf{q}'$ on $\mathbf{q}$. It is convenient to denote precise projection as

$$\langle \mathbf{q_{n'}} \rangle_k \equiv \mp i\varphi^{\pm}_{(k)}\mathbf{q}_{n'}\psi^{\pm}_{(k)} = \cos\angle(\mathbf{q}_{n'}, \mathbf{q}_k) \qquad \text{(no summation by k)}.$$

It is now easy to formulate rule of calculation of projection of a Q-vector a onto arbitrary direction, defined by vector $\mathbf{q}_j$ (e. g. with help of eigenfunctions of positive parity)

$$\langle \mathbf{a} \rangle^{+}_{j} \equiv -ia_{k'}\varphi^{+}_{(j)}\mathbf{q}_{k'}\psi^{+}_{(j)} = a_{k'}R_{k'j} = a_j \qquad \text{(no summation by j)}.$$

Thus quaternionic eigenfunctions with their own interesting properties are more fundamental mathematical objects then Q-units and too can serve as useful tool for practical purposes such as computing projections of Q-vectors.

4. Differential Q-geometry

Quaternionic connection

If vectors of Q-basis are smooth functions of parameters $\mathbf{q}_k(\Phi_\xi)$ (index ξ enumerates parameters), then

$$d\mathbf{q}_k(\Phi) = \omega_{\xi\ kj}\mathbf{q}_j d\Phi_\xi,$$

where an object $\omega_{\xi\ kj}$ is called quaternionic connection. Q-connection is antisymmetric in vector indices

$$\omega_{\xi\ kj} + \omega_{\xi\ jk} = 0,$$

and has the following number of independent components

$$N = Gp(p-1)/2,$$

where G is an number of parameters and $p = 3$ – is a number of space dimensions. If $G = 6$ [a case of group $SO(3, C)$], then $N = 18$; if $G = 3$ [a case of group SO(3,R)], then N = 9. Q-connection can be calculated at least in three ways:

$$\text{using vectors of Q-basis} \qquad \omega_{\xi\ kn} = \left\langle \frac{\partial \mathbf{q}_k}{\partial \Phi_\xi} \right\rangle_n^+,$$

using matrices U from the group SL(2C) (general case) and special representation of constant Q-units $\mathbf{q}_{\tilde{k}} = -i\sigma_k$, where σ_k – Pauli matrices

$$\omega_{\xi\ kn} = \left\langle U^{-1}\frac{\partial U}{\partial \Phi_\xi}\mathbf{q}_{\tilde{k}} - \mathbf{q}_{\tilde{k}} U \frac{\partial U^{-1}}{\partial \Phi_\xi} \right\rangle_n^+,$$

and, finally, using matrices O from $SO(3, C)$ (in a general case)

$$\omega_{\xi\ kn} = \frac{\partial O_{k\tilde{j}}}{\partial \Phi_\xi} O_{n\tilde{j}}.$$

All the formulae of course provide same result.

From the point of view of vector transformations a Q-connection is not a tensor. If $\mathbf{q}_k = O_{kp'}\mathbf{q}_{p'}$, then transformed components of connection are expressed throw original ones with addition of inhomogeneous term

$$\omega_{\xi\ kj} = O_{kp'}O_{jn'}\omega_{\xi\ p'n'} + O_{jp'}\frac{\partial O_{kp'}}{\partial \Phi_\xi}.$$

In 3D space Q-connectivity has clear geometrical and physical treatment as moveable Q-basis with behavior of Cartan 3-frame. Parameters of its ordinary rotations can depend on spatial coordinates $\Phi_\xi = \Phi_\xi(x_k)$, then $\partial_n \mathbf{q}_k = \Omega_{nkj}\mathbf{q}_j$, then components of slightly modified Q-connection

$$\Omega_{nkj} \equiv \omega_{\xi\ kj}\partial_n \Phi_\xi$$

have a sense of Ricci rotation coefficients. Parameters can also depend on the length of line of motion of the Q-basis or on the observer's time. Then $\Phi_\xi = \Phi_\xi(t), \partial_t \mathbf{q}_k = \Omega_{kj}\mathbf{q}_j$, and components of Q-connection

$$\Omega_{kj} \equiv \omega_{\xi\ kj}\partial_t \Phi_\xi$$

became generalized angular velocities of rotations of the frame.

The typical examples of Q-frames and Q-connection are

a) Frene frame. For the smooth curve $x_{\tilde{k}}(s)$ defined in constant basis the Frene frame is represented by the triad $\mathbf{q}_k$, obeying the equations

$$\frac{d}{ds}\mathbf{q}_1 = R_I(s)\mathbf{q}_2,\ \frac{d}{ds}\mathbf{q}_2 = -R_I(s)\mathbf{q}_1 + R_{II}(s)\mathbf{q}_3,\ \frac{d}{ds}\mathbf{q}_3 = -R_{II}(s)\mathbf{q}_2,$$

where the first and the second curvatures are

$$R_I = \Omega_{12}, R_{II} = \Omega_{23}.$$

b) Twisted straight line. For a given straight line $x_{\tilde{1}} = u,\ x_{\tilde{2}} = x_{\tilde{3}} = 0$, one can construct a Q-basis associated with it so that one vector is tangent to the line. In this case Q-connection is not zero and represented the only component describing torsion (or rather twist) of the line about itself.

$$\mathbf{q}_1 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{q}_2 = -i\begin{pmatrix} 0 & -ie^{-i\gamma(u)} \\ ie^{i\gamma(u)} & 0 \end{pmatrix}, \qquad \Omega_{23} = \frac{d\gamma}{du},$$

$\gamma(u)$ is the angle, which is an arbitrary but smooth function of the line length.

Quaternionic spaces

Tangent Q-space [15]. It is known that on every N-dimensional differentiable manifold U_N with coordinates $\{y^A\}$ one can construct a tangent space T_N with coordinates $\{X^{(A)}\}$ so that $dX^{(A)} = g_B^{(A)} dy^B$, where $g_B^{(A)}$ – Lame coefficients. By an extra rotation one can construct a tangent Q-space $T(U, \mathbf{q})$, with coordinates $\{x_k\}$, $k = 1, 2, 3$, which associated with Q-frame vectors.

$$dx_k = h_{k(A)} dX^{(A)} = h_{k(A)} g_B^{(A)} dy^B,$$

where $h_{k(A)}$ are in general non-square matrices normalized by projectors of the basic space onto 3D one or vice versa.

Proper quaternionic space itself $\mathbf{U}_3$ is defined as 3D-space, locally identical to own tangent space $T(\mathbf{U}_3, \mathbf{q})$. The Q-space has the following basic features. Its Q-metric represented by vector part of the Q-multiplication rule $\mathbf{q}_j\mathbf{q}_k = -\delta_{jk} + \varepsilon_{jkn}\mathbf{q}_n$ is nonsymmetric, its antisymmetric part is Q-operator (matrix), so that every point $\mathbf{U}_3$ has internal quaternionic structure. Q-connection $\mathbf{U}_3$ can be: (i) proper (metric) $\Omega_{nkj} \equiv \omega_{\xi kj}\partial_n\Phi_\xi$, for variable Q-basis it is always non zero, and (ii) affine (non-metric), independent from Q-basis. Q-torsion does not vanish in both cases, whereas Q-curvature $r_{knab} = \partial_a\Omega_{bkn} - \partial_b\Omega_{akn} + \Omega_{ajn}\Omega_{bjk} - \Omega_{bjk}\Omega_{ajn}$ for the metric Q-connection identically equals zero, but can be present in the space of affine Q-connection.

Once Q-space is introduced, there appears a new field of investigation of differential manifolds and spaces. Thus in the preliminary classification of Q-spaces based on presence and nature of curvature, torsion and non-metricity at least 10 different families can be distinguish [15]. In addition Q-spaces can be a nontrivial background for classical and quantum theories and problems.

4. Newton mechanics in Q-basis

Dynamics equations in rotating frame [16]

The Q-basis endowed with clock becomes a classical (non-relativistic) reference system. For an inertial observer the dynamic equations of classical mechanics can be written in constant Q-basis

$$m\frac{d^2}{dt^2}x_{\tilde{k}}\mathbf{q}_{\tilde{k}} = F_{\tilde{k}}\mathbf{q}_{\tilde{k}}.$$

$SO(3,R)$-invariance of two Q-vectors, the radius-vector $\mathbf{r} \equiv x_k\mathbf{q}_k$ and force $\mathbf{F} \equiv F_k\mathbf{q}_k$ allow to represent these equations in Q-vector form

$$m\frac{d^2}{dt^2}(x_k\mathbf{q}_k) = F_k\mathbf{q}_k, \qquad \text{or} \quad m\ddot{\mathbf{r}} = \mathbf{F}$$

In explicit form these equations possess enough complicated structure

$$m(\frac{d^2}{dt^2}x_n + 2\frac{d}{dt}x_k\Omega_{kn} + x_k\frac{d}{dt}\Omega_{kn} + x_k\Omega_{kj}\Omega_{jn}) = F_n$$

which nevertheless can be simplified and interpreted from physical points of view. Due to antisymmetry of the connection (generalized angular velocity)

$$\Omega_j \equiv \Omega_{kn}\frac{1}{2}\varepsilon_{knj}, \qquad \Omega_{kn} = \Omega_j\varepsilon_{knj},$$

the dynamic equations can be rewritten in vector components

$$m(a_n + 2v_k\Omega_j\varepsilon_{knj} + x_k\frac{d}{dt}\Omega_j\varepsilon_{knj} + x_k\Omega_j\Omega_m\varepsilon_{jkp}\varepsilon_{mpn}) = F_n$$

or by conventional vector notation

$$m(\vec{a} + 2\vec{\Omega}\times\vec{v} + \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})) = \vec{F}.$$

Among left hand side terms one easily recognizes 4 classical accelerations: linear, Coriolis, angular and centripetal. However this traditional interpretation is good only for simple rotation; in the case of combination of many Q-frame rotations number of components of generalized accelerations highly increases, and the equations become much more complicated. However it is worth noting that derivation of these equations for the most complicated rotations with the help of Q-basis and Q-connection is extremely simple.

Samples of Q-formulation of problems of classical mechanics

'Chasing' Q-basis – is a frame with one of its vectors, say $\mathbf{q}_1$ is always directed to observed particle. Dynamic equations for this case are written in explicit form in following manner

$$\ddot{r} - r(\Omega_2^2 + \Omega_3^2) = F_1/m,$$
$$2\dot{r}\Omega_3 + r\dot{\Omega}_3 + r\Omega_2\Omega_1 = F_2/m,$$
$$2\dot{r}\Omega_2 + r\dot{\Omega}_2 + r\Omega_1\Omega_3 = -F_3/m.$$

Components of Q-connection are defined as functions of angles of two rotations, the first (an angle α) – around vector $\mathbf{q}_3$, the second (an angle β) – around $\mathbf{q}_2$

$$\Omega_1 = \dot{\alpha}\sin\beta, \qquad \Omega_2 = -\dot{\beta}, \qquad \Omega_3 = \dot{\alpha}\cos\beta.$$

The chasing Q-basis approach is convenient to solve a number of mechanical problems related to rotations, some times very complicated, of observed objects and systems of reference. Here is an illustration.

Rotating oscillator. One seeks for motion law $r(t)$ of a harmonic oscillator (mass m, spring elasticity k) which has a freedom of motion along rigid smooth rod rotating in the plane around one of its ends (here one end of the spring is fixed) with angular velocity ω; the equilibrium point is located at the distance l from the rotation center, there is no gravity. Radial and tangent dynamic equations in the chasing Q-basis (F is unknown rod reaction force)

$$\ddot{r} - r\omega^2 = -\frac{k}{m}(r - l), \qquad 2\dot{r}\omega = \frac{1}{m}F,$$

admit the following family of solutions:

$$\text{(i)} \qquad r(t) = r_0 + v_0 t + at^2$$

mass moves away from the center of rotation with quadratic (or linear) law,

$$\text{(ii)} \qquad r(t) = const + Ae^{iwt} + Be^{-iwt}, \qquad w \equiv \sqrt{k/m - \omega^2}$$

here are three different situations depending on a relation of the quantities under the square root:

– $r = const$,
– harmonic oscillators,
– exponential motion away from the center of rotation.

It is interesting that the variants of rotating classical oscillator behavior with $l = 0$ are precisely similar to behavior of four known cosmological models of Einstein-DeSitter-Friedman considered in the General Relativity.

5. Construction of Quaternionic Relativity

Hyperbolic rotations and biquaternions [17]

We noted above, that $SO(3,C)$-transformations of Q-units admit pure imaginary parameters. In this case rotations become hyperbolical (H – from hyperbolic); e.g. simple H-rotation $\mathbf{q}' = H_3^{\psi}\mathbf{q}$ is performed by matrix of the form

$$H_3^{\psi} = \begin{pmatrix} \cosh\psi & -i\sin\psi & 0 \\ i\sin\psi & \cosh\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and 2×2-matrices of Q-units representation are no longer hermitian:

$$\mathbf{q}_{1'} = -i \begin{pmatrix} 0 & e^{\psi} \\ e^{-\psi} & 0 \end{pmatrix}.$$

This is the time to recall the notion of so called biquaternionic vectors (BQ). BQ-vector is defined as Q-vector with complex components $\mathbf{u} = (a_k + ib_k)\mathbf{q}_k$. Obviously for vectors of this type the norm (or modulus) in general can not be defined; but among all BQ-vectors there is a subset of "good" elements with well definable norm by $\mathbf{u}^2 = b^2 - a^2$. These vectors appear to be form-invariant with respect to transformations of subgroup $SO(2,1) \subset SO(3,C)$, and in particular, with respect to simple H-rotations $\mathbf{q}' = H\mathbf{q}\mathbf{u} = u_k\mathbf{q}_k = u_{k'}\mathbf{q}_{k'}$, but only when reciprocally imaginary components $a_k b_k = 0$ are orthogonal to each other.

Quaternionic Relativity

The made above observation allows to suggest a space-time BQ-vector "interval"

$$d\mathbf{z} = (dx_k + idt_k)\mathbf{q}_k,$$

with specific properties:

(i) Temporal interval is defined by imaginary vector,

(ii) space-time of the model appears to have six-dimensional (6D),

(iii) vector of the displacement of the particle and vector of corresponding time change must always be normal to each other $dx_k dt_k = 0$.

In this case BQ-vector-interval is invariant under group $SO(2,1) \subset SO(3,C)$, as well as of course its square (which differs from the square of norm only by sign) $d\mathbf{z}^2 = dt^2 - dr^2$, the latter has precisely the same form as a space-time interval of Special Relativity of Einstein. This 6D-model was initially named the Quaternionic Relativity. Temporal and spatial variables symmetrically enter the expression of BQ-vector-interval, and the Q-triad related to it describes relativistic system of reference $\Sigma \equiv (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$. Transition from one reference system to another is performed with the help of 'rotational equations' of the type $\Sigma' = O\Sigma$ with matrix O from the group $SO(2,1)$ is a product of matrices of real and hyperbolic rotations. So the theory could also be named (may be more correctly) 'Rotational Relativity'. The meaning of a simple H-rotation is immediately revealed from first line of equation $\Sigma' = H_3^{\psi}\Sigma$ in the explicit form

$$i\mathbf{q}_{1'} = i\cosh\psi(\mathbf{q}_1 + \tanh\psi\mathbf{q}_2).$$

If like in Special Relativity $\cosh\psi = dt/dt'$, then

$$idt'\mathbf{q}_{1'} = idt(\mathbf{q}_1 + V\mathbf{q}_2),$$

which describes motion of reference system Σ' relative to Σ with velocity V along direction $\mathbf{q}_2$. It is easy to show that $SO(2,1)$-rotations of Q-reference system enhance Lorenz coordinate transformations and therefore all cinematic effects of Special Relativity.

It should be noted here that parameters of real and hyperbolic rotations can be variable for instance dependent on observer's time. This hints to expect of the discussed theory a possibility to describe non-inertial motions. Analysis of the rotational equations confirms the expectation. Well-known relativistic model of reference system constantly accelerated with respect to the inertial one (hyperbolic motion), frequently found in literature and normally regarded with use of assumption beyond frames of Special Relativity, in quaternionic theory is solved naturally and fast not only from the inertial observer viewpoint, but from position of accelerated frame too [18].

The kinematic problem of other non-inertial motion – relativistic circular motion – can be completely and precisely resolved by means of the rotation equation $\Sigma' = H_2^{\psi(t)} R_1^{\alpha(t)}\Sigma$, where Σ' is reference system rotating along the circle around the immobile frame Σ. This problem also can be solved both from the point of view of inertial observer, in this case the result has the form

$$t = \int dt' \cosh\psi(t'), \qquad \alpha(t) = \frac{1}{R}\int dt' \tanh\psi(t'),$$

$$a_{\tan}(t) = \frac{1}{\cosh^2\psi}\frac{d\psi}{dt}, \qquad a_{norm}(t) = R\left(\frac{d\alpha(t)}{dt}\right)^2,$$

and from the point of view of the observer in the reference system arbitrary moving along circular orbit.

The solution of the problem of "classical" Thomas precession in the framework of Special Relativity also needs additional assumptions, while in the quaternionic theory has a single line form – the first row of the matrix of rotation equation $\Sigma'' = R_1^{-\alpha(t)} H_2^{\psi} R_1^{\alpha(t)} \Sigma$, in this case of course correct value of precession frequency is obtained

$$\omega_T = (1 - \cosh \psi) \approx -\frac{1}{2}\omega V^2.$$

Moreover, the quaternionic theory of relativity appears to be able to describe Thomas precession for the vectors moving along trajectories of general type. The basic rotational equation in this case naturally generalized: $\Sigma'' = R^{-\theta(t)} H^{\psi(t)} R^{\theta(t)} \Sigma$, here $\theta(t)$ – an angle of instant rotation. Requirement that an axis of hyperbolic rotation be normal to the plane formed by the radius-vector of observed frame and its velocity vector, is also significant. In this case formula of variable frequency of general Thomas precession has the form

$$\Omega_T = \frac{d}{dt}(\theta - \theta').$$

An example of such Thomas precession is an apparent displacement of mercurial perihelion, for which calculations give a value $\Delta\varepsilon = 2,7''/100$ years.

Universal character of motion of the bodies (including non-inertial motions) in the Quaternionic Relativity suggests seeking for new cinematic relativistic effects. One is found in Solar System planets' satellites motion. Relative velocity of the Earth and other planets changes with time and sometimes achieves significant value comparable somehow to value of the fundamental velocity. This can lead to discrepancy between calculated and observed from the Earth cinematic magnitudes characterizing cyclic processes on this planet or near it. In particular there must be a deviation of the planetary satellite position. Such an angular difference is surprisingly found to be linearly dependent upon the time of observation

$$\Delta\varphi \approx \frac{\omega V_E V_P}{c^2} t,$$

here ω is an angular velocity of satellite motion around the planet, V – are linear velocities of the Earth and the planet around the sun. The magnitude of the effect is the following for the closest to the Jupiter and "the fasters" Jupiter satellite $\Delta\varphi \cong 12'$ for 100 terrestrial years; for the Mars satellite (Phobos) $\Delta\varphi \cong 20'$ for 100 terrestrial years [19]. Both values are big enough for the effect to be noticed in prolonged and precise observations.

One can say that space-time model and kinematics of the Quaternionic Relativity are nowadays studied in enough details and can be used as an effective mathematical tool for calculation of many relativistic effects. But respective relativistic dynamic has not been yet formulated, there are no quaternionic field theory; Q-gravitation, electromagnetism, weak and strong interactions are still remote projects. However, there is a hope that it is only beginning of a long way, and the theory will "mature". This hope is supported by observation of number of remarkable "Quaternionic Coincidences" forming a discrete mosaic of physical and mathematical facts; probably one day it will turn into a logically consistent picture providing new instruments and extending our insight of physical laws.

6. Remarkable "quaternionic coincidences"

There are, at least, five such coincidences (all of them given below), noted by different authors in various time.

1. *The Maxwell equations as an conditions of the analyticity of functions of quaternionic variable.*

In 1937 year Fueter [20] noted, that Cauchy-Riemann $\partial f/\partial z^* = 0$ equations defining the differentiability of complex variable function and modeling physically a flat motion of liquid without sources and whirls, have the following quaternionic analogue

$$\left(i\frac{\partial}{\partial t} - \mathbf{q}_{\tilde{k}}\frac{\partial}{\partial x_{\tilde{k}}}\right)\mathbf{H} = 0, \qquad \mathbf{H} = (B_{\tilde{n}} + iE_{\tilde{n}})\mathbf{q}_{\tilde{n}}.$$

Surprising fact is that the equations of classic Maxwell electrodynamics in vacuum prove to be corresponding physical model

$$div\vec{E} = 0, \quad div\vec{B} = 0, \quad rot\vec{E} - \frac{\partial\vec{B}}{dt} = 0, \quad rot\vec{B} + \frac{\partial\vec{E}}{dt} = 0.$$

2. *Classical mechanics in the rotating reference systems.*

The compact form of Newton equations in quaternion frame is described above in section 4. Finally it should be stressed that the form of dynamics equations naturally arising and externally primitive

$$m\ddot{\mathbf{r}} = \mathbf{F}$$

hides all possible combinations of rotations of reference systems or observed bodies. Using differential quaternionic objects helps to easily obtain explicit form of the equations whose elements have obvious physical meaning.

3. *The quaternionic theory of relativity.*

1:1 isomorphism of the Lorenz group of Special Relativity and the group of invariance of quaternionic multiplication $SO(3, C)$ leads to non-standard theory of relativity with symmetric six-dimensional space-time. This theory significantly differs from Einstein Special Relativity in origin, model, possibilities and mathematical tools, but predicts absolutely similar cinematic effects. Invariance of specific biquaternionic vector "interval" $d\mathbf{z} = (dx_k n + i\, dt_k)\mathbf{q}_k$ under subgroup $SO(2, 1)$ with in general variable parameters admits calculation of relativistic effects for non-inertial motion of reference systems.

4. *Pauli equations* [21].

Consider the quantum particle with electric charge e, mass m, and generalized momentum

$$P_k \equiv -i\hbar\frac{\partial}{\partial x_k} - \frac{e}{c}A_k$$

in the simplest quaternionic space (all the parameters are constant, connection, non-metricity, torsion and curvature equal to zero). Hamiltonian of such particle in Q-metrics

$$H \equiv -\frac{1}{2m}P_k P_m \mathbf{q}_k \mathbf{q}_m$$

is the exact copy of Hamilton function of Pauli equation

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{2mc}\vec{B}\cdot\vec{\sigma},$$

and the spin term "automatically" acquires a coefficient equal to Bohr magneton.

5. *Yang-Mills field strength.*

If one constructs a "potential" vector in an arbitrary quaternionic space from Q-connection components Ω_{amn} (indices a, b, c enumerate coordinates of basic Q-space, indices j, k, m, n enumerate vectors of tangent triad)

$$A_{ka} \equiv \frac{1}{2}\varepsilon_{kmn}\Omega_{amn},$$

and similarly construct a "field strength" vector

$$F_{kab} \equiv \frac{1}{2}\varepsilon_{kmn}r_{mnab},$$

from quaternionic curvature components

$$r_{knab} = \partial_a\Omega_{bkn} - \partial_b\Omega_{akn} + \Omega_{ajn}\Omega_{bjk} - \Omega_{bjk}\Omega_{ajn}$$

then these two geometrical objects are interconnected in the similar manner as the field strength and potential of the Yang-Mills field

$$F_{kab} \equiv \partial_b A_{ka} - \partial_a A_{kb} + \varepsilon_{kmn}A_{ma}A_{nb}.$$

(formula) It should be stressed that for the Q-spaces with metric (not affine) connection curvature (field strength) identically vanish.

Discussion

Quaternionic numbers of course are first of all mathematical objects, so the problem of development of their algebra, analysis and geometry is self-consistent. But history of modern science states that once the geometry, in particular differential geometry, is discussed the presence of physics is unavoidable. There is a known point of view that Einstein who suggested General Relativity was a pioneer in geometrization of physics. But it is also known that quite earlier Maxwell formulated his electrodynamics in terms of quaternions convenient for description of 'etheric tensions' which were thought to represent field strength vectors. But since that the geometrical language has not been utilized for many decades.

The aspects of quaternionic mathematics given in this review once again draw attention to 'genetic relations' between physics and geometry: from description of frames rotations to quaternionic field structure phenomena in Pauli equations and Yang-Mills theory.

Wide variety of possibilities provided by Q-approach and derived within it non-traditional physical models, like six-dimensional space-time or mentioned above coincidences may lead to opinion that quaternions are still a mathematical play, something like 'lego' elements, from which one can build many exotic constructions.

As a comment there are the following two observations.

1. Producing non-standard physical models Q-method nonetheless allows to successfully solve physical problems thus being a useful tool for practical purposes. A typical example: inherited exponential character of representation of simple rotations helps to simply formulate summation of different rotations, including, of course, imaginary rotations, describing relativistic boosts. Recall that in classical mechanics summation of ordinary rotations is quite a task.

2. All physical quaternionic theories are not heuristically invented, but appear naturally from fundamental mathematical lows, as though confirming Pythagorean idea on "world – number" dependence. Indeed, Q-algebra, the last associative algebra, describes well physical quantities, all of them up to our knowledge being associative with respect to multiplication: from observable kinematic and dynamic one, to tensors and spinors incorporated in the theories. All this gives a hope that further efforts in the research "quaternions – physical laws" relations will once grow into wide scientific programme. Yet another small, but persevering step in this direction has been recently made, when the author of this review succeeded to found an exact solution for relativistic oscillator problem in the framework Quaternionic Relativity. Details of the solution will be published elsewhere.

References

[1] W. R. Hamilton, *Lectures on Quaternions,* Hodges & Smith, Dublin, 1853.

[2] W. R. Hamilton, *Elements of Quaternions,* Chelsey Publ. Co. N. Y., 1969.

[3] D. J. Struik, *A Concise History of Mathematics,* 4th ed., Dover Publ., N. Y., 1987.

[4] N. Burbaki, *Sketches of Mathematics History,* Science, M., 1963.

[5] A. N. Bogolubov, *Mathematicians, Mechanicians,* Science, Kiev, 1983.

[6] F. Klien, *Arithmetic, Algebra, Analyses,* Dover Publ. (Transl. from 3-d German ed.), N. Y., 1924.

[7] P. Rastall, *Quaternions in Relativity,* Review of Modern Physics, July (1964), 820–832.

[8] V. N. Branets, I. P. Shmyuglevscy, *Quaternions in Problems of Solid State Orientation,* Science, M., 1973 (in Russian).

[9] L. P. Horwitz, L. C. Biedenharn, *Quaternionic quantum mechanics: second quantization and gauge fields,* Ann. Phys., 157 (1984), 432–488.

[10] A. V. Berezin, Yu. A. Kurochkin, E. A. Tolkachev, *Quaternions in Relativity Physics,* Science and Technic, Minsk, 1989 (in Russian).

[11] P. S. Bisht, O. P. Negi, B. S. Rajput, *Quaternionic Gauge Theory of Dyonic Fields,* Progr. Theor. Phys., 85, 1 (1991), 157–168.

[12] I. L. Kantor, A. S. Solodovnikov, *Hypercomplex Numbers.* Springer, Berlin, 1989

[13] A. P. Yefremov, *Q-field, variable quaternionic basis. Physics,* Izvestiya vuzov, 12 (1985), 14–18.

[14] A. P. Yefremov, *Tangent Quaternionic Space,* Gravitation & Cosmology, 7, 4 (2001), 273–275.

[15] A. P. Yefremov, *Structure equations and preliminary classification of quaternionic spaces*, Abstracts of the 11-th Int. GRG Conf, 123, Tomsk, 2002.

[16] A. P. Yefremov, *Newton Mechanics in Quaternionic Basis,* RUPF, M., 1995.

[17] A. P. Yefremov, *Quaternionic Relativity. I. Inertial Motion,* Gravitation & Cosmology, 2, 1 (1996), 77–83.

[18] A. P. Yefremov, *Quaternionic Relativity. II. Non-Inertial Motion,* Gravitation & Cosmology, 2, 4 (1996), 335–341.

[19] A. P. Yefremov, *Rotational Relativity,* Acta Phys. Hungarica, New Series – Heavy Ion Physics 11, 1–2 (2000), 147–153.

[20] R. Fueter, Comm. Math. Hel., B7S (1934-1935), 307–330.

[21] A. P. Yefremov, *Quaternionic Multiplication Rule as a local Q-Metric,* Lett. Nuovo Cimento 37, 8 (1983), 315–316.

Algebrodynamics: Primordial Light, Particles-Caustics and Flow of Time

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In the field theories with twistor structure particles can be identified with (spatially bounded) caustics of null geodesic congruences defined by the twistor field. As a realization, we consider the "algebrodynamical" approach based on the field equations which originate from noncommutative analysis (over the algebra of biquaternions) and lead to the complex eikonal field and to the set of gauge fields associated with solutions of the eikonal equation. Particle-like formations represented by singularities of these fields possess "elementary" electric charge and other realistic "quantum numbers" and manifest self-consistent time evolution including transmutations. Related concepts of generating "World Function" and of multivalued physical fields are discussed. The picture of Lorentz invariant light-formed aether and of matter born from light arises then quite naturally. The notion of the Time Flow identified with the flow of primordial light ("pre-Light") is introduced in the context.

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Introduction. The algebrodynamical field theory

Theoretical physics has arrived to the crucial point at which it should fully reexamine the sense and the interrelations of the three fundamental entities: fields, particles and space-time geometry. *String theory* offers a way to derive the low-energy phenomenology from the unique physics at Plankian scale. However, it doesn't claim to find the *origin* of physical laws, the *Code of Universe* and is in fact nothing but one more attempt to *describe* Nature (in a possibly the most effective way) but not at all to *understand* it.

Twistor program of R. Penrose [1, 2] suggests an alternative to string theory in the framework of which one can hope, in principle, to explain the origin of basic physical entities. For this, one only assumes the existence of the primary *twistor space* $\mathbb{C}P^3$ which underlies the physical space-time and predetermines its Minkowsky geometry and, to some extent, the set of physical fields.

The most interesting manifestation of twistor structure is its ability to reduce the resolution of free massless (conformally invariant) equations (both linear and nonlinear ones, specifically of the Yang-Mills type) either to explicit integration in twistor space (the so called Penrose transform) or to resolution of purely algebraic problems (the Kerr theorem, the Ward construction etc. [2]). Making use of the Kerr theorem and of the Penrose's "nonlinear graviton construction", one can also obtain, in a purely algebraic way, the whole set of the self-dual solutions to (complex) Einstein equations.

However, general concept of twistor program as a unified field theory is not at all clear or formulated up to now. Which equations are really fundamental, in which way can the massive fields be described and in which way the particles' spectrum can be obtained? And, finally, why precisely twistor, a rather refined mathematical object, should be taken as a basis of fundamental physics?

In the interim, twistor structure arises quite naturally in the so called *algebrodynamics* of physical fields which has been developed in our works. From general viewpoint, the paradigm of algebrodynamics can be thought of as a revive of Pithagorean or Platonean ideas about *"Numbers governing physical laws"*. As the only (!) postulate of algebrodynamics one admits the existence of a certain unique and exceptional structure, of purely abstract (algebraic) nature, the internal properties of which completely determine both the geometry of physical space-time and the dynamics of physical fields (the latters being also algebraic in nature).

In the most successful realization of algebrodynamics principal structure of the "World algebra" has been introduced via generalization of complex analysis to exceptional noncommutative algebras of quaternion ($\mathbb{Q}$) type [15, 16, 17, 25, 22]. In particular, it was demonstrated that explicit account of noncommutativity in the very definition of functions "differentiable" in $\mathbb{Q}$ inevitably results in the *non-linearity* of the generalized Cauchy-Riemann equations (GCRE) which follow. This makes it possible to regard the GCRE as fundamental dynamical equations of *interacting* physical fields represented by (differentiable) functions of the algebraic $\mathbb{Q}$-type variable.

A wide class of such fields-functions exists only for the complex extension of $\mathbb{Q}$-algebra, i.e. for the algebra of complex quaternions $\mathbb{B}$ (*biquaternions*). Over the $\mathbb{B}$-algebra, the GCRE turn to be Lorentz invariant and acquire, moreover, the gauge and the spinor structures. On this base a self-consistent and unified *algebrodynamical field theory* has been constructed in our works [15, 16, 24, 25, 22, 26, 28, 17].

From the physical viewpoint, the most important property of GCRE is their direct correspondence to a fundamental *light-like* structure. The latter manifests itself in the fact that every (spinor) component $S(x, y, z, t) \in \mathbb{C}$ of the primary $\mathbb{B}$-field must satisfy the *complex eikonal equation* (CEE) [14, 15]

$$\eta^{\mu\nu}\partial_\mu S \partial_\nu S = (\partial_t S)^2 - (\partial_x S)^2 - (\partial_y S)^2 - (\partial_z S)^2 = 0, \qquad (1)$$

where $\eta_{\mu\nu} = diag\{1, -1, -1, -1\}$ is the Minkowsky metric and ∂ stands for the partial derivative by respective coordinate. The CEE (1) is Lorentz invariant, nonlinear and plays the role similar to that of the *Laplace equation* in complex analysis. Each solution to GCRE can be reconstructed from a set of (four or less) solutions to CEE.

In the meantime, in [17] the intrinsic *twistor* structure of CEE has been discovered, and on its base the general solution of the nonlinear eikonal equation has been obtained. It was proved that, in this respect, every CEE solution belongs to one of two classes which both can be obtained from a twistor generating function via a simple and purely algebraic procedure. This construction allows also for definition of singular loci of the null geodesic congruences correspondent to the eikonal field – the *caustics*. Just at the caustics – the envelopes of congruences – the neighbouring rays intersect each other, and the associated physical fields turn to infinity forming, thus, a unique *particle-like* object – a common source of the fields and of the congruence itself. Thus, in the algebrodynamical theory *the particles can be considered as (spatially bounded) caustics of the primordial null congruences.*

On the other hand, null congruences naturally define the universal local "transfer" of the basic twistor field with fundamental constant velocity "c" (in full analogy with the transfer of field by an electromagnetic wave) and point thus to exceptional role of the time coordinate in the algebrodynamical scheme and in twistor theory in general. Existence of the "Flow of Time" becomes therein a direct consequence of the existence of Lorentz invariant "aether" formed by the primordial light-like congruence ("preLight"). In the paper, we underline the principal property of *multivaluedness* of fundamental complex solution to CEE ("World solution") and of physical fields associated with it. As a result, at each space-time point one has a *superposition* of a great number of rays which belong to locally distinct null congruences, and the Time Flow turns to be *multi-directional*, i.e. consists of a number of superposed "subflows" (linked globally by complex structure into a unique physical "corpuscular-field" dualistic complex).

In section 2 we consider the twistor structure of CEE and the procedure of algebraic construction of its two classes of solutions. A few simple illustrative examples are presented. In section 3 we discuss the caustic structure of the CEE solutions, in particular of spatially bounded type (particle-like singular objects), and the properties of associated physical fields. In section 4, we introduce the "World function" responsible for generation of the "World solution" to CEE and discuss the related concept of multivaluedness of physical fields. Final section 5 is devoted to some general issues which bear on the nature of physical time. The notions of the *primordial light* ("pre-Light") and of the *light-formed aether* are introduced, and the Time Flow is actually identified with the Flow of preLight. Intrinsic structure of these fundamental flows is studied which relates to the property of multivaluedness of the basic twistor field.

The article is an extended version of the preceding paper [41] and, as to description of physical picture of the World, continues our paper [60]. In order to simplify the presentation, we avoid to apply the 2-spinor and the other refined mathematical formalisms, for this refering a prepared reader to our recent papers [25, 22, 28, 17].

The two classes of solutions to the complex eikonal equation

The eikonal equation describes the process of propagation of wave fronts (field discontinuties) in any relativistic theory, in Maxwell electrodynamics in particular [4, 5]. Physical and mathematical problems related to the eikonal equation were dealt with in a lot of works, see e.g. [6, 49, 8, 37, 11, 12].

The complex eikonal equation (CEE) arises naturally in problems of propagation of restricted light beams [13] and in theory of congruences related to solutions of Einstein or Einstein-Maxwell system of equations [14]. We, however, interpret the complex eikonal, to the first turn, as a fundamental physical field which describes, in particular, the interacting and "self-quantized" particle-like objects formed by singularities of the CEE solutions. By this, the electromagnetic and the other conventional physical fields can be associated with any solution of the CEE; they are responsible for description of the process of interaction of particles-singularities. Note that particle-like properties of field singularities related to the *5-dimensional* real eikonal field have been studied in [9]; the concept of particles as singularities of electromagnetic and eikonal fields has been incidentically discussed by many authors, in particular by H. Bateman [6] af far as in 1915.

We start with a definition, together with Cartesian space-time coordinates $\{t, x, y, z\}$, of the so called *spinor* or *null* coordinates $\{u, v, w, \bar{w}\}$ (the light velocity is taken to be unity, $c = 1$)

$$u = t + z, \quad v = t - z, \quad w = x - iy, \quad \bar{w} = x + iy \tag{2}$$

which form the *Hermitian* 2×2 matrix $X = X^+$ of coordinates

$$X = \begin{pmatrix} u & w \\ \bar{w} & v \end{pmatrix} \tag{3}$$

In the representation using spinor coordinates the CEE (1) looks as follows:

$$\partial_u S \partial_v S - \partial_w S \partial_{\bar{w}} S = 0. \tag{4}$$

The CEE possesses a remarkable *functional invariance* [15, 16]: for every $S(X)$ being its solution any (differentiable) function $f(S(X))$ is also a solution. The eikonal equation is known also [6] to be invariant under transformations of the full 15-parameter *conformal group* of the Minkowsky space-time.

Let us take now an *arbitrary homogeneous* function Π of two pairs of complex variables $\{\xi, \tau\}$

$$\Pi = \Pi(\xi_0, \xi_1, \tau^0, \tau^1) \tag{5}$$

which are **linearly dependent** at any space-time point via the so called *incidence relation*

$$\tau = X\xi \quad \Leftrightarrow \quad \tau^0 = u\xi_0 + w\xi_1, \quad \tau^1 = \bar{w}\xi_0 + v\xi_1, \tag{6}$$

and which transform as *2-spinors* under Lorentz rotations [1]. The pair of 2-spinors $\{\xi(X), \tau(X)\}$ linked through Eq.(3.8) is known as a (null projective) *twistor* of the Minkowsky space-time [2].

Let us assume now that one of the components of the spinor $\xi(X)$, say ξ_0, is not zero. Then, by virtue of homogeneity of the function Π, we can reduce the number of its arguments to *three projective twistor variables*, namely to

$$\Pi = \Pi(G, \tau^0, \tau^1), \qquad G = \xi_1/\xi_0, \quad \tau^0 = u + wG, \quad \tau^1 = \bar{w} + vG \tag{7}$$

Now we are in order to formulate the main result proved in our paper [17].

Theorem. *Any (analytical) solution of CEE belongs, with respect to its twistor structure, to one of two and only two classes and can be obtained from some generating twistor function of the form (7) via one of the two simple algebraical procedures (described below).*

To obtain the first class of solutions, let us simply resolve the algebraic equation defined by the function (7)

$$\Pi(G, u + wG, \bar{w} + vG) = 0 \tag{8}$$

with respect to the only unknown G. In this way we come to a complex field $G(X)$ which necessarily satisfies the CEE. Indeed, after substitution $G = G(X)$ Eq.(8.2) becomes an *identity* and, in particular, can be differentiated with respect to the spinor coordinates $u, v, w, \bar{w}$. Then we get

$$P\partial_u G = -\Pi_0, \quad P\partial_w G = -G\Pi_0, \quad P\partial_{\bar{w}} G = -\Pi_1, \quad P\partial_v G = -G\Pi_1, \tag{9}$$

where Π_0, Π_1 are the partial derivatives of Π with respect to its twistor arguments τ^0, τ^1 while P is its *total* derivative with respect to G,

$$P = \frac{d\Pi}{dG} = \partial_G \Pi + w\Pi_0 + v\Pi_1 \ , \tag{10}$$

which we thus far assume to be nonzero in the space-time domain considered. Multiplying then Eqs. (9) we prove that $G(X)$ satisfies the CEE in the form (4). It is easy to check that *arbitrary* twistor function $S = S(G, u + wG, \bar{w} + vG)$, under substitution of the obtained $G = G(X)$, also satisfies the CEE (owing to the functional constraint (8.2) it depends in fact on only *two* of three twistor variables).

To obtain the second class of CEE solutions, we have from the very beginning to differentiate the function Π with respect to G and only after this to resolve the resulting algebraic equation

$$P = \frac{d\Pi}{dG} = 0 \tag{11}$$

[1] To simplify the notation, we do not distinguish between the primed and unprimed spinor indices. In the incidence relation (3.8) the standard factor "i" (imaginary unit) is omitted what is admissible under the proper redefinition of the twistor norm

with respect to G again. Now the function $G(X)$ does not satisfy the CEE; however, if we substitute it into (7) the quantity Π becomes an explicit function of space-time coordinates and necessarily satisfies the CEE (as well as any function $f(\Pi(X))$ by virtue of functional invariance of the CEE). Indeed, differentiating the function Π with respect to the spinor coordinates we get

$$\partial_u \Pi = \Pi_0 + P\partial_u G, \;\; \partial_w \Pi = G\Pi_0 + P\partial_w G, \;\; \partial_{\bar{w}} \Pi = \Pi_1 + P\partial_{\bar{w}} G, \;\; \partial_v \Pi = G\Pi_1 + P\partial_v G, \quad (12)$$

and, taking into account the generating condition (11), we immediately find that the function Π itself obeys the CEE (4).

The functional condition (8.2) and, therefore, the CEE solutions of the first class are in fact well known. Indeed, apart from the CEE, the field $G(X)$, if it is obtained by the resolution of Eq.(8.2), satisfies (as it is easily seen from Eqs.(9) for derivatives), the *over-determined* system of differential constraints

$$\partial_u G = G\partial_w G, \quad \partial_{\bar{w}} G = G\partial_v G \quad (13)$$

which define the so called *shear-free (null geodesic) congruences* (SFC). By this, algebraic Eq.(8.2) represents (in implicit form) *general solution* of Eqs.(7.3), i.e. describes the whole set of SFC in the Minkowski space-time. This remarkable statement proved in [18] is known as the *Kerr theorem.*

The second class of CEE solutions generated by algebraic constraint (11), to our knowledge, hasn't been considered in literature previously [2]. It is known, however, that condition (11) defines the *singular locus* for SFC, i.e. for the CEE solutions obtained from the Kerr constraint (8.2). Precisely, condition (11) fixes the *branching points* of the principal complex field $G(X)$ or, equivalently, – the space-time points where Eq. (8.2) has *multiple roots.* As to the CEE solutions of second class themselves, their branching points occur at the locus defined by another condition which evidently follows from generating Eq. (11) and has form

$$\Lambda = \frac{d^2\Pi}{dG^2} = 0. \quad (14)$$

The null congruences (especially the congruences with zero shear), as well as their singularities and branching points, play crucial role in the algebrodynamical approach. They will be discussed below in more details. Here we only repeat that, as it has been proved in [17],

the two simple generating procedures described above exhaust all the (analytical) solutions to the CEE representing, thus, its general solution

(note only that for solutions with zero spinor component, $\xi_0 = 0$, another gauge, in compare with the one used above, should be choosed). The obtained result can be thought of as *a direct generalization of the Kerr theorem.*

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[2] Study of solutions of the *real* eikonal equation by differentiation of generating functions depending on coordinates as parameters is used in general theory of singularities of caustics and wavefronts [11]

To make the exposition more clear, we present below several examples of the described construction.

1. Static solutions. Let the generating function Π depends on its twistor variables in the following way:

$$\Pi = \Pi(G, H), \quad H = G\tau^0 - \tau^1 = wG^2 + 2zG - \bar{w}, \tag{15}$$

where $z = (u - v)/2$, and the time coordinate $t = (u + v)/2$ is, in this way, eliminated. It is evident that the generating ansatz (15) covers the whole class of *static* CEE solutions.

In [21, 14] it was proved that static solutions to the SFC equations (and, therefore, static solutions to the CEE too) with *spacially bounded* singular locus are exhausted, up to 3D translations and rotations, by the *Kerr solution* [18] which follows from generating function of the form

$$\Pi = H + 2iaG = wG^2 + 2z^*G - \bar{w}, \quad (z^* = z + ia) \tag{16}$$

with a real constant parameter $a \in \mathbb{R}$. Explicitly resolving equation $\Pi = 0$ which is quadratic in G we obtain the two "modes" of the field $G(X)$

$$G = \frac{\bar{w}}{z^* \pm r^*} = \frac{x + iy}{z + ia \pm \sqrt{x^2 + y^2 + (z + ia)^2}} \tag{17}$$

which in the case $a = 0$ correspond to the ordinary *stereographic projection* $S^2 \mapsto \mathbb{C}$ from the North or the South pole respectively. It is easy to check that this solution and also its twistor counterpartners

$$\tau^0 = t + r^*, \quad \tau^1 = G\tau^0, \tag{18}$$

satisfy the CEE (as well as any function of them). Correspondent SFC is in the case $a = 0$ *radial* with a point singularity; in general case $a \neq 0$ the SFC is formed by the rectilinear constituents of a system of hyperboloids and has a *ring-like* singularity of a radius $R = |a|$. Using this SFC, a Riemannian metric (of the "Kerr-Schild type") and an electric field can be defined which satisfy together the electrovacuum Einstein-Maxwell system. In the case $a = 0$ this is the Reissner-Nordström solution with Coulomb electric field, in general case – the Kerr-Newman solution with three characteristical parameters: the mass M, the electric charge Q and the angular momentum (spin) Mca, – for which the field distribution possesses also the proper magnetic moment Qa which corresponds to the gyromagnetic ratio specific for the Dirac particle [45, 46]. In the algebrodynamical scheme, moreover, *electric charge of the point or the ring singularity is necessarily fixed in modulus*, i.e. "elementary" [15, 16, 28, 58] (see also [25] where a detailed discussion of this solution in the framework of algebrodynamics can be found).

Now let us obtain, from the same generating function, a solution to CEE of the second class. Differentiating Eq.(16) with respect to G and equating derivarive to zero, we get $G = -z^*/w$ and, substituting this expression into Eq.(16), obtain finally the following solution to CEE (which is univalued everywhere on 3D-space):

$$\Pi = -\frac{(r^*)^2}{w} = -\frac{x^2+y^2+(z+ia)^2}{x-iy}. \tag{19}$$

It is instructive to note that equation $\Pi = 0$, being equivalent to two real-valued constraints $z = 0,\ x^2 + y^2 = a^2$, defines here the ring-like singularity for the Kerr solution (17), as it should be in account of the theorem above presented (for this, see also section 4).

Static solutions of the II class with spatially bounded singularities are not at all exhausted by the solution (19). Consider, for example, solutions generated by the functions

$$\Pi = \frac{G^n}{H}, \quad n \in \mathbb{Z}, \quad n > 2. \tag{20}$$

We'll not write out correspondent solutions in explicit form and shall restrict ourselves by examination of the spacial structure of their singularities which can be obtained from the joint system of equations $P = 0,\ \Lambda = 0$, see Eqs. (11), (14). Eliminating from the latter the unknown field G we find that singularities (branching points of the eikonal field) have again the ring-like form $z = 0,\ x^2 + y^2 = R^2$ with radii equal to

$$R_n = \frac{a(n-1)}{\sqrt{n(n-2)}} \tag{21}$$

The cases $n = 1, 2$ evidently need special consideration. For $n = 1$ equating to zero derivative of the function G/H we find $G = \pm i\bar{w}/\rho$ with $\rho = \sqrt{x^2+y^2}$. This brings us after substitution to the following solution of the CEE:

$$\Pi = (z + ia \pm i\sqrt{x^2+y^2})^{-1} \tag{22}$$

which has the pole at the ring $z = O, x^2 + y^2 = a^2$ but *has a branching point only on the origin* $r = 0$, i.e. *which under any* a *corresponds to the point singularity.*

In the case $n = 2$ via analogous procedure we get $G = \bar{w}/z^*$ and after substitution come to the following solution of the CEE [17]:

$$\Pi = \frac{\bar{w}}{r^*} = \frac{x+iy}{x^2+y^2+(z+ia)^2} \tag{23}$$

which is of the same structure as (the inverse of) the solution (19). As the latter, it has no branching points on the real space-time slice while its pole corresponds to the Kerr ring. Let us take for simplicity $a = -1$; then solution (23) can be rewritten in the following familiar form:

$$\Pi = i\frac{x+iy}{2z+i(r^2-1)} \tag{24}$$

which can be easily identified as the standard *Hopf map.* As the solution of the CEE it has been studied in [22] and especially in the recent paper [23] where its geometrical and topological nature has been examined in detail. We suspect also that generalized Hopf maps considered therein relate (in the case $m = 1$) to the CEE solutions generated by the functions (20) and, as the latters, has the ring singularities correspondent to those represented by Eq.(21). However, this should be verified by direct calculations.

2. Wave solutions. Consider also the class of generating functions dependent on one of the two twistor variables τ^0, τ^1 only, say on τ^0:

$$\Pi = \Pi(G, \tau^0) = \Pi(G, u + wG). \tag{25}$$

Both classes of the CEE solutions obtained via functions (7.12) will then depend on only two spinor coordinates $u = t + z,\ w = x - iy$. This means, in particular, that the fields propagate along the Z-axis with fundamental (light) velocity $c = 1$. A "photon-like" solution of this type, with singular locus spacially bounded in all directions, was presented in [58].

Notice also that an example of the CEE solution with a considerably more rich and realistic structure of singular locus is presented below in section 4 (see also [58]).

Particles as caustics of the primordial light-like congruences

It's well known that a null congruence of rays corresponds to any solution of the eikonal equation; it is orthogonal to hypersurfaces of constant eikonal $S = const$ and directed along the 4-gradient vector $\partial_\mu S$. Usually, these two structures define the *characteristics* and *bicharacteristics* of a (linear) hyperbolic-type equation, e.g. of the wave equation $\Box\Psi = 0$.

In the considered complex case, i.e. in the case of CEE, the hypersurfaces of constant eikonal and the 4-gradient null congruences belong geometrically to the *complex extension* $\mathbb{C}M^4$ of the Minkowski space-time which looks here quite natural in account of the complex structure of the primary biquaternion algebra $\mathbb{B}$. The problem of physical sense of the additional (imaginary) dimensions is much important and nontrivial, and we hope to discuss it in the forthcoming paper.

Here we use another interesting property: existence of a null geodesic congruence defined on a *real* space-time for every of the *complex-valued* solutions to CEE. This remarkable property follows directly from the twistor structure inherent to CEE. Indeed, according to the theorem above-presented, any of the CEE solutions (both of the I and the II classes) is fully determined by a (null projective) twistor field $\{\xi(X), \tau(X)\}$ (in the choosed gauge one has $\xi_0 = 1;\ \xi_1 = G(x)$) subject to the incidence relation (3.8). This latter "Penrose equation" can be explicitly resolved with respect to the space coordinates $\{x_a,\ a = 1, 2, 3\}$ as follows:

$$x_a = \frac{\Im(\tau^+\sigma\xi)}{\xi^+\xi} - \frac{\xi^+\sigma\xi}{\xi^+\xi}\ t\ , \tag{26}$$

with $\{\sigma_a\}$ being the Pauli matrices and the time t remaining a *free parameter*. Eq.(26) manifests that the primordial spinor field $\xi(X)$ *reproduces its value along the 3D rays formed by the unit "director vector"*

$$\vec{n} = \frac{\xi^+\vec{\sigma}\xi}{\xi^+\xi}, \quad \vec{n}^2 = 1 \ , \tag{27}$$

and propagates along these locally defined directions with fundamental constant velocity $c = 1$. In the choosed gauge we have for Cartesian components of the director vector (27)

$$\vec{n} = \frac{1}{(1+GG^*)}\{(G+G^*), -i(G-G^*), (1-GG^*)\}, \tag{28}$$

the two its real degrees of freedom being in one-to-one correspondence with the two components of the complex function $G(X)$.

Thus, for every solution of the CEE the space is foliated by a congruence of *rectilinear* light rays, i.e by a *null geodesic*[3] *congruence* (NGC). Notice that the director vector obeys the *geodesic equation* [60]

$$\partial_t\vec{n} + (\vec{n}\vec{\nabla})\vec{n} = 0 \ . \tag{29}$$

The basic field $G(X)$ of the NGC can be always extracted from one of the two algebraic constraints (8.2) or (11) which at any space-time point possess, as a rule, not one but rather a *finite (or even infinite) set* of different solutions. Suppose that generating function Π is *irreducible*, i.e. can't be factorized into a number of twistor functions of the same structure (otherwise, we should make a choice in favour of one of the multiplies). Then a generic solution of the constraints will be nothing but a *multivalued complex function* $G(X)$. Choose locally (in the vicinity of a particular point X) one of the continious *branches* of this function. Then a particular NGC and a set of physical fields can be associated with this branch, i.e. with one of the "modes" of the multivalued field distribution.

Specifically, for any of the I class CEE solutions the spinor $F_{(AB)}$ of *electromagnetic field* can be defined explicitly in terms of twistor variables of the solution [22, 28, 58]:

$$F_{(AB)} = \frac{1}{P}\left\{\Pi_{AB} - \frac{d}{dG}\left(\frac{\Pi_A\Pi_B}{P}\right)\right\}. \tag{30}$$

where Π_A, Π_{AB} are the first and the second order derivatives of the generating function Π with respect to its two twistor arguments τ^0, τ^1. For every branch of the solution $G(X)$ *this field locally satisfies Maxwell homogeneous ("vacuum") equations.* Moreover, as it has been demonstrated in [16, 24, 22], a complex-valued $SL(2,\mathbb{C})$ *Yang-Mills field* and a *curvature field* (of some effective Riemannian metric) can be also defined through only the same principal function $G(X)$ for any of the CEE solution of the first class.

[3] On the flat Minkowsky background the geodesics are evidently rectilinear

Consider now analytical continuation of the function $G(X)$ up to one of its branching points which corresponds to a multiple root of Eq.(8.2) (or, alternatively, of Eq.(11) for solutions of the II class). At this point $P = 0$, and the strength of electromagnetic field (30) turns to infinity. The same holds for the other associated fields, for curvature field [4] in particular [21]. Thus, the locus of branching points (which can be 0-, 1- or even 2-dimensional, see section 4) manifests itself as a *common source* of a number of physical fields and can be identified (at least, in the case when it is bounded in 3-space) as a unique *particle-like* object.

Such formations are capable of much nontrivial evolution simulating physical interactions or even mutual *transmutations* represented by *bifurcations* of the field singularities (see, e. g., the example in section 4). They possess also a realistic set of "quantum numbers" including a *self-quantized electric charge* and a *Dirac-type gyromagnetic ratio* (equal to that for a spin 1/2 fermion) [45, 46, 25]. Numerous examples of such solutions and their singularities can be found in our works [24, 25, 26, 22].

On the other hand, for the light-like congruences – NGC – associated with CEE solutions via the guiding vector (28) the locus of branching points coincides with that of the principal G-field and represents the familiar *caustic* structure, i.e. the envelope of the system of rays at which the neighbouring rays intersect each other ("focusize"). From this viewpoint, within the algebrodynamical theory *the "particles" are nothing but the caustics of null rectilinear congruences associated with the CEE solutions.*

The World function and the multivalued physical fields

At this point we have to decide which of the two types of the CEE solutions can be in principle taken in our scheme as a representative for description of the Universe structure as a whole. *As a "World solution" we choose a CEE solution of the first class* because a lot of peculiar geometrical structures and physical fields can be associated with any of them [16, 25, 22]. Such a solution can be obtained algebraically from the Kerr functional constraint (8.2) and a generating twistor "World function" Π which is *exceptional* with respect to its internal properties; geometrically it gives rise to an NGC with a special property – zero shear [2, 3].

Moreover, a *conjugated* CEE solution of the II class turns then also to be involved into play since it defines a characteristic hypersurface of the (I class) "World solution". In fact, this is determined as a solution of the joint algebraic system of Eqs.(8.2),(11). Precisely, if we resolve Eq.(11) with respect to G and substitute the result into (8.2), equation $\Pi(G(X)) = 0$ would define then the singular locus (the characteristic hypersurface) of the World solution. On the other hand, the function $\Pi(G(X))$ would necessarily satisfy the CEE representing its II class solution in account of the theorem presented in section 2. Thus,

[4] Associated Yang-Mills fields possess, generically, additional *string-like* singularities

the eikonal field here carries out two different functions being a fundamental physical field (as a CEE solution of the I class) and, at the same time, a characteristic field (as a solution of the II class) which describes the locus of branching points of the basic field (i.e., the discontinuties of its derivatives).

Let us conjecture now that the World function Π is *an irreducible polynomial of a very high but finite order*[5] so that Eq.(8.2) is an algebraic (not a transcendental) one. Note that in this case Eq.(8.2) defines an algebraic surface in the projective twistor space $\mathbb{C}P^3$.

The World solution consists then of a finite number of modes – branches of multivalued complex G-field. A finite number of null directions (represented in 3-space by the director vector (28)) and an equal number of locally distinct NGC would exist then *at every point.*

Any pair of these congruences at some fixed moment of time will, generically, has an envelope consisting of a number of connected one-dimensional components-caustics[6]. Just these spacial structures (in the case they are bounded in 3-space) represent here the "particles" of generic type. Other types of particle-like structures are formed at the focal points of *three or more* NGC where Eq. (8.2) has a root of higher multiplicity. Formations of the latter type would, of course, meet rather rarely, and their stability is problematic. One can speculate on their possible relation to particle's excitations – *resonances.*

Nonetheless, we can model both types of particles-caustics in a simple example based on generating twistor function of the form [58]

$$\Pi = G^2(\tau^0)^2 + (\tau^1)^2 - b^2G^2 = 0, \quad b = const \in \mathbb{R} \,, \tag{31}$$

which leads to the 4-th order polynomial equation for the G-field. At initial moment of time $t = 0$, as it can be obtained analytically, the singular locus consists of a pair of point singularities (with opposite and equal in modulus "elementary" electric charges) and of a neutral 2-surface (ellipsoidal *cocoon*) covering the charges (see [58] for more details). The latter corresponds to the intersection of all of the 4 modes of the multivalued solution while each of the point charges is formed by intersection of a particular pair of (locally radial, Coulomb-like) congruences [58]. Time evolution of the solution and of its singularities is very peculiar: for instance, at $t = b/\sqrt{2}$ the point singularities cancel themselves at the origin $r = 0$ simulating thus the process of *annihilation* of elementary particles. Moreover, this process is accompanied by emission of the *singular light-like wavefront* represented by another 2-dimensional component of connection of the caustic structure.

[5] This conjecture is, in fact, not at all necessary. Indeed, one can easily imagine that the World function leads to the Kerr Eq.(8.2) which possesses an **infinite number of roots** for complex-valued field function $G(X)$ at any space-time point X

[6] In fact, the caustics of **generic type** are determined by one complex condition $\Pi(G(X)) = 0$ (i.e., by two real equations) on three coordinates and, at a fixed moment of time $t = t_0$, correspond to a number of one-dimensional curves ("strings")

Thus, we see that the multivalued fields are quite necessary for to ensure the self-consistent structure and evolution of a complicated (realistic) system of particles-singularities. One only should not be confused by such, much unusual, property of the principal G-field and, especially, by multivalued nature of the other associated fields including the electromagnetic one.

Indeed, in convinient classical theories, the fields are in fact only a tool which serves for adequate description of particle dynamics (including the account of retardation etc.) and for nothing else. In nonlinear theories, as well as in our algebrodynamical scheme, the fields are moreover responsible for *creation* and structure of particles themselves, as regular *solitons* or *singularities* of fields respectively. In the first, more familiar case we, apparently, should consider the fields to be univalued. The same situation occurs in the framework of quantum mechanics where the quantization rules often follow from the requirement for the wave function to be univalued.

However, as we have seen above, in the algebrodynamical construction *the field distributions must not necessarily be univalued!* On the other hand, acception of fields' multivaluedness does not at all prevent to obtain the discrete spectrum of characteristics in a full analogy with quantum mechanics. For example, the requirement of univaluedness of a **particular, locally choosed mode** of the principal G-field and of the associated electromagnetic field (far from the branching points of the first and, consequently, from the infinities of the second!) leads to the general property of *quantization of electric charge of singularities* in the framework of algebrodynamical theory [28, 58].

As to the process of "measurement" of the field strength, say, of electromagnetic field, it directly relates to only the measurements of particles' accelerations, currents etc., and only after the measurements the results are translated into conventional field language. However, this is not at all necessary (in recall, e.g, of the Wheeler-Feynman electrodynamics and of numerous "action-at-a-distance" approaches [31, 32]). In fact, "we never deal with fields but only with particles" (F. Dyson).

In particular, on the classical (nonstochastic) level we can deal, effectively, with the *mean value* of the set of field modes at a point; similar concept based on purely quantum considerations has been recently developed in the works [33]. In our scheme, the true role of the multivalued field will become clear only after the spectrum and the effective mechanics of particles-singularities will be obtained in a general and explicit form.

We hope that a sort of psychological barrier for acception of general idea of the field multivaluedness will be get over as it was with possible *multidimensionality* of physical space-time. The advocated concept seems indeed very natural and attractive. In the purely mathematical framework, multivalued solutions of PDEs are the most common in comparison with the familiar δ-type distributions [34, 8]. From physical viewpoint, this makes it possible to naturally define a dualistic "corpuscular-field" complex of a very rich structure which, actually, gathers all the particles in the Universe into a unique object. The caustics-singularities are well-defined themselves and undergo a collective self-consistent motion free of any ambiguity or divergence (the latters can arise here only in result of incorrect description of the evolution process and can be removed, if arise, on quite legal grounds, contrary, say, to the renormalization procedure in the quantum field theory). Note also that recently accomplished universal local classification of singularities of differentiable maps, in particular of caustics and wavefronts [11], can explicitly bear on the characteristics of elementary particles if the latters are treated in the framework of the algebrodynamical theory.

As to the principal problem of the choice of a particular representative of the generating *World function* Π *of the Universe* we are ready to offer an interesting candidature being in hope to discuss it elsewhere.

The light-formed relativistic aether and the nature of time

Light-like congruences (NGC) are the basic elements of the picture of physical world which arises in the algebrodynamical scheme and, to some extent, in twistor theory in general. The rays of the NGC densely fill the space and consist of a great number of branches – components superposed at each space point and propagating in different directions with constant in modulus and universal (for any branch of multivalued solution, any point and any system of reference) fundamental velocity. *There is nothing in the Universe except this primordial light flow ("pre-Light Flow")* because *the whole Matter is born by pre-Light and from pre-Light* at the caustic regions of "condensation" of the pre-light rays.

In a sense, one can speak here about an exceptional form of *relativistic aether* which is formed by a flow of pre-Light. Such an exceptional form of the World aether has nothing in common with old models of the *light-carrying* aether which had been considered as a sort of elastic medium. Here, the *light-formed* aether consists of structureless "light elements" and is, obviously, in full correspondence with special theory of relativity[7].

At the same time, notions of the aether formed by pre-Light and of the matter formed by its "thickenings" evoke numerous associations with the Bible and with ancient Eastern philosophy. Certainly, there were teologists, philosophers or mystics who were brought to imagine a similar picture of the World. However, in the framework of successive physical theory this picture becomes more truthworthy and, to our knowledge, has not been yet discussed in literature[8].

On the other hand, existence of the primordial light-formed aether and man-

[7] At present, it seems rather strange that A. Einstein didn't come himself to the concept of relativistic aether so consonant with the ideas of STR and with his favourite *Mach principle*. Surprisingly, R. Penrose also overlooked this opportunity which follows naturally from his twistor theory

[8] Similar in some aspects ideas have been advocated in the works [36, 37, 38]. Note, in particular, the concept of the "radiant particle" offered by L. S. Shikhobalov [35]

ifestation of universal property of local "transfer" of the *aether – generating field* $G(X)$ with constant fundamental velocity $c = 1$ points to different status of space and time coordinates and *offers a new approach to the problem of physical time* as a whole. By this, it is noteworthy that since in 1908 H. Minkowski has joined space and time into a unique 4-dimensional continuum, no further understanding of the nature of time has been achieved in fact. Moreover, this synthesis has "shaded" the principal distinction of space and time entities and clarified none of such problems as (micro/macro)irreversibility, (in)homogeneity and (non)locality of time, its dependence on material processes etc.

In the interim, the key problem of Time can be formulated in a rather simple way. *Subjectively*, we perceive time as a continious intrinsic motion, a latent *flow*. Everybody comprehends in a moment, as the ancient Greeks did, what is meant by the "River of Time", the "Flow of Time". As a rule, we consider this intrinsic motion to be independent on our will and on material processes and uniform: not for nothing, in physics the flow of time is modelled by the uniform motion of, say, the record tape etc. Moreover, under variations in time one does not only observe the *conservation* of a particular set of *integral* quantities (which is widely used in the orthodox physics) but perceives subjectively the complete *repetition, reproduction of the* **local states** *of any system;* that's why for measurements of time itself we use *clocks* whose principle of operation is based on reproducible, periodical processes. In other words, whereas one has much ambiguous and diverse distributions of spacial positions of physical bodies, all they and we all have always *one and the same monotonically increasing time coordinate*, i.e. are in a common and permanent motion together with the "Time River".

Surprisingly, almost all these considerations are absent in the structure of theoretical physics and, in particular, in relativity theory. To bring into correspondence the results of calculations with practice (e. g. for the Cauchy problem etc.) one chooses a "time orthogonal hypersurface", i. e. quite ambiguously fixes the unity of the present moment of time, of the moment "now", perceived subjectively by everybody; however, *there are no intrinsic reasons for this choice in the very structure of theoretical physics, including the STR.*

At least partially, such a situation is caused by the following. The notion of everywhere existing, eternal Flow of Time immediately leads to the problem of its (material? pre-material?) carrier. In this connection, the works of N. A. Kozy'rev [39] should be marked, of course, in which the concept of the "active" Flow of Time influencing directly the material processes has been proposed. To our opinion, however, there are no reliable physical grounds at present which confirm the Kozy'rev's ideas, and no mechanism of "interaction" of this exotic form of matter with the ordinary ones. As to the algebrodynamical paradigm, the Time Flow is non-material therein: it does not *interact* or *influence* the Matter at all but just *forms* it. In distinction from the Kozy'rev's concept, we do not deal here with various material entities only one of them being the Time itself: on the contrary, here we have one triply-unique entity – preLight-Time-Matter. Note that more close the approach turns to be to the concept of "Time-generating Flows" developed by A. P. Levich [40].

On the other hand, under consideration of the problem of the carrier of the Time Flow, we inevitably return back to the notion of some form of the *World aether* which has been exiled from physics after the triumph of Einstein's theory. To do without aether, none Flow of Time can be successively included into the structure of theoretical physics and none subjectively perceived properties of time can be precisely formulated and described.

However, in a paradoxical way, just the STR with its postulate of the invariance of light velocity justifies the introduction of the *dynamical Lorentz invariant aether* formed by the light-like congruences as the primary element of physical World. Specifically, the Time Flow can be naturally identified now with the Flow of primordial Light (pre-Light), and the "River of Time" turns to be nothing but the "River of Light". Moreover, it is the universality of light velocity which explains our subjective perception of uniformity and homogeneity of Time Flow.

There is, however, another, the most striking and unexpected feature of the introduced concept of physical time. *The Time Flow manifests here itself as superposition of a great number of distinctly directed and locally independent components – "subflows"*. At any point of 3-dimensional space there exists a (finite) set of directions: each mode of the primordial multivalued field $G(X)$ defines one of these directions and *propagates (reproduces its value) along it* forming thus one of constituents of the (globally unique) Flow of pre-Light identical to the Flow of Time.

One can conjecture that just by virtue of the local multivaluedness we are not capable of to perceive the particular local *direction* of the Time Flow. Apart from this, it is natural to assume that in the tremendously complicated structure of the World solution a *stochastic* component is necessarily present, particularly in the structure of the primordial Light-Time Flow. This results in chaotic variations of local directions of the light-like congruences which are certainly inaccessible for perception. On the other hand, it is the existence of (constant in modulus and the same for all of the branches of the multivalued World solution) *fundamental propagation velocity of the pre-Light rays* which makes it possible to feel the Flow of Time in general and to subjectively regard it as uniform and homogeneous in particular.

Conclusion

Thus, we have examined the realization of the algebrodynamical approach in which, as a base of unified physical theory, the only structure of a purely abstract nature is choosed, namely the algebra of complex quaternions and the generalized CR-equations – the conditions of differentiability in this algebra. Very the same structure can be successively expressed, in fact, on a number of equivalent geometrical languages (of covariantly constant fields, twistor geometry, shear-free congruences etc.).

Primary GCR-equations result directly in the field of complex eikonal which is regarded as a fundamental physical field (alternative in a sense to the linear fields of quantum mechanics). In its turn, the eikonal field is here closely related to the fundamental 2-spinor and twistor fields, on whose language, in particular, the general solution of the complex eikonal equation is formulated. Through the eikonal field also the other ones are defined, namely the electromagnetic and Yang-Mills fields. Singularities of the eikonal and of correspondent null congruences are considered as particle-like formations ("self-quantized" and effectively interacting).

In result, physical picture of the World which arises as a consequence of the only algebraic structure appears to be very beautiful and unexpected. As its basic elements it contains the primordial light flow – "pre-Light" – and the relativistic aether formed by the latter, multivalued physical fields and prelight-born matter (consisting of particles-caustics formed by the superposition of individual branches of the unique pre-light congruence in the points of their "focusization").

As very natural and deep seems to be the here arising connection between the existence of universal velocity (velocity of "light") and of the time flow; connection which permits to understand, in a sense, the origin of the Time itself. *Time is nothing but the primordial Light*; these two entities are undividible. On the other hand, *there is nothing in the World except the preLight Flow* which gives rise to all the "dense" Matter in the Universe.

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References

[1] R. Penrose, in: Quantum gravity: an Oxford Symposium, eds. C. J. Isham, R. Penrose, D. W. Sciama. – Clarendon Press, Oxford, 1975.

[2] R. Penrose and W. Rindler, *Spinors and Space-Time,* Vol. 2. – Cambridge Univ. Press, Cambridge, 1986.

[3] R. Penrose, *Classical and Quantum Gravity*, **14**, A299 (1997).

[4] V. A. Fock, *Theory of Space, Time and Gravitation,* – GITTL, M., 1955 (in Russian).

[5] A. Z. Petrov, *New Methods in General Relativity,* – Nauka, M., 1966 (in Russian).

[6] H. Bateman, *The Mathematical Analysis of Electrical and Optical Wave-Motion,* Dover Publ. Inc., 1955.

[7] S. Fritelli, E. T. Newman and G. Silva-Ortigoza, J. Math. Phys., **40**, 383 (1999); E. T. Newman and A. Perez, J. Math. Phys., **40**, 1089, 1999.

[8] F. Lizzi, G. Marmo, G. Sparano, A. M. Vinogradov, J. Geom. Phys., **14**, 34 (1994).

[9] Yu. P. Pyt'ev, Dokl. AN SSSR, **149**, 298 (1963);
Yu. P. Pyt'ev, Vestnik Mosk. Univ., Fiz. Astron., **2 (102)**, 53 (1965) (in Russian).

[10] V.I. Arnold, *Mathematical Methods of Classical Mechanics,* – Springer-Verlag, Berlin, NY, 1980.

[11] V. I. Arnold, *Singularities of Caustics and Wave Fronts,* – Kluwer, Dordrecht, 1990.

[12] S. N. Kruzhkov, Matemat. Sbornik, **98 (140)**, 450 (1975) (in Russian).

[13] V. P. Maslov, *Complex WKB-Method in Nonlinear Equations,* – Nauka, M., 1977 (rus).

[14] R. P. Kerr and W. B. Wilson, Gen. Rel. Grav., **10**, 273 (1979).

[15] V. V. Kassandrov, *The Algebraic Structure of Space-Time and the Algebrodynamics,* – People's Friend. Univ. Press, M., 1992 (in Russian).

[16] V. V. Kassandrov, Gravitation & Cosmology, **3**, 216 (1995); (gr-qc/0007027).

[17] V. V. Kassandrov, Acta. Applic. Math., **50**, 197 (1998); V. V. Kassandrov, in: *Quasigroups and Nonassociative Algebras in Physics,* eds. J. Lõhmus and P. Kuusk. – Inst. Phys. Estonia Press, 1990, p. 202.

[18] G. C. Debney, R. P. Kerr and A. Schild, J. Math. Phys., **10**, 1842 (1969).

[19] B. Carter, Phys. Rev., **174**, 1559 (1968).

[20] C. A. Lopez, Phys. Rev., **D30**, 313 (1984).

[21] A. Ya. Burinskii, in: *Problems of Theory of Gravitations and Elementary Particles,* No.11, ed. K. P. Stanyukovich. – Atomizdat, M., 1980, p. 47 (in Russian).

[22] C. Adam, B. Muratori and C. Nash, Phys. Lett., **B485**, 314 (2000); (hep-th / 9910139).

[23] C. Adam, *Hopf maps as static solutions of the complex eikonal equation,* – math-ph / 0312031.

[24] V. V. Kassandrov and J. A. Rizcallah, in: *Recent Problems in Field Theory,* ed. A. V. Aminova, Kasan Univ. Press, Kasan, 1998, p. 176; (gr-qc/9809078).

[25] V. V. Kassandrov and J. A. Rizcallah, in: *Fundamental Problems of High Energy Physics and Field Theory,* ed. V. A. Petrov. – Inst. High Energy Phys., Protvino, 2002, p. 199; (gr-qc/9809056).

[26] V. V. Kassandrov and V. N. Trishin, Gravitation & Cosmology, **5**, 272 (1999); (gr-qc / 0007026).

[27] V. V. Kassandrov and J. A. Rizcallah, *Twistor and "Weak" Gauge Structures in the Framework of Quaternionic Analysis,* – gr-qc/0012109.

[28] V. V. Kassandrov, Vestnik People's Friend. Univ., Fizika, **8 (1)**, 34 (2000) (in Russian).

[29] V. V. Kassandrov, *Singular Sources of Maxwell Fields with "Self-Quantized" Electric Charges,* – Arxiv physics/0308045.

[30] V. V. Kassandrov, Gravitation & Cosmology, **8**, Suppl. 2, 57 (2002); (math-ph / 0311006).

[31] *Instantaneous Action-at-a-Distance in Modern Physics. "Pro" and "Contra"*, eds. A. Chubykalo, V. Pope and R. Smirnov-Rueda. – Nova Science Publ. Inc., NY, 1999.

[32] Yu. S. Vladimirov and A. Yu. Tur'ygin, *Theory of Direct Particle Interactions,* – Energoatomizdat, M., 1986.

[33] A. A. Kirillov, Phys. Lett., **B555**, 13 (2003); A. A. Kirillov and D. Turaev, Phys. Lett., **B532**, 185 (2002).

[34] V. V. Lychagin, Acta Applic. Math., **3**, 135 (1985).

[35] L. S. Shikhobalov, Vestnik S.-Petersb. Univ., **1 (3)**, 109 (1997); **1 (4)**, 118 (1999) (in Russian).

[36] I. A. Urusovskii, Zarubezjnaya radioelectronika, **3**, 3 (1996); **6**, 64 (1996); **6**, 66 (2000) (in Russian).

[37] V. V. Smol'yaninov, Uspechi Fiz. Nauk, **170**, 1064 (2000) (in Russian).

[38] I. A. Shelaev, *Introduction to irreversible electrodynamics,* – Dubna, 1999 (in Russian).

[39] N. A. Koz'yrev, *Selected Papers,* – Leningrad, 1991; N. A. Koz'yrev, in: *"History and Methodology of Natural Sciences. No.2.* – Inst. Phil. Press, M., 1963, p. 95 (in Russian).

[40] A. P. Levich, in: *Construction of Time in Natural Sciences: On the Way to Understanding of Time Phenomenon,* ed. A. P. Levich. – M. Univ. Press, M., 1996 (in Russian); A. P. Levich, Gravitation & Cosmology, **1 (3)**, 237 (1995).

[41] V. V. Kassandrov, in: *Physical Interpretations of Relativity Theory,* Proc. Int. Sci. Meeting PIRT-2003, eds. M. C. Duffy, V. O. Gladyshev and A. N. Morozov. – Bauman M. Univ., M., Liverpool, Sunderland, 2003, p. 23.

[42] V. V. Kassandrov, in: Mathematics and Practice. Mathematics and Culture. No. 2, ed. M. Yu. Simakov. – "Self-Education", M., 2001, p. 67 (in Russian).

Quaternionic Analysis and the Algebrodynamics

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We present the "algebrodynamical" approach to field-particle theory based on a nonlinear generalization of the Cauchy-Riemann conditions to non-commutative algebras of quaternion-like type. For complex quaternions the theory is Lorentz invariant and naturally carries some gauge and twistor structures. Point- and string-like singularities are considered as particle-like formations; their electric charge is "self-quantized". A novel "causal Minkowski geometry with additional phase" is presented that is induced by the structure of biquaternion algebra. On its background self-consistent algebraic dynamics of singularities ("ensemble of dublicons") is considered.

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1 On the commutative and non-commutative analysis and the algebrodynamics

History of discovery and investigation of exceptional algebras like quaternions or octonions, as well as of numerous attempts to apply them for "explanation of the structure of the World", is highly dramatic and full of still unjustified hopes [1, 6]. Bibliography on applications of quaternions in theoretical and mathematical physics during only XX century runs to thousands of articles [3]. Considerable part of them is devoted to the problem of construction of quaternionic analysis which in respect of the richness of internal properties and applications can be comparable with complex analysis. However, in opinion of the majority of contemporary mathematicians, this problem has not get its solution till now [4].

Meanwhile, the *commutative* analysis, that is, the analysis for functions taking values in some associative commutative algebra of finite dimension $n \geq 2$ (not necessarily with division), has been constructed by G. Sheffers as far as at the end of XIX century [5] quite in analogy with the complex analysis. At present it is used, in particular, in the conception of *polynumbers* and related Finsler geometries developing in the works of D. G. Pavlov and its group [6, 7]. Generalization of this version of analysis to *superalgebras* has been realized in the works of Yu. S. Vladimirov and I. V. Volovich [8].

Principal distinction of non-commutative and commutative cases has been noted by A. Sudbery [9]: non-commutativity obliterates the difference between an initial q and "conjugated" q^* elements of algebra, making it possible to express them through each other using only constant basic elements ("units") of algebra.

In particular, for the algebra of Hamilton's quaternions $\mathbb{Q}$ for any $q \in \mathbb{Q}$ one gets (see [6], p.121):

$$q^* \equiv -\frac{1}{2}(q + I * q * I + J * q * J + K * q * K), \tag{1.1}$$

where I, J, K are the three "imaginary" units of the quaternion algebra. That is why definition, in analogy with the complex case, of a "quaternionic analytical" ("quaternionic holomorphic") function as that independent on the quaternionic conjugated argument, appears here to be senseless.

On the other hand, natural definition of the "right" ("left") derivative $F'(Z)$ of a quaternionic function $F(Z)$, $F : \mathbb{Q} \mapsto \mathbb{Q}$:

$$F' = dF * dZ^{-1} \quad (F' = dZ^{-1} * dF) \tag{1.2}$$

is also unproductive, since the requirement of existence and uniqueness of the limit (1.2) (that is, of its independence on the path of convergence to zero of the increment dZ in the $\mathbf{E^4}$-space of the algebra $\mathbb{Q}$) leads to a considerably over-determined system of PDE's which appears to be compatible only for the trivial case of a linear function (for details see, e.g., [10]). There exist also additional considerations which convince oneself in the difficulty of construction of quaternionic (and, generally, of non-commutative) analysis (see, e.g., [11]).

Nonetheless, numerous attempts to bypass these difficulties have been undertaken of which most known is the conception of Fueter [9, 11, 12]. In many articles conditions of "quaternionic analyticity" (or their *biquaternionic* extension) have been formally written down in the form of a linear system of equation of Maxwell-like type (together with correspondent wave equation as an expected generalization of the 2D Laplace equation of complex analysis). All these attempts, however, cannot, perhaps, be considered as a successive version of quaternionic analysis. As to the more complicated problem of construction of *non-associative* analysis, say, over the algebra of octonions, none approaches to its solution are seen till now at all (nevertheless, see [10], section 10).

Let us return now to the case of commutative analysis. Modern exposition of the above presented approach of Sheffers may be found, e.g., in the monograph [13]. Therein, instead of definition of (invariant) derivative one exploits the requirement to represent the *differential* of a function of algebraic variable in an invariant "component-less" form. This makes it possible to expand the approach to all the (finite-dimensional) associative commutative algebras $\mathbb{A}$ including those with null divisors, in particular to the algebras of double and dual numbers.

Specifically, let $F(Z)$ be an $\mathbb{A}$-valued function $F : \mathbb{A} \mapsto \mathbb{A}$ of algebraic variable $Z \in \mathbb{A}$. Sheffers formulated condition of its *differentiability in* $\mathbb{A}$ as that of proportionality of linear parts of increments (differentials) dZ, dF of the independent variable and the function respectively:

$$dF = H(Z) * dZ, \tag{1.3}$$

where $H \in \mathbb{A}$ and $(*)$ denotes the operation of multiplication in $\mathbb{A}$. For algebras with division condition (1.3) is evidently equivalent to that of existence and "path-independence" of the ratio of increments, i.e. of the derivative $H(Z) = dF * dZ^{-1} \equiv F'(Z)$ and, in the particular case of the algebra of complex numbers $\mathbb{C}$, immediately leads to the Cauchy-Riemann equations. In general case linear PDE's connecting partial derivatives of the components of $F(Z)$ follow from (1.3) after elimination of the components of $H(Z)$ and are completely analogous to the CR equations for the functions of complex variable. As a whole, the commutative analysis created by Sheffers in many aspects reproduces the 2D complex one, so that a wide class of $\mathbb{A}$-differentiable functions obeying condition (1.3) and containing, in particular, arbitrary polynoms of $\mathbb{A}$-variable can be constructed.

Nonetheless, the transition from commutative case to the non-commutative associative algebras of quaternion type seems rather fascinating since those algebras $\mathbb{A}$, unlike the commutative ones, possess a wide group of continuous symmetries represented by *internal automorphisms*

$$q \mapsto a * q * a^{-1}, \quad a \in \mathbb{A}, \forall q \in \mathbb{A}, \tag{1.4}$$

preserving the multiplication law in $\mathbb{A}$. For algebra $\mathbb{Q}$ the automorphism group is known to be 2:1 isomorphic to the group of 3D rotations $SO(3)$ so that *the exceptional group of Hamilton's quaternions may be treated as the algebra of Euclidean physical space* $\mathbf{E^3}$. Its extension to the field of complex numbers – the algebra of *biquaternions* $\mathbb{B}$ – makes it possible to ensure the transition to the 4D space-time and to write down all the basic equations of relativistic field theory in a very compact and beautiful form (see, e.g., [14]). Finally, the version of (bi)quaternionic analysis earlier suggested by the author and exposed in the article below, made it possible to obtain a nonlinear Lorentz-invariant generalization of the Cauchy-Riemann equations and to built only on this base a self-consistent field-particle theory – the so called *algebrodynamics*. This article is devoted to presentation of this (nonlinear) version of non-commutative analysis and its realization in the framework of the algebrodynamical approach.

2 Quaternionic differentiability and conformal mappings

Correct way to generalize the approach of Sheffers to quaternion-like algebras consists, perhaps, in explicit account of the property of non-commutativity of the algebras like $\mathbb{Q}$ in the very definition of a differentiable function of $\mathbb{Q}$-variable. Specifically, we note that in the right-hand part of the expression (1.3) one finds an invariant $\mathbb{A}$-valued differential 1-form of the most general type which can be constructed using only operations in the algebra $\mathbb{A}$. According to these considerations, in the case of non-commutative (but still associative) algebra $\mathbb{A}$, condition (1.3) may be naturally modified for the following *condition of* $\mathbb{A}$*-differentiability of a function* $F(Z)$ (see [10, 23] and references therein):

$$dF = L(Z) * dZ * R(Z). \tag{2.1}$$

Here $L, R : \mathbb{A} \mapsto \mathbb{A}$ are two the so called *semi-derivative* functions of $F(Z)$, left and right respectively. For a given $F(Z)$ that satisfies (2.1) they are defined non-uniquely, up to a transformation $L \to \alpha L$, $R \to \alpha^{-1} R$ in which the function $\alpha(Z)$ takes values in the *centre* (commutative subalgebra) of the algebra $\mathbb{A}$. Thus, according to this definition, *problem of determination of functions differentiable in a non-commutative associative algebra* $\mathbb{A}$ *is the problem of enumeration of all the triples of functions* $\{F(Z), L(Z), R(Z)\}$ *which satisfy the condition (2.1)* (up to the above mentioned α-equivalence of the semi-derivatives).

For commutative algebras condition (2.1) reduces itself again to (1.3) where now the "derivative" $H(Z)$ is formed from "semi-derivatives" as $H(Z) = L(Z) * R(Z)$. On the other hand, if in the general non-commutative case one takes, say, $R(Z) = E$ (by this expecting the existence of the unity element E in the algebra $\mathbb{A}$ considered), then he returns back to the condition (1.3) with $H(Z) = L(Z)$. However, as it was already noticed, at least for quaternion-like algebras the latter condition are too rigid, since it can be satisfied only by linear functions of the form $F = A * Z + B$ with A, B being some constant elements of algebra at study (see, e.g., [9, 15]).

In general case condition of $\mathbb{A}$-differentiability (2.1) defines a wider class of functions. In particular, for the algebra of Hamilton's quaternions $\mathbb{Q}$ condition (2.1) appears to be algebraically equivalent to the condition of *conformity* of the mapping $Z \mapsto F(Z)$ in the Euclidean space $\mathbf{E^4}$ [25, 26, 24]. Indeed, taking the quaternionic norm $N^2(q) = q_0^2 + q_1^2 + q_2^2 + q_3^2$ of the elements in left- and right-hand parts of the relation (2.1) and using then the property of *multiplicativity* of norms

$$N^2(p * q) = N^2(p) N^2(q), \quad \forall p, q \in \mathbb{Q}, \tag{2.2}$$

one obtains:

$$\overline{ds}^2 \equiv N^2(dF) = N^2(L * R) N^2(dZ) \equiv \Lambda(Z) ds^2, \tag{2.3}$$

so that any $\mathbb{Q}$-differentiable function indeed defines some conformal mapping $ds \mapsto \overline{ds}$ in $\mathbf{E^4}$ with the scale factor $\Lambda(Z) = N^2(L * R)$. Let us notice that in this respect condition (2.1) can be again regarded as a natural generalization of the conditions of complex holomorphy.

However, it is well known (the so called *Liouville theorem*, see, e.g., [16]), that in $\mathbf{E^4}$-space conformal mappings form a finite 15-parametric group, in contrast to the infinite-dimensional group of conformal mappings on a complex plane which are realized by analytical functions of conformal variable. Each of these conformal mappings in $\mathbf{E^4}$ corresponds to some $\mathbb{Q}$-differentiable function obeying condition (2.1). Namely, for inversion $F(Z) = Z^{-1}$ one has $dF = -Z^{-1} * dZ * Z^{-1}$, i.e. expression of the form like (2.1). Analogously, one can verify corresponding statement for other independent conformal mappings in $\mathbf{E^4}$: translations, rotations and dilatation – as well as for arbitrary their sequences. In other words, transformations defined by $\mathbb{Q}$-differentiable functions form the group isomorphic to the conformal group of $\mathbf{E^4}$.

Thus, for exceptional algebra with division $\mathbb{Q}$ the class of $\mathbb{Q}$-differentiable functions defined by the condition (2.1) turns, as before, to be too narrow for applications in fundamental physics for the purpose, say, that such functions could be considered in the capacity of fundamental fields.

3 Biquaternionic differentiability and the equation of $\mathbb{C}$-eikonal

Below we restrict ourselves by the case of the full $N \times N$ matrix algebras $\mathbb{A} = Mat(N)$ over $\mathbb{R}$ or $\mathbb{C}$ (when $N = 2$ we have the isomorphism of the full matrix algebra $Mat(2, \mathbb{C}) \cong \mathbb{B}$ to that of biquaternions). For the equivalent of quaternionic norm – the *determinant* of the matrix of differentials dF in the left-hand part of (2.1) – we get:

$$\det \|dF\| = \det \|L(Z) * R(Z)\| \det \|dZ\| \equiv \lambda(Z) \det \|dZ\|. \tag{3.1}$$

In the case when both matrices L, R are invertible, so that $\lambda(Z) \neq 0$, condition (3.1) defines some conformal mapping with the scale factor $\lambda(Z)$ of the infinitesimal (complex or real indefinite) "metric" represented by determinant in (3.1). In particular, for the algebra $\mathbb{B}$ we deal with conformal mappings in the complexified Minkowski space $\mathbb{C}\mathbf{M}$.

The most interesting, however, seems to be the case when $\det L = 0$ (or, analogously, $\det R = 0$); under this condition the scale factor $\lambda(Z) = 0$, and the relation (3.1) defines a mapping of the full vector space of $\mathbb{A}$ into the subspace of its elements – null divisors (into the complex "light-like cone" in the case of algebra $\mathbb{B}$). Such mappings may be named *degenerate conformal mappings*. They constitute an important and wide class: in the context of algebrodynamical theory presented further in the article just these mappings (and corresponding differentiable $\mathbb{A}$-functions) are identified with physical fields. In particular, *under complexification of quaternions the class of differentiable functions and related mappings considerably extends.*

In the $N \times N$ matrix representation condition of differentiability (2.1) in component notation takes the form $(A, B, ... = 1, ...N)$:

$$\nabla_{AB} F_{CD} = L_{CA} R_{BD} \tag{3.2}$$

where ∇_{AB} corresponds to the operator of differentiation with respect to coordinates Z^{AB}. For some fixed pair of indices C, D denoting $F_{CD} \equiv \Sigma,\ L_{CA} \equiv \phi_A,\ R_{BD} \equiv \psi_B$ one gets instead of (3.2):

$$\nabla_{AB}\, \Sigma = \phi_A \psi_B. \tag{3.3}$$

Determinant of the matrix of semi-derivatives in the right-hand part of the equation, by virtue of the factorized structure, is identically null. Consequently, one gets the equation:

$$\det \|\nabla_{AB} \Sigma\| = 0, \tag{3.4}$$

which is necessarily satisfied by any matrix component $F_{CD} \equiv \Sigma \in \mathbb{R}$ or $\mathbb{C}$ of any function $F(Z)$ differentiable in $\mathbb{A}$.

Equation (3.4) represents itself a *nonlinear* analog of the Laplace equation from complex analysis, and here *nonlinearity arises as a direct consequence of the account of non-commutativity of algebra* in the very definition of $\mathbb{A}$-differentiable functions (2.1). In the case of biquaternion algebra $\mathbb{B}$ equation (3.4) is nothing else but the *equation of complex 4-eikonal.* Indeed, introducing for brevity the following notations for coordinates in matrix representation:

$$Z^{00} = u, \ Z^{11} = v, \ Z^{01} = w, \ Z^{10} = p, \tag{3.5}$$

and computing the determinant (3.4), we come to the equation:

$$(\nabla_u \Sigma)(\nabla_v \Sigma) - (\nabla_w \Sigma)(\nabla_v \Sigma) = 0, \tag{3.6}$$

which in the (complex) Cartesian coordinates $z^0 = (u+v)/2, \ z^3 = (u-v)/2, \ z^1 = (w+p)/2, \ z^2 = i(w-p)/2$ takes the familiar form of eikonal equation:

$$(\frac{\partial \Sigma}{\partial z^0})^2 - (\frac{\partial \Sigma}{\partial z^1})^2 - (\frac{\partial \Sigma}{\partial z^2})^2 - (\frac{\partial \Sigma}{\partial z^3})^2 = 0. \tag{3.7}$$

In accord with results of our paper [17] (see also [18]), *general solution* of the complex eikonal equation (CEE) consists of two different classes both of which can be obtained in an algebraic way with the help of arbitrary (complex analytical) functions of (projective) *twistor variable.* Specifically, let us choose in the formula (3.2) for the 4-gradient of complex eikonal one of the 2-spinors, say, $\psi = \{\psi_B\}$ and define then the 2-spinor $\tau = \{\tau^A\}$ *incident* to it by means of the Klein-Penrose correspondence [19]

$$\tau = Z\psi, \quad \leftrightarrow \quad \tau^A = Z^{AB}\psi_B. \tag{3.8}$$

A couple of spinors $\{\psi_B, \tau^A\}$ connected by the incidence relation (3.8) we shall call the *(projective) twistor* of complex Minkowski space $\mathbb{C}\mathbf{M}$.

Indeed, equation (3.8) as well as the spinor ψ itself in equation (3.2) are defined up to a multiplication to a nonzero complex scalar; therefore, only *three complex ratios* of twistor components are essentially defined. Let, for example, the spinor component ψ_0 is not equal to zero; then, making use of the projective equivalence, one can choose the twistor gauge of the form $\psi_0 = 1$ and to get for the above ratios:

$$\psi_1 = G, \quad \tau^0 = wG + u, \ \tau^1 = vG + p. \tag{3.9}$$

Now let us choose an arbitrary function Π of three complex arguments – components of the projective twistor

$$\Pi(\psi_1, \tau^0, \tau^1) \equiv \Pi(G, wG + u, vG + p) \tag{3.10}$$

Resolving the equation $\Pi = 0$ with respect to the unknown $G(u, v, w, p)$, we obtain some solution of the CEE (of the I class). Further, resolving now equation

$d\Pi/dG = 0$ with respect to G again and substituting the solution into the initial function Π, we come to a "conjugated" solution of the CEE $\Pi(u,v,w,p)$ (of the II class). According to the results of the paper [17], these two classes exhaust all (almost everywhere analytical) solutions of the CEE (see [17, 18] for details). For further needs let us mention only that, for any generating ("World") function Π, solution of the joint system $\Pi = 0, \quad d\Pi/dG = 0$ defines the structure of singular set $\Pi(u,v,w,p) = 0$ – the locus of branching points of the eikonal function G (Π) itself and, correspondingly – of the poles of its 4-gradient. Resolving of this algebraic system makes it possible, sometimes even without explicit expression of the eikonal function itself, to determine the structure of its singularities (which may be extremely complicated). Corresponding examples are presented in [18, 20, 27, 31].

4 Global symmetries and splitting of the equation of $\mathbb{A}$-differentiability

Let us return now back to examine the conditions of $\mathbb{A}$-differentiability (2.1) in general non-commutative case of the matrix algebra $Mat(N,\mathbb{C})$. It is easy to demonstrate that this fundamental relation preserves its form under the following transformations:

$$Z \mapsto PZQ^{-1}, \quad F(Z) \mapsto SF(Z)T^{-1}, \quad L(Z) \mapsto SL(Z)P^{-1}, \quad R(Z) \mapsto QR(Z)T^{-1}, \tag{4.1}$$

where P,Q,S,T are four constant invertible and, in general, distinct matrices $N \times N$ (here and below the symbol of matrix multiplication will be omitted for simplicity). Digressing from dilatations (generally, with different scale factors for coordinates Z and functions $F(Z)$), we shall further on consider determinants of all matrices equal to unity so that $P,Q,S,T \in SL(2,\mathbb{C})$.

In the particular case of equality of the entire matrices one gets the *internal automorphisms* of the algebra at study which leaves invariant both the trace and the determinant of matrices. When $N = 2$, i.e. in the case of biquaternion algebra $\mathbb{B}$, determinant has the structure of the quadratic $\mathbb{C}$-valued form:

$$\det \|Z\| = (z^0)^2 - (z^1)^2 - (z^2)^2 - (z^3)^2 \tag{4.2}$$

Thus, in account of invariance of the trace z^0, automorphisms represent themselves the rotations of 3-dimensional complex space $\mathbb{C}^3$; the automorphism group $Aut(\mathbb{B})$ is 2:1 isomorphic to the group of complex rotations $SO(3,\mathbb{C})$. In general case ($N > 2$) automorphisms look like linear transformations which keep invariant the trace and holomorphic Finsler-like "metrical" form of the N-th order, defined by the structure of matrix determinant.

For simplicity restricting below ourselves by the case $N = 2$, let us consider general symmetries of the *conditions of biquaternionic differentiability* (4.1). The coordinate transformations

$$Z \mapsto PZQ^{-1}, \quad P,Q \in SL(2,\mathbb{C}), \tag{4.3}$$

evidently represent themselves the 6-parametrical rotations of the full vector space of algebra $\mathbb{C}^4$ which leave invariant the holomorphic "metric" (4.2). These transformations form the group 2:1 isomorphic to the group $SO(4,\mathbb{C})$. By this, the law of transformations of the semi-derivatives $L(Z), R(Z)$ and the function $F(Z)$ itself remains, according to (4.1), partially indefinite due to existing voluntarism in the choice of two other matrices $S, T \in SL(2,\mathbb{C})$. This situation is, of course, related to a very wide symmetry group of the conditions of $\mathbb{B}$-differentiability (2.1).

Indeed, one can set, in particular, $S = Q,\ T = P$ considering thus symmetries of the form

$$Z \mapsto PZQ^{-1}, \quad F(Z) \mapsto QF(Z)P^{-1} \quad L(Z) \mapsto QL(Z)P^{-1}, \quad R(Z) \mapsto QR(Z)P^{-1}, \quad (4.4)$$

under which all the "fields" $L(Z), R(Z), F(Z)$ behave themselves as (covariant) vectors realizing in this way vector representation of the group $SO(4,\mathbb{C})$. However, for the same fields another type of transformations preserving the form of basic equations (2.1) is possible. Specifically, let us set the matrices S, T equal to the unit matrix; them we come to the symmetry transformations of the form

$$Z \mapsto PZQ^{-1}, \quad F(Z) \mapsto F(Z) \quad L(Z) \mapsto L(Z)P^{-1}, \quad R(Z) \mapsto QR(Z), \qquad (4.5)$$

so that under these the principal function $F(Z)$ behaves itself as a $SO(4,\mathbb{C})$-scalar, whereas the semi-derivatives $L(Z), R(Z)$ – as a complex of two independently transforming columns (raws), i.e. as the $SO(4,\mathbb{C})$-spinors!

Thus, in the considered case one has a unique situation when one and the same "physical field" can be transformed according to a number of independent representations of the "complex Lorentz group" $SO(4,\mathbb{C})$ manifesting itself at the same as a vector, a couple of spinors or a number of scalars.

The most general symmetries (4.1) form (in the 4:1 ratio) the $12\mathbb{C}$-parametrical group $SO(4,\mathbb{C}) \times SO(4,\mathbb{C})$ which one can imagine himself as the product of *coordinate* and *internal* groups. However, in respect to the transformations of "fields", representation of the full group cannot be uniquely decomposed into representations of each of constituents.

Indeed, matrices S, T can in a unique way be represented in the form $S = \Lambda Q,\ T = \Pi P$ through some new matrices $\Lambda, \Pi \in SL(2,\mathbb{C})$. By this, the field transformations under general symmetries (4.1) take the form:

$$Z \mapsto PZQ^{-1}, \ F \mapsto \Lambda(QFP^{-1})\Pi^{-1}, \quad L \mapsto \Lambda(QLP^{-1}), \quad R \mapsto (QRP^{-1})\Pi^{-1}, \ (4.6)$$

and acquire the following natural interpretation: with respect to the group of "coordinate" transformations $SO(4,\mathbb{C})_{coord}$ all of the fields $L(Z), R(Z), F(Z)$ are (covariant) vectors; at the same time, with respect to the internal "isotopic" group $SO(4,\mathbb{C})_{int}$ each of semi-derivatives $L(Z), R(Z)$ behave itself as a couple of *isospinors* whereas the basic field $F(Z)$ is an *isovector*. However, this interpretation though suitable is not at all the only possible as we have seen above.

Let us notice also that the coordinate space Z can be reduced to the space of *unitary* matrices (Hamilton's quaternions) or to the space of *Hermitian* matrices for which the above introduced rectilinear coordinates z_μ turn to be real and the invariant form (4.2) represents the Minkowski metric. By this, requirement of preservation of the introduced condition (of unitary, Hermitian etc. structure) imposes restrictions on the admissible general symmetry transformations (4.1) so that the symmetry group reduces to a smaller one. All such situations including admissible transformations of "fields" (which generally remain complex-valued) can be easily examined. In particular, on the Hermitian coordinate subspace the *algebrodynamical field theory* based on the conditions of $\mathbb{B}$-differentiability (2.1) will be automatically Lorentz invariant. This case will be discussed in details below.

To conclude the discussion of symmetries, let us note that *linear* transformations (4.1) that contain the $SO(4,\mathbb{C})$-rotations and the dilatation do not exhaust the whole group of symmetries of the $\mathbb{B}$-differentiability conditions (2.1) which are also evidently invariant under the $4\mathbb{C}$ translations as well as under the *inversions* in this space, so that the full group of symmetries includes into itself at least the $15\mathbb{C}$-parametrical group of conformal mappings of the 4D complex space equipped with holomorphic metric (4.2).

Now, in accord with the wide group of their symmetries, conditions of $\mathbb{B}$-differentiability admit various forms of "splitting", i.e. of their reduction to simpler systems of equations. By this, of course, symmetry group of the reduced system will be smaller than initial one. The most important example of the procedure is the row (column) splitting of the matrix of basic function $F(Z)$ [23].

Specifically, let us denote the two *columns* of this matrix as $F = \{\eta_1, \eta_2\}$ and the columns of the right semi-derivative as $R = \{\xi_1, \xi_2\}$. 'hen one reduces the initial matrix system to that of the following two equations of the same type:

$$d\eta_a = \Phi * dZ * \xi_a, \quad a = 1,2 \tag{4.7}$$

and solution of the full system may be build as an arbitrary composition of some two solutions of systems like (4.7) with the same matrix "field" $\Phi(Z)$ (of left semi-derivative). Reduced system (4.7) is form-invariant, in particular, under the following transformations of variables:

$$Z \mapsto QZP^{-1}, \quad \xi \mapsto P\xi, \quad \Phi \mapsto P\Phi Q^{-1}, \quad \eta \mapsto P\eta, \tag{4.8}$$

during which the quantities $\Phi(Z)$ transform themselves as a complex 4-vector and the "fields" $\eta(Z)$, $\xi(Z)$ – as the $SL(2,\mathbb{C})$-spinors [1].

Reduction of the full system of equations of $\mathbb{B}$-differentiability to a simpler system of the form (4.7) for two spinors (basic and additional) and one 4-vector may be called the *spinor splitting* of the primary system of equations (2.1).

[1] It is obvious that transformations (4.8) do not exhaust all the symmetries of each of the systems of equations (4.7) which result from general symmetry transformations (4.1), and are exhibited here only as an example of these

The main class of solutions of the full system (2.1) can in fact be restored from an arbitrary solution of only *one* of the spinor systems (4.7). For this, it is sufficient simply to nullify, say, the spinors η_2 and ξ_2 or to regard them proportional to the initial spinors, i.e. to set:

$$\eta_2 = k\eta_1, \quad \xi_2 = k\xi_1 \tag{4.9}$$

with arbitrary *constant* complex factor of proportionality $k \in \mathbb{C}$. By this, the right semi-derivative will represent itself a degenerate matrix, $\det R(Z) = 0$, and the principal matrix function will differ from a degenerate one to an arbitrary constant matrix C: $F(Z) = C + H(Z)$, $\det H(Z) = 0$. We note that the factor of proportionality k cannot depend on coordinates Z in a nontrivial way what may be easily proved in account of the identical form of the "field" $\Phi(Z)$ for both spinor systems (see [21]).

The *degenerate* case that corresponds to the *degenerate conformal mappings* (see section 3) is in general the only physically nontrivial one. Indeed, in the non-degenerate case correspondent to the canonical conformal mappings in $\mathbb{C}$ the field strengths of gauge fields associated with $\mathbb{B}$-differentiable functions identically turn to zero [10, 23, 22]. On the other hand, when the matrix of, say, the right semi-derivative is degenerate, its two columns are proportional at a point and, by virtue of constancy of the factor k, – globally. Thus, we have shown that physically nontrivial solutions of basic equations of $\mathbb{B}$-differentiability (2.1) all correspond to the case of degenerate matrices and can be all obtained from the solutions of fundamental spinor system

$$d\eta = \Phi dZ\xi \tag{4.10}$$

with the help of trivial completion of the spinors $\xi(Z), \eta(Z)$ to full matrices with zero determinant (by this, any of the spinors can be multiplied to an arbitrary complex number).

5 General solution of fundamental spinor system

As the complex eikonal equation (CEE) for individual components of the principal spinor $\eta(Z)$, general solution of fundamental spinor system (FSS) (" ") (4.10) consists of two different classes and is obtained by analogy with general solution of the CEE itself (section 3). Here we shall only announce its structure (full proof will be given elsewhere).

Solutions of FSS of the I class

Let us define the twistor of the space $\mathbb{C}^4$

$$\mathbf{W} = \{\xi, \kappa\} \equiv \{\xi, Z\xi\}, \tag{5.1}$$

built on a spinor $\xi(Z)$ which satisfies the FSS (4.10) together with some corresponding functions $\eta(Z), \Phi(Z)$. Let three its *projective* components be functionally independent; then one may also consider as functionally independent all four its components

$$\{\xi_A, \quad \kappa^A = Z^{AB}\xi_B\}. \tag{5.2}$$

By this, it may be shown that components of the principal spinor, on the contrary, are functionally dependent and may be considered as dependent on coordinates only through the components of the twistor (5.2):

$$\eta_A(Z) = \eta_A(\sigma), \quad \sigma(Z) = \sigma(\xi,\kappa) \equiv \sigma(\xi(Z), Z\xi(Z)). \tag{5.3}$$

The choice of generating function $\sigma(\xi,\kappa)$ as well as of the functional dependence on it of the components of principal spinor $\eta_A(\sigma)$ may be quite arbitrary (certainly, if one provides necessary smoothness conditions).

It appears that dependence on coordinates of the components of spinor $\xi_A(Z)$ can be by this determined from the solution of algebraic system of two equations of the form:

$$\frac{d\sigma}{d\xi_B} = 0. \tag{5.4}$$

Substituting after this the solution $\xi(Z)$ into (5.3), one obtains expression of the principal spinor $\eta(Z)$. By this, the "field" matrix Φ_{AB} is degenerate and equal to

$$\Phi_{AB} = \frac{d\eta_A}{d\sigma}\frac{\partial\sigma}{\partial\kappa^B}. \tag{5.5}$$

Thus, any differentiable function of twistor variable $\sigma(\xi,\kappa)$ gives rise to a class of equivalent (with respect to functional dependence of the components of spinor η_A) solutions to FSS. These solutions are in evident correspondence to the CEE solutions of the I class described in section 3.

Solutions of FSS of the II class

Let now three projective twistor components (5.1) be functionally dependent; then, again with account of arbitrariness of the choice of the fourth component of *general* twistor, one may consider that there exist *two* functional constraints between its components (5.2) of the form

$$\Pi^{(D)}(\xi_A, \kappa^A) = \Pi^{(D)}(\xi_A, Z^{AB}\xi_B) = 0, \quad (D) = 1,2. \tag{5.6}$$

Resolving this system of algebraic equations, one can find the explicit form of the spinor $\xi_B(Z)$. Differentiating equations (5.6) with respect to coordinates, one can show (for details see [17]) that the components of ξ satisfy differential equations of the form

$$\nabla_{AB}\xi_C = \left[-\frac{\partial\Pi^{(D)}}{\partial\kappa^A}Q^{-1}_{(D)C}\right]\xi_B \equiv \Psi_{CA}\xi_B, \tag{5.7}$$

where the notation Ψ_{CA} for quantities in square brackets is introduced and $Q^{-1}_{(D)C}$ – for the matrix, inverse to

$$Q^{(D)C} := \frac{d\Pi^{(D)}}{d\xi_C}. \tag{5.8}$$

In invariant Pfaffian form system of equations (5.7) may be written down as follows:

$$d\xi = \Psi dZ \xi. \tag{5.9}$$

Under identification of the principal and additional spinors $\eta(Z) \equiv \xi(Z)$ and the function $\Psi(Z)$ with the "field" $\Phi(Z)$ (i.e. with left semi-derivative "field"), this system is evidently itself a solution of FSS correspondent to generating twistor functions (5.6).

Actually, this case is of especial significance for further applications of biquaternionic analysis in algebrodynamical framework; in preceding articles the system (5.9) and corresponding full matrix system

$$dF = \Psi dZ F, \tag{5.10}$$

in which $F(Z)$ is a *degenerate* ($\det F = 0$) biquaternionic field constructed by means of two proportional spinors $\xi(Z)$, has been called the *generating system of equations* (GSE). Indeed, as we shall see later on, any solution of the GSE naturally gives rise to a solution of free equations of Maxwell, Yang-Mills and other fundamental (massless) equations of relativistic fields. We note that from mathematical point of view the GSE represents itself a special case of the $\mathbb{B}$-differentiability conditions under which the *right semi-derivative $R(Z)$ is identified with the principal biquaternionic "field" $F(Z)$.*

Let us present now the *general form* of the FSS solutions of II class that corresponds to some arbitrary composition of the two generating twistor functions $\Pi^{(D)}(\xi_A, \kappa^A)$. From these, resolving the algebraic equations (5.6) for the spinor $\xi(Z)$ and computing the quantities $\Psi(Z)$ by means of formulas (5.7),(5.8), one obtains a complete solution to the GSE (5.9). By this, it turns out that the components of principal spinor $\eta(Z)$ may be arbitrary (and generally different) functions of twistor components (5.2):

$$\eta_A(Z) = \eta_A(\xi, \kappa) \equiv \eta_A(\xi(Z), Z\xi(Z)). \tag{5.11}$$

Let us note also that by virtue of the constraints (5.6) *only two of these twistor components are actually independent.* Finally, corresponding expression for the "field" $\Phi(Z)$ is obtained from the already found solution of GSE $\{\Psi(Z), \xi(Z)\}$ and arbitrarily chosen dependence of components of the principal spinor (5.11) in the following way:

$$\Phi(Z) = (M\Psi(Z) + N(E + Z\Psi(Z)), \quad M := \|\frac{\partial \eta}{\partial \xi}\|, \quad N := \|\frac{\partial \eta}{\partial \kappa}\|, \tag{5.12}$$

where E represents again the 2×2 unit matrix. As the result, one obtains that any pair of independent functions of twistor variable $\Pi^{(D)}(\mathbf{W}) \equiv \Pi^{(D)}(\xi_A, \kappa^A)$ gives rise to a class of equivalent (in respect of the arbitrariness of mutual dependence of the components of the principal spinor $\eta_A(\mathbf{W})$) solutions of the FSS. Certainly, this class corresponds to the II class of solutions to CEE described in section 3.

Thus, we come to the general solution of FSS (4.10). Indeed, since from the three projective components of principal twistor (5.1) either all three or only two are functionally independent [2], any solution to FSS belongs either to the first or to the second class. That is why any (almost everywhere analytical) solution to FSS may be obtained from some generating function of twistor variable (I class) or from a pair of such functions (II class) through the above described *purely algebraic* procedure. In compare with general solution to the complex eikonal equation described earlier in section 3 (in a fixed gauge) and in articles [17, 18], in the case of FSS there exists an additional arbitrariness of the choice (of dependence on twistor variables) of the components of the principal spinor which may be either functionally connected (for the I class solutions) or independent (for solutions of the II class).

Such arbitrariness may be naturally eliminated if one chooses as fundamental the generating system of equations (5.9) or corresponding full-matrix system (5.10). All solutions of the latter belong already to the second class of the FSS solutions and are completely determined by the choice of a pair of generating functions of twistor variable (5.6) (or, under fixing of gauge for projective twistor (see below) – even by a sole generating "World" function). Therefore, we proceed now to the detailed examination of properties and solutions of this universal system of equations.

6 Biquaternionic differentiability and the gauge fields

In the *algebrodynamics*, conditions of biquaternionic differentiability (2.1) and, particularly, principal case of them – the generating system of equations (5.9), (5.10) – are considered as the unique primary equations of physical fields identified with differentiable $\mathbb{B}$-functions. By this, in order to guarantee the theory to be relativistic invariant, one has to restrict the complex coordinates Z to the subspace of Hermitian matrices $Z \mapsto X = X^+$ with the Minkowski metric $\det X = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$. The GSE (5.9) takes then the following form:

$$d\xi = \Psi dX \xi \tag{6.1}$$

and preserves it (including the Hermitian structure of coordinate matrix) under the following symmetry transformations:

$$X \mapsto P^+ XP, \quad \xi \mapsto P^{-1}\xi, \quad \Psi \mapsto P^{-1}\Psi(P^+)^{-1}, \tag{6.2}$$

[2] Statement that at least two components of a generic twistor are always independent is proved, for example, in monograph [22]

where the quantities $\xi(Z)$ and $\Psi(Z)$ behave themselves as an $SL(2,\mathbb{C})$-spinor and a complex 4-vector respectively. Of course, there exists also a more general symmetry group (6.1), namely, the *conformal* group of Minkowski space, and just this fact predetermines the existence of *twistor* structure introduced above.

It should be noted, however, that the property of Hermitiance represents itself some superfluous requirement which is not motivated by the internal structure of initial algebra of biquaternions. In the last section we shall demonstrate in which way the structure of Minkowski space is actually *encoded* in the structure of the full vector space $\mathbb{C}^4$ of the $\mathbb{B}$-algebra. In account of this circumstance, in this and subsequent sections we preserve, as a rule, the *holomorphic* structure of theory dealing, as before, with complex coordinates $Z = \{z_\mu\}$ and, correspondingly, – with GSE in its previous form (5.9), (5.10). When only theory acquires an explicit physical interpretation, we accomplish transition to the real coordinates $\{x_\mu\}$ or, in other words, – to the Hermitian matrix of Minkowski space coordinates $X = X^+$.

Let us recall now that the GSE (5.9) is *over-determined* (8 differential equations for 6 unknown functions). Therefore, some conditions of compatibility (integrability etc.) must be fulfilled which let to obtain from (5.9) some restrictions on both the spinor $\xi(Z)$ and the vector field $\Psi(Z)$. However, before we start to consider these, it is necessary to examine the *gauge* nature of the field $\Psi(Z)$ that turns to be essentially distinct form generally accepted one. Let us also note that further in this and subsequent sections we follow mostly the exposition of the discussed questions presented in [22, 26].

It is easy to see that well known from the field theory gauge $U(1)$-transformations of the form

$$\xi \mapsto \exp i\alpha(X)\xi, \quad \Psi \mapsto \Psi - i\nabla \ln \alpha, \quad \alpha \in \mathbb{R} \tag{6.3}$$

or their natural complexification, do not leave the GSE form-invariant. Nonetheless, in our papers [23, 22] it was shown that this system possesses the so called "weak" (or "restricted") gauge symmetry under which the gauge parameter α depends on coordinates implicitly, only through the components of the transformed spinor $\xi(Z)$ itself and the spinor $\kappa(Z) = Z\xi(Z)$ twistor-conjugated to it:

$$\alpha = \alpha(\mathbf{W}) = \alpha(\xi, \kappa) \equiv \alpha(\xi(Z), Z\xi(Z)). \tag{6.4}$$

Such transformations that correspond to the projective transformations of twistor components, form a group which is a (proper) subgroup of the full gauge group $\mathbb{C}$ (the latter being the complexification of $U(1)$) [22]). By this, the quantities $\Psi(Z)$ transform gradient-wise, that is, behave themselves as the *potentials* of some gauge field. As we shall see below, this field may be naturally associated with (complexified) *electromagnetic* field.

Indeed, the GSE (5.9) can be considered as the *condition for the spinor* $\xi(Z)$ to be covariantly constant (absolutely parallel) with respect to the $\mathbb{B}$-valued differential 1-form of effective connection:

$$\Omega = \Psi dZ. \tag{6.5}$$

Interestingly, in the 4-vector representation the $\mathbb{B}$-connection (6.5) gives rise to the affine connection of the form [10, 23]:

$$\Gamma^{\mu}_{\nu\rho} = \delta^{\mu}_{\nu}\Psi_{\rho} + \delta^{\mu}_{\rho}\Psi_{\nu} - \eta_{\rho\nu}\Psi^{\mu} - i\epsilon^{\mu}_{.\nu\rho\lambda}\Psi^{\lambda}, \tag{6.6}$$

that defines actually the effective complex *Weyl-Cartan* geometry. In such $\mathbb{B}$-induced geometry the non-metricity Weyl vector and the vector of the pseudotrace of the skew-symmetric torsion are proportional to each other and are expressed both through the components of the principal gauge field $\Psi(Z)$ [3].

Making now use of the definition (6.5), let us rewrite the initial GSE (5.9) in the form

$$d\xi = \Omega\xi \tag{6.7}$$

Dynamics of the connection $\Omega(Z)$ may be obtained through external differentiation of (6.7) that results in the condition of integrability of the form

$$R\xi \equiv (d\Omega - \Omega \wedge \Omega)\xi = 0, \tag{6.8}$$

where (in parentheses) a *curvature 2-form* R appears. Since the spinor ξ is not arbitrary but subject to (6.7), conditions of integrability (6.8) do not result in the zero value of curvature [4]. Quite remarkably, instead of trivial "zero curvature" requirement, integrability conditions (6.8) result in the *self-duality* of curvature [10, 23].

In order to demonstrate this, let us note that for connection of the type (6.5) the curvature R is of the following, rather special form:

$$R = (d\Psi - \Psi dZ\Psi) \wedge dZ \equiv \pi \wedge dZ, \tag{6.9}$$

in which a novel $\mathbb{B}$-valued 1-form π arises, with components

$$\pi_{AC} = \pi_{ACBD}dZ^{BD} = (\nabla_{BD}\Psi_{AC} - \Psi_{AB}\Psi_{CD})dZ^{BD}. \tag{6.10}$$

Now the integrability conditions (6.8) take the form $(\pi \wedge dZ)\xi = 0$, or, in matrix representation:

$$\pi_{ACBD}dZ^{BD} \wedge dZ^{CE}\xi_E = 0.$$

With account of the symmetry properties of 2-spinors from the last relation one obtains:

$$\pi_{A\ \ C(B}^{\ \ C}\xi_{E)} = 0,$$

[3] Absolutely parallel fields in the framework of Weyl geometry free of torsion have been studied in [31]; their properties are closely related to the symmetries of Weyl manifolds [32]. For real connections of such type relations between the non-metrical and torsion parts were the object of consideration in [29]

[4] At this point our approach considerably differs from that accepted in the works of Buchdahl [33], Penrose [34] or Plebanski [35] who conjectured that integrability conditions resembling (6.8) should be fulfilled for arbitrary spinor field (or for a wide class of solutions to the so called "exact" systems of field equations)

so that for any nontrivial solution $\xi(Z)$ one has:

$$\pi_{A\ CB}^{\ C} \equiv \nabla_{CB}\Psi_A^{\ C} + \Psi_{BC}\Psi_A^{\ C} = 0. \tag{6.11}$$

Further, making use of the standard procedure and decomposing the curvature (6.9) into self- and antiself-dual parts one finds that equations (6.11) represent themselves just the *conditions for the self-dual part of curvature to vanish.* By this, another antiself-dual its part $\bar{R}$ has the form:

$$\bar{R}_{A\ (BC)}^{\ D} = \nabla^C_{\ (B}\Psi_{AC)} - \Psi^C_{\ (B}\Psi_{AC)} \tag{6.12}$$

and satisfies the additional integrability conditions $\bar{R}\xi = 0$ (later in the article we do not make use of these conditions).

Thus, though the curvature 2-form (6.9) of the connection 1-form (6.5) is not (anti)self-dual by itself (i.e. (anti)self-dual in the "strong" sense), it necessarily becomes antiself-dual *on the solutions to GSE.* For this reason this property of the effective curvature of GSE has been called *weak (anti)self-duality* [39].

From physical viewpoint, expression (6.12) defines the field strength of some matrix gauge field; in particular, its diagonal part

$$F_{BC} = \bar{R}_A^{\ A}{}_{(BC)} = \nabla^A_{\ (B}\Phi_{AC)} \tag{6.13}$$

corresponds to the strength of (complexified) electromagnetic field whereas the trace-tree part (6.12) defines the strength of a complex field of the *Yang-Mills* type [5]. Indeed, in account of the *Bianchi identities*

$$dR \equiv \Omega \wedge R - R \wedge \Omega, \tag{6.14}$$

self-duality of curvature $R + iR^* = 0$ immediately implies the fulfillment of free Maxwell equations for diagonal (electromagnetic) part of the 2-form $F = Tr(R) = R_A^{\ A}$:

$$dF^* = 0 = dF \equiv 0, \tag{6.15}$$

as well as of Yang-Mills equations for trace-free part of curvature form $\mathbf{F}_A^{\ B} = R_A^{\ B} - \frac{1}{2}F\delta_A^B$.

By this, though the electromagnetic 2-form F is, generally speaking, $\mathbb{C}$-valued, by virtue of its self-duality it is reduced to the real 2-form $\mathtt{F}$ connected with F in the following way:

$$F = \mathtt{F} - \mathtt{i}\mathtt{F}^*. \tag{6.16}$$

[5] In fact, here introduced field is not exactly what is generally accepted as the Yang-Mills one with the gauge group $SL(2,\mathbb{C})$ if one takes in account the restricted (weak) gauge symmetry. However, the form of gauge equations is completely identical to that generally accepted. Restrictions take place only with respect to the class of the admissible solutions and their transformations into each other under the action of the "weak" gauge group

Certainly, for this form homogeneous Maxwell equations are satisfied too so that the number of independent degrees of freedom turns to be equal to that for the ordinary real electromagnetic field. In explicit form for $\mathbb{C}$-valued strengths of "electric" $\vec{E}$ and "magnetic" $\vec{H}$ fields one has from (symmetric part of) the integrability conditions (6.11):

$$\vec{E} + i\vec{H} = 0, \tag{6.17}$$

from where one gets $\Im(\vec{H}) = \mathbb{R}(\vec{E})$, $\Im(\vec{E}) = -\mathbb{R}(\vec{H})$ so that a pair $\{\mathbb{R}(\vec{E}), \mathbb{R}(\vec{H})\}$ represents the $\mathbb{R}$-valued electromagnetic field subject to Maxwell equations. In addition, from (skew symmetric with respect to the spinor indices part of) equations (6.11) one obtains the following "inhomogeneous Lorentz condition" [10, 23] for the $\mathbb{C}$-valued electromagnetic potentials $A_\mu \leftrightarrow \Phi_{AD}$:

$$\partial_\mu A^\mu + 2A_\mu A^\mu = 0, \tag{6.18}$$

which must also hold identically on the solutions of GSE. Certainly, condition (6.18) is not gauge invariant by itself, in the accepted "strong" sense; nonetheless, it *is* invariant with respect to the "weak" gauge transformations (6.4), under the requirement that the transformed potentials (together with some corresponding spinor field $\xi(Z)$) really satisfy the GSE.

As to the Yang-Mills fields, they can be here always expressed through the strengths of electromagnetic field and the spinor ξ_A itself and, therefore, cannot be considered as independent. Note that separately the real and imaginary parts of the trace-less component of the curvature $\mathbf{F}_A{}^B$ will no longer satisfy free Yang-Mills equations by virtue of non-linearity of the latters. That is why here the Yang-Mills fields are *essentially complex-valued*. Other properties and peculiarities of the Yang-Mills fields arising in the framework of algebrodynamical approach can be found, say, in [23].

7 Null shear-free congruences associated with GSE

Let us now consider restrictions on the principal spinor ξ_A arising through elimination of the gauge fields of potentials from the GSE (5.9). For this purpose, let us write out the given Pfaffian system of differential equations in components:

$$\nabla_{BA}\xi_C = \Psi_{CB}\xi_A. \tag{7.1}$$

Multiplying the latter by the orthogonal spinor ξ^A with account of skew-symmetry of the spinor norm $\xi_A\xi^A = 0$ we get:

$$\xi^A \nabla_{BA}\xi_C = 0, \tag{7.2}$$

i.e. the system of nonlinear equations for the components of the spinor $\xi(Z)$. Let us note that under the restriction of complex coordinates to the Minkowski subspace $\mathbf{M}$, as a consequence of (7.2), one obtains a (well known in the framework of GTR) system of equations for the principal spinor of the so called *shear-free null (geodesic) congruence* (SFC) of 4-dimensional rectilinear “rays”:

$$\xi^B \xi^C \nabla_{AB} \xi_C = 0. \tag{7.3}$$

At this point we must warn the reader that here and below, in contrast to the generally accepted formalism, we make no difference between the primed and unprimed spinor indices under the restriction of coordinates to **M**. This is made to preserve as much as possible the notations specific for the full complex space and surely will not lead to any misunderstanding.

In our articles [22, 56] it was shown that initial system (7.2) differs from the SFC system (7.3) only in a more rigid fixing of the gauge of the principal spinor ξ, and is completely equivalent to the latter in what is related to the *ratio* of spinor's components. In particular, *general solution* of the SFC system (and, therefore, – complete description of all such congruence on the background of Minkowski space **M**) is explicitly related to its twistor structure and is represented by the famous *Kerr theorem* [40, 19] in the form of implicit algebraic equation:

$$\Pi(\xi, \kappa) = \Pi(\xi, Z\xi) = 0, \tag{7.4}$$

where Π is an arbitrary *homogeneous* function of twistor arguments. From the constraint (7.4) the ratio of spinor components may be found which only is defined by the SFC system of equations (7.3). Analogously, the more rigid system of equations for the principal spinor of GSE (7.2) has, as it has been shown earlier (section 4), general solution (5.6) in the form of two equations that contain some arbitrary and independent twistor functions $\Pi^{(D)}(\xi, \kappa)$. From these equations now both spinor components $\xi(Z)$ can be defined altogether.

It is well known that, in order to draw geometrically on **M** a SFC, one has to define, via the principal spinor $\xi(Z)$, the field of a (real-valued) null 4-vector $k_\mu(X)$ tangent to the (rectilinear) rays of the congruence as follows:

$$k_\mu(X) = \xi^+ \sigma_\mu \xi, \quad \sigma_\mu = \{E, \sigma_a\}, \tag{7.5}$$

where $\{\sigma_a\},\ a = 1, 2, 3$ are the Pauli matrices and E – the unit 2×2-matrix.

It is well known that, from the resolving of the condition of spinor incidence (3.8) restricted to **M**,

$$\kappa = X\xi, \tag{7.6}$$

with respect to the space-time points X, it follows [19] that twistor field $\{\xi, \kappa\}$ together with the SFC tangent vector k_μ *is transported in parallel along rectilinear null directions defined by the vector itself.* By this, as a parameter of transportation along the rays one may choose the time coordinate itself [60, 18, 61].

Let us note now that for physical applications only *projective* components of the GSE twistor are of importance that are defined, say, by the ratio of spinor components $\xi_1/\xi_0 = G$ and equal then to

$$\kappa^0 = wG + u, \quad \kappa^1 = vG + p, \tag{7.7}$$

where u, v, w, p are the complex matrix coordinates (3.5) two of which (u, v) become real under the restriction to $\mathbf{M}$ whereas the two others (w, p) become complex-conjugated. By this, both systems (7.2) and (7.3) for fundamental spinor field G turn to be equivalent to a pair of PDE's of the following form:

$$\nabla_w G = G \nabla_u G, \quad \nabla_v G = G \nabla_p G, \tag{7.8}$$

General analytical solution of equations (7.8) for function $G(X)$ now follows immediately from the gauge invariant its representation (5.6) in the form of a unique algebraic equation [6]

$$\Pi(G, \kappa^0, \kappa^1) = \Pi(G, wG + u, vG + p) = 0, \tag{7.9}$$

that implicitly defines the function $G(X)$. Here Π is an arbitrary holomorphic function of three complex twistor variables. Equation (7.9) expresses itself the fact of functional dependence of the three components $\{G, \kappa^0, \kappa^1\}$ of the projective twistor $\mathbf{W}$ associated with solutions to GSE. For the SFC equations (7.3) this equation is well known representing the Kerr theorem in a fixed gauge.

Let us notice now that solutions of (7.8) are defined almost everywhere except the branching points of the $G(X)$-function that correspond to *multiple* roots of the Kerr equation (7.9) and are defined by the condition of the form:

$$P := \frac{d\Pi}{dG} = 0. \tag{7.10}$$

Now, multiplying the two equations (7.8) one can verify once more the fact of fulfillment of the complex 4-eikonal equation for the field $G(X)$ in the form:

$$\nabla_u G \nabla_v G - \nabla_w G \nabla_p G = 0, \tag{7.11}$$

On the other hand, differentiating these equations one can check that $G(X)$ satisfies also the linear *wave* (d'Alembert) equation [41, 39, 56]

$$\Box G \equiv (\nabla_u \nabla_v - \nabla_w \nabla_p) G = 0. \tag{7.12}$$

We mention also that in account of (7.11) any C^2-function $\lambda(G)$ is also harmonic on the solutions of GSE:

$$\Box \lambda(G) = 0. \tag{7.13}$$

Further, making use of the expression (5.7) for potentials Ψ_{AB} and taking in account equation (7.11), we can express the strengths of electromagnetic field (6.13) through the second order derivatives of $\ln G$:

$$F_{00} = \nabla_u \nabla_p \ln G, \quad F_{11} = \nabla_v \nabla_w \ln G, \quad F_{01} = \nabla_w \nabla_p \ln G, \tag{7.14}$$

[6] It may be compared with general solution of the complex eikonal equation of the II class, see section 3

so that the fulfillment of free Maxwell equations for the strengths (7.14) follows directly from the wave equation (7.13) for $\lambda = \ln G$. Now, differentiating twice the identity (7.9) with respect to the coordinates $\{u, v, w, p\}$, we obtain a very important (and having none analogues in literature) representation of the strengths of electromagnetic field (7.14) through the twistor variables [22, 26]:

$$F_{AB} = \frac{1}{P}\left(\Pi_{AB} - \frac{d}{dG}(\frac{\Pi_A \Pi_B}{P})\right), \tag{7.15}$$

where the function P is defined by (7.10) and $\{\Pi_A, \Pi_{AB}\}$, $A, B = 0, 1$ denote the (first and second order) derivatives of the function Π with respect to its twistor arguments κ^0, κ^1. Below we shall return back to this compact expression of the strengths of the associated electromagnetic field.

Close connections between the GSE and SFC equations gives us an opportunity to introduce one more geometrophysical structure – an effective *Riemannian metric*. Indeed, it is well known [40, 42] that it is possible to deform the flat space-time metric $\eta_{\mu\nu}$ into a metric $g_{\mu\nu}$ of the *Kerr-Schild type*:

$$g_{\mu\nu} = \eta_{\mu\nu} + h k_\mu k_\nu \tag{7.16}$$

so that all the defining characteristics of the SFC – geodesity, twist and shear-free property – are preserved under such a deformation. Here h is some scalar function of coordinates, and the null (with respect to both the flat and deformed metrics) congruence $k(X)$ defined in (7.5) has the following projective invariant form:

$$k = du + \bar{G}dw + Gd\bar{w} + G\bar{G}dv, \tag{7.17}$$

where as $\bar{G}$ the quantity complex conjugated to G is denoted.

Let us turn now to the results of classical paper [40] in which it has been proved that metric (7.16) satisfies the electrovacuum Einstein-Maxwell system of equations for functions G obtained as the solutions of the Kerr algebraic equation (7.9) with *linear* with respect to the twistor arguments κ^0, κ^1 generating functions Π :

$$\Pi = \varphi + (qG + s)\kappa^1 - (pG + \bar{q})\kappa^0. \tag{7.18}$$

Here $\varphi = \varphi(G)$ is an arbitrary analytical function of the complex variable G, s, p are real and q – complex constants. Not going in details, we note that according to the results of paper [40] scalar function h in (7.16) is defined, up to an arbitrary constant, by initial generating function Π and another function $\Psi(G)$ independent on $\varphi(G)$ and related to the electromagnetic field of the solution of Einstein-Maxwell system. Such fields are defined in the curved space with metric (7.16) and are, generally, different from those arising in our approach and satisfying Maxwell equations on the *flat* space-time background [7]. Nonetheless, for the most

[7] At the same time both these types of fields are, generally, different also from the fields which may be defined for any SFC through the twistor Penrose transform, see, e.g., [19], chapter 6

physically interesting solutions like those of Reissner-Nördstrem or Kerr-Newman expressions for both these fields are nearly identical differing only in respect that in our approach electric charge is fixed in absolute value by existing "master" structure of GSE (see section 8 below).

It was also shown in [40, 41] that singularities of curvature of the effective Kerr-Schild metric (7.16) are defined just by the condition (7.10). On the other hand, it follows from expression (7.15) that the same equation $P = 0$ defines the locus of singular points of associated electromagnetic field. The very same condition may be checked to define singularities of the Yang-Mills (YM) field associated with solutions of GSE [8].

Thus, to any solution of the GSE it can be naturally put in correspondence some electromagnetic, complex YM and curvature (effective gravitational) fields. These satisfy respectively the free (complexified) equations of Maxwell, Yang-Mills and, at least in the basic stationary case – the electrovacuum Einstein-Maxwell system [9]. Singularities of all these fields are defined by one and the same condition (7.10) and completely coincide in space and time. This remarkable fact makes it possible, in the framework of algebrodynamical approach based on the GSE, *to consider particles as common singularities of all the associated fields.* We shall develop this conception in the subsequent section.

8 Singular "particle-like" solutions of GSE with self-quantized electric charge

We present here a brief review of the main classes of solutions of the GSE and of the associated Maxwell equations known for the present. All these can be obtained through the choice of a generating function Π with subsequent resolving of algebraic Kerr equation (7.8) and calculation of derivatives. If one restricts himself by the simplest case of solutions that can be obtained in *explicit* form, he has to consider only functions Π *quadratic* in twistor arguments (linear functions lead to solutions with zero field strengths (7.14)) of associated electromagnetic field.

Fundamental *static* solution is generated by the function Π of the form

$$\Pi = G\kappa^0 - \kappa^1 + 2ia \equiv G(wG + u) - (vG + p) + 2ia, \tag{8.1}$$

($a = Const \in \mathbb{R}$) which does not contain the time coordinate. Equating the function to zero and resolving the quadratic equation with respect to the unknown G one gets (under restriction of coordinates to the real Minkowski space):

$$G = \frac{p}{(z + ia) \pm r_*} \equiv \frac{x + iy}{(z + ia) \pm \sqrt{x^2 + y^2 + (z + ia)^2}}. \tag{8.2}$$

[8] Additional singularities of the YM field strengths correspond to *poles* of function $G(X)$ [21, 22]

[9] Correspondence between shear-free null congruences and gauge fields has been studied for the case of a curved (algebraically special) space-time background in our paper [57]

Electromagnetic field (7.14) corresponding to the above solution,

$$\vec{E} - i\vec{H} = \pm\frac{\vec{r}_*}{4(r_*)^{3/2}}; \qquad (\vec{E} + i\vec{H} = 0), \tag{8.3}$$

where $\vec{r}_* = \{x, y, z+ia\}$, possesses the singular locus in the form of a *ring* of radius a, the only possible value of electric charge $q = \pm 1/4$ (in the dimensionless units used) and a dipole magnetic and quadruple electric moments equal respectively to qa and qa^2 [39, 56]. It one digresses from the restrictions on charge, the electromagnetic field (8.3) together with the Riemannian metric (7.16) corresponding to the SFC (7.17), precisely reproduces the field and metrics of the Kerr-Newman solution (in the coordinates used in [40]). In the particular case, when $a = 0$, solution (8.2) corresponds to the *stereographic projection* $S^2 \to \mathbb{C}$ and the fields turn into the Coulomb electric field and the Reissner-Nördstrem metric.

Self-quantization of electric charge is a fundamental property of the GSE solutions discovered in [10, 23]. This property follows from the self-duality conditions (6.17) which, together with the property of gauge invariance of GSE, leads to the restriction $q = N/4, N \in \mathbb{Z}$ on the admissible values of electric charge of the field associated with any solution of GSE [10]. This property has both topological and purely dynamical reasons, the latters being connected to the over-determined structure of the GSE. Proof of general theorem on charge quantization in the framework of algebrodynamics is presented in the articles [26, 58].

By this, it is necessary to mention that, in contrast to some other, purely topological approaches to the problem of the charge quantization [43, 44], in the context of the GSE the charge of fundamental static solution (8.2) can possess only a unit in modulus value and, consequently, can be naturally treated as *elementary charge*. Together with the known property of the Kerr-Newman solution to have the gyromagnetic ratio $g = 2$, equal to that for the Dirac particle [45], appearance of elementary charge in the theory justifies the numerous attempts to interpret the ring singularity of fundamental solution (8.2) in capacity of the classical model of electron. Such attempts have been undertaken, say, in the models of Lopes [46], Israel [47] or Burinskii [48] based exclusively on the properties of the solutions of Einstein-Maxwell system [11].

According to the general theorem proved in [41] (see also [48]), all *static* solutions of the SFC equations (and, consequently, – of the GSE) for which the singular locus is bounded in 3D-space (below we call them *particle-like solutions* [27]) reduce

[10] In the $\mathbb{B}$-electrodynamics invariant with respect to the so called *duality transformations* actually not electric but the effective *magneto-electric* charge is physically significant and quantized, and the problem of magnetic monopole gets a natural solution [58]

[11] However, recently in our work [59] it has been proved that the Kerr congruence is *unstable* in the "Arnold's sense", i.e. with respect to a small alteration of parameters of the generating function (8.1) under which the static singular Kerr ring transforms into the ring uniformly expanding and "irradiates to infinity". Resolving of the instability problem requires, perhaps, the transition to a novel "causal Minkowski geometry with phase", see discussion in section 9 below

themselves (up to the 3D rotations and translations) to the Kerr solution (8.2). If, however, one would remove requirement on a solution to be static and leave the class of functions (7.18) considered in [40], he can find a lot of time-dependent "particle-like" solutions with bounded singularities of different dimensions, temporal dynamics and spatial shapes.

In particular, the *axisymmetric* solution of the particle-like type generating by the function

$$\Pi = \kappa^0 \kappa^1 + b^2 G^2 = 0, \quad b = Const, \tag{8.4}$$

has been found in [39, 21]. For real-valued b it describes *two point-like singularities with elementary unlike charges* $+1/4,\ -1/4$ accomplishing a counter hyperbolic motion (i.e., uniformly accelerated). Electromagnetic field for such solution

$$E_\rho = \pm\frac{8b^2\rho z}{\Delta^{3/2}}, \quad E_z = \mp\frac{4b^2 M}{\Delta^{3/2}}, \quad H_\varphi = \pm\frac{8b^2\rho t}{\Delta^{3/2}}, \tag{8.5}$$

corresponds to that of the well known *Born solution.* By this, the following notations are used:

$$\rho^2 = x^2 + y^2, \quad s^2 = t^2 - z^2, \quad M = s^2 + \rho^2 + b^2, \quad \Delta = M^2 - 4s^2\rho^2,$$

and the field singularities are defined by the condition $\Delta = 0$. For purely imaginary $b = ia,\ a \in \mathbb{R}$ initially, at $t = 0$, one has an *electrically neutral* ring-like singularity of radius a which in the course of time turns into an expanding *torus.* After the time passed $t = |a|$ singular locus transforms itself into a *self-intersecting torus* represented at Fig.1.

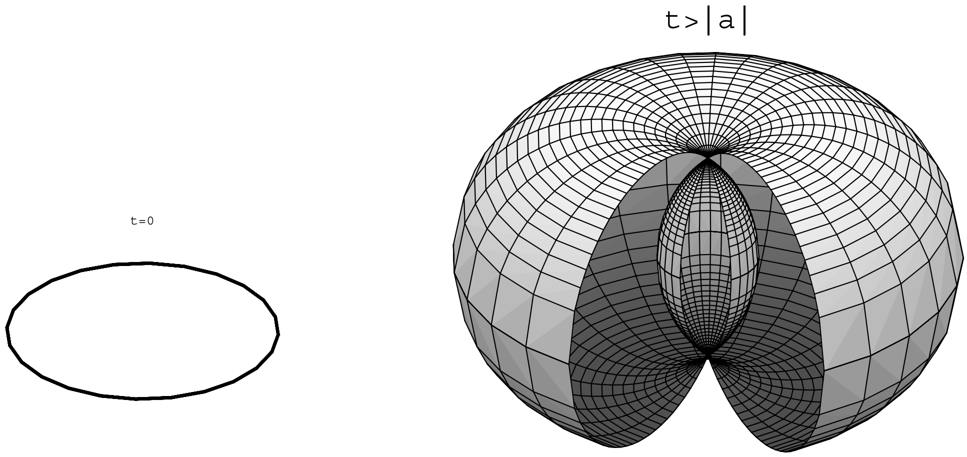

Figure 1: The shape of singular locus for electromagnetic field (8.5) of electrically neutral solution (8.4) at initial ($t = 0$) and final ($t > |a|$) instants

Let us here mention also the particle-like solution whose singular locus is *8-shaped* at initial moment, and the wave-like solution with a *helix-like* singularity [27]. The latter (corresponding to the generating function more complicated than the quadratic one) represents itself an analogue of electromagnetic wave in the algebrodynamical context.

If one gives up the condition for generating function to be quadratic, he comes to a wider class of the GSE solutions and corresponding solutions of Maxwell equations with extremely complicated structure of singular locus. In particular, in [18, 20, 61] a solution of such type has been presented which describes the *process of annihilation* of a pair of oppositely charged point-like singularities. We have also found therein a solution with a "photon-like" singularity (in the form of a pair of crossed rings) moving uniformly and rectilinearly with the speed of light.

Thus, in a purely algebraic way a wide class of explicitly or implicitly given solutions of *free* Maxwell equations with complicated and combined structure of point-like or extended singularities has been obtained. Considerable part of these solutions has not been known previously, and even their very existence has not been discussing. These solutions are well defined everywhere except the points at which the electromagnetic field strength turns to infinity. Locus of these singular points (at a given instant) may be 0-, 1- and even 2-dimensional (as it takes place for the case of the torus-like singularity (8.5)); moreover, it may dynamically change its dimension (say, for the same solution (8.5)). However, for solutions of *general type*, free of any type of symmetry, this singular locus is always one-dimensional and consists of a number of closed or infinite curves ("strings") [20]. For solutions of *particle-like* type singular locus is bounded in the 3D physical space.

Despite the initial "vacuumness" of gauge equations arising from the structure of GSE, the field singularities define spatial distribution and temporal dynamics of an effective *field source* at the points of whose location the property of analyticity of solutions becomes broken. Therefore, in contrast to the ordinary approach where an initially posed source defines electromagnetic field in surrounding space, in here presented conception, on the contrary, *almost everywhere analytical field subject to free Maxwell (Yang-Mills) equations predetermines itself the location of its singular sources.* The considered solutions are well defined in the whole infinite space-time except at a singular set of zero measure. They are obtained algebraically from an arbitrary generating function and *do not require any initial or boundary conditions.*

Moreover, it appears to be impossible, generally, to reduce this singular locus to a standard description covering it by a family of δ-like distributions, owing to essential *multi-valuedness* of charged solutions of the Kerr type. Nonetheless, the whole set of "quantum numbers", the shape and dynamics of such singularities are correctly defined and quite nontrivial, this being related to the so called *hidden nonlinearity* [51, 43] of Maxwell (and Yang-Mills) equations in the framework of algebrodynamics, that is, to their *secondariness* with respect to nonlinear structure of the primary GSE (as integrability conditions of the latter).

It is just the presence of "master" equations – the GSE – that ensures the existence of a number of "selection rules" even for solutions of linear Maxwell equations, restrictions on admissible values of electric charge among them, and results also in the breakdown of the superposition principle (since, say, a sum of solutions satisfies the linear Maxwell equations but quite not necessarily – the primary GSE itself!)

The over-determined primary GSE is, generally, not also invariant with respect to the spatial reflections (and, perhaps, – to the time reversal) [23]. These invariances are restored only at the level of *consequences, integrability conditions* of these primary equations, namely – at the level of Maxwell, Yang-Mills and like equations. Such situation seems to be unique for the field theory and, on the other hand, is completely adequate for the observed physical reality and seems to be much perspective in this respect to describe the P-violation and the time irreversibility.

A more detailed discussion of the status of singular particle-like solutions in algebrodynamics the reader can find in our works [23, 27, 58, 18].

9 $\mathbb{B}$-induced complex space-time geometry and the ensemble of dublicons

A beautiful representation of the solutions of SFC equations (and, consequently, – of our biquaternion-induced GSE) has been suggested in the works of E. T. Newman et al. [49, 52, 53] and developed later in the article [50] and in a series of subsequent works of A. Ya. Burinskii and of E. T. Newman with collaborators. In this representation one considers a "virtual" point-like charge "moving" along an arbitrary curve $\{z_\mu(\tau)\}, \ \ \tau \in \mathbb{C}$ in the *complexification* of Minkowski space $\mathbb{C}\mathbf{M}$. In this case the "trace" of the *complex null ("light-like") cone* of the "moving" charge on the *real* Minkowski "slice" of complex space $\mathbf{M}$ forms there a null congruence of rays which appears always to be shear-free.

The Kerr congruence represents itself only a simplest example of such representation (its generating point source is "at rest" at some place of "imaginary" subspace of $\mathbb{C}\mathbf{M}$ supplementary to $\mathbf{M}$). The above presented solutions of GSE and corresponding SFC can all be obtained from such *Newman's representation.* On the other hand, these examples demonstrate that for such *"complexified" Lienard-Wiehert fields* the structure of singular locus can be very complicated and consists, generally, of a great number of one-dimensional curves – strings.

In the algebrodynamical context complexified Minkowski space $\mathbb{C}\mathbf{M}$ arises unavoidably as the full vector space of the biquaternion algebra $\mathbb{B}$. At the same time, the above used procedure of restriction of coordinates to the real space-time $\mathbf{M}$ looks artificial and motivated only via physical considerations. Indeed, this subspace does not even form a subalgebra in $\mathbb{B}$ and is invariant neither under the $\mathbb{B}$-automorphisms nor under the full group of symmetry transformations (4.1).

On the other hand, the group of $\mathbb{B}$-automorphisms $SO(3,\mathbb{C})$ consists of 6 real parameters and is 2:1 isomorphic to the Lorentz group $SO(3,1)$. One does not know any other group with properties like these. Quite reasonably, the algebra $\mathbb{B}$ and its symmetry group $SO(3,\mathbb{C})$ have been used in the works of A. P. Yefremov [54] for construction of the so called *quaternionic theory of relativity* in context of which the invariant subspace $\mathbb{C}^3$ has been considered in capacity of the primordial space-time with three space-like and three time-like coordinates. In order to reduce three-dimensional time to physical one-dimensional, some additional requirements of orthogonality have been imposed.

From the author's viewpoint, such "exotic" interpretation of the properties of biquaternion algebra is quite unnecessary. The matter is that its invariant subspace $\mathbb{C}^3$ can be in a natural way mapped into the "causal" domain of the physical Minkowski space-time equipped with additional internal fibre-like variables [55]. Specifically, the principal invariant of initial complex space

$$\sigma = (z_1)^2 + (z_2)^2 + (z_3)^2 \tag{9.1}$$

can be separated into a non-compact "modulus-like" and compact "phase-like" parts. It is just the first part represented by the real-valued nonnegative invariant

$$S^2 := \sigma\sigma^* \geq 0, \tag{9.2}$$

that predetermines the observable "spatially extended" physical macro-geometry. whereas the second "phase" part of invariant σ is perceived as defining the internal geometry of the "fibre". By this, the most important result of the above cited paper consists in the fact that the positively definite (or null) $SO(3,\mathbb{C})$-invariant (9.2) can be identically represented in the form of Minkowski-like interval:

$$S^2 = \sigma\sigma^* \equiv T^2 - |\vec{X}|^2, \tag{9.3}$$

where the *real-valued* quantities

$$T := (\vec{z} \cdot \vec{z}^*), \quad \vec{X} := i[\vec{z} \times \vec{z}^*] \tag{9.4}$$

under $SO(3,\mathbb{C})$-rotations transform themselves as time and space coordinates of Minkowski space under the Lorentz-like transformations do. In definition (9.4) parentheses and square brackets denote the scalar and vector product of 3D (complex) vectors respectively.

Thus, one can really consider the algebra of biquaternions $\mathbb{B}$ as the *space-time algebra*, and the Minkowski geometry is induced by this via the *quadratic mapping* of complex coordinates of the invariant subspace $\mathbb{C}^3$ of the full vector space of $\mathbb{B}$ *into the internal, "causal" domain of the light cone of* $\mathbf{M}$ including its null boundary. In this connection, apart of the *positively definite Minkowski interval (!)* (9.3) there arises another *phase invariant* of Lorentz transformations (precisely, of $SO(3,\mathbb{C})$-rotations) that might turn to be closely related to universal quantum properties of matter and to manifestations of quantum interference in particular.

In accord to the here discussed notions, the "true" primordial dynamics of matter-like formations (singularities, solitons etc.) takes place just in the initial complex space whereas the observable, "shadow-like" dynamics – in the "causal Minkowski space-time" it induces. Such approach makes it possible, in particular, to successfully realize the beautiful old idea of Wheeler-Feynman about "reproduction of electrons from one sole electron-germ".

Specifically, let the point particle, in accord with the Newman's representation, "moves" in $\mathbb{C}\mathbf{M}$ along a "trajectory" $\{z_\mu(\tau)\}, \quad \tau \in \mathbb{C}$ of general (sufficiently complicated) form. Then it can be shown [61] that any position of this particle will be strictly correlated with other its positions on its own Worldline. Precisely, this correlation is established through equal values of fundamental twistor field of the null (complex) congruence generated by the particle-source and, correspondingly, – through the equation of *complex null cone*.

The situation strongly resembles the known procedure for the *Lienard-Wiehert fields* in the framework of classical electrodynamics. However, in contrast to the case of real space-time, in complex space the "light-like cone equation" always have a considerable (if not infinite) countable number of roots. Consequently, a particle will "see" and "receive signals" "from itself" at different its locations on a unique trajectory. The arising set of identical but differently located and moving particles has been named in our paper [61] the ensemble of *duplicons*.

Apart of the idea of duplicons, a problem of *complex time* unavoidably arises in the context of complex dynamics which turns to be related to general conception of physical time in the algebrodynamical paradigm [60, 18, 61]. Specifically, to each of the GSE solutions there corresponds some shear-free null congruence of rays (section 7). This can be considered as the basic element of the pattern of the World arising in the algebrodynamics, namely, – as a flow of primordial light, the so called *Prelight flow* [60, 18]. In this connection, the whole "matter" represented in the theory by particle-like singular formations of associated fields appears as a set of *caustics* or *focal lines* of the fundamental Prelight flow.

Returning now back to the problem of time, let us note that on the real space-time $\mathbf{M}$ the time coordinate plays the role of the *parameter along the rays of fundamental congruence* so that the defining property of time in this approach is the property of *reproduction*, of preservation of the primordial twistor field that takes place along the congruence rays (the "rays of time").

Speaking figuratively, in the presented theory *time manifests itself as an automorphism of the primary field.* However, the electromagnetic and other associated fields expressible via the derivatives of fundamental twistor field are, of course, not preserved along the rays as well as the caustics-particles themselves. Just this circumstance defines another fundamental function of time that is related to the motion and *variability* of different forms of physical matter.

Situation drastically changes in complex space $\mathbb{C}\mathbf{M}$ where the twistor field is defined up to a pair of arbitrary complex parameters and remains constant along the 2D *complex planes* [19, 61]. If, however, one requires in addition the property of preservation of the "matter-like" structure of *caustics*, then only *one* complex parameter remains free which thus can be interpreted as *complex time* [61]. Under this situation, however, there remains indefinite the *succession of the occurrences of events* (of the "states" of the Universe), and in absence of any grounds for its fixing it is the most natural to regard the alterations of complex time as *completely random* (casual). Then the arising for the ensemble of duplicons stochasticity can, apparently, be closely connected with quantum uncertainty and quantum theory in the Feynman formulation in general. However, we are only coming to find the correct realization of these ideas.

10 Conclusion

In this article we did not regard as our principal goal to present a novel field model or a powerful algebraic method to obtain new complicated solutions of generally known equations of classical field theory. Instead we here attempted to successively reveal the properties of differentiable (analytical) functions of biquaternionic variable, that is, to develop a novel version of noncommutative analysis. Nonetheless, general conditions of $\mathbb{B}$-differentiability [10, 23, 24] reduce themselves to the generating system of equations (5.9) which possesses an innate gauge and 2-spinor (twistor) structures and shows remarkable connections with the structures and language generally accepted in the formalism of relativistic field theory.

Essentially, it is sufficient to formulate only three principle conjectures in order to physically interpret the initially abstract mathematical scheme:
1) on the space-time as a (real or invariant complex) subspace of vector space of $\mathbb{B}$-algebra,
2) on physical fields as differentiable functions of $\mathbb{B}$-variable,
3) on particles as (bounded in 3-space) singularities of strengths (curvatures) of the gauge and metrical fields directly associated with the primary $\mathbb{B}$-differentiable functions-fields.

From the physical viewpoint, the GSE may be considered as a rather specific system of field equations (nonlinear, non-Lagrangian, over-determined) for an effectively coupling 2-spinor and electromagnetic (Yang-Mills) fields so that equations for both are not postulated but follow directly from integrability conditions or "contractions" of the GSE itself.

Twistor structure also arises in a quite natural, "dynamical" way in the course of integration of GSE and makes it possible to obtain the whole set of its solutions as well as those of equations for associated gauge fields in a perfectly simple algebraic way[12].

[12] Note that in the Penrose's twistor approach [19, 30] in order to obtain the solutions of wave-like (massless) equations it is sufficient to carry out an integration in twistor space; as to the presented scheme, even such integration is redundant therein

Particularly, from algebraic Kerr equation (7.9) a wide class of exact solutions of linear Maxwell equations with spatially extended yet bounded structure of singularities can be directly obtained. In this connection, condition (7.10) plays the role of the *equation of motion* for these particle-like formations and, at the same time, defines their characteristics ("quantum numbers") and spatial distribution, realizing thus the Einstein's conception of *super-causality* [38].

In consequence of breakdown of the superposition principle for solutions of "master" equations – the GSE – temporal evolution of such particle-like objects simulates the process of physical interaction whereas dynamical reconstructions (*bifurcations*) of the structure of singular locus can be treated as *transmutations* of particles, in particular, as emission / absorption processes. All these processes obviously manifest close relationship to the theory of singularities of differentiable mappings and the catastrophe theory [36].

We also hope that at least a number of remarkable properties of the GSE can be interesting in the general context of field theory. Let us note here, in particular:
1) an opportunity to extend the class of physically important gauge field models with account of the "weak" gauge symmetry (6.4) discovered for GSE or using exceptional connections (6.5),(6.6) of the Weyl-Cartan type;
2) a natural opportunity to obtain proper "selection rules" for electric charge, spin and other physical characteristics starting from an over-determined system of field equations of the GSE type;
3) complete algebraization of the primary PDE's for fundamental fields possessing twistor structure;
4) possibility to define the spatial distribution and the law of evolution of the field singularities without explicit resolving of field equations themselves (using instead algebraic method of elimination of the principal field $G(X)$ from the joint system of equations (7.9), (7.10)).

One may imagine himself at least three possible points of view on the meaning of algebraic structures presented in this article and on the fundamental GSE in particular: as on a beautiful mathematical "toy", on a powerful method to obtain the solutions of the familiar field equations or, finally, as on a fundamental dynamical system of equations primary with respect to generally accepted Lagrangian structures. In this connection, the construction of classical dynamics on the base of over-determined systems like GSE requires also quite new methods of quantization. On the other hand, at this point one can try to explain quantum properties as a whole via, say, the stochastic behaviour of an ensemble of particle-like field objects (dublicons, solitons, etc.) or invoking other yet purely classical and algebraic in nature methods and ideas.

In any case, in order to find a correct approach to quantization and explanation of quantum properties of matter in general, it is necessary at the beginning to carefully study the properties of classical solutions on the background of ordinary Minkowski space-time and over the "phase extension of the Minkowski geometry" directly induced by the internal properties of the $\mathbb{B}$-algebra and briefly considered in the last section. We think that just the underlying complex geometry can actually turn to be the true pregeometry of physical space-time and, moreover, to be responsible for universal quantum properties of matter and quantum uncertainty in particular (in general context of an initially classical and deterministic theory).

To conclude, the already discovered properties of the $\mathbb{B}$-differentiable functions-fields and numerous geometrophysical structures they give rise to, looks like so unusually and, on the other hand, to such a great extent correlate with models and mathematical formalism of theoretical physics that force ourselves to ponder over possible *numerical* origin of fundamental laws of nature [60, 62] and to turn again, at a modern mathematical and physical level of comprehension, to the ancient philosophy of Pythagor, Plato and their followers.

* * *

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References

[1] W. R. Hamilton, *Elements of Quaternions*, London: Longmans Green, 1866.

[2] R. Penrose, *The Road to Reality. A Complete Guide to the Laws of the Universe,* London: Jonathan Cape, 2004.

[3] A. Gsponer, J.-P. Hurni, *Quaternions in mathematical physics (1): Alphabetical bibliography*, www.arXiv.org/math-ph/0510059; *(2): Analytical bibliography*, www.arXiv.org/math-ph/0510092.

[4] M. Atiah, *The Geometry and Physics of Knots,* Cambridge Univ. Press, 1990.

[5] G. Scheffers, *Verallgemeinerung der Grundlagen der gewöhnlichen komplexen Functionen,* Berichte Sächs. Acad. Wiss., **Bd.45**, 1893, 828–842.

[6] D. G. Pavlov, *Philosophical and mathematical reasons for Finsler extensions of relativity theory,* Hypercomplex Numbers in Geometry and Physics, **2 (4)**, 2005, 12–18.

[7] G. I. Garas'ko, *Generalized analytical functions of polynumber variable,* Hypercomplex Numbers in Geometry and Physics, **1 (1)**, 2004, 70–83.

[8] V. S. Vladimirov, I. V. Volovich, *Superanalysis I. Differential calculus,* Theor. Math. Phys., **59**, 1984, 317–335;
V. S. Vladimirov, I. V. Volovich, *Superanalysis II. Integral calculus,* Theor. Math. Phys., **60**, 1984, 743–765.

[9] A. Sudbery, *Quaternionic Analysis,* Math. Proc. Camb. Phil. Soc., **85**, 1979, 199–225.

[10] V. V. Kassandrov, *The Algebraic Structure of Space-Time and the Algebrodynamics.* People's Friendship Univ. Press, Moscow, 1992 (in Russian).

[11] M. Evans, F. Gürsey, V. Ogievetsky, *From 2D conformal to 4D self-dual theories: the quaternionic analyticity,* Physical Review, **D47**, 1993, 3496–3524.

[12] R. Fueter, *Zur Theory der Regulären Functionen einer Quaternionenvariablen,* Monatsh. Math. Phys., **43**, 1936, 69–74 ;
R. Fueter, *Über dir Analytische Darstellung der Regulären Functionen einer Quaternionenvariablen*, Comment. Math. Helv., **8**, 1936, 371–378.

[13] V. V. Vishnewski, A. P. Shirokov, V. V. Shur'ygin, *Spaces over Algebras.* Kasan Univ. Press, 1985, Chapter 5 (in Russian).

[14] A. V. Berezin, Yu. A. Kurochkin, E. A. Tolkachev, *Quaternions in Relativity Physics,* Science and Technic, Minsk, 1989 (in Russian).

[15] A. Deavours, *The Quaternionic Calculus,* Amer. Math. Monthly, **80**, 1973, 995–1008.

[16] B. A. Dubrovin, S. P. Novikov, A. T. Fomenko, *Contemporary Geometry,* Science, Moscow, 1979 (in Russian).

[17] V. V. Kassandrov, *General solution of the complex 4-eikonal equation and the "algebrodynamical" field theory,* Gravitation & Cosmology, **8**, Suppl.2, 2002, 57–62; www.arXiv.org/math-ph/0311006.

[18] V. V. Kassandrov, *The Algebrodynamics: Primordial Light, Particles-Caustics and the Flow of Time,* Hypercomplex Numbers in Geometry and Physics, **1 (1)**, 2004, 84–99; www.arXiv.org/hep-th/0312278.

[19] R. Penrose, W. Rindler, *Spinors and Space-Time. V. 2. Spinor and Twistor Methods in Space-Time Geometry,* Cambridge Univ. Press, 1986.

[20] V. V. Kassandrov, *Biquaternions as the algebra of extended physical space-time,* In: *Proc. Int. Conf. on Non-Euclidean Geometries*, ed. V. I. Redkov. Institute of Physics of Belarus Press, Minsk, 2006; http://dragon.bus-net.by/bgl5/proc.htm.

[21] J. A. Rizcallah, *Geometrization of Electromagnetism on the Base of Spaces with Weyl-Cartan Connections.* Ph. D. Thesis. Russian People's Friendship University, Moscow, 1999 (in Russian).

[22] V. V. Kassandrov, J. A. Rizcallah, *Twistor and "weak" gauge structures in the framework of quaternionic analysis,* www.arXiv.org/math-ph/0311006.

[23] V. V. Kassandrov, *Biquaternion electrodynamics and the Weyl-Cartan geometry of space-time,* Gravitation & Cosmology, **1**, No. 3, 1995, 216–222; www.arXiv.org/gr-qc/0007026.

[24] V. V. Kassandrov, *Conformal mappings, hyper-analyticity and field dynamics,* Acta Applic. Math., **50**, 1998, 197–206.

[25] V. V. Kassandrov, *Physical fields as super-analytic mappings on the algebraic structure of space-time,* In: *Quasigroups and Non-Associative Algebras in Physics*, ed. J. Lõhmus and P. Kuusk. Institite of Physics of Estonia Press, Tallinn, 1990, 202–212.

[26] V. V. Kassandrov, *The Algebrodynamics: quaternions, twistors, particles,* Vestnik of Russian Peop. Frien. Univ. Phys., **8(1)**, 2000, 34–45 (in Russian).

[27] V. V. Kassandrov, V. N. Trishin, *"Particle-like" singular solutions in Einstein-Maxwell theory and in algebraic dynamics,* Gravitation & Cosmology, **5**, 1999, 272–276; www.arXiv.org/gr-qc/0007027.

[28] I. Robinson, *Null electromagnetic fields,* J. Math. Phys., **2**, 1961, 290–291.

[29] K. P. Tod, *Local heterotic geometry and self-dual Einstein-Weyl spaces,* Class. Quant. Grav., **13**, 1996, 2609–2616.

[30] R. Penrose, *Twistor quantization and curved space-time,* Int. J. Theor. Phys., **1**, 1968, 61–99; R. Penrose, *Solutions of the zero-rest-mass equations,* J. Math. Phys., **10**, 1969, 38–39.

[31] V. V. Kassandrov, J. A. Rizcallah, *Covariantly constant fields and geometrization of electromagnetism,* In: *Geometrization of Physics II, Proc. Int. Conf. in memory of A. Z. Petrov*, ed. V. I. Bashkov. Kasan Univ. Press, 1996, 137–143.

[32] G. S. Hall, *Covariantly constant tenzors and holonomy structure,* J. Math. Phys., **32**, 1991, 181–187.

[33] H. A. Buchdahl, *On the compatibility of relativistic wave equations in riemann spaces,* Nuovo Cimento, **25**, 1962, 486–496.

[34] R. Penrose, *Golden Oldie,* Gen. Rel. Grav., **12**, 1980, 225–264.

[35] J. Plebanski, *The "vectorial" optics of fields with arbitrary spin, rest-mass zero,* Acta Polon., **27**, 1965, 361–393.

[36] V. I. Arnold, A. N. Varchenko, S. M. Gusein-Zade, *Singularities of Differentiable Mappings.* Moscow Indep. Centre Math. Educ. Press, 2004 (in Russian).

[37] V. I. Arnold, *Singularities of Caustics and Wave Fronts,* Science, M., 1996 (in Russian).

[38] A. Einstein, *Collected Papers,* V. 4, Science Moscow, 1967, p. 109 (in Russian); A. Einstein, *Ansprache von Prof. Einstein an Prof. Planck,* Forschun. und Forschr., **5**, No. 21, 1929, 248–249.

[39] V. V. Kassandrov, J. A. Rizcallah, *Algebrodynamical approach in field theory,* In: *Recent Problems in Field Theory*, ed. A. V. Aminova. Kasan Univ. Press, 1998, 176–186; www.arXiv.org/gr-qc/9809078.

[40] G. C. Debney, R. P. Kerr, A. Schild, *Solutions of the Einstein and Einstein-Maxwell equations,* J. Math. Phys., **10**, 1969, 1842–1854.

[41] R. P. Kerr, W. B. Wilson, *Singularities in the Kerr-Schild metrics,* Gen. Rel. Grav., **10**, 1979, 273–281.

[42] D. Cramer, H. Stephani, M. MacCallum, E. Herlt, *Exact solutions of the Einstein field equations,* Deutsch. Verlag Wissens, Berlin, 1980.

[43] A. F. Ranãda, *A Topological model of electromagnetism: quantization of electric charge,* An. Fis. (Madrid), **A87**, 1991, 55–59; www.arXiv.org/hep-th/9802166.

[44] V. N. Zhuravlev, *Electrodynamics with integer-valued charge and topology,* Izvestia Vuzov. Physics, 2002, 134–140 (in Russian).

[45] B. Carter, *Global structure of the Kerr family of gravitational fields*, Phys. Rev., **174**, 1968, 1559–1571.

[46] C. A. Lopez, *Extended model of the electron in general relativity,* Phys. Rev., **D30**, 1984, 313–316.

[47] W. Israel, *Source of the kerr metric,* Phys. Rev., **D2**, 1970, 641–647.

[48] A. Ya. Burinskii, *Strings in the Kerr-Schild metrics,* In: *Problems of Theory of Gravitation and Elementary Particles. No.11*, ed. K. P. Stanyukovich. Atomizdat, Moscow, 1980, 47–60, (in Russian).

[49] E. T. Newman, *Maxwell's equations and complex Minkowski space,* J Math. Phys., **14**, 1973, 102–107.

[50] A. Ya. Burinskii, In: *Proc. IV Hungarian Relativity Workshop* ed. P. R. Kerr and Z. Perjés. Budapest: Académiai Kiadó, 1992, 149–158;
A. Ya. Burinskii, R. P. Kerr, Z. Perjés, www.arXiv.org/gr-qc/9501012.

[51] A. F. Ranãda, J. L. Trueba, *Electromagnetic knots,* Phys. Lett., **A202**, 1995, 337–342; A. F. Ranãda, J. L. Trueba, *Two properties of electromagnetic knots,* Phys. Lett., **A232**, 1997, 25–33.

[52] R. W. Lind, E. T. Newman, *Complexification of the algebraically special gravitational fields,* J. Math. Phys., **15**, 1974, 1103–1112.

[53] E. T. Newman, *Classical, geometrical origin of magnetic moments, spin-angular momentum and the Dirac gyromagnetic ratio,* Physical Review, **D65**, 2002, p. 104005; www.arXiv.org/gr-qc/0201055.

[54] A. P. Yefremov, *Quaternionic relativity I. Inertial motion,* Gravitation & Cosmology, **2**, 1, 1996, 77–83; *Quaternionic relativity II. Non-inertial motion,* Gravitation & Cosmology, **2**, 4, 1996, 335–341;
A. P. Yefremov, *Quaternionic Spaces, Frames of References and Fields,* Russian Peop. Frien. Univ. Press, Moscow, 2005 (in Russian).

[55] V. V. Kassandrov, Gravitation & Cosmology, **11**, 2005, p. 354; www.arXiv.org/ gr-qc/0602088.

[56] V. V. Kassandrov, J. A. Rizcallah, *Particles as singularities in the unified algebraic field dynamics,* In: *Geometrical and Topological Ideas in Modern Physics*, ed. V. A. Petrov. Institute for high energy physics, Protvino, 2002, 199–211.

[57] V. V. Kassandrov, V. N. Trishin, *Effective connections and fields associated with shear-free null congruences,* General Relativity and Gravitation, **36**, 2004, 1603–1612; www.arXiv.org/gr-qc/0401120.

[58] V. V. Kassandrov, *Singular sources of Maxwell fields with self-quantized electric charge,* In: *Has the Last Word Been Said on Classical Electrodynamics?*, eds. A. Chubykalo, V. Onoochin, A. Espinoza, R. Smirnov-Rueda. Rinton Press, 2004, 42–66; www.arXiv.org/physics/0308045.

[59] V. V. Kassandrov, *On the structure of general solution of the qquations of shear-free null congruences,* In: *Proc. Int. Sch. on Geometry and Analysis, in memory of N. V. Efimov.* Rostov Univ. Press, 2004, 65–68; www.arXiv.org/ gr-qc/0602046.

[60] V. V. Kassandrov, *Number, time and light,* In: *Mathematics and Practice. Mathematics and Culture. No. 2*, ed. M. Yu. Simakov. Moscow: Self-Education, 2001, 61–76; www.chronos.msu.ru (in Russian).

[61] V. V. Kassandrov, *Twistor algebraic dynamics in complex space-time and physical meaning of hidden dimensions,* In: *Proc. Int. Conf. "Physical Interpretations of Relativity Theory"*, eds. V. O. Gladyshev, A. N. Morozov, M. C. Duffy. Bauman Tech. Univ. Press, Moscow, 2005, 42–53; www.arXiv.org/gr-qc/0602064.

[62] V. V. Kassandrov, *Created by the word, appeared through the light, Delphis (Moscow)*, **2**, 2005, 61–70 (in Russian).

Finslerian 4-Spinors as a Generalization of Twistors

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This paper formulates main results of the geometry of Finslerian 4-spinors. First, it is shown that R. Penrose's twistors are form a special case of Finslerian 4-spinors of 16-dimensional vector space equipped with a metric form, and can be associated with Finsler geometry. Also, is formulated the procedure of dimensional reduction which allows to rewrite the expression of the Finslerian length of a 16-vector in terms of 4-dimensional geometric objects, and is described the corresponding isometry group.

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1 Introduction

In the works [1, 2], *hyperspinors* and their basic properties were considered. The same mathematical objects were independently studied under the name of N-component spinors in the papers [3, 4]. Finally, in the work [5], the general algebraic theory of *Finslerian N-spinors* was constructed. The last term is more suitable because it reflects the close connection between hyperspinors and Finslerian geometry.

This paper is devoted to formulating the main facts of the geometry of Finslerian 4-spinors. We show that *twistors* of R. Penrose [6] are a special case of Finslerian 4-spinors and can be associated not only with pseudo-Euclidean geometry, but also with Finslerian one. After deducing the expression for the length of a vector in the 16-dimensional Finslerian space, we describe the corresponding isometry group. We also formulate the procedure of dimensional reduction which allows us to rewrite the expression for the Finslerian length of a 16-vector in terms of 4-dimensional geometric objects.

2 The geometry of Finslerian 4-spinors

Let $\mathbb{C}^4$ be the vector space of 4-component columns of complex numbers with respect to the standard matrix operations of addition and multiplication by elements of the field $\mathbb{C}$. Let us consider the antisymmetric 4-linear form

$$[\xi, \eta, \lambda, \mu] = \varepsilon_{abcd}\, \xi^a \eta^b \lambda^c \mu^d, \tag{1}$$

where ξ, η, λ, $\mu \in \mathbb{C}^4$, ε_{abcd} is the Levi-Civita symbol with the ordinary normalization $\varepsilon_{1234} = 1$, the indices a, b, c, d run independently from 1 to 4, and ξ^a, η^b, λ^c,

$\mu^d \in \mathbb{C}$. Here and in the following formulas, the summation is taken over all the repeating indices.

The space $\mathbb{C}^4$ equipped with the form (1) is called the *space of Finslerian 4-spinors*. The complex number $[\xi, \eta, \lambda, \mu]$ is respectively called the *symplectic scalar 4-product* of the Finslerian 4-spinors ξ, η, λ, and μ.

Since (1) is the determinant

$$[\xi, \eta, \lambda, \mu] = \begin{vmatrix} \xi^1 & \eta^1 & \lambda^1 & \mu^1 \\ \xi^2 & \eta^2 & \lambda^2 & \mu^2 \\ \xi^3 & \eta^3 & \lambda^3 & \mu^3 \\ \xi^4 & \eta^4 & \lambda^4 & \mu^4 \end{vmatrix} \tag{2}$$

with the columns ξ, η, λ, μ, the symplectic scalar 4-product $[\xi, \eta, \lambda, \mu]$ vanishes if and only if the Finslerian 4-spinors ξ, η, λ, and μ are linearly dependent [7]. In particular, $[\xi, \xi, \xi, \xi] = 0$ for any $\xi \in \mathbb{C}^4$.

Let us find isometries of the space of Finslerian 4-spinors, i.e., the linear transformations

$$\xi' = D\xi \quad \Longleftrightarrow \quad \xi'^a = d^a_b \xi^b \quad (D = \|d^a_b\|; \quad d^a_b \in \mathbb{C}; a, b = \overline{1,4}) \tag{3}$$

which preserve the symplectic scalar 4-product:

$$[\xi', \eta', \lambda', \mu'] = [\xi, \eta, \lambda, \mu] \quad \text{for any} \quad \xi, \eta, \lambda, \mu \in \mathbb{C}^4. \tag{4}$$

Substituting (3) and the similar expressions for η', λ', μ' into the condition (4), we obtain

$$[\xi, \eta, \lambda, \mu] \det D = [\xi, \eta, \lambda, \mu] \tag{5}$$

with regard to (2). Due to arbitrariness of ξ, η, λ, $\mu \in \mathbb{C}^4$, the equation (5) implies $\det D = 1$. Thus, the isometries of the space of Finslerian 4-spinors form the group $\mathrm{SL}(4, \mathbb{C})$.

If to equip $\mathbb{C}^4$ with the additional geometric structure, then the space of Finslerian 4-spinors becomes the twistor space. Namely, let us consider Hermitian form

$$\langle \xi, \eta \rangle = \xi^1 \overline{\eta^1} + \xi^2 \overline{\eta^2} - \xi^3 \overline{\eta^3} - \xi^4 \overline{\eta^4}, \tag{6}$$

where ξ, $\eta \in \mathbb{C}^4$ and the over-lines denote complex conjugating. The complex number $\langle \xi, \eta \rangle$ is usually called the pseudounitary scalar product of ξ and η. With respect to the scalar product (6), $\mathbb{C}^4$ is the twistor space [6]. It is evident that the transformations (3), which preserve the forms (1) and (6) simultaneously, make up the so-called twistor group $\mathrm{SU}(2, 2) \subset \mathrm{SL}(4, \mathbb{C})$. In this sense, twistors are a special case of Finslerian 4-spinors.

Let us consider the subspace of the vector space $\mathbb{C}^4 \otimes \overline{\mathbb{C}^4}$ which consists of Hermitian tensors. This subspace is isomorphic to the 16-dimensional *real* vector space $\mathrm{Herm}(4) = \{X \mid X = X^+\}$ of all Hermitian 4×4 matrices with complex elements. Here, the cross denotes Hermitian conjugating.

As a basis of the space Herm(4), we choose the following linearly independent matrices

$$\tau_0 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0&-i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},$$

$$\tau_4 = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \tau_5 = \begin{pmatrix} 0&0&-i&0\\0&0&0&0\\i&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \tau_6 = \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}, \quad \tau_7 = \begin{pmatrix} 0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0 \end{pmatrix},$$

$$\tau_8 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, \quad \tau_9 = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0 \end{pmatrix}, \quad \tau_{10} = \begin{pmatrix} 0&0&0&-i\\0&0&0&0\\0&0&0&0\\i&0&0&0 \end{pmatrix}, \quad \tau_{11} = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0 \end{pmatrix},$$

$$\tau_{12} = \begin{pmatrix} 0&0&0&0\\0&0&0&-i\\0&0&0&0\\0&i&0&0 \end{pmatrix}, \quad \tau_{13} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}, \quad \tau_{14} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0 \end{pmatrix}, \quad \tau_{15} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}. \tag{7}$$

Then, for any $X \in \mathrm{Herm}(4)$, we have the expansion

$$X = X^A \tau_A \quad (A = \overline{0,15}), \tag{8}$$

where $X^A \in \mathbb{R}$ are components of the 16-vector X with respect to the basis (7). Along with the matrices (7), we introduce another set of the Hermitian 4×4 matrices: $\tau^B = \tau_B$ $(B \neq 8, 15)$, $\tau^8 = 2\tau_8$, $\tau^{15} = 2\tau_{15}$. Under such a choice of the matrices, the remarkable relations

$$\mathrm{Tr}(\tau^A \tau_B) = 2\delta^A_B \quad (A, B = \overline{0,15}) \tag{9}$$

are fulfilled. Here, δ^A_B is the Kronecker symbol. Because of (8) and (9),

$$X^A = \frac{1}{2}\mathrm{Tr}(\tau^A X). \tag{10}$$

Let us equip Herm(4) with the structure of the Finslerian space. To this end, we define the *length* $|X|$ of the 16-vector $X \in \mathrm{Herm}(4)$ in the following way: $|X| \equiv \sqrt[4]{\det X}$. Computing the determinant of (8), we obtain the expression for $|X|^4$ in the basis (7):

$$|X|^4 = G_{ABCD} X^A X^B X^C X^D =$$

$$
\begin{aligned}
&= X^{15}\big\{[(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2]X^8 - \\
&- [(X^4)^2 + (X^5)^2 + (X^6)^2 + (X^7)^2]X^0 + 2[X^4X^6 + X^5X^7]X^1 + \\
&+ 2[X^5X^6 - X^4X^7]X^2 + [(X^4)^2 + (X^5)^2 - (X^6)^2 - (X^7)^2]X^3\big\} - \\
&- [(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2][(X^{13})^2 + (X^{14})^2] + \\
&+ [(X^4)^2 + (X^5)^2][(X^{11})^2 + (X^{12})^2] + [(X^6)^2 + (X^7)^2]\times \\
&\times [(X^9)^2 + (X^{10})^2] - X^0X^8[(X^9)^2 + (X^{10})^2 + (X^{11})^2 + (X^{12})^2] + \\
&+ X^3X^8[(X^9)^2 + (X^{10})^2 - (X^{11})^2 - (X^{12})^2] + 2\big\{[X^0 - X^3]\times \\
&\times [X^4X^9X^{13} + X^4X^{10}X^{14} - X^5X^9X^{14} + X^5X^{10}X^{13}] + \\
&+ [X^0 + X^3][X^6X^{11}X^{13} + X^6X^{12}X^{14} - X^7X^{11}X^{14} + X^7X^{12}X^{13}] \\
&- X^1[X^4X^{11}X^{13} + X^4X^{12}X^{14} - X^5X^{11}X^{14} + X^5X^{12}X^{13} + X^6X^9X^{13} \\
&+ X^6X^{10}X^{14} - X^7X^9X^{14} + X^7X^{10}X^{13} - X^8X^9X^{11} - X^8X^{10}X^{12}] \\
&- X^2[X^4X^{11}X^{14} - X^4X^{12}X^{13} + X^5X^{11}X^{13} + X^5X^{12}X^{14} - X^6X^9X^{14} + \\
&+ X^6X^{10}X^{13} - X^7X^9X^{13} - X^7X^{10}X^{14} + X^8X^9X^{12} - X^8X^{10}X^{11}] \\
&- X^4[X^6X^9X^{11} + X^6X^{10}X^{12} + X^7X^9X^{12} - X^7X^{10}X^{11}] \\
&+ X^5[X^6X^9X^{12} - X^6X^{10}X^{11} - X^7X^9X^{11} - X^7X^{10}X^{12}]\big\}. \qquad (11)
\end{aligned}
$$

Here, G_{ABCD} are components of the covariant symmetric tensor on Herm(4). Thus, the Finslerian length of the 16-vector $X \in \mathrm{Herm}(4)$ in the basis (7) is the form of degree 4 with respect to its components (10). It should be noted that the form (11) is indefinite, i.e., the cases $|X|^4 > 0$, $|X|^4 < 0$ or $|X|^4 = 0$ are possible. Since $|X|^4 = \det X$, we have $|X|^4 = 0$ if and only if $\det X = 0$.

Any linear transformation (3) of the space of Finslerian 4-spinors induces the transformation

$$X' = DXD^+ \quad \Longleftrightarrow \quad X'^{a\dot b} = d^a_c \overline{d^{\dot b}_{\dot e}} X^{c\dot e} \quad (X' = \|X'^{a\dot b}\|; X = \|X^{c\dot e}\|) \qquad (12)$$

in Herm(4). Here, all the indices run from 1 to 4 and $X \in \mathrm{Herm}(4)$. It is evident that the transformation (12) has the following properties:

1. If $X = X^+$, then $X' = X'^+$.
2. The transformation (12) is linear with respect to X.
3. If $\det D = 1$, then $\det X' = \det X$ for any $X \in \mathrm{Herm}(4)$.

Since $|X| = \sqrt[4]{\det X}$, the last property means that the linear transformation (12) with $D \in \mathrm{SL}(4,\mathbb{C})$ is a Finslerian isometry of the space Herm(4), i.e., $|X'| = |X|$. It is clear that all such isometries form a group. We will give the explicit matrix description of this group in the basis (7).

Let us substitute the expansions $X' = X'^A\tau_A$ and $X = X^B\tau_B$ into (12). We then multiply the resulting equality by τ^A from the left, compute its trace, and use the relations (9). As a result, we obtain

$$X'^A = L(D)^A_B X^B \quad (A, B = \overline{0,15}), \qquad (13)$$

where

$$L(D)^A_B = \frac{1}{2}\mathrm{Tr}(\tau^A D \tau_B D^+) \tag{14}$$

are elements of the matrix of the linear transformation (12) in the basis (7). It should be noted that $L(D)^A_B \in \mathbb{R}$. Thus, for any $D \in \mathrm{SL}(4,\mathbb{C})$, the transformation (13)–(14) preserves the form (11) so that $G_{ABCD}X'^A X'^B X'^C X'^D = G_{ABCD}X^A X^B X^C X^D$.

Since the group $\mathrm{SL}(2,\mathbb{C}) \subset \mathrm{SL}(4,\mathbb{C})$ is locally isomorphic to the group $\mathrm{O}^{\uparrow}_{+}(1,3)$ [8], it is interesting to consider the transformation (13)–(14) with $D \in \mathrm{SL}(2,\mathbb{C})$, i.e., from the point of view of a "4-dimensional observer". This will allow us to represent the expression (11) for the Finslerian length of the 16-vector X completely in the 4-dimensional form.

Let

$$D_2 = \begin{pmatrix} d^1_1 & d^1_2 & 0 & 0 \\ d^2_1 & d^2_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \det D_2 = 1 \quad (d^{\hat a}_{\hat b} \in \mathbb{C}; \hat a, \hat b = 1,2). \tag{15}$$

The matrices (15) form a subgroup of $\mathrm{SL}(4,\mathbb{C})$ which is isomorphic to the group $\mathrm{SL}(2,\mathbb{C})$. Let us substitute the matrix D_2 from (15) into (14) instead of D. Direct computations show that

$$\begin{aligned}
L(D_2)^0_0 &= \frac{1}{2}(d^1_1\overline{d^1_1} + d^1_2\overline{d^1_2} + d^2_1\overline{d^2_1} + d^2_2\overline{d^2_2}), & L(D_2)^0_1 &= \frac{1}{2}(d^1_1\overline{d^1_2} + d^2_1\overline{d^2_2} + d^1_2\overline{d^1_1} + d^2_2\overline{d^2_1}),\\
L(D_2)^0_2 &= \frac{i}{2}(d^1_2\overline{d^1_1} + d^2_2\overline{d^2_1} - d^1_1\overline{d^1_2} - d^2_1\overline{d^2_2}), & L(D_2)^0_3 &= \frac{1}{2}(d^1_1\overline{d^1_1} + d^2_1\overline{d^2_1} - d^1_2\overline{d^1_2} - d^2_2\overline{d^2_2}),\\
L(D_2)^1_0 &= \frac{1}{2}(d^1_1\overline{d^2_1} + d^2_1\overline{d^1_1} + d^1_2\overline{d^2_2} + d^2_2\overline{d^1_2}), & L(D_2)^1_1 &= \frac{1}{2}(d^1_1\overline{d^2_2} + d^2_1\overline{d^1_2} + d^1_2\overline{d^2_1} + d^2_2\overline{d^1_1}),\\
L(D_2)^1_2 &= \frac{i}{2}(d^1_2\overline{d^2_1} + d^2_2\overline{d^1_1} - d^1_1\overline{d^2_2} - d^2_1\overline{d^1_2}), & L(D_2)^1_3 &= \frac{1}{2}(d^1_1\overline{d^2_1} + d^2_1\overline{d^1_1} - d^1_2\overline{d^2_2} - d^2_2\overline{d^1_2}),\\
L(D_2)^2_0 &= \frac{i}{2}(d^1_1\overline{d^2_1} - d^2_1\overline{d^1_1} + d^1_2\overline{d^2_2} - d^2_2\overline{d^1_2}), & L(D_2)^2_1 &= \frac{i}{2}(d^1_1\overline{d^2_2} - d^2_1\overline{d^1_2} + d^1_2\overline{d^2_1} - d^2_2\overline{d^1_1}),\\
L(D_2)^2_2 &= \frac{1}{2}(d^1_1\overline{d^2_2} + d^2_2\overline{d^1_1} - d^1_2\overline{d^2_1} - d^2_1\overline{d^1_2}), & L(D_2)^2_3 &= \frac{i}{2}(d^1_1\overline{d^2_1} - d^2_1\overline{d^1_1} - d^1_2\overline{d^2_2} + d^2_2\overline{d^1_2}),\\
L(D_2)^3_0 &= \frac{1}{2}(d^1_1\overline{d^1_1} - d^2_1\overline{d^2_1} + d^1_2\overline{d^1_2} - d^2_2\overline{d^2_2}), & L(D_2)^3_1 &= \frac{1}{2}(d^1_1\overline{d^1_2} - d^2_1\overline{d^2_2} + d^1_2\overline{d^1_1} - d^2_2\overline{d^2_1}),\\
L(D_2)^3_2 &= \frac{i}{2}(d^1_2\overline{d^1_1} - d^2_2\overline{d^2_1} - d^1_1\overline{d^1_2} + d^2_1\overline{d^2_2}), & L(D_2)^3_3 &= \frac{1}{2}(d^1_1\overline{d^1_1} - d^1_2\overline{d^1_2} - d^2_1\overline{d^2_1} + d^2_2\overline{d^2_2}),
\end{aligned}$$

$$L(D_2)^{3+i}_{3+j} = L(D_2)^{8+i}_{8+j} = M(D_2)^i_j \ (i,j=\overline{1,4}), \qquad \text{where} \tag{16}$$

$$\begin{aligned}
M(D_2)^1_1 &= \frac{1}{2}(\overline{d^1_1} + d^1_1), & M(D_2)^3_1 &= \frac{1}{2}(\overline{d^2_1} + d^2_1), & M(D_2)^1_2 &= \frac{i}{2}(\overline{d^1_1} - d^1_1),\\
M(D_2)^3_2 &= \frac{i}{2}(\overline{d^2_1} - d^2_1), & M(D_2)^1_3 &= \frac{1}{2}(\overline{d^1_2} + d^1_2), & M(D_2)^3_3 &= \frac{1}{2}(\overline{d^2_2} + d^2_2),\\
M(D_2)^1_4 &= \frac{i}{2}(\overline{d^1_2} - d^1_2), & M(D_2)^3_4 &= \frac{i}{2}(\overline{d^2_2} - d^2_2), & M(D_2)^2_1 &= \frac{i}{2}(d^1_1 - \overline{d^1_1}),
\end{aligned}$$

$$M(D_2)^4_1=\frac{i}{2}(d^2_1-\overline{d^2_1}),\quad M(D_2)^2_2=\frac{1}{2}(d^1_1+\overline{d^1_1}),\quad M(D_2)^4_2=\frac{1}{2}(d^2_1+\overline{d^2_1}),$$
$$M(D_2)^2_3=\frac{i}{2}(d^1_2-\overline{d^1_2}),\quad M(D_2)^4_3=\frac{i}{2}(d^2_2-\overline{d^2_2}),\quad M(D_2)^2_4=\frac{1}{2}(d^1_2+\overline{d^1_2}),$$
$$M(D_2)^4_4=\frac{1}{2}(d^2_2+\overline{d^2_2}), \tag{17}$$

$L(D_2)^8_8 = L(D_2)^{13}_{13} = L(D_2)^{14}_{14} = L(D_2)^{15}_{15} = 1$, while the other elements of the matrix of the transformation $X'^A = L(D_2)^A_B X^B$ vanish. Thus, for $D = D_2$, the Finslerian isometry (13) has the form

$$\begin{aligned}
X'^\alpha &= L(D_2)^\alpha_\beta X^\beta \qquad (\alpha,\beta=\overline{0,3}),\\
\theta'^i &= M(D_2)^i_j\theta^j \qquad (i,j=\overline{1,4}),\\
X'^8 &= X^8,\\
\vartheta'^i &= M(D_2)^i_j\vartheta^j \qquad (i,j=\overline{1,4}),\\
X'^{13} &= X^{13},\quad X'^{14}=X^{14},\quad X'^{15}=X^{15},
\end{aligned} \tag{18}$$

where $L(D_2)^\alpha_\beta$, $M(D_2)^i_j$ are given by (16)–(17) and the notation $\theta^i = X^{3+i}$, $\vartheta^j = X^{8+j}$ is used.

It was shown in the paper [5] that (16) and (17) are the elements of the matrices of the transformations for a Lorentz 4-vector and a Majorana 4-spinor respectively. Therefore, the result (18) asserts that, for $D = D_2$, the 16-vector X^A splits into the Lorentz 4-vector X^α, the Majorana 4-spinors θ^i, ϑ^j, and the Lorentz 4-scalars X^8, X^{13}, X^{14}, X^{15}.

This is the essence of the procedure of dimensional reduction allowing to display the "4-dimensional structure" of 16-dimensional expressions. Let us apply this procedure to the cumbersome formula (11) for the Finslerian length of the 16-vector X^A. Taking into consideration (18), we obtain

$$\begin{aligned}
|X|^4 &= X^{15}[X^8 g_{\mu\nu}X^\mu X^\nu - g_{\mu\nu}X^\mu\overline{\theta}\gamma^\nu\theta]-\\
&- [(X^{13})^2+(X^{14})^2]g_{\mu\nu}X^\mu X^\nu - X^8 g_{\mu\nu}X^\mu\overline{\vartheta}\gamma^\nu\vartheta+\\
&+ 2X^{13}g_{\mu\nu}X^\mu\overline{\theta}\gamma^\nu\vartheta + 2X^{14}g_{\mu\nu}X^\mu\overline{\theta}\gamma^5\gamma^\nu\vartheta + \tfrac{1}{2}g_{\mu\nu}\overline{\theta}\gamma^\mu\theta\,\overline{\vartheta}\gamma^\nu\vartheta,
\end{aligned} \tag{19}$$

where $\mu,\nu=\overline{0,3}$, $\|g_{\mu\nu}\| = \mathrm{diag}\,(1,-1,-1,-1)$ is the matrix of components of the Minkowski metric tensor in a pseudoorthonormal basis,

$$\gamma^0=\begin{pmatrix}0&0&i&0\\0&0&0&-i\\-i&0&0&0\\0&i&0&0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}i&0&0&0\\0&-i&0&0\\0&0&-i&0\\0&0&0&i\end{pmatrix},\quad \gamma^2=\begin{pmatrix}0&i&0&0\\i&0&0&0\\0&0&0&i\\0&0&i&0\end{pmatrix},$$

$$\gamma^3=\begin{pmatrix}0&0&-i&0\\0&0&0&i\\-i&0&0&0\\0&i&0&0\end{pmatrix},\quad \gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}$$

are the Dirac matrices in the Majorana representation [5], $\theta, \vartheta \in \mathbb{R}^4$ are 4-component columns of real numbers, and $\overline{\theta} = \theta^\top \gamma^0$, $\overline{\vartheta} = \vartheta^\top \gamma^0$ ($\top$ means the matrix transposition). Thus, the expression (11) is written in the compact 4-dimensional form (19).

Conclusion

Summarizing, we make some remarks concerning the obtained results.

First of all, we should note the dual nature of twistors. Those are spinors of the 6-dimensional pseudo-Euclidean space with two time-like dimensions [6]. On the other hand, as it was shown in this paper, twistors are a special case of Finslerian 4-spinors of the 16-dimensional vector space equipped with the metric form (11).

In addition, the paper contains the explicit description of isometries of the above 16-dimensional Finslerian space and the procedure of dimensional reduction which allows us to write (11) in the 4-dimensional form (19). The latter is important because it demonstrates the correspondence of our constructions to the standard relativistic theory on the level of geometry.

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References

[1] D. Finkelstein. *Hyperspin and hyperspace.* Physical Review Letters **56**, 1532–1533 (1986).

[2] D. Finkelstein, S. R. Finkelstein, C. Holm. *Hyperspin manifolds.* International Journal of Theoretical Physics **25**, 441–463 (1986).

[3] Yu. S. Vladimirov, A. V. Solov'yov. *Physical structure of rang* $(4, 4; b)$ *and 3-components spinors.* "Systemology and methodological problems of information-logic systems". – Novosibirsk: Institute of mathematics SD AS USSR, 1990. – Computative systems, vyp. 135, 44–66 (in Russian).

[4] A. V. Solov'yov. *On theory of binary physical structures of rang* $(5, 5; b)$ *and above.* "Systemology and methodological problems of information-logic systems". – Novosibirsk: Institute of mathematics SD AS USSR, 1990. – Computative systems, vyp. 135, 67–77 (in Russian).

[5] A. V. Solov'yov and Yu. S. Vladimirov. *Finslerian N-spinors: Algebra.* International Journal of Theoretical Physics **40**, 1511–1523 (2001).

[6] R. Penrose and W. Rindler, *Spinors and space-time. Spinor and twistor methods in space-time geometry.* Cambridge University Press, 1986.

[7] A. I. Kostrikin, *Introduction in algebra.* – Physmatlit, M., 1994 (in Russian).

[8] M. M. Postnikov, *Lecures on Geometry. II Semester. Linear algebra.* — Nauka, M., 1986 (in Russian).

To the Question on Quartic Geometry

P. D. Suharevsky

Emerging from A. D. Saharov's ideas on Universe and applications to modern Cosmology, are provided arguments for employing quartic symmetric forms as metric tensors. A non-associative algebra of anti-commuting 4-order matrices whose squares are the anti-commuting Pauli and Dirac matrices, is built. Further, are determined the equations of motion, which are quartic analogues of the Dirac equations; using the introduced quadratic spinors, is derived the associated to the motion Lagrangian. As well, are defined the infinite-dimensional extension of quaternions and their matrix representations, a prerequisite for solving problems in the multilinear background.

MSC: 11R16, 70S99.

Sakharov multisheet models of the Universe and "Big bang"

In 70–80s of the past century academician of the USSR Academy of Sciences Andrey Dmitrievich Sakharov, the well-known nuclear physicist, who made the determining contribution to the creation of the world first thermonuclear bomb, published a number of articles about "multisheet models of the Universe" [1]. The articles discussed "pulsing" and "oscillating" cosmologic models, which for a long time attracted attention of the scientists. Sakharov proposed the term "The multisheet model of the Universe" in 1969. He considered it "more expressive, more appropriate to the emotional and philosophic meaning of the immense picture of multiple repetitions of cycles of existence". It is possible but not known for sure that he had some another deep understanding on the term connected with sheets of the Riemann surface, which appear as a result of integration of the quadric interval.

In 1970 his first preprint "The multisheet model of the Universe" was published [2]. In the preprint in his proof of the hypothesis of the multisheet structure of the Universe Sakharov referred to the ideas of astrophysicist I. D. Novikov [3] on "cross-linking" during a gravitation collapse of two four-dimensional spaces, one of which is in the state of compression while the other is in the state of dilatation. Sakharov considered an infinite sequence of such coupled linked spaces, which he called sheets. His second prime assumption was the use of the extreme case of the Freedman model of the Universe with the metrics:

$$ds^2 = dt^2 - \{a(t)\}^2(dx^2 + dy^2 + dz^2) \tag{1}$$

and the law of variation of the scale: $a(t) \sim \mid t \mid^{2/3}$. Taking into account that the singularity of the metrics (1) at $t \to \infty$ is the growth of arbitrarily small agitations of density in unlimited number of times, in [2] the conclusion is made on the possibility of gravitation collapses at $\Delta\varepsilon/\varepsilon \sim 1$, where ε is the average density of energy including energy of gravitation nature. In the case when the Universe full of the dust $\varepsilon \sim 1/a^3$. Sakharov in [2] thinks that the so-called "premature" collapses in the compressing world earlier discussed in [4, 5], which correspond to anticollapses of inhibited nuclei in the expanding world, are the same physical processes.

Recently the hypothesis of Caldwell, Kamionkovsky and Weinberg, Phantom Energy and Cosmic Doomsday [6], has been actively discussed in literature. It proves that existence of the dark energy in the Universe, which at the values of the characteristic parameter ("quintessence") $w \equiv p/\varepsilon$ in the interval $-1 < w < -1/3$ leads to observable accelerated expansion of the Universe ($\varepsilon \sim 1/a^{3(1+w)}$), in the case of $w < -1$ will lead to the rip of the Universe within a finite period. This catastrophe of the Universe got the special name *"Big Rip"*. It is easy to notice that this process is in the essence similar to the case Sakharov discussed for the fluctuations of the Freedman Universe at $t \to \infty$ in [2] and his subsequent articles listed in [1]. The difference is only in the time of the beginning of the catastrophe. Therefore one may suppose that Big Rip will not lead to destruction of the Universe but to linking of its two sheets and flowing of the matter to the other sheet with subsequent compression of the Universe and possible turn of the time arrow. As far as according to Sakharov at $t < 0$ the static laws with time reversal act there is no paradox of inversion in his model. Hence the ideas of the multisheet structure of the Universe again become topical.

We note that Sakharov in fact determined the coupled sheets of the Universe as far as he used the quadric metrics. Because of the recent discussion of multi-dimensional models of the Universe a consideration of spaces with Finsler metrics presents great interest. Due to various reasons the preference is given to spaces with quartic metrics sometimes called the quadraspaces [7]. It is interesting that as noted by one of the most initiative authors of articles on the theme [8] among the explored spaces with quadric metrics the spaces with two dimensions are marked out as far as according to the theorem of Liuvill [9] for the two-dimensional case the range of transformations attributed to as conformal is significantly larger, which leads to significant diversity of analytic functions of the complex variable, each of them having the corresponding conformal map on the Euclidean plane. Therefore the quadric metrics is the most adequate for the spaces with two but not with four dimensions. The natural question arises (and the corresponding answer suggests itself): which metrics is the most adequate to spaces with four dimensions?

Before answering the question let's note basing on the work [10] from digest [7] that from the purely mathematical point of view the dimension four appears the most complex as far as additional $d - 4$ dimensionalities grant new freedom of action. For example, as it is noted in [10] at dimensionality ≥ 5, when self-crossing of complexes inside manifolds appear, they may be eliminated by small stir. While in small dimensionalities it is impossible.

According to the opinion of the author of work [10] "the dimensionality four from the topological point of view is the only dimensionality, where such different techniques come across and the questions emerge that do not correspond at first glance. Solution of many of them will require development of even more surprising techniques of algebra, geometry and topology".

Considering the above notes (the last of them is the most inspiring) let's assume that for four dimensions the most adequate should be the quartic metrics. In this case the idea of Sakharov multisheet models of the Universe will change. As it will be shown below this won't be a sequence of linked (or pasted together) in pair's sheets of four-dimensional spaces, but a more complex cyclical construction with a replace of four sheets of the four-dimensional spaces.

Axiomatically basing quartic geometry

Let's discuss some mathematic ideas that confirm these suppositions. First of all we will advert to the axiomatic construction of the differential-geometric manifold, which allows exploration of the physical structure of space basing on the first principals.

As it is known there are three main directions of classical geometry construction on the whole, which in principal may be adapted to construction of geometry on the small cases:

The first direction origins from the famous book of David Gilbert, The Bases of Geometry [11], which was written in the end of the 19th century. This was the most comprehensive axiomatic construction of Euclidean geometry. In this construction it was static as in the Euclid's treatise, Principia, which had appeared about 300 years B. C. After abandoning the axiom about parallel lines one may transform it into hyperbolic geometry of Lobachevsky but it is impossible to generalize it to elliptic geometry of Riemann.

The second direction is the vectorial construction of geometry, which was for the first time demonstrated in the well-known book of Herman Veil, Space. Time. Matter. (1918) [12]. The great role of the vectorial space in modern theoretical research and ease of axiom generalization to tensor analysis make this direction very attractive.

The third direction is geometry construction on the basis of the symmetry concept, which was demonstrated by Frederik Bachman in the post-war years and completed by publication of a monograph in 1959 [13]. The importance of the group theory in modern physics and especially of the group of movement and the possibility to transform the axiom system into both hyperbolic and elliptic geometry promote this direction to the leading position in the sphere of construction of axiomatic geometry between classic approaches.

However as it is noted in the foreword of the book's editor [13] I. M. Yaglom the system of the Bachman axioms is not full unlike the Gilbert's axiomatic. This does not reduce in principal its scientific value but stimulates further development. In particular it says nothing about congruence, order and continuity, which have further use in group relations, as well as about speed of mapping.

Taking into account that congruence is included in the concept of the group of movement while lack of relations and axioms of order and continuity allows including elliptic geometry into consideration one may regard it as some compensation of imperfection of the Bachman's axiom system. However there is nothing to compensate the lack of an axiom about the speed of mapping, which impairs understanding of physical processes and the problem of geometrization of real motions on the whole.

In this connection let's consider the two statements, which seem to represent axioms [14] forming the basis of the physical (material) geometry. Here the term "physical (material) geometry" will mean an abstract object, which along with spatial relations has attributes described by equations and fields, which are specific for real physical bodies. From the above listed directions of the axiomatic construction of geometry follows that such understanding of geometry meets the historical trend.

(I) *The axiom of mapping (measurement).* In the physical geometry there exists such a parameter of mapping (measurement) τ, that the speed of mapping $ds/d\tau = const$, where ds is the infinitely small interval between two points (events). The parameter τ allows making uniform marking of the grid chart in the differential-geometric manifold.

This axiom defines existence of the speed of movement in a differential-continuous form (or mapping of symmetries according to Bachman). Though the Bachman's axiom system has no concept of differential continuity it tolerates this concept trough the group theory. However in the group theory and hence in the Bachman's axiomatics motions mean only purely geometric mutually single-valued maps of ensembles of object's points and lines on themselves keeping the relations of incidence (attachment) and order and transforming segments and angles into congruent (identically equal) segments and angles. This allows avoiding the need to answer on, for example, the paradox (aporria) of Zenon, "Achilles and a Tortoise", about inconsistency of some attributes of motion but it is inadequate from the point of view of physical understanding of processes.

At the same time in fact at any speed of real movement (constant or variable) the parameter τ may be defined as far as it is not set by any prior conditions. On the other hand if linking the parameter τ with the parameter of time of real movement t the axiom of mapping transforms into the postulate about existence of some constant speed of real movement - lack of forces in physical geometry. Unlike the axiom this postulate may get broken in connection with slow evolution of the scale of physical geometry. But it allows constructing a theory, which geometrizes the observed movements using in general case the differential interval of the m-th degree of Finsler geometry, e.g. using the algorithm:

$$ds = cd\tau = [g_{\mu\nu\ldots\varepsilon} dx^\mu dx^\nu \ldots dx^\varepsilon]^{1/m} = [g_{\mu\nu\ldots\varepsilon} dx^\mu/dt \; dx^\nu/dt \; \ldots \; dx^\varepsilon/dt]^{1/m} dt. \quad (2)$$

In principal constructions of differential intervals of Finsler geometry may be of less complex form. Specifically at $m = 4$ string, membrane and hyper membrane objects may be easy introduced in the form:

$$(ds')^4 = g_{1\mu\nu\gamma\xi} dx^\mu dx^\nu dx^\gamma d\lambda^\xi + g_{2\mu\nu\gamma\xi} dx^\mu dx^\nu d\sigma^{\gamma\xi} + \\ + g_{3\mu\nu\gamma\xi} dx^\mu dv^{\nu\gamma\xi} + g_{4\mu\nu\gamma\xi} d\sigma^{\mu\nu} d\sigma^{\gamma\xi} + g_{5\mu\nu\gamma\xi} d\omega^{\mu\nu\gamma\xi}. \quad (3)$$

as well as objects with $m < 4$, including square intervals, which may have dominating weight at speeds of movement less than a half of the light speed.

(II) *The axiom of hardness (ordering).* The elements of physical geometry possess hardness (ordering), i.e. noncondensability and single-valued sequence of objects, relative to which measurement is conducted.

Axiom (II) generalizes and adds Bachman axiom of hardness on a plane, which states: if h is a beam coming from point A and S is a half-plane limited by the line carrying beam h while h' is a beam coming from point A' and S' is a half-plane limited by the line carrying beam h' then there exist no two different movements which would transform A into A', h into h' and S into S'.

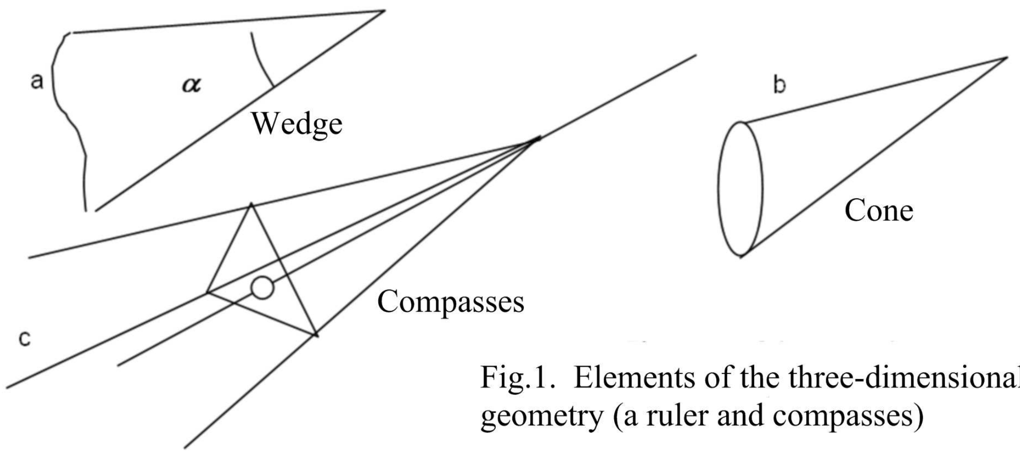

Fig.1. Elements of the three-dimensional geometry (a ruler and compasses)

Axiom (II) means that in n-dimensional geometry after carrying out $n - 1$ random shifts (transforms) keeping the interval ds constant the n-th transform may be defined by elements of geometry itself. The counter images of such elements in two-dimensional geometry are a ruler and compasses. For example, the problem of cube rooting using a ruler and compasses comes to angle trisecting. In plane geometry it has no solution. However using three-dimensional analogs of a ruler and compasses (there may be several modifications of them) the problem of angle trisecting is solved elementary.

Indeed, let's choose a piece of a plane of a wedge form with the angle at the vertex equal to α, turn it into a cone and glue the borders (Fig. 1a, 1b). Let's also choose the analog of compasses in the form of a tripod with the central bar and the guide frame of an equilateral triangle (Fig. 1c) and put the cone on it fixing it at the vertex.

Pulling the frame by the central bar in the direction of the vertex of the solid angle let's achieve tight abutment of the cone's material to the legs of the compasses such a way that the cone transforms into a pyramid. The angle at the vertex of the pyramid on each bound will be precisely equal to $\alpha/3$.

One of the effects of axiom (II) as it is seen from further consideration is ambiguity of the n-th transform, which does not result from the Bachman axioms. The general effect of axioms (I) and (II) for physical geometry of $n = 4 + k$ dimensions appears the limitation on introduction of the differential form of the m-th order by the value $m \leq 4$.

Introduction of these axioms corresponds with the general idea on the need of physics axiomatization and the wish to understand the fundamental principles of outward things. It is natural that axiomatics becomes constructive if it does not lead up a blind alley or overcomplicates the problem. For example introduction of the differential form:

$$ds^m = g_{\mu\nu\ldots\varepsilon} dx^\mu dx^\nu \ldots dx^\varepsilon, \tag{4}$$

where $\mu\nu\ldots\varepsilon \in \{1, 2, \ldots, n\}$, $n = m + k$, $k \geq 0$, corresponds with the conception about physical structures underlying burbakanization of physics in the spirit of Kulakov and Vladimirov [15, 16], which was fundamentally developed in the work of the second author [21], i.e. setting as source concepts the relations between different objects – here between ds and $dx^\mu, dx^\nu, \ldots, dx^\varepsilon$ - through $g_{\mu\nu\ldots\varepsilon}$. To achieve full source concept about this physical structure one should define relations for each of these objects.

Taking this into account let's define relations between one of the objects, for example dx^μ, and other objects by means of solving the algebraic equation of degree m at fixed values of $ds, dx^\nu, \ldots, dx^\varepsilon$.

As it is known such algebraic equation at $m \leq 4$ is solved in radicals. Let's discuss the consequence of such solution.

First of all it meets the requirement of hardness, i.e. axiom (II). Indeed, it is known that at $m = 1$ or 2 the solution of an algebraic equation is found by the use of a ruler and compasses. If $m = 3$ then (using the Cardano method) after reducing the source cubic equation to the form $y^3 + py + q = 0$ and replacement $y_0 \equiv \alpha + \beta$, where y_0 is the root of the modified cubic equation, it reduces to a quadratic, the roots of which are the values α^3 and β^3. The latter is also solved at a plane by means of a ruler and compasses. The cubic roots of the values α^3 and β^3 are solved by the use of three-dimensional analogs of compasses and a ruler as it was shown above. A similar situation appears in solution of a quartic equation $m = 4$ by Ferrari method. Therefore solution of an algebraic equation for each object of this physical structure at $m \leq 4$ meets geometric formulation.

Second, this solution is ambiguous. Multiple-valued function of a complex variable expressing this solution at $m \leq 4$ corresponds with Galua group [17]. In other words, Galua group underlies geometrization of physical structures at $m \leq 4$.

Third, a random function, which may be expressed in radicals, can be defined at continuity along any continuous curve C not passing through the points, in which this function is not defined. If at the same time the curve C does not pass through the points of bifurcation and ambiguity it is defined at continuity along the curve C unambiguously [18]. I.e. this function possesses the valuable feature of monodromy.

Fourth, for any function expressed in radicals one may construct a Riemann surface. In this case it will be m-sheeted ($m \leq 4$) for each object $ds, dx^{\mu}, dx^{\nu}, \ldots, dx^{\varepsilon}$, $\mu\nu\ldots\varepsilon \in \{1, 2, \ldots, n\}$, $n = m + k$, $k \geq 0$. Following group transforms at keeping ds the Riemann surfaces for each individual object from n transfer one into another forming a single Riemann hypersurface with four n-dimensional hypersheets. It is evident that these properties remain at transition from geometry on the small to geometry on the whole.

Therefore turning back to the ideas of the multisheet Universe one may identify at $m = 2$, $n = 4$ the Riemann hypersurface of four dimensions as two sheets (two four-dimensional spaces) of the Sakharov's multisheet model of the Universe. Then at $m = 4$, $n = 4 + k$, $k \geq 0$ one easy gets the generalization of the Sakharov's two-sheet model to the four-sheet Riemann hypersurface of n dimensions, which by analogy may be called a fragment of the four-sheet cyclic structure of the Universe (or simply the four-sheet model of the Universe if not assuming its doubling).

In other words the ground is enough to consider differential forms with $m \leq 4$, $n \geq m$ as the basis for geometrization of physical structures, which one can't say about differential forms with $m \geq 5$.

Particular case of quartic generalization of the Dirac equations

Let's turn to the simplest (canonic) differential form of the fourth order:

$$ds^4 = g_{\mu\nu\gamma\epsilon} dx^{\mu} dx^{\nu} dx^{\gamma} dx^{\epsilon} \quad \Rightarrow \quad ds^4 = dx_1^4 + dx_2^4 + dx_3^4 + dx_4^4 \tag{5}$$

and solve a problem for the characteristic constant of the quartic differentiation operator in such plane Finsler geometry:

$$(\partial_1^4 + \partial_2^4 + \partial_3^4 + \partial_4^4)\psi = m^4 \psi, \tag{6}$$

This operator is marked out in such a kind that its dimensionality and the order of derivation are congruent and equal to 4 while the dimensionality 4 is marked out by nature itself. Some grounds of why it is so result from the above consideration.

Is it possible to find such a root of the fourth order from the quartic differentiation operator that

$$(M_i \partial_i)^4 = (\gamma_i \partial_i^2)^2 = \sum_i \partial_i^4, \tag{7}$$

where γ_i are the gamma-matrices of Dirac or their generalization and $\sum\limits_i$ is the sum at the i index?

It is easy to show that there exist no anti-commutation matrices of any order with ordinary rules of multiplication, the square of which is also an anti-commutation matrix. Indeed, if $M_iM_j = -M_jM_i$, then $M_i^2M_j^2 = M_iM_iM_jM_j = M_iM_jM_jM_i = M_j^2M_i^2$. More complex combinations with mixing of commutation and anti-commutation matrices also give no result.

However this obstacle appears because in quartic geometry one should use more complex mathematic instruments. Therefore let's introduce quartic matrices generalizing Pauli matrices and Dirac matrices with special rules of multiplication. Namely, at first using ordinary rules let's multiply four-square matrices on one hyper-bound of the quart and then using constants of structure that define the rules of multiplication on orthogonal hyper-bounds reduce the rest of indices of matrix elements. By the very definition algebra of these matrices will be nonassociative.

Let's denote the quartic matrix generalizing the Pauli matrix ${\sigma_i}^a{}_b$ as ${\zeta_i}^a{}_b{}^c{}_d$, where in this case the index $i(j,k)$ is connected with the index of the coordinate, the left pair of indices (superscript a and subscript b) are the indices of the two-dimensional matrix elements on the hyper-bound, while the right pair (superscript c and subscript d) are the indices on the orthogonal hyper-bounds (in the third and the fourth dimensions respectively). The indices a, b, c and d pass the values 1 and 2. It is natural that further both the left pair and the right pair of indices may be denoted by other Latin letters. The sense of the innovation is that after multiplication of anti-commutation matrices on one hyper-bound they won't be anti-commutation on it but they may be re-defined in such a way that they become anti-commutation on other orthogonal hyper-bounds of the quart.

Let's give the following values for the elements of matrices ζ_1 and ζ_2 (${\zeta_i}^a{}_b{}^c{}_d :\equiv {\sigma_i}^a{}_b({\sigma_i}^c{}_d)^{1/2}$) :

$$
\begin{aligned}
&{\zeta_1}^a{}_b{}^1{}_1 = {\zeta_1}^a{}_b{}^2{}_2 = 0\,, \quad {\zeta_1}^a{}_b{}^1{}_2 = {\sigma_1}^a{}_b\,, \quad {\zeta_1}^a{}_b{}^2{}_1 = {\sigma_1}^a{}_b\,, &(8a)\\
&{\zeta_2}^a{}_b{}^1{}_1 = {\zeta_2}^a{}_b{}^2{}_2 = 0\,, \quad {\zeta_2}^a{}_b{}^1{}_2 = (-i)^{1/2}{\sigma_2}^a{}_b\,, \quad {\zeta_2}^a{}_b{}^2{}_1 = (i)^{1/2}{\sigma_2}^a{}_b\,. &(8b)
\end{aligned}
$$

It is evident that matrices ζ_1 and ζ_2 are hermitian conjugate, i.e. $\zeta_i^+ = \zeta_i$.

As a result of multiplication of the elements of the quartic matrices ζ_1 and ζ_2 we receive:

$$
\begin{aligned}
&{\zeta_1}^a{}_b{}^2{}_1\,{\zeta_2}^b{}_c{}^1{}_2 = {\zeta_1}^a{}_b{}^1{}_2\,{\zeta_2}^b{}_c{}^1{}_2 = (-i)^{1/2}{\sigma_1}^a{}_b\,{\sigma_2}^b{}_c = (i)^{1/2}{\sigma_3}^a{}_c\,; &(9a)\\
&{\zeta_1}^a{}_b{}^2{}_1\,{\zeta_2}^b{}_c{}^2{}_1 = {\zeta_1}^a{}_b{}^1{}_2\,{\zeta_2}^b{}_c{}^2{}_1 = (i)^{1/2}{\sigma_1}^a{}_b\,{\sigma_2}^b{}_c = (-i)^{1/2}{\sigma_3}^a{}_c\,. &(9b)
\end{aligned}
$$

At ordinary multiplication of the matrices one had to keep the second member of (9b) and the first member of (9a) while the two residuary members one had to cast aside. On the contrary, in the proposed algebra one retains the first member of (9b) and the second of (9a) while ignoring the members that are retained at ordinary way of matrices multiplication.

According to the new rules of multiplication at squaring of the quartic matrix ζ_1 the members $({\zeta_1}^a{}_b{}^1{}_2)^2 = I^a{}_b$ and $({\zeta_1}^a{}_b{}^2{}_1)^2 = I^a{}_b$ are preserved. While at squaring of the matrix ζ_2 the members $(\zeta_2{}^a{}_b{}^1{}_2)^2 = -iI^a{}_b$ and $(\zeta_2{}^a{}_b{}^2{}_1)^2 = iI^a{}_b$ are preserved. If placing the results of squaring of the matrices ζ_1 and ζ_2 on one hyper-bound according to the right pair of indices of the members situated in parenthesis and reducing unitary matrices we get the two required matrices σ_1 and σ_2.

At the same time this order of placing the results of squaring is unsuitable for getting the third Pauli matrix at multiplication in (9a) and (9b). In order to get the matrix σ_3 multiplied by virtual value i after squaring the respective members from (9a) and (9b) of these results one has to put the member $\zeta_1{}^a{}_b{}^1{}_2\zeta_2{}^b{}_c{}^1{}_2$ into the cell $({}^1{}_1)$ while $\zeta_1{}^a{}_b{}^2{}_1\,\zeta_2{}^b{}_c{}^2{}_1$ into $({}^2{}_2)$, where the right pair of indices is indicated in parenthesis, and at squaring keep in the same cells. However in a two-dimensional space this operation is excess so it is not used below.

As far as $(\zeta_1{}^a{}_b{}^1{}_2\zeta_2{}^b{}_c{}^1{}_2)^2 = iI^a{}_c$, and $(\zeta_1{}^a{}_b{}^2{}_1\zeta_2{}^b{}_c{}^2{}_1)^2 = -iI^a{}_c$, multiplication of the squares of the matrices $\zeta_{1,2}$ will correspond with the algebra of Pauli matrices. It is natural that multiplication of matrices i on a unitary matrix I does not change position of their elements. Hence the squares of the matrices $\zeta_{1,2}$ produce homomorphism of the group $SU(2)$ on the quartic space.

In order to formalize this rule of multiplication of matrices $\zeta_{1,2}$ and to "forget" about the considerations connected with finding grounds for the innovation let's introduce the matrix of constants of structure (or operator) $F := F^k{}_d{}^c{}_f{}^e{}_m$. Note that all indices of this matrix are connected only with the right pair of indices of the quartic matrix. The elements of this matrix are equal to 1 when $k = c = e,\ d = f = m,\ m \neq k$; in other cases they are equal to zero. They are called structure because they locate the elements of the results of multiplication of matrices ζ_1 and ζ_2 in proper cells.

Such multiplication of matrices ζ_i and ζ_j may be written in the following form:

$$(\zeta_i\zeta_j)^{ak}{}_{gm} = F^k{}_d{}^c{}_f{}^e{}_m\,\zeta_i{}^a{}_b{}^c{}_d\,\zeta_j{}^b{}_g{}^e{}_f\,. \tag{10}$$

However one may simply remember the rule of reduction of the right indices at multiplication of quartic matrices, which will be used below at keeping the sign of the operator F.

It is also convenient to present the operator F in the form of the product of two four-symbol matrices:

$$F = \Phi_L\Phi_R := \Phi_L{}^k{}_d{}^c{}_r\Phi_R{}^r{}_f{}^e{}_m\,, \tag{11}$$

the elements of which are equal to 1, when Φ_L: $k = r = c$, $d \neq k$, , Φ_R: $m = f$, $r = e$, $m \neq r$ and equal to 0 in other cases.

The quartic spinor will naturally have two superscripts in this approach, i.e. $\psi := \psi^{ab}$. Then the member with the partial derivative by the i-th coordinate in the matrix form for the direct and the hermitian conjugate spinor may be written in the following form:

$$\Phi_R\zeta_i\,\partial_i\psi := \Phi_R{}^r{}_f{}^e{}_m\,\zeta_i b_g{}^e{}_f\,\partial_i\psi^{gm} \equiv A_{Ri}{}^r{}_g{}^b{}_m\,\partial_i\psi^{gm} \tag{12a}$$

$$\partial_j\psi^+\,\zeta_j\Phi_L := \partial_j\psi^*{}_{ka}\,\zeta_j{}^a{}_b{}^c{}_d\,\Phi_L{}^k{}_d{}^c{}_r \equiv \partial_j\psi^*{}_{ka}\,A_{Lj}{}^k{}_b{}^a{}_r\,. \tag{12b}$$

Generalization of this algebra on the quartic four-dimensional space presents no complexity.

Indeed, let's choose presentation of anti-commutation four-row matrices in the form:

$$\gamma_k = i\sigma_k\tau_2,\ \gamma_4 = \tau_1,\ \gamma_5 = -i\tau_3, \tag{13}$$

where τ_1, τ_2, τ_3 are the Pauli matrices, which instead of units contain unit two-row matrices, and let's write sixteen-row analogs of matrices ζ_i the following way:

$$Z_\mu{}^a{}_b{}^c{}_d = \gamma_\mu{}^a{}_b(\gamma_\mu{}^c{}_d)^{1/2}, \tag{14}$$

where Greek indices μ and Latin indices a, b, c and d pass the values $1, 2, 3$ and 4.

It is easy to ascertain that through introducing a matrix of constants of structure F, the same as in the two-dimensional case but with Latin indices passing the values $1, 2, 3$ and 4, we also get similar algebra of anti-commutation matrices, which transform after the turn on the quart in four Dirac matrices in presentation (13). The elements of these matrices are equal to 1 at $k = c = e$, $d = f = m$, $m = 5 - k$. For matrices Φ_L we respectively have $k = r = c, d = 5 - k$, for Φ_R have $m = f$, $r = e$, $m = f = 5 - e$ and $f = e \pm 2$. The rest of the values are equal to zero. Instead of the matrices A_{Li} and A_{Rj} we respectively get the matrices:

$$B_{L\mu} := B_{L\mu j}{}^k{}_b{}^a{}_r = Z_\mu{}^a{}_b{}^c{}_d\, \Phi_L{}^k{}_d{}^c{}_r \quad\text{and}\quad B_{R\nu} := B_{R\nu}{}^r{}_g{}^b{}_m = \Phi_R{}^r{}_f{}^e{}_m\, Z_\nu{}^b{}_g{}^e{}_f \tag{15a,b}$$

Then the generalization of the Dirac equation in quartic geometry is:

$$iB_{R\mu}\partial_\mu\psi - (m^4)^{1/4}\psi = 0 \tag{16a}$$

and its conjugate:

$$i\partial_\mu\underline{\psi}B_{L\mu} + (m^4)^{1/4}\underline{\psi} = 0, \qquad \text{where} \tag{16b}$$

$$\underline{\psi} := \psi^+\Gamma_4 \equiv \psi^*{}_{ab}\Gamma_4{}^b{}_c, \tag{17}$$

while Γ_4 is the Dirac matrix γ_4 with units replaced by four-row unit matrices I, indices b and c evidently belong to the right pair of indices. Then the spinor quartic Lagrangian has the form:

$$L_4 = (i/2)[\underline{\psi}B_{R\mu}\partial_\mu\psi - \partial_\mu\underline{\psi}B_{L\mu}\psi] - (m^4)^{1/4}\underline{\psi}\psi. \tag{18}$$

Therefore the problem of getting the root of the forth order from the differential operator $\sum\limits_i \partial_i{}^4$ may be solved in the form:

$$\sum_\mu \partial_\mu{}^4 = (\gamma_\mu\partial_\mu{}^2)(\gamma_\nu\partial_\nu{}^2) = [(Z_\mu\partial_\mu)F(Z_\nu\partial_\nu)][(Z_\alpha\partial_\alpha)F(Z_\beta\partial_\beta)],\ \text{where } F = \Phi_L\Phi_R. \tag{19}$$

At this non-associative property of multiplication is necessary only in the right part of the equation (19).

As far as $(m^4)^{1/4} = (\pm, \pm i)\, m$, then the theory constructed with the use of such algebra should describe particles, antiparticles, pseudo-particles and antipseudo-particles. Moreover, gravitation interaction of two pseudo-particles leads to anti-gravitation, which in global scale may lead to observed acceleration of enlargement of the Universe, which in alternative ([19], [20]) may be interpreted as accelerated reduction of the scale of length and time in the area surrounding the observer as well as coherent with this reduction changes of fundamental physical constants. Both alternatives lead to the collapse of the Universe with subsequent rip or flowing to another sheet of the hyper-Riemann surface. Coalescence of pseudo-particles and antipseudo-particles leads to birth of antiparticles. Vice versa, antiparticles may decay into pseudo- and antipseudo-particles, which should contribute to baryon asymmetry and dark matter of the Universe.

Naturally that all above-mentioned effects as a whole must have an experimental or observed in cosmic phenomena verification, which are able become apparent solely attached to very high energy or extraordinary small mass.

If for instance a differential form for flying along coordinate x1 particle have air:

$$ds^4 = (dx^4)^4 + (dx^1)^4 - [(dx^4)^2 + (dx^1)^2)]a^2, \tag{20}$$

where a^2 - too small but a finite value, $dx^4 = icdt$, after dividing each part of the equality (20) by a^2 attached to $dx^1 < cdt$ practically always first and second members in (20) one may disregard and use metric of Minkowski space-time.

Then the size ds^2/a will be fan analogue of the interval. The quartic metric begins become apparent solely attached to condition:

$$2|(dx^4)^2|/a^2 + (dx^1)^2/(dx^4)^2 = 2(c^2/a^2)dt^2 + \beta^2 \geq 1, \qquad \text{where} \quad \beta \equiv v/c \tag{21}$$

Consequently, observable metric effects expects really very small. But they will be able to play important role in understanding physical picture of the universe and in a development of the modern physical theory.

Polinions

Unfortunately, having considered higher the task appears solely a particular case of all totality problems, arose attached to attempts to discover worker instruments for researchies in the quartic space. It is obvious in particular from the articles, publishes in the magazine "Hipercomplex numbers in geometry and physics" references on that one cited in this article, in particular. In order to somewhat undo knot these problems we shall bring introduction in some new mathematical apparatus as author hope suitable for investigations in cubic and quartic n-measure space.

It is found handy introduce infinite extension of algebra the quaternions in wich commutation correlations for components of vector part looks the following image:

$$q_j q_i = \exp\{2\pi i/m\} q_i q_j, \tag{22}$$

where $m \in \{N\}$ – the power of the differential form interval determining metric; indexes $i, j\{1, \ldots, n\}, i < j, qi = qi(m)$. By that accomplishes equality:

$$(q_i \partial_i)^m = \sum_i \partial_i{}^m \tag{23}$$

Let us name these quantities $q_i(m)$ by *polinions.* Building the matrix representation for the polinions with the commutation correlations (22) is found not trivial even attached to $m = 3$ and 4. Indeed, so as for any two matrices with usual rules of multiplication $Sp(M_i M_j) = Sp(M_j M_i)$, then for satisfaction (22) both members this equality must be equal zero. With another side as easy prove for attempts representation of polinions by Pauli matrices, with are exact matrix representation of usual quatenions, for existence matrix representation unit $q_i{}^m = I$ and for formation of group transformation ($q_i \circ q_j = q_k$, $q_i, q_j, q_k \in G$) under $m = 3$ and 4 the conditions $Sp(M_i M_j) \neq 0$ and $Sp(M_i) \neq 0$ are necessary (if $m = 2$ this demand do not).

However solution of the task is found unexpectedly simple for non-associative algebra with taking into account offered above algorithm for special case extracted root fourth power from quartic canonical differential which permit generalization at any value m. Indeed, let us consider generalization of Pauli matrices in appearance:

$$\sigma_1(m) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2(m) \to \begin{pmatrix} 0 & (-i)^{2/m} \\ (i)^{2/m} & 0 \end{pmatrix};\ \sigma_3(m) \to \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{2/m} \end{pmatrix}; \tag{24}$$

Then if multiplication of matrices hold as such Riemann surface, when unit may factorize at the multipliers:

$$1 = 1^{2/m} 1^{2/m} = (-1)^{2/m} (-1)^{2/m} = (i)^{2/m} (-i)^{2/m}, \tag{25}$$

that, generally speaking, do not coincide (for example by $m = 4$) with usual algebra of complex numbers, then fulfill of the condition (22).

It already stand for that we pass on to non-associative algebra multiply firstly numbers being in parentheses and solely after raise the result to the fractional power. The sheets of Riemann surface are gluing together attached to arguments 0 and 2π. For all that evidently generalized Pauli matrices by such non-associative multiply satisfies some analogue unimodular group $SU(2)$ with following commutation correlations:

$$\sigma_i(m) \sigma_j(m) = (i)^{1/m} \varepsilon_{ijk} \sigma_k(m). \tag{26}$$

For all that under the modulus of vector $\sigma_i x_i$ need understand value $(\sigma_i(m) x_i(m))^{1/m}$.

Apparently non-associative and so irreversibility time characteristically for cubic an quartic spaces and also for one large dimenshions. That by the way take off question about paradox of turning time in the Sakharov multisheet models Universe attached to it generalization at the cubic and the quartic geometries.

References

[1] A. D. Sakharov, *Scientific Works.* CentrCom, Moscow, 1995 (in Russian).

[2] A. D. Sakharov, *Multisheet model of the universe,* Preprint #7 of the Institute of applied mathematics of the Russian Academy of Sciences, Moscow, 1970 (in Russian).

[3] Ya. B. Zeldovich, I. D. Novikov, *Relativistic Astrophysics.* Science, Moscow, 1967 (in Russian).

[4] I. D. Novikov, AG 41. 1964, P 1075 (in Russian).

[5] Y. Ne'eman, Ap. J. 141. 1965, P. 1303.

[6] Robert R. Caldwell. Mark Kamionkowski, Nevin N. Weinberg, *Phantom energy and cosmic doomsday,* Phys. Rev. Lett. 91. 2003. 071301.

[7] D. G. Pavlov, *Four-dimensional time,* Hypercomplex Numbers in Geometry and Physics, 1 (2004), 31–39.

[8] D. G. Pavlov, *Generalization of scalar product axioms,* Hypercomplex Numbers in Geometry and Physics, 1 (2004), 5–18.

[9] B. A, Rosenfeld, *Multidimensional Spaces.* Science, Moscow, 1966 (in Russian).

[10] R. V. Mikhailov, *On some questions of four-dimensional topology: survey of modern research,* Hypercomplex Numbers in Geometry and Physics, 1 (2004), 106–110.

[11] David Hilbert, *Gesametle Abhandlungen.* Bd 1–3, B., 1932–1935.

[12] Von Hermann Veyl, *Raum. Zeit. Materie.* Berlin: J. Springer. 1923.

[13] Von Friedrich Bachmann, *Aufbau der Geometrie aus dem Spiegelungsbegriff.* Berlin-Heidelberg: Springer-Verlag. 1959.

[14] P. D. Suharevsky, *Two axioms of geometry and their effects,* Proc. of the Conference "Engineering and physics problems of new techniques", Moscow, 1994, 144–145 (in Russian).

[15] Yu. I. Kulakov, *Elements of Physical Structures.* Novosibirsk University Publ., 1968 (in Russian).

[16] Yu. S. Vladimirov, *Space-Time: Apparent and Hidden Dimensionalities.* Science, Moscow, 1989 (in Russian).

[17] A. I. Kostrikin, *Introduction into Algebra. Part III. Major Structures.* Physmatlit, Moscow, 2001 (in Russian).

[18] V. B. Alexeev, *Abel's Theorem in Problems and Solutions.* MCCME, Moscow, 2001 (in Russian).

[19] K. P. Stanyukovich, P. D. Sukharevsky, N. P. Scherbinskaya, *About one possibility of unified classification of space objects and ultimate particles,* Issues of Nuclear Science and Techniques. Theoretical and Applied Physics Series. Issue 2 (1987), 20–28 (in Russian).

[20] P. D. Sukharevsky, N. P. Scherbinskaya, *Interpretation of quasars in TVG*, Issues of Nuclear Science and Techniques. Theoretical and Applied Physics Series. Issue 1 (1992), 3–5 (in Russian).

[21] Yu. S. Vladimirov, *Relation Theory of Space-Time and Interactions.* Moscow University Publ. Part I (Theory of systems of relations), 1996. Part II (Theory of Physical interactions), 1998 (in Russian).

Theory of the Zero Order Effect Suitable to Investigate the Space-Time Geometrical Properties

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The applicability of Einstein's relativity theory on galactic scale and the role of geometry for the solution of the problems of observational astrophysics are discussed. The theory of the zero order effect to study the geometrical properties of space-time in experiment is given.

"I will try to escape discussing questions that though providing mathematicians with the possibility to reveal their skills, will not be helpful in broadening our field of knowledge"
J. C. Maxwell "Treatise on Electricity"

The problem of measurement and its interpretation starts to play specific role,
since there is no possibility to perform a measurement in such a way
that the current state of the system and the prediction of its behavior
become simultaneously known with the desired accuracy.
On the W. Heisenberg's Uncertainty principle.

MSC2000: 83C99, 83D05.

Introduction

The declinations of the planets' orbits from ideal circles experimentally discovered by I. Kepler in XVII century posed a dilemma. Either the laws of Nature and Mathematics were not identical, and the mathematical harmony did not rule the Universe, or our knowledge was not complete not only in Science but in Mathematics as well.

In the end of XVIII century C. Gauss became the first who approached the problem of the applicability of the Euclidean geometry to the World in a constructive way. He measured the sum of the interior angles of the triangle in situ directly. The vortices of the triangle were at the peaks of the surrounding mountains. Gauss did not find any deviation in the geometry of the world and Euclidean geometry within the accuracy of his measurements.

In the beginning of the XIX century N. Lobachevsky considered and evaluated the principal possibilities of the astronomical measurements, and this inevitably lead him to the construction of the first non-Euclidean geometry.

In the middle of the XIX c. W. Clifford proclaimed and successively defended his idea that no physical phenomenon can be experimentally distinguished from the corresponding change of the geometrical curvature of the World.

Following these ideas A. Einstein in the beginning of XX c. reduced the general but qualitative Clifford's statement to the more narrow but quantitative theory. He demanded the general covariance of physical laws, postulated the invariance of the light speed and the equivalence principle and produced a theory according to which the unavoidable gravitation could not be distinguished from the geometrical properties of space-time. In his fundamental paper Einstein considered the space-time described by Riemann-Minkowski geometry, predicted the effects that could be experimentally observed in this case and gave the calculation of them. The experiments revealed the good correlation with the calculation, and geometry became the full right part of physical theory. On the macro level, it made the Newton theory of Solar system gravity more precise. On the micro level, the Dirac theory was introduced into quantum mechanics. On mega level, the cosmology obtained the expanding Universe theory and the accompanying circumstances.

When we discuss the experimental data that have to be compared with the theory, we should mention the scale. There are three such scales in astrophysics: Solar system, galaxy and all the observable Universe. The known achievements of the general relativity theory (GRT) based on the introduction of the new (Riemann) geometry provided the link between the laboratory physics and the first of these scales. In the end of the XX c. there appeared the astrophysical data that can not be explained by the theory without new notions like dark matter or dark energy or without the modification of the foundations of the theory including the geometry of space-time. When choosing the new geometry one should start with the analysis of the problems appearing already on the galactic scale.

Suggesting the physical effect demonstrating the geometrical properties of space-time, one should pay attention to the fact that the static spherically symmetric solutions of the gravity equations both for Riemann geometry and for its generalization, Finsler geometry, give the same observable predictions. The effect that could be used for such investigation is the gravitational radiation, i.e. gravitational waves (GW). The existence of the GW predicted by the GRT has an indirect experimental support – the change of the orbital period in some double star systems [1]. For the different geometries of space-time the GW would possess different properties. But though there are several physical effects that could be used to investigate these properties, the problem of the direct observation of the GW is not solved up to now. This is due to the utmost smallness of the perturbation that the GW produce on any of the known physical effects already in the first order of magnitude. Since the GW are waves, we can use the resonance phenomena that could appear not in the first order of the perturbation theory as it was suggested in various approaches up to now but in the zero order.

The further material is organized as follows. We consider the metrical approach to the gravitation theory to be valid. Since the geometry appears to be closely connected to the mass distribution, let us first give some results of the recent astrophysical observations and discuss their possible interpretations. Then we will point out some additional details concerning the space-time geometry apart from those that follow from the experimental data. Then the theory of the optic-metrical parametric resonance (OMPR) will be discussed and its results and interpretations for various cases will be analyzed. Finally, the examples of the astrophysical systems suitable for the observations are given.

1. Experimental data and its interpretation

The results of the astrophysical observations are the following. On the galactic scale, the rotation curves, i.e. the dependencies of the stars orbital velocities on their distances to the galaxies' centers were measured for some galaxies [2–5]. On the Universe scale, the GRT effect of the gravitational lensing on the galactic clusters is found. This supports the Einstein idea about the link between the metric and gravity, but the result is several times larger than the GRT prediction. The acceleration of the Universe expansion is ascertained [6-7], and this leads to the notion of the dark energy.

The review of the theoretical results is given in [8]. According to the Introduction let us give a brief list of the results and ideas concerning only the galactic scale phenomena. To illustrate them let us give a figure from [5], Fig. 1. The experimental points obtained when measuring the orbital velocities, v, of stars of the spiral galaxies vis. their distances to the centers of those galaxies, R, can be described by the empirical formula [9]

$$v^2 = \frac{\beta^* c^2 N^*}{R} + \frac{\gamma^* c^2 N^* R}{2} + \frac{\gamma_0 c^2 R}{2}\,, \tag{1}$$

where c is the light speed, N^* is the number of stars in the galaxy (usually about 10^{11}), β^* for the Sun is $\beta^* = \frac{M_S G}{c^2} = 1.48 \cdot 10^5$ cm (M_S is the Solar mass, G is the gravity constant), γ^* and γ_0 are universal parameters $\gamma^* = 5.42 \cdot 10^{-41}$ cm^{-1}, $\gamma_0 = 3.06 \cdot 10^{-30}$ cm^{-1}.

All the three parameters become of the same order at the border of a galaxy, while the result of the Newton theory as well as the Schwarzschild's solution of the GRT equations predict only the decrease of the velocity corresponding to the first term in Eq. (1). The calculations were performed with regard to the exponential distribution of stars in a galaxy. To provide the observed motion of the gleaming stars, the existence of additional matter interacting with the stars gravitationally is suggested. The mass of this matter must be thrice as much as the mass of the visible stars, it must be located at the periphery of a galaxy and it neither emits, nor absorbs the electromagnetic radiation. In this paper it is essential to underline that the same effects take place for the clusters of galaxies too [8], that is on a

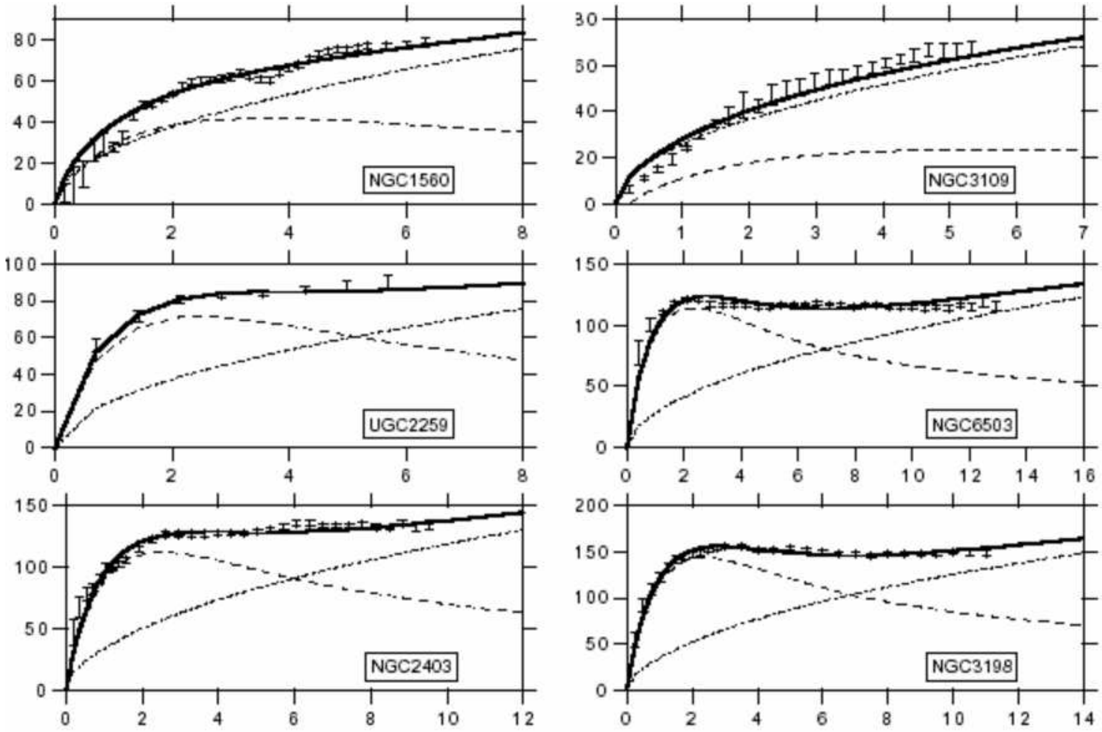

Figure 1: [4] Orbital velocities (km/s) as functions of R/R_0, where R_0 is a characteristic scale for each galaxy. Dashed line is the Newtonian potential (coinciding with the Schwarzschild's solution), produced by the observable gleaming matter with regard to the exponential distribution of stars inside the galaxy.

Universe scale. That is why it is desirable to have the same explanation for both scales and not involve additional reasons.

The efforts of the theoreticians aimed at the solution of the problem have two directions. The first is the construction of a theory of the hypothetical elementary particles forming the dark matter. The second suggests modifying the existing theory of space-time and gravitation in such a way that there is no need for the extra type of matter. For any change of the theory the natural test is the preservation of the existing phenomenology, particularly, Newton gravity law for the Solar system scale and two other GRT effects following from the Schwarzschild's solution.

Let us now briefly mention some approaches belonging to the second direction.

I. The most straightforward approach is the successive complication of the quadratic expression for the Einstein-Hilbert action

$$S_{EH} = -\frac{c^3}{16\pi G}\int d^4x(-g)^{1/2}R^{\alpha}{}_{\alpha} \tag{2}$$

with account to the metric terms of the higher orders. For example [10],

$$S_{W_1} = -\frac{c^3}{16\pi G}\int d^4x(-g)^{1/2}(R^{\alpha}{}_{\alpha})^2 \tag{2a}$$

or

$$S_{W_2} = -\frac{c^3}{16\pi G}\int d^4x(-g)^{1/2}R_{\alpha\beta}{}^{\alpha\beta} \qquad (2b)\,.$$

The corrections due to Eqs. (2.a) or (2.b) must give a negligibly small contribution to the Schwarzschild's solution. Besides, already this approach makes it possible to regard the cosmological constant in a way Einstein tried to do it himself.

II. Another natural approach is the introduction of an additional macroscopic gravitational field, S, usually the scalar one. Fort example [11]

$$S_{BD} = -\frac{c^3}{16\pi G}\int d^4x(-g)^{1/2}\Big(SR^{\alpha}{}_{\alpha} - w\frac{S_{;\mu}S^{;\mu}}{S}\Big), \qquad (3)$$

where w is a constant.

III. The third approach is the increase of the number of the space-time dimensions with the subsequent transfer to the Planck's scale of lengths. The corresponding works began from [12] and then lead to the mathematically developed modern theories of strings [13] and then of branes [14].

Let us now mention the approaches providing not only the specification of the already existing structures in order to get the solution that is closer to the observations, but the approaches aimed at the revision of the structures themselves presumably giving the same result.

IV. The classical foundation can be also revised. The MOND phenomenological approach (MOdified Newton Dynamics) was suggested in [15] to introduce the new world constant with the dimension of an acceleration

$$\mu\Big(\frac{a}{a_0}\Big)\vec{a} = \vec{f} \qquad \text{or} \qquad \vec{a} = \nu\Big(\frac{f}{a_0}\Big)\vec{f}\,. \qquad (4)$$

It was suggested to find such functions $\mu(x)$ or $\nu(x)$ and such value of a_0 that they match the classical result for the Solar system scale and give Eq.(1) for the galaxy scale. The relativistic generalization of MOND was performed in [16] where the scalar field ψ was introduced to give an additional term to the expression of Einstein-Hilbert action in the form

$$S(\psi) = -\frac{1}{8\pi GL^2}\int d^4x(-g)^{1/2}f(L^2g^{\alpha\beta}\psi_{;\alpha}\psi_{;\beta}), \qquad (5)$$

(f is a scalar function, L is constant). After that the MOND theory can not be regarded as a pure phenomenology. Naturally, this approach gives a good fit for the observed rotation curves described by Eq. (1).

In fact, it does not matter if we speak about the dark matter or a scalar field in the gravitation theory, or about the ether in electrodynamics - in both cases the object of discussion acts on observable bodies but can not be detected itself. But the same can be said about the geometry of the world. The principal idea of relativity stemming from Lobachevsky's work and formulated by Einstein is that one should not oppose gravitation and geometry but regard them in the non-separable connection.

V. The geometry of space-time can be also modified. The rejection of symmetry in metric's indices [17, 18] can also lead to the suitable description of the rotation curves while dark matter is not needed.

VI. Already in 1918 G.Weyl stepped aside from the Riemannian geometry suggested by Einstein in order to unify gravitation and electromagnetism with the help of metrics. He suggested the transformations of the following form

$$\begin{aligned} g_{\mu\nu}(x) &\to e^{2\alpha(x)} g_{\mu\nu}(x) && (a) \\ A_\mu(x) &\to A_\mu(x) - e\partial_\mu \alpha(x) && (b) \end{aligned} \qquad (6)$$

Here the gravitation and electromagnetism are united by the common function $\alpha(x)$, and this leads to the new – Weylian – geometry. The equations that can be obtained in this approach do not give the regular Einstein equations; nevertheless, they contain the Schwarzschild's solution for the Solar system scale. Weyl called Eq.(6.1) the gauge transformation, i.e. dependent on scale, but later this term was adopted by the other fields of physics mainly for the cases when the exponent was imaginary. In gravitation theory such transformations are now called conformal.

VII. The further evolution of these ideas lead to the theories of conformal gravitation where the metrics has an additional symmetry, corresponding to Eq. (6.1), the electromagnetic variables are not involved and this means that the geometry remains Riemannian. Formally such approach is analogous to I, but the choice of coefficients in Eqs. (2.a) and (2.b) is specific. The Einstein equations that appear in this approach are [20]

$$4\alpha_g W^{\mu\nu} = 4\alpha_g (2C^{\mu\lambda\nu\kappa}{}_{;\,\lambda;\,\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}) = T^{\mu\nu}, \qquad (7)$$

where α_g is a dimensionless constant, $C^{\mu\lambda\nu\kappa}$ is the so called Weyl tensor which doesn't change with transformations Eq. (6.1). Then they change [20]

$$\begin{aligned} W^{\mu\nu}(x) &\to e^{-6\alpha(x)} W^{\mu\nu}(x) \\ T^{\mu\nu}(x) &\to e^{-6\alpha(x)} T^{\mu\nu}(x), \end{aligned}$$

transform the coordinates with the use of a certain function $B(r)$ and introduce the source function $f(r)$. As a result the stationary version of Eq. (7) gives the Poisson equation but not of the second order as usual, but of the fourth order

$$\nabla^4 B(r) = f(r). \qquad (8)$$

If there is a spherical symmetry, the Eq. (8) has an exact solution. And this solution not only contains the term corresponding to the Newton-Schwarzschild's solution but also the terms corresponding to Eq. (1).

$$\begin{aligned} B(r>R) = -g_{00} = 1 - \frac{2\beta}{r} + \gamma r \\ 2\beta = \frac{1}{6}\int\limits_0^R dr' r'^4 f(r'); \quad \gamma = \frac{1}{2}\int\limits_0^R dr' r'^2 f(r') \end{aligned} \tag{9}$$

Solid lines on Fig. 1 correspond to the results of the conformal gravitation approach to the galactic rotation curves. The fits are good. The described approach does not need the introduction of the additional (dark) matter, i.e. the additional scalar field. Instead it uses another choice of the scalar function when formulation of the variation principle. This preserves the Riemannian geometry of space-time but leads to the Einstein equation of the form of Eq. (7) which by the way does not have the structure of the wave equation for the empty space. This means that the GW do not exist, and the effect described in [1] which coincide with the prediction of the traditional GRT within accuracy of 2% must be explained in some other way.

The material discussed in this Section suggests the following conclusion. The successful modifications of the theory that correlate with the experimental data point at the possible existence of the additional terms in the gravitation law, their role depending on the chosen scale. Preserving Riemannian geometry one has to chose one of the following:

- either to search for an additional - dark - matter, located at the periphery of a galaxy;
- or to describe the gravitation on the scale of a galaxy using another scalar when formulating the variation principle for the action.

2. Finslerian geometry of the anisotropic space-time

Apart from the scale, one has to pay attention to another important thing. The data present on Fig. 1 and those analogous to them do mainly concern the spiral galaxies that have expressed (space) anisotropy. But the notion of an isotropy could be regarded in a broader sense. The generalization of the GRT for the anisotropic space-time in which, for example, the light speed varies and depends on the direction, was performed in [21] where the theory is based on Finsler geometry. The metrics in Finsler geometry depends not only on the coordinate of a point (x^α) as in Riemannian geometry, but on a certain tangent vector too, $(\dot{x}^\alpha) = \frac{dx^\alpha}{dt}$ (t is a parameter). Usually [22] this metrics is presented as

$$g_{\mu\nu}(x,\dot{x}) = \frac{1}{2}\frac{\partial^2 F^2(x,\dot{x})}{\partial \dot{x}^\mu \partial \dot{x}^\nu}, \tag{10}$$

where $F(x,\dot{x})$ is a smooth, scalar, homogeneous of the first order, positive function with determinant $\det|\frac{\partial^2 F^2(x,\dot{x})}{\partial \dot{x}^\mu \partial \dot{x}^\nu}| \neq 0$. One of the principal results obtained in [21] is the proof that the analogues of Einstein equation in Finsler case (for various metrics) have Schwarzschild's solutions. It is also shown that within the same accuracy of measurements performed in the Solar system, it is impossible to distinguish these solutions from those of the GRT. Two other effects (the light bending when passing close to the Sun and the red shift) are present both in Riemann and Finsler cases though for different reasons. That is why these rffects can't be used to make a justified choice of geometry to describe the real space-time.

Finsler geometry can be involved into the traditional approach by the use of a special metrics in tangent space. This metrics consists of two parts one of which can depend not only on the coordinate but on the vector direction as well. If one performs a conformal transformation with the "horizontal" part, the corresponding corrections of the "vertical" part would affect the Einstein equations. In this case they present a system of equations for the corresponding tensors [23].

There are several additional reasons to turn to a special case of Finsler geometry - to the spaces with the Berwald-Moor metrics which corresponds to

$$F(y) = \sqrt[4]{y^1 y^2 y^3 y^4}. \tag{11}$$

In [24–26] it is shown that the well-known (physical) problem of the spontaneous symmetry break in the fermion-antifermion condensate corresponds to the (geometrical) partial or complete isotropy break of the space-time if its metrics can be described as

$$\begin{aligned} ds' = {} & (dx_0 - dx_1 - dx_2 - dx_3)^{(1+r_1+r_2+r_3)/4}(dx_0 - dx_1 + dx_2 + dx_3)^{(1+r_1-r_2-r_3)/4} \\ & \cdot(dx_0 + dx_1 - dx_2 + dx_3)^{(1-r_1+r_2-r_3)/4}(dx_0 + dx_1 + dx_2 - dx_3)^{(1-r_1-r_2+r_3)/4}, \end{aligned} \tag{12}$$

Here the non-dimensional parameters r_i characterize the rate of anisotropy. If we take the simplest case $r_i = 0$ and introduce the new coordinates $\xi_i = A_{ij}x_j$, where

$$A_{ij} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \tag{13}$$

then, the interval (12) takes the form

$$ds' = ds_{BM} = \sqrt[4]{d\xi_1 d\xi_2 d\xi_3 d\xi_4}\,. \tag{11a}$$

The difference of this approach from the standard theory is the following: the spontaneous symmetry break is accompanied not by the appearance of the cosmological constant, but by the appearance of the space-time anisotropy.

The similar expression for the metrics which factually uses the notion of a volume was used in [27] to construct the theory of gravitation. In [28] independently of [21] there was obtained the conclusion that it is impossible to observe the effects pointing at the difference in metrical properties between Riemann space-time and Finsler space-time on the Solar system scale.

In [29] and the subsequent series of papers the Berwald-Moor metrics is connected with the fundamental mathematical properties of the little known number-like object – hyper complex numbers H_4. The use of the H_4 or other algebra of the kind might lead to the change in the description of phenomena not only in mega scale but on a micro scale of quantum phenomena, and this has a ring with the ideas [30].

It should be underlined that though there are certain promising perspectives in the theory dealing with Finsler geometry, the connection of this direction with observations is insufficient. Moreover, the experiment that could make it possible to judge upon the geometrical properties of space-time has not been suggested up to now.

3. Optic-metrical parametric resonance

As it was mentioned in the previous Section, the experiments dealing with static case don't suite, that's why the GW were suggested as a proper effect to study the space-time geometrical properties. But all the methods suggested up to now to detect the GW (eighteen in number [31]) deal with the registration of the GW effects as the first order perturbations. For the Solar system it means the accuracy of 10^{-24} which is not yet achieved in spite of long lasting efforts and expensive projects. And even in case of success, the extremely small value of the supposed effect would give a small confidence in the results while the problems of registering and processing would be hard to overcome if one intends to use this effect for further investigations.

Let us take the semi-classical model to describe the interaction between the atom and the electromagnetic field which is well known in theoretical spectroscopy [32]. We are going to apply it to describe the action of the GW on the atom of a space maser.

Let us first regard a two-level atom in the monochromatic quasi-resonant strong field with frequency, Ω, which is close to the atomic frequency ω. "Strong" field means that the stimulated transitions dominate. This system is described in terms of the density matrix on the one hand, and the field is described classically, on the other hand. As a result we get a system of Bloch equations for the density matrix components

$$\begin{aligned}
&\frac{d}{dt}\rho_{22} = -\gamma\rho_{22} + 2i\alpha_1\cos(\Omega t - k_1 y)(\rho_{21} - \rho_{12}) \\
&\Big[\frac{\partial}{\partial t} + v\frac{\partial}{\partial y}\Big]\rho_{12} = -(\gamma_{12} + i\omega)\rho_{12} - 2i\alpha_1\cos(\Omega t - k_1 y)(\rho_{22} - \rho_{11}) \\
&\rho_{22} + \rho_{11} = 1
\end{aligned} \tag{14}$$

Here ρ_{22}, ρ_{11} are the level populations; ρ_{12}, ρ_{21} are the polarization terms; γ, γ_{12} are the longitudinal and transversal decay rates (if the lower level is the ground one, then $\gamma_{12} = \gamma/2$); $\alpha_1 = \frac{\mu E}{\hbar}$ is the Rabi parameter proportional to the intensity of the electromagnetic wave (EMW); μ is the dipole momentum; E is the electric stress; $\hbar = 1.05 \cdot 10^{-27}$ erg·s is Planck's constatnt; k_1 is the wave vector of the EMW; v is the component of the atom velocity along the Oy-axis; $\gamma \ll \alpha_1$ is the condition of the strong field.

In the series of papers [33–36] the phenomenon of the optic-mechanical parametric resonance was theoretically investigated. If a component of the velocity of such a two-level atom parallel to the wave-vector of the field varies periodically with time at frequency related to the Rabi frequency, then the scattered radiation obtains the so called non-stationary component at the frequency close to the frequency of the atomic transition. In other words, the signal at this frequency will be periodically amplified and attenuated with the frequency of the mechanical oscillations of the atom. This effect is due to the redistribution of the energy between the frequencies due to the parametric resonance. In regular observations this signal can't be registered because of the time averaging, but if a special device known in spectroscopy as a gate detector is used, or the signal is registered in time domain and then processed in a special way, then the non-stationary component can be detected and measured. It turns out that the amplitude of such signal is comparable to the height of the regular peak characterizing the interaction between the atom and the resonant field. That is this signal is large.

Turning to the investigation of the astrophysical system, we see that the sources of the monochromatic EMW are known in space. These are the space masers whose atoms are in the ground states and the transitions take place from the metastable levels, i.e. in this case they fit the two-level model. The saturated space masers realize the conditions of the strong field. On the other hand, we can suppose that there exists the reason due to which the distance and consequently the atom velocity component in the direction at the detector on the Earth would periodically change. This reason is the action of the periodical GW emitted by a pulsar located as shown on Fig. 2.

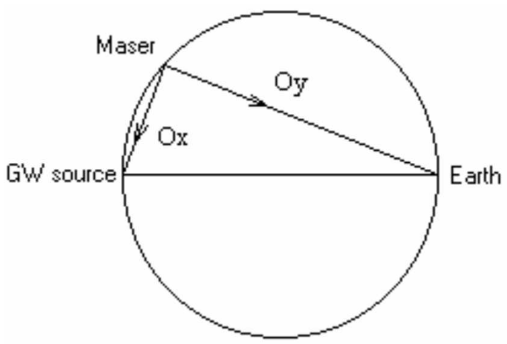

Figure 2:

The GW acts on the atomic levels, on the EMW of the maser and on the atom location. In [37] it was shown that the first effect is negligibly small in comparison with the other two. The action of the GW on the monochromatic EMW is accounted for when solving the eikonal equation

$$g^{ik}\frac{\partial\psi}{\partial y^i}\frac{\partial\psi}{\partial y^k}=0\,. \tag{15}$$

The law of the atomic motion must be obtained from the solution of the geodesic equation

$$\frac{d^2x^i}{ds^2}+\Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds}=0, \tag{16}$$

(and not from the solution of the geodesic declination equation as in the corresponding calculations for the relative displacement of the parts of the laboratory set up). Equations (14-16) describe the behavior of the two-level atom of the saturated space maser in the field of the GW. Solving them and demanding the conditions of the parametric resonance to be fulfilled, we can calculate the signal. This effect is of the zero order and its detection on the Earth is possible with the help of the already existing radio telescopes that are able to detect the space maser signal.

Such experiment can be used to investigate the space-time geometrical properties in the following way. The theoretical expressions that must be compared to the experiment results should be obtained with the help of the various suggestions about the space-time geometry. The suggestion that gives the best fit with the experimental results will correspond to the geometrical properties of real space-time.

4. Isotropic perturbation of the Minkowsky metrics

Let us consider the geometry to be Riemannian and use the regular Einstein equations in the approximation of the weak field far from masses $g^{ik}=g^{(0)ik}+h^{ik}$. The corrections to the metric tensor of the flat space-time suffice the wave equation. In the simplest case for the plane waves it has the form

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)h^k{}_i=0\,. \tag{17}$$

The solution can be naturally taken as [38]

$$h^k{}_i=Re[A^k{}_i\exp(ik_\alpha x^\alpha)]\,, \tag{17a}$$

that suffice the equation if $k_\alpha k^\alpha=0$, i.e. k^α is a light-like vector. That is why the metric tensor can be written as

$$g^{ik}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1+h\cos\frac{D}{c}(x^0-x^1)&0\\0&0&0&-1-h\cos\frac{D}{c}(x^0-x^1)\end{pmatrix} \tag{18}$$

where h is the dimensionless amplitude of the GW ($h \ll 1$), D is the frequency of the GW.

Solving Eq. (15) with regard to Eq. (18), we see that the GW leads to the phase modulation of the EMW. Since h is small, the latter can be presented [39] as a superposition

$$E(t) = E\cos(\Omega t - ky) + E\frac{\omega}{4D}h\big[\cos((\Omega - D)t - ky) - \cos\big((\Omega + D)t - ky\big)\big]\,. \quad (19)$$

The solution of Eq.(16) with regard to the Eq.(18) gives [37]

$$y(t) \sim h\frac{c}{D}\sin(Dt + k_g x), \quad (20)$$

where k_g is the wave vector of the GW. The expression Eq. (20) gives the following formula for the component of the atomic velocity in the direction of the Earth

$$\begin{aligned} v &= v_0 + v_1\cos Dt \\ v_1 &= hc \end{aligned} \quad (21)$$

Substituting Eq. (21) and Eq. (19) into Eq. (14), we get

$$\begin{aligned} \frac{d}{dt}\rho_{22} &= -\gamma\rho_{22} + 2i[\alpha_1\cos(\Omega t - ky) + \alpha_2\cos((\Omega - D)t - ky) - \\ &\quad -\alpha_2\cos((\Omega + D)t - ky)](\rho_{21} - \rho_{12}) \\ \frac{d}{dt}\rho_{12} &= -(\gamma_{12} + i\omega)\rho_{12} - 2i[\alpha_1\cos(\Omega t - k_1 y) + \alpha_2\cos((\Omega - D)t - ky) - \\ &\quad -\alpha_2\cos((\Omega + D)t - ky)](\rho_{22} - \rho_{11}) \\ \rho_{22} + \rho_{11} &= 1 \end{aligned} \quad (22)$$

where $\alpha_2 = \frac{\omega h}{4D}\alpha_1$, and where the relation (21) was taken into account in the expression for the full time derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + kv$. The system (22) can be solved by the asymptotic expansion method. If certain conditions on the parameters are fulfilled, we can speak of the optic-metrical parametric resonance (OMPR). These conditions formulated in [37, 39] have the form:

- The EMW is spectroscopically strong

$$\frac{\gamma}{\alpha_1} = \Gamma\varepsilon; \quad \Gamma = O(1); \quad \varepsilon \ll 1\,. \quad (22.1)$$

- The amplitude condition of the OMPR related to the trichromatic field

$$\frac{\alpha_2}{\alpha_1} = \frac{\omega h}{4D} = a\varepsilon; \quad a = O(1); \quad \varepsilon \ll 1\,. \quad (22.2)$$

- The amplitude condition of the OMPR related to the periodic change of the atomic velocity

$$\frac{kv_1}{\alpha_1} = \frac{\omega h}{\alpha_1} = \kappa\varepsilon; \quad \kappa = O(1); \quad \varepsilon \ll 1\,. \tag{22.3}$$

- The frequency condition of the OMPR

$$(\omega - \Omega + kv_0)^2 + 4{\alpha_1}^2 = D^2 + O(\varepsilon) \quad \Rightarrow \quad D \sim 2\alpha_1\,. \tag{22.4}$$

If the conditions (22.1–22.4) are fulfilled, then solving Eqs. (22) by the asymptotic expansion method for the small parameter, ε, we get the principal term of the expansion for $Im(\rho_{21})$ which characterizes the scattered energy flow. At the frequency shifted by D from the central peak, the flow is proportional to ε^0 and has the form

$$Im(\rho_{21}) \sim \frac{\alpha_1}{D}\cos 2Dt + O(\varepsilon). \tag{23}$$

The negative values correspond to the amplification, the positive values correspond to the attenuation of the energy flow at the mentioned frequency due to the redistribution of the energy of the maser radiation in the conditions of the OMPR. Similarly to [33–36], in the regular observations of the space maser signal it is impossible to observe the non-stationary component because of the time averaging, but the use of the gate detector, or the appropriate processing of the signal in the time domain would provide the observable OMPR signal. This result means that the GW whose existence follows from the GRT and is indirectly supported in [1] could be observed in the direct way with the help of the OMPR based method. More detailed discussion and the analysis of the feasibility of the OMPR conditions for the real astrophysical systems are given in [37, 39]. Here we will only mention that if this type of a signal is detected in the process of purposeful observations, the reason for it can be undoubtedly identified as the GW.

If the GW emitted by the pulsars and the short-period doubles do exist, the OMPR based method can become the foundation of the gravitational astronomy for the inner region of the Milky Way disk. Appendix 1 contains the coordinates of the astrophysical systems suitable for observations both for the galactic vicinity of the Sun and for the periphery of our galaxy (see pulsar 3) which also belongs to the class of spiral galaxies.

But it could happen that the signal in the proposed experiment would be absent or would differ from the predicted one for some of the observation points. This will mean that some essential factors were not taken into account. And the space-time geometrical properties are among these factors.

5. Anisotropic perturbation of the Minkowsky metrics

All the calculations leading to Eq.(17) could be repeated, if we change the expression for the metrics to

$$g_{ij}(x) \ \rightarrow \ g_{ij}(x,\dot{x}) = \eta_{ij}(x) + h_{ij}(x,\dot{x}), \tag{24}$$

where

$$\eta_{ij}(x) = \eta^{(0)}{}_{ij}(x) \tag{25}$$

is the Minkowsky metrics for the flat space, $h_{ij}(x,\dot{x})$ is a small perturbation such that $h^k{}_i(x,\dot{x}) = \eta^{(0)kj}h_{ij}(x,\dot{x})$. The structure of Einstein equations will remain the same and the perturbation will still suffice the wave equation similar to Eq. (17). But the expression Eq. (17.1) will look like

$$h^k{}_i(x,\dot{x}) = Re\big[A^k{}_i(\dot{x})\exp(ik_\alpha x^\alpha)\big]\,. \tag{26}$$

This means that the amplitude of the GW will vary in various directions of their propagation. From the point of view of observations based on the OMPR method this difference can not be observed directly since the effect is of the zero order. But it will reveal itself in the indirect way, for example, the conditions of the OMPR will be sufficed at different distances in different directions from one and the same GW source. Then the conditions (22.2) and (22.3) will transform to the following

- The amplitude condition of the OMPR related to the trichromatic field

$$\frac{\alpha_2\zeta_1(\dot{x})}{\alpha_1} = \zeta_1(\dot{x})\frac{\omega h}{4D} = a\zeta_1(\dot{x})\varepsilon; \quad a = O(1); \quad \varepsilon \ll 1\,. \tag{27}$$

- The amplitude condition of the OMPR related to the periodic change of the atomic velocity

$$\frac{kv_1\zeta_2(\dot{x})}{\alpha_1} = \zeta_2(\dot{x})\frac{\omega h}{\alpha_1} = \kappa\zeta_2(\dot{x})\varepsilon; \quad \kappa = O(1); \quad \varepsilon \ll 1\,, \tag{28}$$

Here the functions $\zeta_1(\dot{x}), \zeta_2(\dot{x})$ are related to the expressions for the amplitudes, $A^k{}_i(\dot{x})$, of the GW.

6. Investigations of the space-time properties with the help of the OMPR effect

In this Section we will analyze the possible results of the OMPR based experiment with regard to the problems mentioned in Sections 1 and 2. It was found [37, 39] that the distances between the GW sources (pulsars or doubles) and space masers are not small but are of interstellar scale. This means that one and the same GW source could affect several masers. Such a source is a kind of a beacon with the frequency now known to the eight decimal digits, while this or that maser is a receiver. The Milky Way scale experiment should be performed in the following way. Let us chose the GW sources in various places of our galaxy and regard several masers together with each of them paying attention to the conditions (22.1–22.4). Then we will try to detect the OMPR signal according to the method described in [39] for the GW sources closer to the inner part of the galaxy (IPG) and for the GW sources closer to the periphery part of the galaxy (PPG). One may check that these experiments can give only nine possible outcomes that will have the meanings given below.

$$1.\quad \begin{array}{lcc} & I & P \\ \textit{All} & 0 & 0 \\ \textit{Some} & - & - \end{array} \Longleftrightarrow \begin{array}{ccc} I & P & \\ 0 & 0 & R \\ - & - & F \end{array}$$

Result: no OMPR signal for all the masers corresponding to the IPG GW sources, no OMPR signal for all the masers corresponding to the PPG GW sources. Interpretation: no gravitational waves (and no possibility for the GW astronomy) $\rightarrow$ Einstein equations for the empty space don't have the structure of the wave equation $\rightarrow$ no need for dark matter $\rightarrow$ Riemannian geometry suits. Problems: choice of the scalar in the variation principle, interpretation of the results in [1].

$$2.\quad \begin{array}{lcc} & I & P \\ \textit{All} & 1 & 0 \\ \textit{Some} & - & - \end{array} \Longleftrightarrow \begin{array}{ccc} I & P & \\ 1 & 0 & R \\ - & - & F \end{array}$$

Result: OMPR signal is present for all the masers corresponding to the IPG GW sources, no OMPR signal for all the masers corresponding to the PPG GW sources. Interpretation: scale dependence (possibly, conformal gravity outside the galaxy), Riemannian geometry suits, GRT in the IPG where GW astronomy is possible.

$$3.\quad \begin{array}{lcc} & I & P \\ \textit{All} & 0 & 1 \\ \textit{Some} & - & - \end{array} \Longleftrightarrow \begin{array}{ccc} I & P & \\ 0 & 1 & R \\ - & - & F \end{array}$$

Result: no OMPR signal for all the masers corresponding to the IPG GW sources, OMPR signal is present for all the masers corresponding to the PPG GW sources. Interpretation: scale dependence (possibly, conformal gravity in the IPG), Riemannian geometry suits, GRT outside the galaxy where GW astronomy is possible.

$$4.\quad \begin{array}{lcc} & I & P \\ \textit{All} & 1 & 1 \\ \textit{Some} & - & - \end{array} \Longleftrightarrow \begin{array}{ccc} I & P & \\ 1 & 1 & R \\ - & - & F \end{array}$$

Result: OMPR signal is present for all the masers corresponding to the IPG GW sources, OMPR signal is present for all the masers corresponding to the PPG GW sources. Interpretation: Riemannian geometry suits, GRT works and GW astronomy is possible. Problems: dark matter problem.

The rest corresponds to the situation when we have to use Eqs.(27, 28) instead of Eqs.(22.2, 22.3), that is only some of the selected masers behave as they should when the OMPR conditions are fulfilled. This will point at the anisotropy effects mentioned in Section 5.

5.
$$\begin{array}{lcccccc} & I & P & & I & P & \\ \mathit{All} & - & - & \Longleftrightarrow & - & - & R \\ \mathit{Some} & 1 & 1 & & 1 & 1 & F \end{array}$$

Result: OMPR signal is present for some of the masers corresponding to the IPG GW sources, OMPR signal is present for some of the masers corresponding to the PPG GW sources. Interpretation: Finslerian geometry suits, GW astronomy is possible. Problems: dark matter problem.

6.
$$\begin{array}{lcccccc} & I & P & & I & P & \\ \mathit{All} & - & - & \Longleftrightarrow & - & - & R \\ \mathit{Some} & 1 & 0 & & 1 & 0 & F \end{array}$$

Result: OMPR signal is present for some of the masers corresponding to the IPG GW sources, no OMPR signal for all the masers corresponding to the PPG GW sources. Interpretation: scale dependence (possibly, conformal gravity outside the galaxy), Finslerian geometry suits in the IPG where GW astronomy is possible.

7.
$$\begin{array}{lcccccc} & I & P & & I & P & \\ \mathit{All} & - & - & \Longleftrightarrow & - & - & R \\ \mathit{Some} & 0 & 1 & & 0 & 1 & F \end{array}$$

Result: no OMPR signal for all the masers corresponding to the IPG GW sources, OMPR signal is present for some of the masers corresponding to the PPG GW sources. Interpretation: scale dependence (possibly, conformal gravity in the IPG), Finslerian geometry suits in the PPG where GW astronomy is possible.

8.
$$\begin{array}{lcccccc} & I & P & & I & P & \\ \mathit{All} & 1 & 0 & \Longleftrightarrow & 1 & 0 & R \\ \mathit{Some} & - & 1 & & 0 & 1 & F \end{array}$$

Result: OMPR signal is present for all the masers corresponding to the IPG GW sources, OMPR signal is present for some of the masers corresponding to the PPG GW sources. Interpretation: Riemannian geometry suits in the IPG, Finslerian geometry suits in the PPG, GW astronomy is possible. Problems: dark matter problem.

$$9. \quad \begin{array}{rcc} & I & P \\ \textit{All} & 0 & 1 \\ \textit{Some} & 1 & - \end{array} \Longleftrightarrow \begin{array}{ccc} I & P & \\ 0 & 1 & R \\ 1 & 0 & F \end{array}$$

Result: OMPR signal is present for some of the masers corresponding to the IPG GW sources, OMPR signal is present for all the masers corresponding to the PPG GW sources. Interpretation: Finslerian geometry suits in the IPG, Riemannian geometry suits in the PPG, GW astronomy is possible. Problems: dark matter problem.

The coordinates of the pairs of masers corresponding to the GW sources located both in the IPG (pulsar 6) and in the PPG (pulsar 7) are given in Appendix 2.

If in the observations we find that the situations 5–9 are realized, then the systematic observations interpreted with the help of expression (26) could give function in the expression for the metrics corresponding to Eq. (10)

$$g_{ij}(x,\dot{x}) = {\eta^{(0)}}_{ij}(x) + h_{ij}(x,\dot{x}) = {\eta^{(0)}}_{ij}(x) + \frac{1}{2}\frac{\partial^2 F^2(x,\dot{x})}{\partial \dot{x}^k \partial \dot{x}^j} = {\eta^{(0)}}_{ij}(x) + {\eta^{(0)}}_{kj}(x){h^k}_i(x,\dot{x}). \tag{29}$$

Thus, if the space-time anisotropy takes place on the galactic scale, then its quantitative characteristic could be obtained in OMPR based experiment.

7. Berwald-Moor metrics

The natural continuation of this approach is the consideration of the situation when the space-time anisotropy is not a small correction as in the previous Section but is described by Finsler geometry. In accord with the experimental approach dealing with the GW described above, one should again use the small linear correction for the empty space, but the unperturbed metrics now is not the Minkowsky one, but some Finsler space metrics

$$g_{ij}(x,\dot{x}) = h_{ij}(x,\dot{x}) + \chi_{ij}(x,\dot{x}). \tag{30}$$

It seems appealing to choose the Berwald-Moor metrics for the unperturbed metrics. To speak about the OMPR effect, one should find out explicitly if the GW are possible in such a space-time and write down the corresponding correction to the metrics; then also find out how the description of the electromagnetic processes (16, 19) change and write down the geodesics equation.

One could expect that the structure of Einstein equations remains that of the wave equation and, thus, the GW would be possible though maybe become more complicated. The geodesics equation seems also to become more complicated, but its solution will still present a technical problem. But the description of the electromagnetic processes and the description of the GW-EMW interaction will present a different kind of a problem.

An essential feature discovered and underlined in [21] is the following: the notion of simultaneity which is the base of any relativistic theory might belong not to the causal structure but to the structure of Lagrangean. This remark causes a profound methodological problem. The choice of Riemann geometry for the description of space-time is closely connected with the invariance of Maxwell equations – the foundation of the majority of experiments. It was this fact that Einstein considered while formulating the relativity principle and while constructing the SRT. Rejecting Riemann geometry, we reject the Maxwell equations' invariance, and this means the appearance of the terms that have the metric origin. These we will have to interpret in frames of the known phenomenology. The analogous problem was posed in the end of [40]. The situation becomes even more complicated, if we consider the relation between the gravitation and electromagnetism both for classical GRT effects such as light bending and gravitational red shift and for the direct transformation of gravity and electromagnetism into each other [41]. Finally, we see that the transfer to Finsler geometry demands a detailed physical consideration.

Conclusion

The goal of this paper was to suggest an experiment suitable for the investigation of the space-time geometrical properties and to give the corresponding theory. The physical effect underlying such experiment is the optic-metrical parametric resonance described in Sections 3-5 and in papers [37, 39]. The possible results of the OMPR based observations analyzed in Section 6 could give an answer to the question which geometry suits best for the description of the physical space-time. Moreover, these results could also be used to choose the direction of the further fundamental research. If it turns out that Riemann geometry is suitable in the galaxy scale, then astrophysics will confront either the problem of the choice of the variation principle scalar lying in the base of the axiomatic theory, or the problem of the dark matter which has to be solved in frames of the elementary particles theory (and corresponding experiment). In the last case the GW astronomy can appear and be developed. If it turns out that the geometry must be modified and, for example, must become Finsler one, then instead of the mentioned problems the foundations of the electrodynamics must be carefully examined, and this might have far going consequences on all the levels from quantum mechanics to cosmology.

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Appendix 1

Coordinates and parameters of the astrophysical systems suitable for the OMPR based detection of the GW [42–43]

	Name	RaJ	DecJ	d (pc)	D (Hz)
1. Pulsar	J1022+1001	10:22:58.006	$+10°01'52.8''$	300	60.7794489280
Maser	AF Leo	11:25:16.4	$+15°25'22''$	270	
2. Pulsar	B0656+14	06:59:48.134	$+14°14'21.5''$	290	2.59813685751
Maser	U ORI	05:52:51.0	$+20°10'06.0''$	280	
3. Pulsar	J0538+2817	05:38:25.0632	$+28°17'9.07''$	1770	6.9852763480
Maser	HH 4	05:37:21.8	$+23°49'24.0''$	1700	
4. Pulsar	B0031-07	00:34:08.86	$-07°21'53.4''$	720	1.0605004987
Maser	U CET	02:31:19.6	$-13°22'02.0''$	660	
5. Double	RXJ0806.3+1527	08:06.3	$+15°27'$	100	0.00311526
Maser	RT Vir	13:00:06.1	$+05°27'14''$	120	

Appendix 2

Coordinates and parameters of the astrophysical systems suitable for the OMPR based detection of the GW (space-time anisotropy test) [42-43]

	Name	RaJ	DecJ	d (pc)	D (Hz)
6. Pulsar	J1908+0734	19:08:17.01	$+07°34'14.36''$	580	4.70914721426
Maser-1	IRC+10365	18:34:59.0	$+10°23'00.0''$	500	
Maser-2	RT AQL	19:35:36.0	$+11°36'18.0''$	530	
7. Pulsar	J0205+6449	02:05:37.92	$+64°49'42.8''$	3200	15.223855772
Maser-1	IRAS00117+6412	00:11:44.6	$+64°12'04.0''$	3170	
Maser-2	W3 (1)	02:21:40.8	$+61°53'26.0''$	3180	

References

[1] R. A. Hulse, J. H. Taylor, 1975, Astrophys J., 195, L-51
[2] M. S. Roberts and R. N. Whitehurst, Astrophys. J. 201 (1975) 327
[3] A. Bosma, Astronom. J. 86 (1981) 1791
[4] K. G. Begeman, Astronom. and Astrophys. 223 (1989) 47
[5] K. G. Begeman, A. H. Broeils, R. H. Sanders, Mon. Not. R. Astron. Soc. 249 (1991) 523
[6] A. G. Reiss et. al., Astrophys. J. 560 (2001) 49

[7] A. G. Reiss et. al., Astrophys. J. 607 (2004) 665
[8] P. D. Mannheim, Prog.Part.Nucl.Phys. 56, p. 340, (2006)
[9] P. D. Mannheim, Astrophys. J. 479 (1997) 659
[10] B. S. DeWitt, *Relativity, Groups and Topology,* Gordon and Breach New York (1964)
[11] C. Brans and R. H. Dicke, Phys. Rev. 124 (1961) 925
[12] Th. Kaluza, Sitzungsber, d. Berl. Akad., 1921, p. 966
[13] A. V. Marshakov, Uspekhi Fiz.(in Russian), 172, p. 977, 2002
[14] L. Randall, and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690
[15] M. Milgrom, Astrophys. J. 270 (1983) 384
[16] J. D. Bekenstein, and M. Milgrom, Astrophys. J. 286 (1984) 7
[17] B. N. Kursunoglu, Jour. Phys. Essays 4 (1991) 439
[18] J. W. Moffat, Phys. Lett. B 355 (1995) 447
[19] H. Weil, Sitzsungber. Preuss. Akad. Wiss. (1918) 465
[20] P. D. Mannheim, and D. Kazanas, Gen. Rel. Gravit. 26 (1994) 337
[21] R. I. Pimenov. *Anisotropic Finslerian Generalization of the Relativity Theory as an Order Structure.* Syktyvkar, 1987 (in Russian).
[22] H. Rund, *The Differential Geometry of Finsler Spaces,* Springer, Berlin, 1959.
[23] V. Balan, N. Brinzei, Hypercomplex Numbers in Geometry and Physics, 2005, 4, 114
[24] G. Yu. Bogoslovsky, Doklady (in Russian), 1973, 213, p. 1055
[25] G. Yu. Bogoslovsky, H. F. Goenner, Phys. Lett. A, 2004, v. 323, 40
[26] G. Yu. Bogoslovsky, Hypercomplex Numbers in Geometry and Physics, 2005, 4, 27
[27] G. S. Asanov, Vestnik MSU, 1976, v. 17, p. 288
[28] G. S. Asanov, *Appendices.* In: H. Rund, *The Differential Geometry of Finsler Spaces,* Springer, Berlin, 1959.
[29] D. G. Pavlov, Hypercomplex Numbers in Geometry and Physics, 2004, v. 1, p. 5
[30] P. Rowlands, Proc. Conf. PIRT-06, London, 2006
[31] V. O. Gladyshev, *Irreversible Electromagnetic Processes in the Astrophysical Problems.* Bauman MSTU, 2000
[32] S. Stenholm, 1984, *Foundations of Laser Spectroscopy.* (New York: Wiley)
[33] S. V. Siparov, Phys. Rev. A 55, 3704, 1997
[34] A. Ya. Kazakov, S. V. Siparov, Opt. i Spektrosk. (in Russian), 83, 961, 1997
[35] S. V. Siparov, J. Phys. B 31, 415, 1998
[36] S. V. Siparov, J. Phys. B 34, 2881, 2001
[37] S. V. Siparov, 2004, Astron. & Astrophys., 416, 815
[38] E. Amaldi, G. Pizzella, 1979, in Astrofisika e Cosmologia Gravitazione Quanti e Relativita, (Firenze, Guinti Barbera)
[39] S. V. Siparov, Proc. Conf. PIRT-06, London, 2006
[40] S. V. Siparov, Hypercomplex Numbers in Geometry and Physics, 2005, 4, 51.
[41] M. E. Hertsenshtein, JETP (in Russian), v.41, p. 113, 1961
[42] *The ATNF Pulsar Database* http://www.atnf.csiro.au/research/pulsar/psrcat/
[43] *Galactic H2O masers (Palagi+ 1993)* http://vizier.u-strasbg.fr/cgi-bin/Vizier?-source=3DJ/A+AS/101/153

Experimental Investigation of Spinning Massive Body Influence on Fine Structure of Distribution Functions of α-Decay Rate Fluctuations

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The present investigation is dedicated to study of physical basis of macroscopic fluctuations effect [1]. In particular experimental investigation of possible influence of rapidly spinning massive body on distribution function of the α-decay rate fluctuations was carried out. Possible anisotropy of such influence was tested. The paper also contains fundamentals of the macroscopic fluctuations effect, method of experimental data processing and short review of phenomenology collected during more than fifty-years history of the macroscopic fluctuation effect investigations.

1 Fundamentals of macroscopic fluctuations effect. Method of experimental data processing

To understand the essence of macroscopic fluctuation effect let us consider a simple example. Suppose an electrical direct-current circuit. Also suppose that we provide a set of consecutive measurements of the current value, every time with more and more sensitive device. Then, at some point during such measurements we will be able to see that measured value (which was a constant at the beginning) is subjected to some fluctuations. Apparently obtaining fluctuations is possible this way in practically any process. Time series of fluctuations obtained in different processes are basic raw data for investigation of macroscopic fluctuations effect. Below we consider a method of experimental data processing, which is the basis for further investigations of macroscopic fluctuations effect.

This method can be divided into two stages. The first one is illustrated at the Fig. 1. Here Fig. 1A presents initial time series of fluctuations of some process. These initial time series are divided onto short intervals ordinarily of 30–60 points in length, Fig. 1B. For every such interval a histogram (distribution function of fluctuating values) is calculated, Fig. 1C. After this we smooth every histogram by w-points rectangular windows, Fig. 1D. The most often value of w is $w = 4$. As a result of the first stage of procedures applied to initial time series, Fig. 1A, we obtain a set of smoothed histograms, Fig. 1D. These histograms are subject of subsequent data processing procedure illustrated by Fig. 2 where the result of the second stage of data processing is presented.

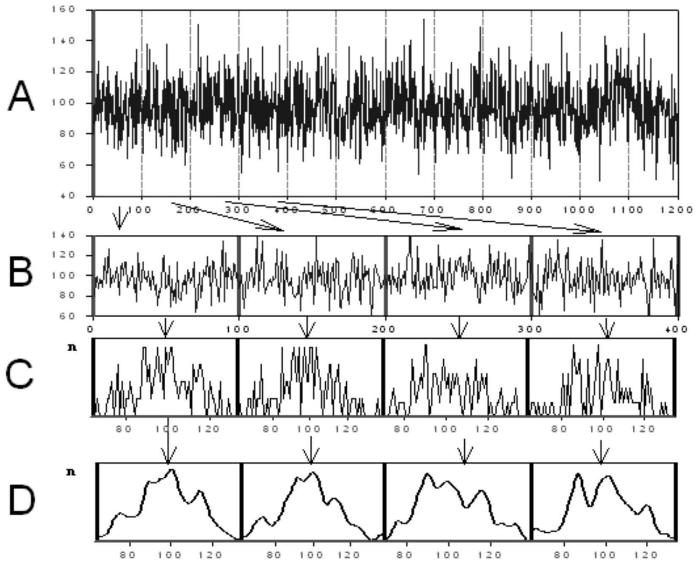


Figure 1: Processing of initial time series. The method of smoothed histograms set obtaining.

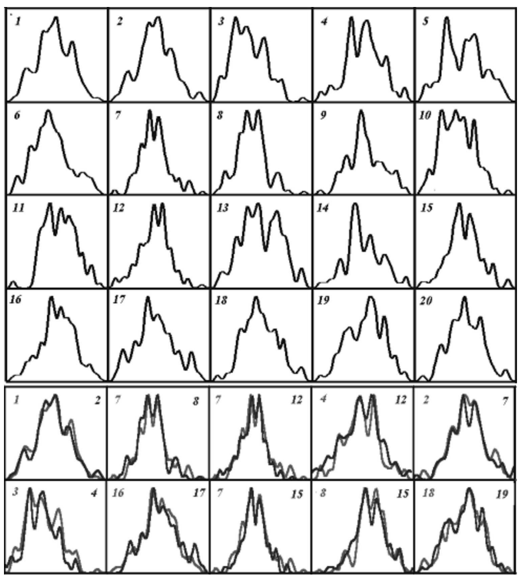


Figure 2: Initial set of smoothed histograms and pairs of similar histograms.

Upper part of Fig. 2 gives an example of $n = 20$ histograms set, which serves as initial material for the process of visual comparison of histogram pairs by an expert. This set is obtained in the same way as the set at Fig. 1D. Every histogram of this set is compared with all other histograms of this set, or some other. In case when we compare histograms in the same set we need $n(n-1)/2$ comparisons of histogram pairs. In case when we compare histograms between different sets we need n^2 comparisons. For set displayed in Fig. 2 we need 190 comparisons. Lower part of Fig. 2 presents 10 pairs histograms, which were found similar by expert.

It is possible to see from Fig. 2 that the process of histogram comparison by expert consists of evaluation of similarity of shape for histogram pairs. The process of expert histogram comparison is very sensitive to peculiarities of histograms shape. Usually the results of expert comparisons cannot be repeated by traditional methods of correlation analysis, spectral analysis, or using different measures of similarity, etc. [2]. Multiple attempts to create algorithm for automatic comparisons of histograms made clear that complete or partial automation of the process of expert histogram comparison is possible only using complex algorithms simulating some aspects of human perception, especially its whole nature.

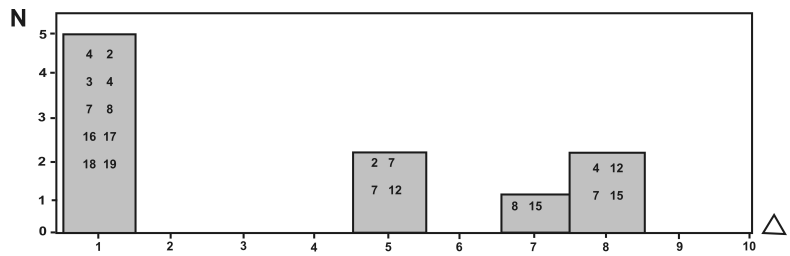


Figure 3: An example of distribution of similar histograms by intervals between them for the set of histograms presented on the fig. 2A.

The end of the second stage is construction of distribution of number of similar histograms on time intervals between them. An example of such distribution for the set of histograms at Fig. 2 is presented at Fig. 3. Interval Δ is duration of time between each two histograms in time series. Expert estimation of similarity of histograms in pair is "1" or "0" value ("yes" or "no"). If histograms are similar, then similarity is equal to "1", in opposite case similarity is equal to "0". For example, the set of $n = 20$ histograms presented in Fig. 2 have $n - \Delta = 15$ pairs of histograms separated by interval $\Delta = 5$. From all of them only pairs #2–#7 and #7–#12 are similar by expert's opinion. Consequently in the distribution, based on the set of histograms presented in Fig. 2, number of similar histograms for interval $\Delta = 5$ will be equal $N = 2$.

Construction of interval distribution graph completes processing of experimental data by expert. On the basis of this distribution all main properties of macroscopic fluctuations effect are obtained. Following chapter gives a short revue of phenomenology of the effect.

2 Basic phenomenology of Macroscopic Fluctuations Effect.

The most general result of many-years investigations of macroscopic fluctuations effect is a proof of non-randomness of fine structure of histograms shapes built on the base of short samples of time series of fluctuations of different processes of any nature – from biochemical reactions and noises in gravitational antenna to fluctuations in α-decay rate. Below we consider basic phenomenology of macroscopic fluctuations effect.

1. The Near Zone Effect

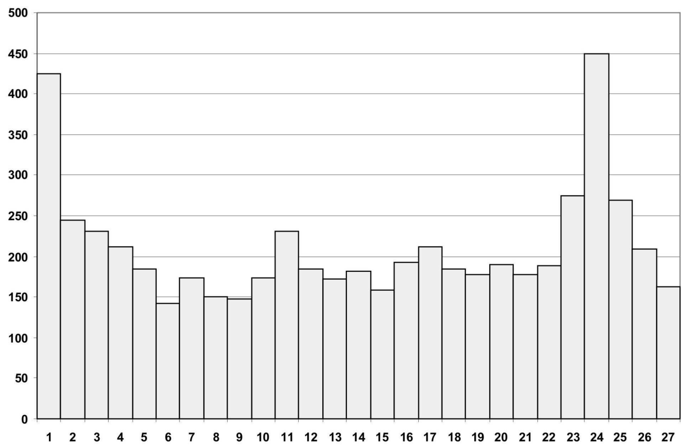
The Near Zone Effect consists in higher probability of meeting similar pair of histograms in the nearest (neighboring) non-overlapping segments of time series of the results of measurements, Fig. 4 *a*). The effect leads to the notion of 'life-time' of histogram's definite shape. But at the present day it is not possible to point out time interval during which the shape of histogram is still invariable. The Near Zone Effect was tested for time intervals from several hours to seconds. Physical meaning of such a fractality needs further investigations [1–3].

2. Universal Nature of Macroscopic Fluctuations Effect

Universal nature of Macroscopic Fluctuations Effect means that the effect is invariant in relation to the qualitative nature of the fluctuation process. The facts of similarity of fine structure of histogram's shape in processes with energies differing in many orders (for example, energy of α-decay rate fluctuations and energy of noise in gravitational antenna differ approximately in 40 orders) mean that physical nature of this similarity is non-energetic. All above mentioned also represents a quite common reason of histogram's similarity [1, 2, 4].

3. Periodical Manifestations of Macroscopic Fluctuations Effect

Important evidence of non-randomness of histogram's shape is a regular character of its changes with time. This regularity manifests itself in the following phenomena.



3.1. Near-daily periods of changes in histograms shape similarity. They consist of two well-resolvable sidereal (1436 min) and solar (1440 min) periods. Existence of the periods means dependence of histograms shape on the rotations of the Earth around its axis.

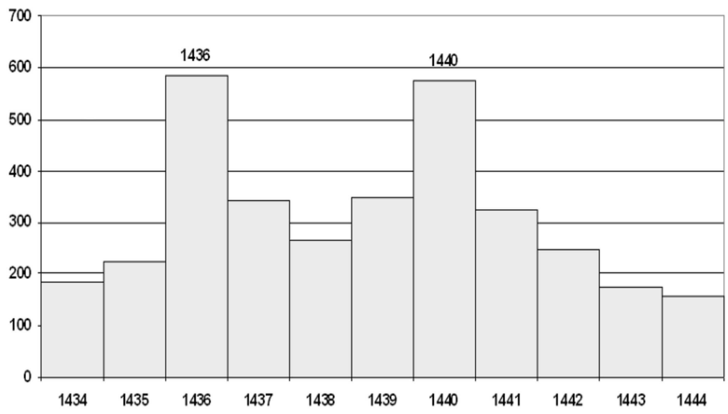


Figure 4: *a*) Example of Near Zone Effect and daily period [6], *b*) splitting of daily period on solar and star periods [7]. X-axis – time intervals between pairs of similar histograms, *a*) – hours, *b*) – minutes; Y-axis – number of similar histogram pairs found by expert.

3.2. Near-27-days periods of changes in histograms shape similarity. The periods probably mean dependence of histograms shape on the relative position of Earth, Moon, Sun and, probably, the planets [7].

3.3. Yearly periods of changes in histograms shape similarity. They consist of solar (365 solar days) and sidereal (365 solar days, 6 hours, 9 minutes) periods [8].

All above-mentioned periods mean dependence of histograms shape on the rotations of the Earth around its axis and movements of the Earth along its circumsolar orbit.

4. Local-Time Effect

Dependence of histogram's shape on the rotations of the Earth around its axis manifests itself in the local-time effect. The effect consists in synchronous changes of histogram's shape similarity for different geographical locations at the same local time. It was tested many times for different geographical locations around the Globe. It was found that the effect works for maximally possible distances (about 15000 km) between the places of measurements. For some cases absolute-time synchronism (synchronous changes of histogram's shape similarity for different geographical locations at the same moments) can be observable.

Fig. 5 presents two intervals distributions constructed on the base of time series of α-decay rate fluctuations of ^{239}Pu. The time series were obtained in Moscow region, (Pushchino, latitude 54°50′ North and longitude 37°38′ East) and in Antarctica (Novolazarevskaja station, latitude 70°02′ South and longitude 11°35′ West). The distance between the points of measurements is about 14500 km and difference in local time is 103 min.

Left side of fig. 5 presents interval distribution, which illustrates the effect of histogram similarity by absolute time. Right side of fig. 5 presents the effect of similarity by local time. It can be seen that local time effect appears more clearly [1, 2].

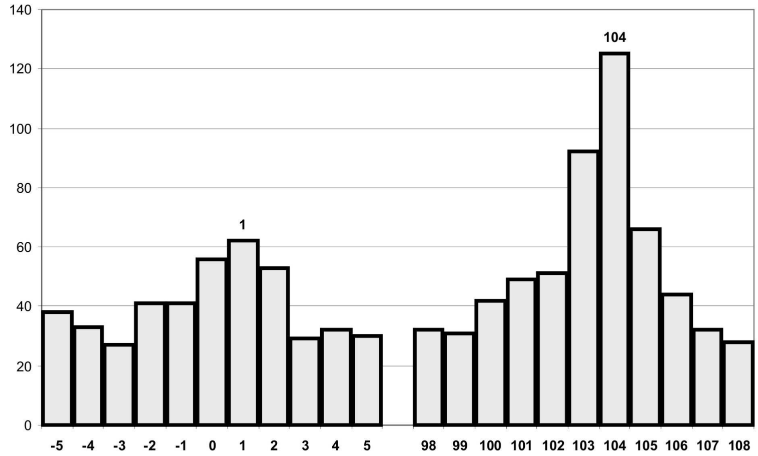


Figure 5: Synchronous changes of histogram's shape similarity in different geographical locations. Left intervals distribution presents the effect of histogram similarity by absolute time, right distribution presents similarity by local time.

5. Disappearance of daily periods for measurements near the North Pole

The dependence of histogram's shape on the Earth rotation around its axis is also revealed in disappearance of daily periods in measurements conducted close to the North Pole. Such measurements were carried out at the latitude 82° North in 2000. Near-daily periods disappeared for histograms in 15-minute and 60-minute length. But for 1-minute histograms the periods were found. For such histograms a local-time effect was also found [7].

Above-mentioned results lead to necessity of measurements as close as possible to the North Pole. Impossibility of such measurements stimulates us to start measurements with collimators cutting out a stream of α-particles at radioactive decay of ^{239}Pu. Results of these experiments radically change our understanding of macroscopic fluctuations effect [9].

6. Motionless collimator directed at the Polar Star

Measurements were taken with the collimator of α-particles directed at the Polar Star. For these measurements the near-daily periods and near zone effect was not observed. The measurements were made in Pushchino at latitude 54° North, but the effect was as would be expected at latitude 90° North, i.e. at the North Pole. This indicates the dependence of histograms shape on the direction in space. Such a dependence in its turn leads to the conclusion about anisotropy of space itself [7, 9].

7. Motionless collimators directed to the East and to the West

The conclusion about anisotropy of space was confirmed by measurements with two collimators. One of them was directed to the East; the other one to the West. In those experiments two important effects were discovered.

7.1. The histograms registered in the experiments with the East-directed collimator are similar to histograms from West-directed collimator with delay of 718 min, i.e. half of the sidereal day.

7.2. No similar histograms were observed in the simultaneous measurements with the East and West collimators. Without collimators, it is highly probable for similar histograms to appear at the same place and time. This space-time synchronism disappears when α-particles streaming in the opposite directions are counted.

These results are in agreement with the concept that the histogram shape depends on the direction of the α-particle emission i.e. with the concept of space anisotropy [10].

8. Experiments with the rotating collimators

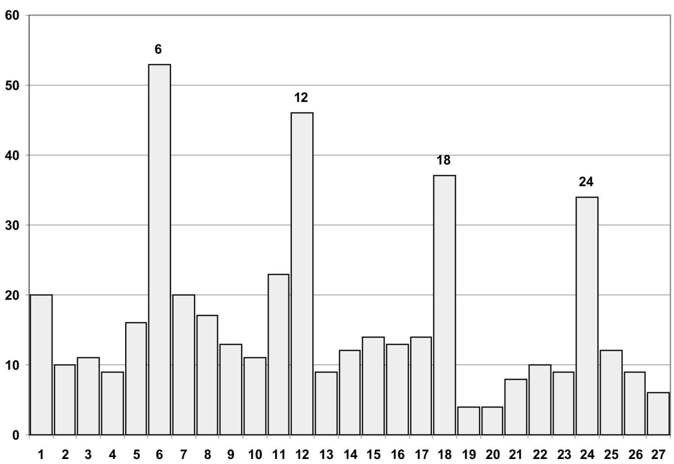


Figure 6: Interval distribution obtained on the base of 60-min histograms constructed from measurements of α-decay rate fluctuations of collimated ^{239}Pu source.

Experiments with rotating collimators were a natural development of above-mentioned investigations [11].

8.1. Collimator rotating counter-clockwise scans coelosphere with the period equal to number of collimator rotations plus one rotation made by Earth itself. The dependence of the probability of appearance similar histograms on the number of collimator rotations per day was studied. Just as expected, the probability jumps with periods equal to 1440 min divided by the number of collimator rotations per day plus 1. Examination of experimental data at $1, 2, 3, 4, 5, 6, 7, 11$ and 23 rotations per day reveals periods equal to $12, 8, 6$ etc. hours. The analysis of 1-min histograms shows that each of these periods has two extremes: "sidereal" and "solar". These results indicate that the histogram pattern is indeed determined by direction of α-particle emission [11]. An example of 6-hours period obtained with 60-min histograms at three counter-clockwise rotations of collimator per day is presented in fig. 6.

8.2 For collimator, which made 1 clockwise rotation per day, the rotation of the Earth was compensated (α-particles always emitted in the direction of the same region of the coelosphere) and, correspondingly, the daily periods disappeared. This result was completely analogous to the results of measurements near the North Pole and measurements with the immobile collimator directed towards the Polar Star [10].

8.3 With the collimator placed at the ecliptic plane, directed toward the Sun and making 1 clockwise rotation per day, α-particles are constantly emitted in the direction of the Sun. As it was expected, the near-daily periods, both solar and sidereal, disappeared in such conditions.

9. Characteristic histogram's shapes at new Moon and solar eclipses

All the results presented above have probabilistic character and were obtained by the evaluation of tens of thousands of histogram pairs in every experiment. A completely different approach is used in the search for characteristic histogram shapes in the periods of the new Moon and solar eclipses. In these cases the histogram's shape is examined at a certain predetermined moment of new Moon or solar eclipse. In such a way it was discovered that at the moment of the new Moon, a certain characteristic histogram appears practically simultaneously at different longitudes and latitudes – all over the Earth. This characteristic histogram corresponds to a time segment of 0.5–1.0 min [12]. When the solar eclipse reaches maximum (as a rule, this moment does not coincide with the time of the new Moon), a specific histogram also appears; however, it has a different shape. Such specific shapes emerge not only at the moments of the new Moon or solar eclipses, though the probability of their appearance at these very moments at different places and on different dates (months, years) is extremely low. These specific histogram's shapes neither relate to tidal effects nor depend on position on the Earth's surface, where the Moon's shadow falls during the eclipse or the new Moon.

10. Characteristic histograms shapes at rise and set of Sun and Moon

The shape of histograms is determined by a complex set of cosmo-physical factors. As it follows from the existence of the near-27-day periods, amongst these factors may be the relative positions and states of the Sun, the Moon and the Earth. We repeatedly observed similar histograms during the risings and settings of the Sun and the Moon. A very large volume of work has been carried out. Yet we have not found a histogram shape, which would be characteristic for those instants. A review and analysis of the corresponding results will be given in a special paper.

11. Mirror symmetry of histograms

Very often (up to 30%) shape of histograms in the similar pair has "mirror" symmetry. This means an existence of left and right shapes. This phenomenon possibly signifies that chirality is an immanence property of space-time [1].

3 The possible nature of the Macroscopic Fluctuations Effect. Idea of the present investigation

Above-mentioned properties #3 − #4 of the macroscopic fluctuations effect pointed out the dependence of the effect phenomenology on the space position of the Earth, Moon, and the Sun and property #2, which states independence of the phenomenology from qualitative nature of fluctuating process, lead to supposition that phenomenology can be determined by only a such common factor as space-time heterogeneity. The space-time heterogeneity can be connected with gravitational interaction. On the other hand, properties #5 − #8 indicating space anisotropy of acting agent and property #9 – synchronous arising of similar histograms shapes in different geographical locations at certain moments in dynamics of Sun, Earth, and Moon leads to conclusion about wave nature of acting agent.

To sum up we can suppose that acting agent determining above-mentioned properties of macroscopic fluctuations have gravity-wave nature. According to this, shape of histograms, can be sensitive to gravitational wave influence. This is the base idea of experiment, which is presented at fig. 7.

4 Experimental setup

At fig. 7 simplified diagram of experimental setup on detecting gravitational wave influence on the shape of α-decay rate histograms is presented. The left side of the diagram schematically presents generator of gravitational influence. A centrifuge K70 ("JANETZKI") symmetrically loaded with two bottles of water was used as such generator. The weight of every bottle was 1.5 kg.

Gravitational radiation of the generator, schematically presented by parallel arrows, influences on two-channel registration system, showed as Ch. 1 and Ch. 2. The system consists of two recorders of α-decay rate from ^{239}Pu-sources. Average α-decay rate for Ch. 1 is 272 decays per second and 174 decays per second for Ch. 2. The recorders lie in the plane of centrifuge rotor and are placed at a distance of 1.5 m from its axle. For every recorder the angle φ between wave vector of generating gravitational wave and direction of α-particles emitting is different. For Ch. 1-recorder the angle is $\varphi = 180°$ and for Ch. 2 $\varphi = 90°$. In view of wave nature of expected influence it must be angle-sensitive. So, recorders Ch. 1 and Ch. 2 must be of different sensitivity to generate wave influence. Values of α-decay rate per second from every channel and speed of rotation of centrifuge rotor were registered by a special computer system.

Experiments were carried out as 5-minute cycles of running and turning off the

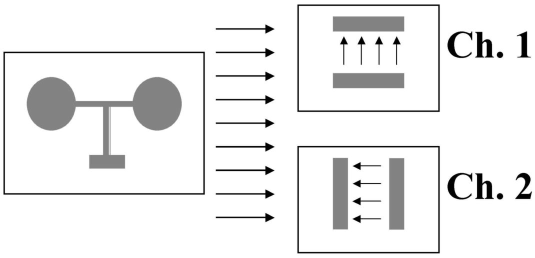


Figure 7: Simplified diagram of experimental setup on detecting gravitational wave influence on the shape of α-decay rate histograms.

centrifuge. So, period of influence was equal to 10 min. The rotation speed of running centrifuge rotor was 3000 revolutions per minute. The rotor of turning off centrifuge at the end of 5-minute cycle keeps the rotation speed about 300 rpm.

5 Simulation of expected results

Fig. 8 *a*) shows idealized diagram presenting change of rotation speed of centrifuge rotor with time. We expect, that all histograms constructed from pieces of time series corresponding to running centrifuge are similar between themselves, but non-similar to histograms constructed from pieces of time series corresponding to turning off centrifuge. In the same way all histograms constructed from pieces of time series corresponding to turning off centrifuge are similar between themselves, but non-similar to histograms constructed from pieces of time series corresponding to running centrifuge.

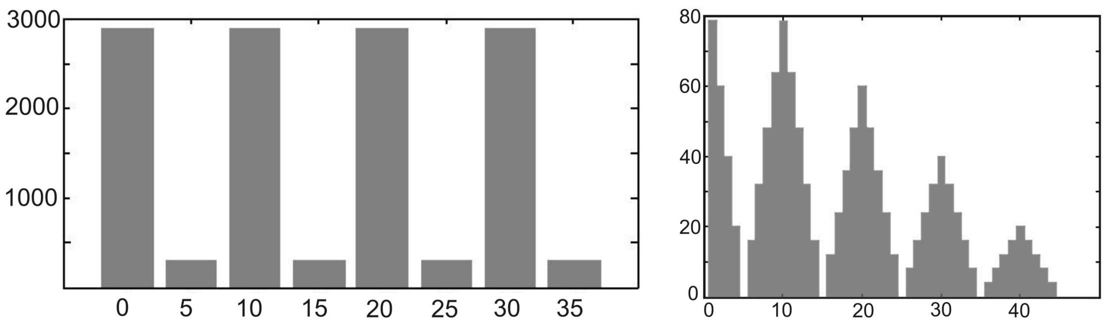


Figure 8: Idealized diagram of rotation speed of centrifuge rotor with time, *a*); expected interval distribution, *b*).

Above supposition allows us to calculate expected interval distribution, fig. 8 *b*). As it can be seen at fig. 8 *b*) interval distribution for experimental record of fixed length consists of finite number of decreasing peaks repeating with period, which equal to period of centrifuge alternating.

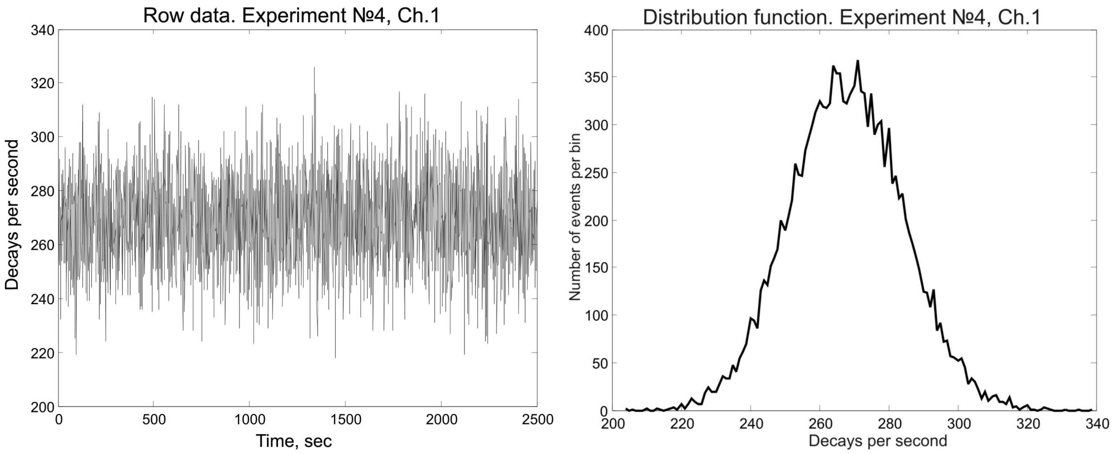


Figure 9: An example of experimental record of α-decay rate fluctuations obtained from ^{239}Pu-source (Ch. 1, series No. 4), *a*; and corresponding distribution function, *b*.

6 Experimental results

According to above described method five series of measurements were carried out. An example of experimental record No. 4 obtained from Ch. 1-recorder is given at fig. 9 *a*). This graph presents a piece in 2500 sec length of time series in 26400 one-second measurements. Fig. 9 *b*) presents distribution function for this time series. As it is possible to see from fig. 9 *a*) and fig. 9 *b*) time series of α-decay rate fluctuations and its distribution function are typical for this process. Absence of any peculiarities in the presented time series and the distribution function are expected and is evidence of good quality of experimental registration. As it was noted at the beginning, traditional methods of time series processing are not sensitive to macroscopic fluctuations effect manifestations.

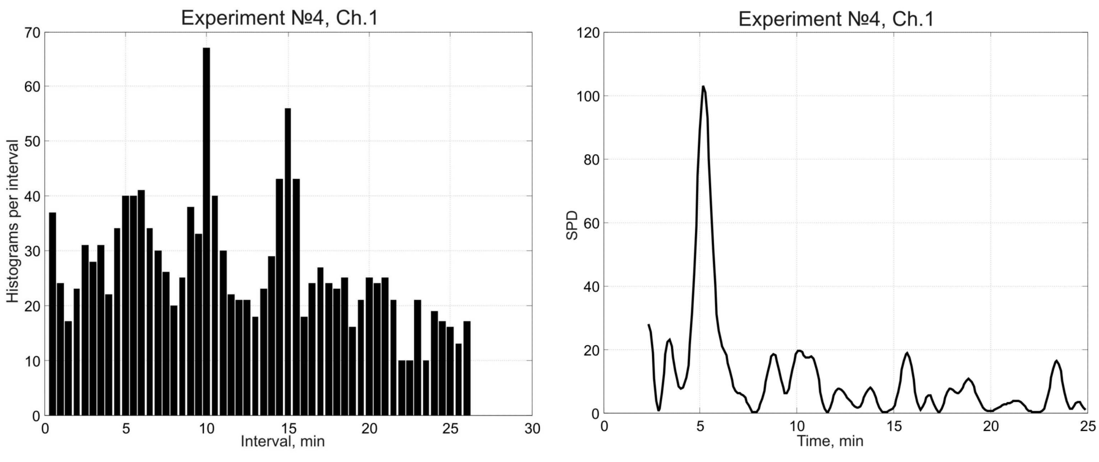


Figure 10: An example of interval distribution (Ch. 1, series No. 4), *a*);
and corresponding density function of power spectrum, *b*).

According to the methods described in first chapter, on the base of obtained experimental records five sets of 0.5-min histograms were constructed. Expert tested the pairs of histograms from every set for similarity. Typical example of interval distribution constructed on the base of result of expert comparison (Ch. 1, series No. 4) is presented at fig. 10 *a*).

As it is possible to see from fig. 10 the interval distribution consists of quite distinct periodic peaks. The period of peak repetition is 5 min. Fig. 10 *b*) presents spectral power density corresponding to the interval distribution. As it can be expected from interval distribution the spectrum has distinct 5-minute peak. Appearance of the 5-minute period is quite unexpected from the point of view of above developed model. The meaning of the period will be considered below.

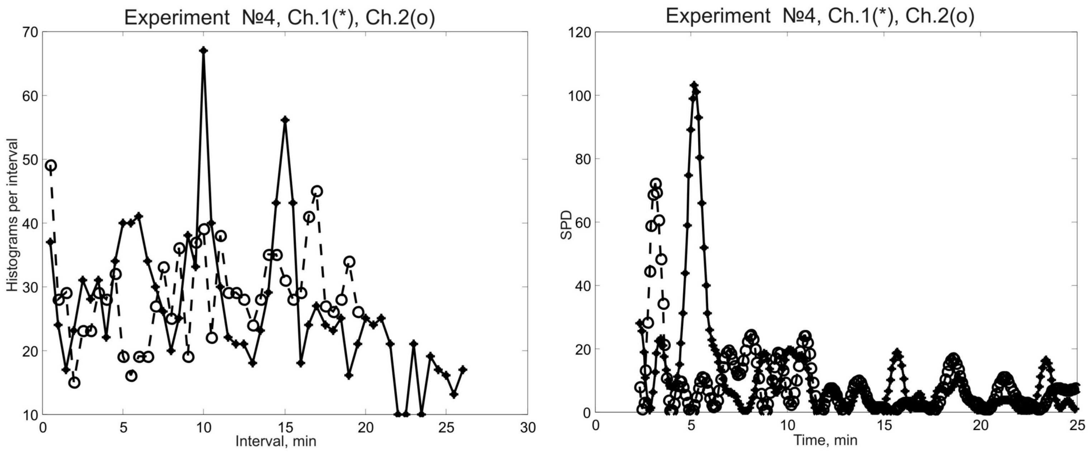


Figure 11: An Example of interval distribution for Ch. 1 (*) and Ch. 2 (°) (series No. 4), *a*); and corresponding densities of power spectrum, *b*).

For convenience in fig. 11 *a*) interval distributions for Ch. 1 (marked by asterisks) and Ch. 2 (marked by little circles) are given. As it is possible to see, periodical pattern typical for Ch. 1 is absent in Ch. 2. Fig. 11 *b*) presents spectral power densities corresponding to interval distributions in fig. 11 *a*). It is clear that 5-minute peak is absent for Ch.2 spectrum. This result validates supposition that registration system is angle-sensitive in relation to generated influence. At the same time for some cases spectrum for Ch. 2 contains $2.5 \div 3$- minute peak, which can also be seen in fig. 11 *b*). The physical nature of this peak and its correspondence to centrifuge dynamic is unknown.

7 Acceleration modes

Results, illustrated above by data for series No. 4 were also obtained for other series. This allows us to make a statement about sensitivity of histogram's shape to influence of rotating centrifuge rotor. This influence reveals itself by higher probability of meeting similar pair of histograms with period, which equals 5 min.

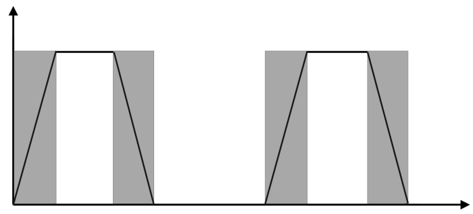


Figure 12: Simplified diagram for centrifuge rotor acceleration and braking.

Appearance of 5-minute period instead of 10-minute one indicates that histograms shape is sensitive not to rotation speed of centrifuge rotor, as it was suggested in the model presented at fig. 8, but to its accelerations. Fig. 12 illustrates this supposition. Here gray rectangles mark intervals of acceleration and braking of centrifuge rotor. Every period includes two such intervals. If histograms shape is sensitive to accelerations, we will obtain double frequency, i.e. 5-minute period instead of 10-minute one. Interval distribution in this case will be the same as presented in fig. 8 *b*) but with 5-minute peaks period, which is observed in interval distribution obtained from experimental data.

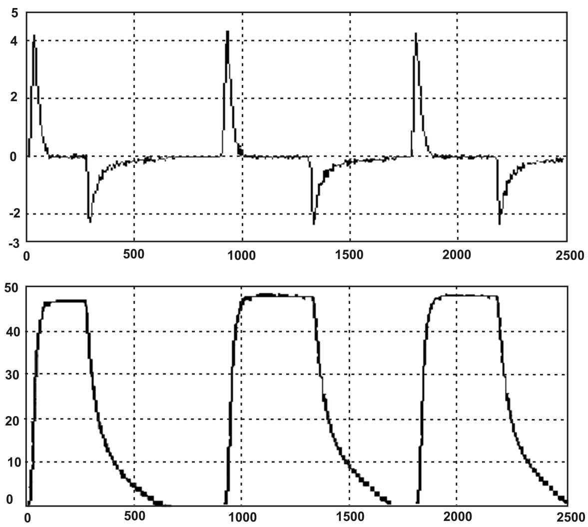


Figure 13: Upper graph: tests record of three acceleration-braking periods of rotation speed of centrifuge rotor; lower graph: derivative of rotation speed (acceleration), presented at the upper graph.

Experiments with rotation speed of centrifuge rotor confirm this supposition. The upper graph in fig. 13 presents test record of three acceleration / braking periods of rotation speed of centrifuge rotor. The lower graph in fig. 13 presents derivative of rotation speed, which corresponds to rotor accelerations. Narrow peaks in this graph correspond to acceleration mode. We suppose that histograms of α-decay fluctuations, which correspond to acceleration modes in centrifuge operations determine 5-minute period observable in above described experiments.

8 Discussion

In favor of the supposition that acceleration modes can define the shape of histograms we shall consider works [13, 14]. These works used a registration system providing acceleration modes, which are in some way complementary to such modes in our works. This registration system used as a sensor rapidly spinning massive body with artificially created acceleration mode by means of special braking pulse. Duration of the pulse is 18-30% of rotation period [13]. Registering parameter of the system is angular velocity of spinning body. It turned out, that such a system is sensitive to the same events, which are noted in properties No. 9 and No. 10 of macroscopic fluctuation effect phenomenology [14]. All this events have a relation to certain extrema in the velocity of change of the space-time position of the Sun, the Earth and the Moon; in this respect, the situation can be considered as a regime with acceleration and thus, it can determine the form of histograms describing the fluctuations in various processes.

As the second example of experimental investigation, where acceleration modes play an important role, we will consider work [15]. In this work a pair of generator with identical crystal oscillators was used as a registration system. The generators were placed in such a way that the positions of the crystals were orthogonal. Registering parameter of this system is relative change of resonant frequencies of crystals of the generators. Authors named this parameter as T-signal. A study of daily changes of T-signal shows its anisotropy with extrema at local noon and midnight. Authors note non-electromagnetic nature of T-signal, and its biological activity. The T-signal changes were considered as consequence of gravitation waves emission of the Sun.

It is possible to note the general moments typical for works [13–15] and experiment considered in the present work. The first important feature is presence of "acceleration modes" in the registering system, allowing distinguishing some direction in space. In our experiments it is set by a direction of outgoing α-particles, in [13–14] – by the moment of breaking impulse, in [15] – by a perpendicular to a plane of the crystal plate oscillations.

"Acceleration modes", causing anisotropic properties of registering system at the same time make it sensitive to the same "acceleration modes" which are external in relation to it and, presumably, are connected with gravity-wave radiation.

Summarizing we shall note, that as a result of the present experimental investigation influence of quickly rotating massive body on the shape of fine structure of constructed upon small samples distribution functions of fluctuations of α-decay rate, appearing in higher probability of similarity of shape of the histograms for the moments corresponding to "acceleration modes" is fixed. The influence possesses anisotropic properties and, presumably, has the gravity-wave nature.

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References

[1] S. E. Shnoll, V. A. Kolombet, E. V. Pozharskii, T. A. Zenchenko, I. M. Zvereva, A. A. Konradov, *Realization of discrete states during fluctuations in macroscopic processes,* Physics-Uspekhi 41 (10) (1998), 1025–1035.

[2] S. E. Shnoll, T. A. Zenchenko, K. I. Zenchenko, E. V. Pozharskii, V. A. Kolombet, A. A. Konradov, *Regular variation of the fine structure of statistical distributions as a consequence of cosmophysical agents,* Physics – Uspekhi 43 (2) (2000), 205–209.

[3] S. E. Shnoll, N. V. Udaltsova, V. A. Kolombet, V. A. Namiot, and N. B. Bodrova, *Regularities in the discrete distributions of the results of measurements (cosmophysical aspects),* Biophysics 37 (3) (1992), 378–398.

[4] S. E. Shnoll, V. A. Namiot, V. E. Zhvirblis, V. N. Morozov, A. V. Temnov, and T. Ya. Morozova, *Possible common nature of macroscopic fluctuations of the rates of biochemical and chemical reactions, electrophoretic mobility of the cells and fluctuations in measurements of radioactivity, optical activity and flicker noise,* Biophysics 28 (1) (1983), 164–168.

[5] S. E. Shnoll, *The form of the spectra of states realized in the course of macroscopic fluctuations depends on the rotation of the Earth about its axis,* Biophysics, 40 (4) (1995), 857–866.

[6] S. E. Shnoll, *Discrete distribution patterns: arithmetic and cosmophysical origins of their macroscopic fluctuations,* Biophysics, 2001, v. 46 (5), 733–741.

[7] S. E. Shnoll, *Periodical changes in the fine structure of statistic distributions in stochastic processes as a result of arithmetic and cosmophysical reasons,* Time, Chaos, and Math. Problems, No. 3, University Publ. House, Moscow, 2004, 121–154.

[8] S. E. Shnoll, K. I. Zenchenko, N. V. Udaltsova, *Cosmo-physical effects in structure of the daily and yearly periods of change in the shape of the histograms constructed by results of measurements of alpha-activity Pu^{239},* http://arxiv.org/abs/physics/0504092.

[9] S. E. Shnoll, K. I. Zenchenko, I. I. Berulis, N. V. Udaltsova, S. S. Zhirkov, I. A. Rubinstein, *The dependence of "macroscopic fluctuations" on cosmophysical factors. Spatial anisotropy,* Biophysics, 49 (1) (2004), 129–139.

[10] Simon E. Shnoll, Konstantin I. Zenchenko, Iosas I. Berulis, Natalia V. Udaltsova, I. A. Rubinstein, *Fine structure of histograms of alpha-activity measurements depends on direction of alpha particles flow and the Earth rotation: experiments with collimators,* http://arxiv.org/abs/physics/0412007.

[11] S. E. Shnoll, I. A. Rubinshtejn, K. I. Zenchenko, V. A. Shlekhtarev, A. V. Kaminsky, A. A. Konradov, N. V. Udaltsova, *Experiments with rotating collimators cutting out pencil of α-particles at radioactive decay of ^{239}Pu evidence sharp anisotropy of space,* Progress in Physics, V. 1, 2005, 81–84. http://arxiv.org/abs/physics/0501004

[12] S. E. Shnoll, K. I. Zenchenko, S. N. Shapovalov, S. N. Gorshkov, A. V. Makarevich, O. A. Troshichev, *The specific form of histograms presenting the distribution of data of α-decay measurements appears simultaneously in the moment of New Moon in different points from Arctic to Antarctic,* http://arxiv.org/abs/physics/0412152.

[13] B. Yu. Bogdanovich, I. S. Shchedrin, V. N. Smirnov, N. V. Yegorov, *Special method of massive body spinning as instrument for astrophysical research,* Scientific Session of MIFI, vol. 7, 2003, 45–46.

[14] B. Yu. Bogdanovich, I. S. Shchedrin, V. N. Smirnov, N. V. Yegorov, *Preliminary estimations of kinetic energy changes of spinning massive body on space-time position of Sun and Moon,* Scientific Session of MIFI, vol. 7, 2003, 47–48.

[15] N. V. Klochek, L. E. Palamarchuk, M. V. Nikonova, *Preliminary results of investigation of the influence of cosmophysical radiation of non-electromagnetic nature on physical and biological systems,* Biophysics, 40 (4) (1995), 889–896.

Local-Time Effect on Small Space-Time Scale

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The existence of the local time effect was studied for different places of measurement located at different positions on Earth at various distances ranging from hundreds of kilometers up to the largest possible ones. The paper studies the existence of the local time effect for relatively small distances between the places of measurements. The distribution of time intervals in the neighborhood of the local time peak was studied, and the splitting of the peak was evidentiated.

1 Introduction

The present work was carried out as further investigations of macroscopic fluctuations phenomena [1 – 4]. The local time effect, which is the main subject of this paper, is synchronous in local time appearance of pairs of histograms with similar fine structure constructed on the base of measurements of fluctuations in processes of different nature fulfilled in different geographical locations. The effect points out on the dependence of fine structure of the histograms on the Earth rotations around its axis and around the Sun. The existence of local time effect was studied for different distances between places of measurement from hundred kilometers up to highest possible on the Earth distances ($\sim$ 15000 km). The goal of the present work is the investigations of the existence of the local time effect for relatively small distances between places of measurements.

The main problem of experimental investigations of local-time effect at the small space distances is resolution enhancement of the macroscopic fluctuations method. As a rule all above-mentioned investigations of local-time effect were carried out by using α-decay rate fluctuations of ^{239}Pu source. But such source of fluctuations becomes uselessness for distances in tens of kilometers or less when histograms duration must be about one second or less. By this reason in the present work we refuse α-decay sources of fluctuations. As such a source was chosen noise generated by germanium semiconductor diode. Such source gives noise signal with frequency band up to tens of megahertz and because of this satisfies the requirements of present investigations.

To check convenience of selected noise source for local-time effect investigations it was tested on distances for which existence of the effect was proved [5]. In cited work was shown appropriateness of semiconductor noise diode for studies of local-time effect.

2 Investigation of local-time effect for longitudinal distance between locations of measurements in 15 km

First series of synchronous measurements were carried out in Pushchino (Lat. $54°50.037'$ North, Lon. $37°37.589'$ East) and Bolshevik (Lat. ($54°54.165'$ North, Lon. $37°21.910'$ East). Longitudinal difference α between places of measurements is $\alpha = 15.679'$. This value of α corresponds to difference of local time $\Delta t = 62.7$ sec and longitudinal distance Δl, equal $\Delta l = 15$ km.

Fluctuations from noise generator were digitized with sampling frequency equal to 44100 Hz. In this way in Pushchino and Bolshevik were obtained 10-minute time series. From this initial time series with three different steps (pointed in second line of Table 1) were extracted single measurements and obtained three time series with frequency pointed in third line of Table 1. On the base of this time series in a standard way [1–3] were constructed three sets of histograms. All histograms were constructed using sample length pointed in fourth line of Table 1. Number of histograms per second and durations of histogram for every set are given in the fifth and sixth line of Table 1 correspondingly.

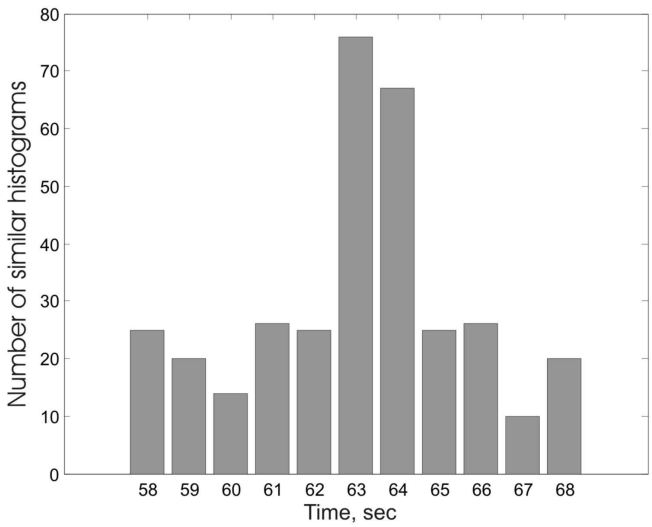
Table 1. Parameters used for calculating sets of 1-, 0.2-, and 0.02-sec histograms.

1	Sampling frequency, Hz	44100	44100	44100
2	Step, points	735	147	14
3	Frequency of histogram time series, Hz	60	300	3150
4	Histogram sample length, points	60	60	63
5	Histograms per 1 sec	1	5	50
6	Duration of histogram, sec	1	0.2	0.02

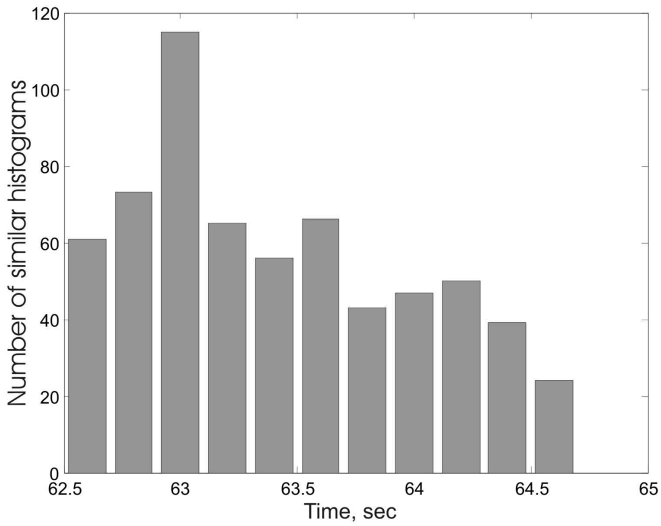
Fig. 1 presents intervals distribution obtained after expert comparisons of 1-sec histogram sets. The distribution has a peak, which corresponds to time interval equal to 63 ± 1 sec. Taking into account accuracy of synchronization of measurements beginning ($0.1 - 0.2$ sec) and duration of histograms one can consider this peak to be corresponding with good accuracy to local time difference $\Delta t = 62.7$ sec between places of measurements.

Local time peak ordinary obtained on the interval distributions is very sharp and consists of 1–2 histograms [1–3] i.e. is practically structureless. Peak on the Fig. 1 *a*) also can be considered as structureless. This fact leads us to the problem of further investigating of structure of local time peak.

The fact that all sets of histograms were obtained on the base of the same initial time series enables enhancement of time resolution of the method of investigation. Using of 0.2-sec histograms set (forth column of Table 1) increase resolution in five times and allows more detailed investigations of local-time peak structure. Since the positions of the peak on the intervals distribution (Fig. 1) are known it is possible to select their neighborhood by means of 60 sec relative shift of initial time series and prepare after this 0.2-sec histograms set for further expert comparison.



a)



b)

Intervals distribution obtained in result of expert comparisons for 0.2-sec histograms set is presented on Fig. 1*b*). One can see that maximum similarity of histograms shape is observed for pairs of histograms separated by interval in 63?0.2 sec. This value is the same as for 1-sec histograms intervals distribution, but in latter case it is defined with accuracy in 0.2 sec.

It's easy to see from intervals distribution, Fig. 1*b*), that after 5-times enhancement of resolution the distribution has single sharp peak again. So, change of time scale in this case doesn't lead to change of intervals distribution. This means that

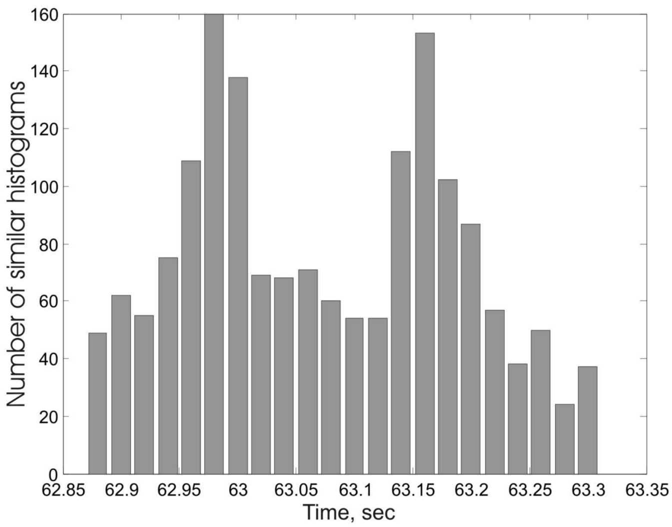


Figure 1: Intervals distributions obtained after expert comparisons of 1-sec (a), 0.2-sec (b), and 0.002-sec (c) histogram sets. Y-axis presents number of histograms, which were found similar; X-axis – time interval between pairs of histograms, sec.

we must enhance time resolution again to study the local time peak. We can do this by using of 0.02-sec histograms (third line of Table 1).

Intervals distribution for case of 0.002-sec histograms is presented on the Fig. 1c). Unlike to intervals distributions on the Fig. 1a) and Fig. 1b) distribution on the Fig. 1c) consists of two distinct peaks. The first peak corresponds to local time difference equal 62.98 ± 0.002 sec, the second one to 63.16 ± 0.002 sec. The difference between the peaks is $\Delta t' = 0.18 \pm 0.002$ sec.

Splitting of local-time peak on the Fig 1 c) is similar to splitting of daily period on two peaks with periods, which equal to solar and sidereal days [6–8]. This fact will be discussed below.

3 Investigation of local-time effect for longitudinal distance between locations of measurements from 6 km to 0.5 km. Mobile experiment

Above presented experiment demonstrates the existence of local-time effect for longitudinal distance between locations of measurements in 15 km and splitting of local-time peak corresponding to the distance. It is natural to investigate the question: which is the minimal distance of local time effect existence? Next step in this direction is the experiment presented below.

Table 2. Locations of mobile measurement system and corresponding parameters.

#	Locations of mobile measurement system	α	Δt, sec	Δl, km	$\Delta t'$, sec
1	Lat. 54°48.16′ N, Lon. 37°43.54′ E	5.95′	23.8	6	0.066
2	Lat. 54°49.28′ N, Lon. 37°41.44′ E	3.85′	15.4	3.9	0.043
3	Lat. 54°50.126′ N, Lon. 37°39.21′ E	1.618′	6.47	1.6	0.018
4	Lat. 54°49.989′ N, Lon. 37°38.13′ E	0.538′	2.152	0.5	0.006

In the experiment two measurement systems were used: stationary with location in Pushchino (Lat. 54°50.037′ N, Lon. 37°37.589′ E) and mobile one. Four series of measurements were carried out. Locations of mobile measurement system for every series of measurements are given in second column of Table 2. Angular longitudinal difference of locations of measurements, α, is presented in third column of the table. Local time difference Δt and longitudinal difference of locations of measurements Δl, are given in fourth and fifth columns of Table 2 correspondingly. Last column gives splitting value of local-time peak, $\Delta t'$.

Method of experimental data processing was the same as for experiment presented in second section of the paper. Was found that within accuracy of experiment the local time value Δt and the local-time peak splitting value $\Delta t'$ can be observed.

4 Discussions

Local-time effect as pointed in [1], is linked to rotatory movement of Earth. The simplest explanation of the fact can be following. Due to the rotatory movement of the Earth after time Δt measurement system No. 2 appears in the same places where was system No. 1. The same places cause the same shape of fine structure of histograms. Actually such explanation is incorrect because of orbital motion of Earth, which noticeably exceeds rotatory movements. Therefore measurement system No. 2 cannot appear in the same places where was system No. 1. But if we consider two directions defined by center of Earth and two points of measurement then after time Δt measurement system No. 2 take the same directions in the space as system No. 1 before. From this it follows that similarity of histograms shapes in some way is connected with the same space directions. This supposition also agrees with experimental results presented in [9–10].

Four-minute splitting of daily period of repetition of histograms shape on solar and stellar sub-periods [3] is the evidence of existence of two preferential directions: to the Sun and to the coelosphere. Really after time interval equal 1436 min the Earth makes one complete revolution and measurement system plane has the same direction in the space as one stellar day before. After four minutes from this moment measurement system plane will be directed to the Sun. This is the cause of solar-day period – 1440 min.

Let us suppose that splitting described in the present paper has the same nature as splitting of daily period. Then from daily period splitting ΔT , which equal $\Delta T = 4$ min its possible to obtain proportionality coefficient k:

$$k = \frac{240 \text{ sec}}{86400 \text{ sec}} \approx 2.78 \cdot 10^{-3}. \tag{1}$$

Longitudinal difference between places of measurements presented in second section is $\Delta t \approx 62.7$ sec and we can calculate splitting of local-time peak for this value of Δt:

$$\Delta t' = k\Delta t = 62.7 \times 2.78 \cdot 10^{-3} \approx 0.17 \text{ sec}. \tag{2}$$

As it is easy to see from Fig. 1c) splitting of local-time peak is equal to 0.18 ± 0.02 sec. This value agrees with estimation (2). Values of splitting of the local-time peak, which are presented in last column of Table 2 also were calculated by help of formula (2). Experiment described in third section shows good agreement of the values with experimentally obtained.

This result allows us to consider sub-peaks of local-time peak as stellar and solar and suppose that in this case the cause of splitting is the same as for daily-period splitting.

Speaking about preferential directions we implicitly supposed that measurement system is directional and because of this can resolve these directions. Such supposition is quit reasonable for the case of daily period splitting but for splitting of local-time peak observed on the small distances becomes problematic since an angle, which must be resolved by the measurement system is neglible. Most likely that in this case we deal with space-time fluctuations, which in some way are connected with preferential directions to the Sun and coelosphire. In other words we can speak about sharp anisotropy of near-earth space-time.

Results obtained in the present work prove possibility of local-time effect investigation on small space scale up to 0.5 km. Farther decreasing of this scale is our immediate task. In the same time sub-peaks obtained as result of splitting of local-time peak also consist of one-two histograms, so are structureless. This fact poses a problem of more detailed investigations of local-time peak structure.

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References

[1] S. E. Shnoll, V. A. Kolombet, E. V. Pozharskii, T. A. Zenchenko, I. M. Zvereva, A. A. Konradov, *Realization of discrete states during fluctuations in macroscopic processes,* Physics-Uspekhi 41 (10) (1998), 1025–1035.

[2] S. E. Shnoll, T. A. Zenchenko, K. I. Zenchenko, E. V. Pozharskii, V. A. Kolombet, A. A. Konradov, *Regular variation of the fine structure of statistical distributions as a consequence of cosmophysical agents,* Physics-Uspekhi 43 (2) (2000) 205–209.

[3] S. E. Shnoll, *Periodical changes in the fine structure of statistic distributions in stochastic processes as a result of arithmetic and cosmophysical reasons.* Time, Chaos, and Math. Problems, No. 3, University Publ. House, Moscow, 2004, 121–154.

[4] S. E. Shnoll, *Changes in the fine structure of stochastic distributions as consequence of space-time fluctuations,* Progress in Physics 6 (2006), 39–45.

[5] V. A. Panchelyuga, V. A. Kolombet, A. V. Kaminsky, M. S. Panchelyuga, S. E. Shnoll, *The local time effect observed in noise processes,* Bulletin of Kaluga University, 2006.

[6] S. E. Shnoll, *Discrete distribution patterns: arithmetic and cosmophysical origins of their macroscopic fluctuations,* Biophysics 46 (5) (2001), 733–741.

[7] S. E. Shnoll, K. I. Zenchenko, N. V. Udaltsova, *Cosmophysical Effects in the Structure of Daily and Yearly Periods of Changes in the Shape of Histograms Constructed from the Measurements of ^{239}Pu alpha-activity,* Biophysics 49, 1 (2004), p. 155.

[8] S. E. Shnoll, K. I. Zenchenko, N. V. Udaltsova, *Cosmo-physical effects in structure of the daily and yearly periods of change in the shape of the histograms constructed by results of measurements of alpha-activity ^{239}Pu,* 2005, http://arxiv.org/abs/physics/0504092.

[9] Simon E. Shnoll, Konstantin I. Zenchenko, Iosas I. Berulis, Natalia V. Udaltsova, Ilia A. Rubinstein, *Fine structure of histograms of alpha-activity measurements depends on direction of alpha particles flow and the Earth rotation: experiments with collimators,* http://arxiv.org/abs/physics/0412007.

[10] S. E. Shnoll, I. A. Rubinshtejn, K. I. Zenchenko, V. A. Shlekhtarev, A. V. Kaminsky, A. A. Konradov, N. V. Udaltsova, *Experiments with Rotating Collimators Cutting out Pencil of α-Particles at Radioactive Decay of ^{239}Pu Evidence Sharp Anisotropy of Space*, Progress in Physics 1 (2005), 81–84, http://arxiv.org/abs/physics/0501004.

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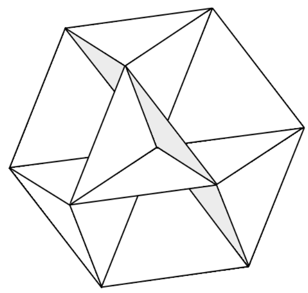

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