

CONSTRUCTION OF THE PSEUDO RIEMANNIAN GEOMETRY ON THE BASE OF THE BERWALD-MOOR GEOMETRY

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The space of the associative commutative hyper complex numbers, H_4 , is a 4-dimensional metric Finsler space with the Berwald-Moor metric. It provides the possibility to construct the tensor fields on the base of the analytical functions of the H_4 variable and also in case when this analyticity is broken. Here we suggest a way to construct the metric tensor of a 4-dimensional pseudo Riemannian space (space-time) using as a base the 4-contravariant tensor of the tangent indicatrix equation of the Berwald-Moor space and the World function. The Berwald-Moor space appears to be closely related to the Minkowski space. The break of the analyticity of the World function leads to the non-trivial curving of the 4-dimensional space-time and, particularly, to the Newtonian potential in the non-relativistic limit.

1 Introduction

The fascinating beauty of the theory of the functions of complex variable reveals itself, for example, in the harmony of the algebraic fractals on the Euclidian plane. It makes many researches look for the analogous number systems, the elements of which could be correlated not to the points on the plane but to the points of the 4-dimensional space-time. In case of the success of such a search, we could really trust the famous Pythagoras saying 'all the existing is number'. On this way, the interesting results were obtained for quaternions [1], biquaternions [2-4], octaves [5] and so forth. Nevertheless, none of these number system theories can be compared even to the theory of the relatively simple 2-component complex numbers. The main reason for this seems to be the lack of the commutativity (and sometimes even of the associativity) of the multiplication in these algebras. Although the authors of this paper realize the conceptual bases of all the variety of algebras, the commutativity of the multiplication is the integral property of all the principal number systems that contain natural, integer, rational, real and complex numbers. Finally, the commutativity and the associativity of the multiplication are among the axioms of arithmetic which presents the foundation of mathematics, and it would be strange if the algebraic system which is the most natural for our real world does not correspond to the rules of regular counting.

One of the systems free from this drawback is the algebra of the commutative and associative hyper complex numbers, related to the direct sum of the four real algebras, which will be denoted as H_4 . The algebra of these numbers is isomorphic to the algebra of the 4-dimensional square real diagonal matrices, and the corresponding space is a linear Finsler space with the Berwald-Moor metric (the last fact was proved by the authors in [6]).

It should be mentioned that Finsler space with the Berwald-Moor metric has been known and partially investigated for a long time [7–8].

One of the main properties of this space is the existence of such a range of the parameters that the 3-dimensional distances (from the point of view of the observer who uses the radar method to measure them [9]) correspond to the positively defined metric function the limit of which is the quadratic form [10]. In other words, the 3-dimensional world observed by an " H_4 inhabitant" is Euclidian within certain accuracy. Moreover, when one passes to the relativistic velocities, the 4-dimensional intervals between the H_4 events present the Minkowski space correlations [11]. All this makes possible to suggest that the H_4 space and the corresponding Finsler geometry can be used as a mathematical model of the real space-time, and maybe this model would be even more productive than the pseudo Riemannian constructions prevailing in Physics now.

Any hyper complex algebra is completely defined by the multiplication rule for the elements of a certain fixed basis. In the H_4 number system there is a special – isotropic – basis e_1, e_2, e_3, e_4 , such that

$$e_i e_j = p_{ij}^k e_k \quad p_{ij}^k = \begin{cases} 1, & \text{if } i = j = k, \\ 0, & \text{else.} \end{cases} \quad (1)$$

Any analytical function in this basis can be given as

$$F(X) = f^1(\xi^1)e_1 + f^2(\xi^2)e_2 + f^3(\xi^3)e_3 + f^4(\xi^4)e_4, \quad (2)$$

where

$$H_4 \ni X = \xi^1 e_1 + \xi^2 e_2 + \xi^3 e_3 + \xi^4 e_4, \quad (3)$$

and f^i are four arbitrary smooth functions of a single real variable.

In H_4 there is one more – orthogonal – selected basis $1, j, k, jk$, which is related to the isotropic basis by the following formulas

$$\left. \begin{aligned} 1 &= e_1 + e_2 + e_3 + e_4, \\ j &= e_1 + e_2 - e_3 - e_4, \\ k &= e_1 - e_2 + e_3 - e_4, \\ jk &= e_1 - e_2 - e_3 + e_4, \end{aligned} \right\} \quad (4)$$

where 1 is the unity of algebra, and the corresponding component of the analytical function of the H_4 variable is defined by the formula

$$u = \frac{1}{4} [f^1(\xi^1) + f^2(\xi^2) + f^3(\xi^3) + f^4(\xi^4)]. \quad (5)$$

If X is a radius vector, then the coordinate space $\xi^1, \xi^2, \xi^3, \xi^4$ is a Berwald-Moor space with the length element

$$ds = \sqrt[4]{d\xi^1 d\xi^2 d\xi^3 d\xi^4} \equiv \sqrt[4]{g_{ijkl} d\xi^i d\xi^j d\xi^k d\xi^l}, \quad (6)$$

where

$$g_{ijkl} = \begin{cases} \frac{1}{4!}, & (i \neq j \neq k \neq l), \\ 0, & (else). \end{cases} \quad (7)$$

For this geometry the tangent indicatrix equation is

$$g^{ijkl} p_i p_j p_k p_l - 1 = 0, \quad (8)$$

where

$$g^{ijkl} = \begin{cases} \frac{4^4}{4!}, & (i \neq j \neq k \neq l), \\ 0, & (\text{else}), \end{cases} \quad (9)$$

$$p_i = \frac{g_{ijkl} d\xi^j d\xi^k d\xi^l}{(g_{mrst} d\xi^m d\xi^r d\xi^s d\xi^t)^{3/4}} \quad (10)$$

are the components of the generalized momentum or generalized momenta.

If we have tensors p_{ij}^k , g_{ijkl} , g^{ijkl} and vector fields of the analytical functions $F_{(A)}(X)$ of the H_4 variables, we could construct the metric tensors in the 4-dimensional space-time in many ways. For example,

$$g_{ij}(\xi) = g_{ijkl} f_{(1)}^k f_{(2)}^l, \quad (11)$$

Now one can investigate the obtained Riemannian geometry. The main drawback of this approach is the variety of the ways to construct it.

It is known [12] that if the tangent indicatrix equation is defined as

$$\Phi(p; \xi) = 0, \quad (12)$$

then the geodesics will be the solutions of the canonical system of differential equations

$$\dot{\xi}^i = \frac{\partial \Phi}{\partial p_i} \cdot \lambda(p; \xi), \quad \dot{p}_i = -\frac{\partial \Phi}{\partial \xi^i} \cdot \lambda(p; \xi), \quad (13)$$

$\lambda(p; \xi) \neq 0$ is an arbitrary smooth function, and a dot above ξ^i and p_i means the derivation by the evolution parameter, τ .

2 Construction of the metric function of the pseudo Riemannian space

Let us regard a space which is conformally connected to the H_4 space, that is to the space with the length element

$$ds' = \kappa(\xi) \cdot \sqrt[4]{g_{ijkl} d\xi^i d\xi^j d\xi^k d\xi^l}, \quad (14)$$

where $\kappa(\xi) > 0$ is a scalar function which is a contraction-extension coefficient depending on the point.

Let there be a normal congruence of geodesics (world lines). Then there is a scalar function $S(\xi)$ (see, e.g. [12]) such that its level hyper surfaces are transversal to this normal congruence of the world lines and this function is a solution of the equation

$$g^{ijkl} \frac{\partial S}{\partial \xi^i} \frac{\partial S}{\partial \xi^j} \frac{\partial S}{\partial \xi^k} \frac{\partial S}{\partial \xi^l} = \kappa(\xi)^4, \quad (15)$$

while the generalized momenta along this congruence of the world lines are related to $S(\xi)$ by

$$p_i = \frac{\partial S}{\partial \xi^i}, \quad (16)$$

The equations for the world lines obtain the form

$$\dot{\xi}^i = g^{ijkl} \frac{\partial S}{\partial \xi^j} \frac{\partial S}{\partial \xi^k} \frac{\partial S}{\partial \xi^l} \cdot \lambda(\xi), \quad (17)$$

were $\lambda(\xi) \neq 0$.

In Physics the function $S(\xi)$ is called "action as a function of coordinates" and (15) is known as the Hamilton-Jacoby equation. In [10] the function $S(\xi)$ was called the *World function*.

If there is a congruence of the world lines, then the evolution of every point in space is known, particularly, the velocity field is known, but the energy characteristics of the material objects (observers) corresponding to a given world line are not known. The knowledge of the World function $S(\xi)$ makes it possible to calculate the generalized momenta p_i , corresponding to the energy characteristics, and the invariant energy characteristic, $\kappa(\xi)$, which has also the meaning of the local contraction-extension coefficient of the plane H_4 space.

So, if our world view is the classical mechanics, then any pair out of the three: World function, congruence of the world lines, Finsler geometry - gives us the complete knowledge of the World.

Let us construct a twice contravariant tensor $g^{ij}(\xi)$ in the following way:

$$g^{ij}(\xi) = \frac{1}{\kappa(\xi)^4} \cdot g^{ijkl} \frac{\partial S}{\partial \xi^k} \frac{\partial S}{\partial \xi^l}. \quad (18)$$

Since

$$\det(g^{ij}(\xi)) = -\frac{4^4}{3^3 \kappa(\xi)^8} \neq 0, \quad (19)$$

then everywhere where the geometry (14) is defined, one can construct a tensor $g_{ij}(\xi)$ such that

$$g^{ik}(\xi) g_{kj}(\xi) = \delta_j^i, \quad (20)$$

$$g_{ij}(\xi) = 4 \cdot \begin{pmatrix} -2 \left(\frac{\partial S}{\partial \xi^1} \right)^2 & \frac{\partial S}{\partial \xi^1} \frac{\partial S}{\partial \xi^2} & \frac{\partial S}{\partial \xi^1} \frac{\partial S}{\partial \xi^3} & \frac{\partial S}{\partial \xi^1} \frac{\partial S}{\partial \xi^4} \\ \frac{\partial S}{\partial \xi^1} \frac{\partial S}{\partial \xi^2} & -2 \left(\frac{\partial S}{\partial \xi^2} \right)^2 & \frac{\partial S}{\partial \xi^2} \frac{\partial S}{\partial \xi^3} & \frac{\partial S}{\partial \xi^2} \frac{\partial S}{\partial \xi^4} \\ \frac{\partial S}{\partial \xi^1} \frac{\partial S}{\partial \xi^3} & \frac{\partial S}{\partial \xi^2} \frac{\partial S}{\partial \xi^3} & -2 \left(\frac{\partial S}{\partial \xi^3} \right)^2 & \frac{\partial S}{\partial \xi^3} \frac{\partial S}{\partial \xi^4} \\ \frac{\partial S}{\partial \xi^1} \frac{\partial S}{\partial \xi^4} & \frac{\partial S}{\partial \xi^2} \frac{\partial S}{\partial \xi^4} & \frac{\partial S}{\partial \xi^3} \frac{\partial S}{\partial \xi^4} & -2 \left(\frac{\partial S}{\partial \xi^4} \right)^2 \end{pmatrix}. \quad (21)$$

No doubt that in the same coordinate space $\xi^1, \xi^2, \xi^3, \xi^4$ such tensor $g_{ij}(\xi)$ defines a Riemannian or pseudo Riemannian geometry with the length element

$$ds'' = \sqrt{g_{ij}(\xi) d\xi^i d\xi^j}. \quad (22)$$

The construction of tensor $g_{ij}(\xi)$ leads directly to the conclusion: the change of geometry (14) to the geometry (22) does not lead to the change of the initial congruence of the world lines and corresponding World function $S(\xi)$.

Therefore, in our concept one and the same World, i.e. the pair {World function; congruence of the world lines}, corresponds to a whole class of related but qualitatively different Finsler geometries.

3 Analyticity condition and the Minkowski space

Let the World function $S(\xi)$ be the (unity) component of an analytical function of the H_4 variable in the orthogonal basis (4), that is

$$S(\xi) = \frac{1}{4} [f^1(\xi^1) + f^2(\xi^2) + f^3(\xi^3) + f^4(\xi^4)] . \quad (23)$$

Then

$$g^{ijkl} \frac{\partial S}{\partial \xi^i} \frac{\partial S}{\partial \xi^j} \frac{\partial S}{\partial \xi^k} \frac{\partial S}{\partial \xi^l} = \frac{\partial f^1(\xi^1)}{\partial \xi^1} \frac{\partial f^2(\xi^2)}{\partial \xi^2} \frac{\partial f^3(\xi^3)}{\partial \xi^3} \frac{\partial f^4(\xi^4)}{\partial \xi^4} = \kappa(\xi)^4 > 0, \quad (24)$$

and this leads to the limitation on the functions, f^i :

$$\frac{\partial f^1(\xi^1)}{\partial \xi^1} \frac{\partial f^2(\xi^2)}{\partial \xi^2} \frac{\partial f^3(\xi^3)}{\partial \xi^3} \frac{\partial f^4(\xi^4)}{\partial \xi^4} > 0. \quad (25)$$

It follows from (24) that the space with the length element (14) can be obtained from the space with the length element (6) with the help of the conformal transformation, which means that the condition of the analyticity of the World function can be treated in a sense as the condition of the conformal symmetry.

Let us construct tensor $g_{ij}(\xi)$ following the algorithm developed in the previous section. It turns out that in a region where functions f^i have no singularities there will always be such a coordinate system x^0, x^1, x^2, x^3 in which the length element ds'' has a form

$$ds'' = \sqrt{(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2}. \quad (26)$$

Let us express the coordinates x^0, x^1, x^2, x^3 in terms of the initial coordinates $\xi^1, \xi^2, \xi^3, \xi^4$:

$$\left. \begin{aligned} x^0 &= \frac{1}{4} (f^1(\xi^1) + f^2(\xi^2) + f^3(\xi^3) + f^4(\xi^4)) , \\ x^1 &= \frac{\sqrt{3}}{4} (f^1(\xi^1) + f^2(\xi^2) - f^3(\xi^3) - f^4(\xi^4)) , \\ x^2 &= \frac{\sqrt{3}}{4} (f^1(\xi^1) - f^2(\xi^2) + f^3(\xi^3) - f^4(\xi^4)) , \\ x^3 &= \frac{\sqrt{3}}{4} (f^1(\xi^1) - f^2(\xi^2) - f^3(\xi^3) + f^4(\xi^4)) . \end{aligned} \right\} \quad (27)$$

Therefore, to obtain the non-trivial curving of the space-time one should use the World functions with the broken conformal symmetry.

4 Newtonian potential

Let us show that there are World functions that lead to the non-trivial pseudo Riemannian 4-dimensional spaces. Let us regard a function

$$S(\xi) = \frac{1}{4} (\xi^1 + \xi^2 + \xi^3 + \xi^4) + \alpha \cdot \psi(\varrho), \quad (28)$$

where α is the parameter characterizing the break of the analyticity of the World function (the break of the conformal symmetry in the H_4 space), ψ is an arbitrary function of a single argument

$$\varrho = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}, \quad (29)$$

and y^0, y^1, y^2, y^3 are the coordinates in the orthogonal basis $1, j, k, jk$:

$$\left. \begin{aligned} y^0 &= \frac{1}{4}(\xi^1 + \xi^2 + \xi^3 + \xi^4), \\ y^1 &= \frac{1}{4}(\xi^1 + \xi^2 - \xi^3 - \xi^4), \\ y^2 &= \frac{1}{4}(\xi^1 - \xi^2 + \xi^3 - \xi^4), \\ y^3 &= \frac{1}{4}(\xi^1 - \xi^2 - \xi^3 + \xi^4). \end{aligned} \right\} \quad (30)$$

Then the derivatives of the World functions over the coordinates ξ^i can be expressed in the following way:

$$\left. \begin{aligned} \frac{\partial S}{\partial \xi^1} &= \frac{1}{4} \left[1 + \frac{\alpha}{\varrho} \frac{d\psi}{d\varrho} (y^1 + y^2 + y^3) \right], \\ \frac{\partial S}{\partial \xi^2} &= \frac{1}{4} \left[1 + \frac{\alpha}{\varrho} \frac{d\psi}{d\varrho} (y^1 - y^2 - y^3) \right], \\ \frac{\partial S}{\partial \xi^3} &= \frac{1}{4} \left[1 + \frac{\alpha}{\varrho} \frac{d\psi}{d\varrho} (-y^1 + y^2 - y^3) \right], \\ \frac{\partial S}{\partial \xi^4} &= \frac{1}{4} \left[1 + \frac{\alpha}{\varrho} \frac{d\psi}{d\varrho} (-y^1 - y^2 + y^3) \right]. \end{aligned} \right\} \quad (31)$$

Let us calculate the components of the metric tensor in coordinates y^0, y^1, y^2, y^3 using the invariance of the square of the length element

$$g_{ij}(\xi) d\xi^i d\xi^j = \tilde{g}_{ij}(y) dy^i dy^j \quad (32)$$

Grouping the terms, one gets

$$\tilde{g}_{00} = 1 - 3\alpha^2 \left(\frac{d\psi}{d\varrho} \right)^2, \quad \tilde{g}_{\beta\beta_-} = -3 \left\{ 1 + \alpha^2 \left(\frac{d\psi}{d\varrho} \right)^2 \left[1 - \frac{4(y^\alpha)^2}{3\rho^2} \right] \right\}, \quad (33)$$

$$2\tilde{g}_{0\beta} = -4 \left[\alpha \frac{d\psi}{d\varrho} \frac{y^\beta}{\varrho} + 3\alpha^2 \left(\frac{d\psi}{d\varrho} \right)^2 \cdot \frac{y^1 y^2 y^3}{y^\beta \varrho^2} \right], \quad (34)$$

$$2\tilde{g}_{\beta\gamma} = -4 \left[3\alpha \frac{d\psi}{d\varrho} \frac{y^\delta}{\varrho} + \alpha^2 \left(\frac{d\psi}{d\varrho} \right)^2 \cdot \frac{y^\beta y^\gamma}{\varrho^2} \right], \quad (35)$$

where $\beta, \gamma, \delta, = 1, 2, 3$; $\beta \equiv \beta_-$ but no summation is performed here; in the last formula all the indices β, γ, δ are different.

If $\alpha = 0$, then

$$(\tilde{g}_{ij}) = \text{diag}(1, -3, -3, -3). \quad (36)$$

This means that the real physical coordinates x^0, x^1, x^2, x^3 of the space-time are expressed by the coordinates y^0, y^1, y^2, y^3 in the following way

$$x^0 = y^0, \quad x^\beta = \sqrt{3} \cdot y^\beta. \quad (37)$$

Let us pass to the physical coordinates x^0, x^1, x^2, x^3 :

$$\tilde{g}_{ij}(y) dy^i dy^j = \bar{g}_{ij}(x) dx^i dx^j, \quad (38)$$

where

$$\bar{g}_{00} = \tilde{g}_{00}, \quad \bar{g}_{0\beta} = \frac{1}{\sqrt{3}} \cdot \tilde{g}_{0\beta}, \quad \bar{g}_{\beta\gamma} = \frac{1}{3} \cdot \tilde{g}_{\beta\gamma}. \quad (39)$$

Let us denote

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \equiv \sqrt{3} \cdot \varrho, \quad (40)$$

Then

$$\bar{g}_{00} = 1 - 9\alpha^2 \left(\frac{d\psi}{dr} \right)^2, \quad \bar{g}_{\beta\beta} = - \left\{ 1 + 3\alpha^2 \left(\frac{d\psi}{dr} \right)^2 \left[1 - \frac{4(x^\alpha)^2}{3r^2} \right] \right\}, \quad (41)$$

$$2\bar{g}_{0\beta} = -4 \left[\alpha \frac{d\psi}{dr} \frac{x^\beta}{r} + 3\sqrt{3}\alpha^2 \left(\frac{d\psi}{dr} \right)^2 \cdot \frac{x^1 x^2 x^3}{x^\beta r^2} \right], \quad (42)$$

$$2\bar{g}_{\beta\gamma} = -4 \left[\sqrt{3}\alpha \frac{d\psi}{dr} \frac{x^\delta}{r} + \alpha^2 \left(\frac{d\psi}{dr} \right)^2 \cdot \frac{x^\beta x^\gamma}{r^2} \right]. \quad (43)$$

The metric tensor $\bar{g}_{ij}(x) = \bar{g}_{ij}(x^1, x^2, x^3)$ depends only on the space coordinates x^1, x^2, x^3 , and this corresponds to the stationary gravitational field, stationary Universe. The probe particle of mass m moves along the geodesic of the pseudo Riemannian space with metric tensor $\bar{g}_{ij}(x^1, x^2, x^3)$.

Let a particle move in a fixed frame and have velocity much less than the light velocity, c :

$$\frac{dx^\beta}{dt} = v^\beta, \quad |v^\beta| \ll c, \quad (44)$$

The gravitational fields are weak, that is the condition $|v^\beta| \ll 1$ remains valid for all the time of the particle motion. Let us obtain the Lagrange function, L , to describe such non-relativistic motion of the probe particle in the weak gravity field. To do this, develop the right hand side of the expression

$$L = -mc \cdot \frac{\sqrt{\bar{g}_{ij}(x^1, x^2, x^3) dx^i dx^j}}{dt} \quad (45)$$

Within the accuracy of $\left(\frac{v}{c}\right)^2$

$$L = -mc^2 \sqrt{\bar{g}_{00}} \cdot \sqrt{1 + \frac{1}{\bar{g}_{00}} \left(2\bar{g}_{0\beta} \frac{v^\beta}{c} + \bar{g}_{\beta\gamma} \frac{v^\beta v^\gamma}{c^2} \right)}, \quad (46)$$

$$L \simeq -mc^2 \sqrt{\bar{g}_{00}} \cdot \left\{ 1 + \frac{1}{2\bar{g}_{00}} \left(2\bar{g}_{0\beta} \frac{v^\beta}{c} + \bar{g}_{\beta\gamma} \frac{v^\beta v^\gamma}{c^2} \right) - \frac{1}{8\bar{g}_{00}^2} \left(2\bar{g}_{0\beta} \frac{v^\beta}{c} \right)^2 \right\}. \quad (47)$$

Opening the brackets in the right hand side, we get an additive term which is the full time derivative of a certain function $f(r)$, it depends linearly on the velocity components and, thus, it can be omitted. Leaving the same designation for the Lagrange function, we get

$$L \simeq -mc^2 \sqrt{\bar{g}_{00}} \cdot \left\{ 1 + \frac{1}{2\bar{g}_{00}} \cdot \bar{g}_{\beta\gamma} \frac{v^\beta v^\gamma}{c^2} - \frac{1}{8\bar{g}_{00}^2} \left(2\bar{g}_{0\beta} \frac{v^\beta}{c} \right)^2 \right\}. \quad (48)$$

Our goal is the Lagrange function of the form

$$L = \frac{m\vec{v}^2}{2} - U(\vec{x}), \quad (49)$$

where $U(\vec{x})$ is the potential energy of the probe particle, $\vec{x} \equiv (x^1, x^2, x^3)$, $\vec{v} \equiv (v^1, v^2, v^3)$, $r^2 = \vec{x}^2$, $\vec{v}^2 = (v^1)^2 + (v^2)^2 + (v^3)^2 \equiv v^2$. To reach it we have to make some assumptions about the correlation between the parameter, α and light velocity:

$$\alpha = \frac{\nu}{c}, \quad \text{when } c \rightarrow \infty \quad \alpha \rightarrow 0. \quad (50)$$

Besides, let α be of the same order (or smaller) with the relation $\left| \frac{\nu}{c} \right|$. Then leaving only the terms that don't disappear at $c \rightarrow \infty$ in the (48), one gets

$$L \simeq -mc^2 + mc^2 \frac{9\nu^2}{2c^2} \left(\frac{d\psi}{dr} \right)^2 + m \cdot \frac{v^1 v^1 + v^2 v^2 + v^3 v^3}{2}. \quad (51)$$

Since $(-mc^2)$ is a full time derivative of function $(-mc^2 \cdot t)$, we omit it and get

$$L \simeq \frac{m\vec{v}^2}{2} + \frac{9m\nu^2}{2} \left(\frac{d\psi}{dr} \right)^2. \quad (52)$$

Let a mass M be motionless in the frame origin, and then the potential energy of the probe particle with mass m located at x^1, x^2, x^3 is equal to

$$U(r) = -\gamma \frac{mM}{r}, \quad (53)$$

where γ is the gravitational constant. Comparing (49) and (52), we get the equation for $\psi(r)$:

$$\frac{9m\nu^2}{2} \left(\frac{d\psi}{dr} \right)^2 = \gamma \frac{mM}{r} \quad \Rightarrow \quad \frac{d\psi}{dr} = \pm \frac{\sqrt{2\gamma M}}{3\nu} \frac{1}{r^{1/2}}. \quad (54)$$

Therefore,

$$\psi(r) = \pm \frac{2\sqrt{2\gamma M}}{3\nu} \cdot r^{1/2} + \psi_0 \quad (\psi_0 = \text{const}). \quad (55)$$

Finally, the World function is equal to

$$S = x^0 \pm \frac{2\sqrt{2\gamma M}}{3c} \cdot r^{1/2} + C_0 \quad (C_0 = \text{const}), \quad (56)$$

When it performs a conformal transformation of the length element of the plane Berwald-Moor space, it induces a pseudo Riemannian geometry in the Minkowski space. For a non-relativistic probe particle of mass m , this geometry gives the motion equations for the Kepler problem for the point mass M located in the origin of the space frame.

The more complicated World function, maybe also leading to the stationary Universe, has the form

$$S(\xi) = \frac{1}{4} (\xi^1 + \xi^2 + \xi^3 + \xi^4) [1 + \alpha_1 \cdot \psi_1(\varrho)] + \alpha_2 \cdot \psi_2(\varrho), \quad (57)$$

where α_A are the parameters of the analyticity break of the World function (parameters of the conformal symmetry break in the H_4 space), ψ_A are the arbitrary functions of single argument ϱ (29), (30).

Conclusion

The results obtained in this paper point at the deep correlation between the Einstein geometries and Finsler spaces with Berwald-Moor metric. We managed to find the concrete Finsler space with the Berwald-Moor metric which in the limit appeared to be related to the curved pseudo Riemannian space with the Newtonian gravitational potential. This fact points at the principal possibility to built more interesting constructions, particularly, such Finsler spaces whose limit cases would be the known relativistic solutions.

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