BERWALD-MOOR-TYPE (h,v)-METRIC PHYSICAL MODELS

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In the framework of vector bundles endowed with $(h, v)-$metrics several physical models for relativity are presented. A characteristic of these models is that the vertical part is provided by the flag-Finsler Berwald-Moor (fFBM) metric, while the horizontal part is specialized to the conformal and to Synge-relativistic optics metrics. As well, the particular case of $h-$Riemannian $v-$fFBM metric of Riemann-Minkowski type is examined, considering as nonlinear connection both the trivial canonical connection, and the one induced by the Lagrangian of electrodynamics. For all these models, basic properties are described and the extended Einstein and Maxwell equations are determined.

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1 Introduction

The recent attempts of modeling relativity based on metrical structures include two notable trends: one originates in the theory of bundles endowed with Ehresmann connection (e.g. via osculating structures and their duals, R. Miron [7–10]) and one based on a palette of physical models relying on the Berwald-Moor metric (D.G. Pavlov, G.S. Asanov [1, 12, 13]). The present work proposes several relativistic models of Miron type which emerge naturally from this metric. The basic geometric structure is an $(h, v)-$metric on a vector bundle (in particular the tangent bundle of a Space-Time), where the horizontal part is of Generalized Lagrange type ([8]) and the vertical one is of Finslerian Berwald-Moor type. For these models ($h-$conformal, h-relativistic optic, $h-$electromagnetic and $h-$classical Riemannian) the GR formalism is developed, and the Einstein and relativistic Maxwell equations are described.

2 The flag-Finsler Berwald-Moor metric

Let M be a 4-dimensional differential manifold of class $\mathcal{C}^\infty$, TM its tangent bundle and (x^i, y^a) the coordinates in a local chart on TM. If $F : TM \to \mathbf{R}$, $F = F(y)$ is a Finsler function, we denote by

$$h^*_{ab} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^a \partial y^b}, \quad a, b = \overline{1, 4},$$

the associated metric tensor. For $F(y) = \sqrt[4]{|y^1y^2y^3y^4|}$, Pavlov has studied the "4-pseudoscalar product" related to the Berwald-Moor metric ([13])

$$(X, Y, Z, T) = G_{abcd}X^aY^bZ^cT^d, \tag{2.1}$$

where

$$G_{abcd} = \frac{1}{4!}\frac{\partial^4 \mathcal{L}}{\partial y^a \partial y^b \partial y^c \partial y^d}, \tag{2.2}$$

and $\mathcal{L} = F^4$. We denote

$$\langle X, Y\rangle = \frac{1}{F^2}(X, Y, y, y,), \quad X, Y \in \mathcal{X}(M), \tag{2.3}$$

where $y = y^a \dfrac{\partial}{\partial y^a}$ is the Liouville vector field ([9]), the vector fields X, Y being considered at some $x \in M$. Then $\langle\ ,\ \rangle$ is a pseudo-scalar product; locally we have

$$\langle X, Y\rangle = \frac{1}{F^2} G_{abcd} X^a Y^b y^c y^d = \frac{G_{ab00}}{F^2} X^a Y^b, \tag{2.4}$$

where the null index represents transvection with y. The coefficients of the scalar product (2.4) are hence

$$h_{ab} = \frac{G_{ab00}}{F^2} = \frac{1}{12F^2}\frac{\partial^2 F^4}{\partial y^a \partial y^b}, \tag{2.5}$$

providing a tensor which coincides with the one $\widetilde{y}^{(4)}_{ij}$ proposed by Lebedev ([6]). Then, h_{ab} is a 2-covariant tensor field, and (M, h) thus becomes a generalized Lagrange space. Its absolute energy, $\mathcal{E} = h_{ab} y^a y^b$, is

$$\mathcal{E} = \frac{G_{ab00}}{F^2} y^a y^b = \frac{1}{4F^2}\frac{\partial F^4}{\partial y^b} y^b = \frac{F^4}{F^2} = F^2,$$

this is, $\mathcal{E} = F^2$. The Lagrange metric associated to h is exactly

$$\frac{1}{2}\frac{\partial^2 \mathcal{E}}{\partial y^a \partial y^b} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^a \partial y^b} = h^*_{ab},$$

and taking into account that F is a Finsler function, h^* is nondegenerate and of constant signature, which shows that $(M, \mathcal{E} = F^2)$ is a Lagrange space. From the homogeneity of F it also follows that

$$\frac{1}{2}\frac{\partial \mathcal{E}}{\partial y^a} = h_{ab} y^b. \tag{2.6}$$

Consequently, we can state

Theorem 1. *The space (M, h) with h given by (2.5) is a generalized Lagrange space with regular metric. The associated Lagrange metric h^*_{ab} coincides with the Finsler metric generated by F and the two metrics provide the same energy,*

$$\mathcal{E} = F^2 = h_{ab} y^a y^b = h^*_{ab} y^a y^b.$$

Remark. The considerations above hold true for an arbitrary Finsler space whose fundamental function is of locally Minkowski type.

3 A Riemann-locally Minkovski model

Let TM be endowed with a nonlinear connection N with coefficients $N^a_{\ i} = N^a_{\ i}(x, y)$ and let

$$\left\{ \delta_i = \frac{\delta}{\delta x^i}, \dot{\partial}_a = \frac{\partial}{\partial y^a} \;\middle|\; i, a = \overline{1,4} \right\}$$

denote the corresponding adapted basis, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^b_{\ i}\frac{\partial}{\partial y^b}, \ i = \overline{1,4}.$$

We also denote the dual basis by $\{dx^i, \delta y^a \mid i, a = \overline{1,4}\}$, with $\delta y^a = dy^a + N^a_{\ j} dx^j$. If D is a linear d-connection on TM ([9]), then it is described by its adapted coefficients $D\Gamma(N) = \{L^i_{\ jk}, L^a_{\ bk}, C^i_{\ jc}, C^a_{\ bc}\}$, where:

$$D_{\delta_k}\delta_j = L^i_{\ jk}\delta_i,\ D_{\delta_k}\dot{\partial}_b = L^a_{\ bk}\dot{\partial}_a,$$

$$D_{\dot{\partial}_c}\delta_j = C^i_{\ jc}\delta_i,\ D_{\dot{\partial}_c}\dot{\partial}_b = C^a_{\ bc}\dot{\partial}_a.$$

We shall further denote by $_{|}$ and $|$ the $h-$ and $v-$ covariant derivatives induced by D respectively.

As well, the torsion T of the linear connection D has the adapted components

$$\begin{array}{ll} hT(\delta_k, \delta_j) = T^i_{\ jk}\delta_i, & vT(\delta_k, \delta_j) = R^a_{\ jk}\dot{\partial}_a, \\ hT(\dot{\partial}_c, \delta_j) = C^i_{\ jc}\delta_i, & vT(\dot{\partial}_c, \delta_j) = P^a_{\ jc}\dot{\partial}_a, \\ hT(\dot{\partial}_c, \dot{\partial}_b) = 0, & vT(\dot{\partial}_c, \dot{\partial}_b) = S^a_{\ bc}\dot{\partial}_a, \end{array}$$

while the adapted components of the curvature R are

$$\begin{array}{ll} R(\delta_l, \delta_k)\delta_j = R^i_{jkl}\delta_i, & R(\delta_l, \delta_k)\dot{\partial}_b = R^a_{bkl}\dot{\partial}_a, \\ R(\dot{\partial}_c, \delta_k)\delta_j = P^i_{jkc}\delta_i, & R(\dot{\partial}_c, \delta_k)\dot{\partial}_b = P^a_{bkc}\dot{\partial}_a, \\ R(\dot{\partial}_c, \dot{\partial}_b)\delta_j = S^i_{jbc}\delta_i, & R(\dot{\partial}_d, \dot{\partial}_c)\dot{\partial}_b = S^a_{bcd}\dot{\partial}_a. \end{array}$$

Now, let us consider on TM the following Riemann-locally Minkovski $(h, v)-$metric:

$$\mathcal{G} = g_{ij}(x)dx^i \otimes dx^j + h_{ab}(y)\delta y^a \otimes \delta y^b, \tag{3.1}$$

which we shall use in our further considerations. Together with N, this metric produces the *canonical* metrical $d-$connection $C\Gamma(N)$ ([9]),

$$\begin{cases} L^i_{\ jk} = \dfrac{1}{2}g^{ih}\left(\dfrac{\delta g_{hj}}{\delta x^k} + \dfrac{\delta g_{hk}}{\delta x^j} - \dfrac{\delta g_{jk}}{\delta x^h}\right), \\ L^a_{\ bk} = \dfrac{\partial N^a_k}{\partial y^b} + \dfrac{1}{2}h^{ac}\left(\dfrac{\delta h_{bc}}{\delta x^k} - \dfrac{\partial N^d_k}{\partial y^b}h_{dc} - \dfrac{\partial N^d_k}{\partial y^c}h_{bd}\right), \\ C^i_{\ jc} = \dfrac{1}{2}g^{ih}\dfrac{\partial g_{jh}}{\partial y^c}, \\ C^a_{\ bc} = \dfrac{1}{2}h^{ad}\left(\dfrac{\partial h_{db}}{\partial y^c} + \dfrac{\partial h_{dc}}{\partial y^b} - \dfrac{\partial h_{bc}}{\partial y^d}\right). \end{cases} \tag{3.2}$$

For h given in (2.5), the $(h, v)-$metric $\mathcal{G}$ given in (3.1) is v-regular, which implies that the coefficients of the canonical (Kern [4,9]) nonlinear connection $\tilde{N}$ vanish,

$$N^i_a(x, y) = 0, \quad i, a = \overline{1,4}. \tag{3.3}$$

The canonical metrical linear d-connection $C\Gamma(\tilde{N})$ associated to $\mathcal{G}$, is given by ([9])

$$L^i_{\ jk} = \gamma^i_{\ jk},\ L^a_{\ bk} = 0,\ C^i_{\ jc} = 0,\ C^a_{\ bc} = \frac{1}{2}h^{ad}\left(\frac{\partial h_{db}}{\partial y^c} + \frac{\partial h_{dc}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d}\right),$$

where $\gamma^i_{\ jk}$ denote the Christoffel symbols of g. It is worth mentioning that, for the canonic d-linear connection in the Kern case (3.3), the torsion vanishes,

$$T^i_{\ jk} = 0,\ R^a_{\ jk} = 0,\ C^i_{\ jc} = 0,\ P^a_{\ jb} = 0,\ S^a_{\ bc} = 0.$$

4 Locally v-Minkovskian metrics

In general, an $(h, v)-$metric

$$\mathcal{G} = g_{ij}(x,y)dx^i \otimes dx^j + h_{ab}(x,y)\delta y^a \otimes \delta y^b \tag{4.1}$$

which has the property that in the neghborhood of any point $(x, y) \in TM$ there exists a local map in which $h(x, y) = h(y)$, is called *v-locally Minkovski.* A known result provides consequences specific to this case, as follows

Theorem 2. ([9]) *If $\mathcal{G}$ is a v-locally Minkovski metric and $h = h(y)$ is weakly regular, then the Kern nonlinear connection $\tilde{N}$ and the canonic linear d-connection D (3.2) given by $C\Gamma(\tilde{N}) = \{L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\}$ obey the properties*

1. $N^a_j = 0, \quad L^i{}_{jk} = \{^{\,i}_{jk}\}, \quad L^a{}_{bk} = 0;$

2. $T^i{}_{jk} = 0, \quad S^a{}_{bc} = 0, \quad R^a{}_{jk} = 0, \quad P^a{}_{jb} = 0.$

3. $R^a_{b\ jk} = 0, \quad P^a_{b\ kc} = 0,$

where $\{^i{}_{jk}\}$ are the Christoffel symbols corresponding to $g = g(x, y)$.

Remark 1. In our case, the following consequences hold true:

1. The equality $N^a_j = 0$ yields $\dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i}$.
2. The torsion of the canonic linear d-connection has a single non-vanishing component, namely the coefficient $P^i{}_{jc} = C^i{}_{jc}$ of $hT(\dot{\partial}_c, \delta_j)$.
3. $C^a{}_{bc}$ are the Christoffel symbols of second kind associated to $h_{ab} = h_{ab}(y)$ and they depend on y only.

We shall assume further that $h = h(y)$ is the metric (2.5) from [3]; this satisfies

$$h_{ab} = \frac{1}{12\mathcal{E}}\frac{\partial^2\mathcal{E}^2}{\partial y^a \partial y^b}.$$

In this case, the *deflection tensor fields* attached to the nonlinear connection above are

$$D^a{}_j = y^a{}_{|j} = \frac{\partial y^a}{\partial x^j} + y^b L^a{}_{bj} = 0, \quad d^a{}_b = y^a|_b = \delta^a_b + y^c C^a{}_{cb}.$$

From the definition of C^a_{cb}, (since h is $0-$homogeneous in y), it follows that

$$y^c C^a{}_{cb} = \frac{1}{2}h^{ad}\left(\frac{\partial h_{bd}}{\partial y^c} + \frac{\partial h_{dc}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d}\right)y^c = \frac{1}{2}h^{ad}\left(\frac{\partial h_{dc}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d}\right)y^c.$$

Taking into account the particular form (2.5) of h, and taking into account the homogeneity of $\mathcal{E}$, we get by deriving the product w.r.t. y^b that

$$\frac{\partial h_{dc}}{\partial y^b}y^c = \frac{1}{12}y^c\frac{\partial}{\partial y^b}\left(\frac{1}{\mathcal{E}}\frac{\partial^2\mathcal{E}^2}{\partial y^c \partial y^d}\right) = -\frac{1}{2\mathcal{E}}\frac{\partial \mathcal{E}}{\partial y^b}\frac{\partial \mathcal{E}}{\partial y^d} + 2h_{bd},$$

is a geometric object symmetrical in the indices b and d, whence

$$y^c C^a{}_{cb} = 0 \Rightarrow d^a{}_b = \delta^a{}_b.$$

Hence, the canonic linear d-connection is *of Cartan type* ([9]) and the deflection tensors are

$$D_{ij} = 0, \quad d_{ab} = h_{ab},$$

where the indices were raised/lowered using the corresponding parts of the $(h, v)-$metric. We obtain subsequently that the *electromagnetic tensors identically vanish*,

$$\begin{cases} F_{ij} = \dfrac{1}{2}(D_{ij} - D_{ji}) = 0, \\ f_{ab} = \dfrac{1}{2}(d_{ab} - d_{ba}) = 0. \end{cases}$$

and, since D is of Cartan type, we have ([9])

$$S^a_{d\ bc}y^d = S^a_{\ bc} = 0, y^d R^a_{d\ jk} = R^a_{\ jk} = 0,\ y^d P^a_{d\ kc} = P^a_{\ kc} = 0.$$

5 Einstein equations for the Riemann-locally Minkovski model

The curvatures of the canonical metrical linear d-connection associated to $\mathcal{G}$ in (3.1) with (2.5) are, according to [9],

$$\begin{cases} R_j{}^i{}_{kh} = r_j{}^i{}_{kh}, R_b{}^a{}_{kh} = 0, \quad P_j{}^i{}_{kc} = 0, P_b{}^a{}_{kc} = 0,\ S_j{}^i{}_{bc} = 0, \\ S^a_{b\ cd} = \dfrac{\partial C^a_{\ bc}}{\partial y^d} - \dfrac{\partial C^a_{\ bd}}{\partial y^c} + C^f_{\ bc}C^a_{\ fd} - C^f_{\ bd}C^a_{\ fc}, \end{cases} \tag{5.1}$$

where $r^i_{j\ kh}$ are the components of the curvature tensor of the horizontal metric. Taking into account the relations (5.1), it follows, as in [9], that the Einstein equations of the canonical metrical linear d-connection $C\Gamma(\tilde{N})$ (3.2)-(3.3) can be written as

$$\begin{cases} r_{ij} - \dfrac{1}{2}(r + S)g_{ij} = T^H_{ij}, \\ T^{M_1}_{bj} = 0, \quad T^{M_2}_{jb} = 0, \\ S_{ab} - \dfrac{1}{2}(r + S)h_{ab} = T^V_{ab}, \end{cases} \tag{5.2}$$

where r_{ij} denotes the Ricci tensor $r_{ij} = r_i{}^h{}_{jh}$ attached to the Riemannian metric g, S_{ab} is the Ricci tensor attached to the vertical metric h_{ab}, r is the scalar curvature of r^i_{jkl} and $T_{\alpha\beta}$ are the components of the energy-momentum tensor field. If it is to compare (5.2) with the (classical) Einstein equations of the Riemannian manifold (M, g), we have to notice in the h-part of the above equations the "perturbation"introduced by the term $-\dfrac{1}{2}Sg_{ij}$. According to [9], the energy conservation law is identically satisfied by $C\Gamma(\tilde{N})$.

6 The electrodynamic case

If we consider the Lagrangian of electrodynamics ([10]),

$$L_0(x, y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i, \tag{6.1}$$

where γ_{ij} is a Lorentz metric tensor, $A_i(x)$ is a covector field and m, c, e are physical constants, then, the attached Lagrange metric tensor is

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} = mc\gamma_{ij}.$$

On the other hand, from the variational problem associated to 6.1, there arises a nonlinear connection $\hat{N}$, whose coefficients are given by ([10])

$$N^a_j = \gamma^a{}_{jb}(x)y^b - \overset{\circ}{F}{}^a{}_j, \tag{6.2}$$

where F is the electromagnetic field

$$\overset{\circ}{F}{}^i{}_j = \frac{e}{2m} g^{ik}(A_{j;k} - A_{k;j}),$$

the symbol ";" denotes the covariant derivative defined by means of the Christoffel symbols $\gamma^i{}_{jk}(x)$ of the Lorentz metric tensor γ_{ij}, and we denoted for simplicity, $\gamma^a{}_{jb} = \delta^a_i \delta^k_b \gamma^i{}_{jk}$.

If we consider now TM endowed with the (h,v)-metric

$$\mathcal{G} = g_{ij}(x)dx^i \otimes dx^j + h_{ab}(y)\delta y^a \otimes \delta y^b,$$

then the canonical metrical linear $d-$connection $C\Gamma(\hat{N})$ associated to $\mathcal{G}$ is given by

$$\begin{cases} \hat{L}^i{}_{jk} = \gamma^i{}_{jk}, \\ \hat{L}^a{}_{bk} = \gamma^a{}_{bk} - \frac{1}{2}h^{ac}(N^d_k \frac{\partial h_{bc}}{\partial y^d} + \gamma^d{}_{kb}h_{dc} + \gamma^d{}_{kc}h_{bd}), \\ \hat{C}^i{}_{jc} = 0, \quad C^a{}_{bc} = \frac{1}{2}h^{ad}\left(\frac{\partial h_{db}}{\partial y^c} + \frac{\partial h_{dc}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d}\right). \end{cases}$$

By direct computation, one obtains that this time, the torsions of $C\Gamma(\hat{N})$ are

$$T^i{}_{jk} = 0, \; C^i{}_{jc} = 0, \; S^a{}_{bc} = 0,$$

while $P^a{}_{jb}$ and $R^a{}_{jk}$ do not vanish. Its curvatures are

$$R_j{}^i{}_{kh} = r_j{}^i{}_{kh}, \; R_b{}^a{}_{kh}, \quad P_j{}^i{}_{kc} = 0, \; P_b{}^a{}_{kc}, \quad S_j{}^i{}_{bc} = 0, \; S^a_b{}_{cd},$$

where the expression of $S^a_b{}_{cd}$ is similar to that one in the previous section. The Ricci tensor has the properties

$$R_{ij} = r_{ij}, \overset{2}{P}_{jb} = P^h_j{}_{hb} = 0.$$

The Einstein equations take the particular form

$$\begin{cases} r_{ij} - \frac{1}{2}(r+S)g_{ij} = \overset{h}{T}_{ij}, \\ \overset{1}{T}_{bj} = \overset{1}{P}_{bj}, \; \overset{2}{T}_{jb} = 0, \\ S_{ab} - \frac{1}{2}(r+S)h_{ab} = \overset{v}{T}_{ab}, \end{cases}$$

while the energy conservation law writes as:

$$\begin{cases} \left(r^i{}_j - \frac{1}{2}r\delta^i{}_j\right)_{|i} + \overset{1}{P}{}^a{}_j|_a = 0, \\ \left(S^a{}_b - \frac{1}{2}S\delta^a{}_b\right)|_a = 0, \end{cases}$$

where $r^i_j = g^{ih} r_{hj}$, $S^a_b = h^{ac} S_{cb}$, $\overset{1}{P}{}^a_j = h^{ac} \overset{2}{P}_{cj}$.

We shall study further two particular cases of v-locally Minkowski metrics, by preserving $h = h(y)$ from (2.5) and particularizing $g = g(x, y)$. In these cases the results in Section 4 still hold true, and the nonlinear connection used throughover is according to Theorem 2, the trivial one.

7 The relativistic Miron-Kawaguchi optic $h-$metric case

Let $\gamma_{ij} = \gamma_{ij}(x)$ be a Riemannian metric on M. We denote

$$y_i = \gamma_{ij} y^j, \ \|y\|^2 = \gamma_{ij} y^i y^j.$$

We consider now the metric $\mathcal{G}$ from (4.1), in which the $h-$metric is given by

$$g_{ij} = \gamma_{ij} + c^{-2} y_i y_j,$$

where c is a nonzero real constant. The coefficients $C^i{}_{jd}$ of the linear $d-$connection are

$$C^i{}_{jd} = \frac{1}{2} g^{ih} \frac{\partial g_{jh}}{\partial y^d} = \frac{g^{ih}}{2c^2} (\gamma_{jd} y_h + \gamma_{hd} y_j),$$

and $C^a{}_{bc} = C^a{}_{bc}(y)$ are determined in [3].

From the theorem above, it results that the Ricci tensor field has the components

$$\begin{aligned} R_{ij} &= R^h_{i\ jh}, \ \overset{1}{P}_{bj} = P^a_{b\ ka} = 0, \\ \overset{2}{P}_{jb} &= P^h_{j\ hb}, \ S_{bc} = S^a_{b\ ca}. \end{aligned}$$

The Einstein equations write then

$$\begin{cases} R_{ij} - \dfrac{1}{2}(R+S) g_{ij} = \overset{h}{T}_{ij}, \\ \overset{1}{T}_{bj} = 0, \ \overset{2}{T}_{jb} = -\overset{2}{P}_{jb}, \\ S_{ab} - \dfrac{1}{2}(R+S) h_{ab} = \overset{v}{T}_{ab}, \end{cases}$$

and the energy conservation law is described by the system of PDEs

$$\begin{cases} \left(R^i{}_j - \dfrac{1}{2} R \delta^i{}_j\right)_{|i} = 0, \\ \left(S^a{}_b - \dfrac{1}{2}(R+S)\delta^a{}_b\right)|_a - \overset{2}{P}{}^i_{b|i} = 0, \end{cases}$$

where $R^i{}_j = g^{ih} R_{hj}$, $S^a{}_b = h^{ac} S_{cb}$, $\overset{2}{P}{}^i{}_b = g^{ij} \overset{2}{P}_{jb}$.

The first equality from above is identically satisfied (see [8]), since it coincides with the horizontal part of the energy conservation law for the canonical linear $d-$connection of the generalized Lagrange space (M, g) (which is infered straightforward by the Bianchi identity).

8 The $h-$conformal metric case

In the $h-$conformal metric case, i.e. for the horizontal metric given by

$$g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x),$$

the coefficients $L^i{}_{jk}$ are given by ([2])

$$L^i{}_{jk} = \gamma^i{}_{jk} + \delta^i_j\sigma_k + \delta^i_k\sigma_j - \gamma_{jk}\sigma^i,$$

where $\sigma^i = \gamma^{il}\sigma_l$, $\gamma^i{}_{jk}$ are the Christoffel symbols of $\gamma_{ij}(x)$ and for h given by (2.5) we have $\sigma_k = \dfrac{\delta\sigma}{\delta x^k} = \dfrac{\partial\sigma}{\partial x^k}$. Obviously, $L^a{}_{bk}$ and $C^a{}_{bc}$ are as in Theorem 2 and Remark 1. By direct computation, we get

$$C^i{}_{jc} = \frac{1}{2}g^{ih}\frac{\partial g_{jh}}{\partial y^c} = \delta^i_j\dot{\sigma}_c,$$

where $\dot{\sigma}_c$ denotes the derivative of σ w.r.t. $y : \dot{\sigma}_c = \dfrac{\partial\sigma}{\partial y^c}$. As well, the torsion components vanish, except $P^i{}_{jc} = C^i{}_{jc}$ and the curvature components are

$$\begin{cases} R^i_{j\ kl},\ R^a_{b\ jk} = 0, P^i_{j\ kc},\ P^a_{b\ kc} = 0,\ S^a_{b\ cd.} \\ S^i_{j\ bc} = \dfrac{\partial C^i{}_{jb}}{\partial y^c} - \dfrac{\partial C^i{}_{jc}}{\partial y^b} + C^h{}_{jb}C^i{}_{hc} - C^h{}_{jc}C^i{}_{hb} = 0 \\ P^i_{j\ kb} = \delta^i_k\sigma_{jb} - \gamma_{jk}\gamma^{il}\sigma_{lb}, \end{cases}$$

where $\sigma_{jb} = \dfrac{\partial^2\sigma}{\partial x^j\partial y^b}$, $\sigma_{lb} = \dfrac{\partial^2\sigma}{\partial x^l\partial y^b}$. The Ricci tensor has the properties: $\overset{1}{P}_{bj} = P^a_{b\ ka} = 0$ and

$$\overset{2}{P}_{jb} = P^h_{j\ hb} = \delta^h_h\sigma_{jb} - \gamma_{jh}\gamma^{hl}\sigma_{lb} = 4\sigma_{jb} - \delta^l_j\sigma_{lb} = 3\sigma_{jb}.$$

Then the Einstein equations are

$$\begin{cases} R_{ij} - \dfrac{1}{2}(R+S)g_{ij} = \overset{h}{T}_{ij}, \\ \overset{1}{T}_{bj} = 0,\ \overset{2}{T}_{jb} = -3\sigma_{jb}, \\ S_{ab} - \dfrac{1}{2}(R+S)h_{ab} = \overset{v}{T}_{ab}. \end{cases}$$

Taking into account that $S = S(y)$, the conservation law is described by

$$\begin{cases} \left(R^i{}_j - \dfrac{1}{2}R\delta^i{}_j\right)_{|i} = 0, \\ \left(S^a{}_b - \dfrac{1}{2}(R+S)\delta^a{}_b\right)|_a - 3\sigma^j{}_{b|j} = 0, \end{cases}$$

where the first equality is identically satisfied.

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References

[1] G.S. Asanov, *Finsleroid space supplemented by angle and scalar product*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 40-62.

[2] V.Balan, *Generalized Einstein-Yang Mills Equations for the Space* $GL^n = (M, g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x))$, Tensor N.S., 52 (1993), 199-203.

[3] V. Balan, N. Brinzei, *Einstein equations for the Berwald-Moor type Euclidean-locally Minkowski relativistic model*, preprint.

[4] J. Kern, *Lagrange geometry*, Arch. Math. 25 (1974), 483-443.

[5] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry I, II*, Interscience Publishers, New York, (1963), (1969).

[6] S.V. Lebedev, *Properties of spaces associated with commutative-associative* H_3 *and* H_4 *algebras*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 63-69.

[7] R. Miron, *The Geometry of Higher Order Finsler Spaces*, Hadronic Press, 1998.

[8] R. Miron, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publishers, 1994.

[9] R. Miron, M.Anastasiei, *Vector Bundles. Lagrange Spaces. Applications to Relativity*, Geometry Balkan Press, 1996.

[10] R. Miron, D. Hrimiuc, H. Shimada, S.V. Sabău, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer, FTPh. 198, 2001.

[11] Gh. Munteanu, V. Balan, *Lectures of Relativity Theory* (Romanian), Bren Eds., Bucharest, 2000.

[12] D.G. Pavlov, *Chronometry of the three-dimensional time*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 19-30.

[13] D.G. Pavlov, *Four-dimensional time*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 31-39.

[14] D.G. Pavlov, *Generalization of scalar product axioms*, Hypercomplex Numbers in Geometry and Physics, Ed. "Mozet", Russia, 1, 1 (2004), 5-18.