THE 2-COTANGENT BUNDLE WITH BERWALD-MOOR METRIC

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On the total space of the dual bundle $(T^{*2}M, \pi^{*2}, M)$ of the $2-$tangent bundle (T^2M, π^2, M), the paper develops results related to the notions: of nonlinear connection, distinguished tensor fields, almost contact structure, Riemannian structures, $N-$linear connections and associated convariant derivations. The Ricci identities are derived and the local expressions of the corresponding $d-$tensors of torsion and curvature are provided. Further, the metric structures and the metric $N-$linear connections are studied, and the obtained results are specialized to the case when the metric tensor field is of Berwald-Moor type.

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1 The dual bundle $(T^{*2}M, \pi^{*2}, M)$ of the 2-tangent bundle (T^2M, π^2, M)

Let M be a real differentiable manifold of dimension n. A point of M will be denoted by x and its local coordinates in a chart (U, φ), as $\varphi(x) = (x^a)$. The indices $a, b, ...$ will further run over the set $\{1, ..., n\}$ and the Einstein convention of transvection will be adapted all over this work. Let (TM, π, M) be the tangent bundle of the manifold M and let (T^*M, π^*, M) be its cotangent bundle ([7], [9]).

Definition 1.1. We call *the dual bundle* of the $2-$tangent bundle (T^2M, π^2, M), the differentiable bundle $(T^{*2}M, \pi^{*2}, M)$ whose total space is

$$T^{*2}M = TM \ \times_M \ T^*M \tag{1.1}$$

Sometimes we shall denote $(T^{*2}M, \pi^{*2}, M)$ briefly by $T^{*2}M$. A point $u \in T^{*2}M$ will be denoted by $u = (x, y, p)$, having the local coordinates (x^a, y^a, p_a). The projection is given by $\pi^{*2}(u) = \pi^{*2}(x, y, p) = x$. Evidently, we take the projections on the factors of the fibered product of (1.1): $\pi^{*2}_1 : T^{*2}M \to TM$, $\pi : TM \to M$ as being $\pi^{*2}_1(x, y, p) = (x, y)$ and $\pi^*(x, y) = x$; also, $\overline{\pi}^* : T^{*2}M \longrightarrow T^*M$ is given by $\overline{\pi}^*(u) = \overline{\pi}^*(x, y, p) = (x, p)$.

The change of local coordinates on the manifold $T^{*2}M$ is:

$$\begin{cases} \tilde{x}^a = \tilde{x}^a(x^1, ..., x^n), \quad \det\left(\dfrac{\partial \tilde{x}^a}{\partial x^b}\right) \neq 0, \\ \tilde{y}^a = \dfrac{\partial \tilde{x}^a}{\partial x^b} y^b, \\ \widetilde{p}_a = \dfrac{\partial x^b}{\partial \tilde{x}^a} p_b. \end{cases} \tag{1.2}$$

The dimension of the manifold $T^{*2}M$ is $3n$.

The null section $O : M \to T^{*2}M$ of the projection π^{*2} is defined by $O : (x) \in M \to (x, 0, 0) \in T^{*2}M$, where we denote $\widetilde{T^{*2}M} = T^{*2}M \setminus \{0\}$.

Let us consider the tangent bundle of the differentiable manifold $T^{*2}M$, $(TT^{*2}M, \tau^{*2}, T^{*2}M)$, where τ^{*2} is the canonical projection and the vertical distribution $V : u \in T^{*2}M \longrightarrow V(u) \subset T_uT^{*2}M$ generated by the vector fields $\left\{\frac{\partial}{\partial y^a}|_u, \frac{\partial}{\partial p_a}|_u\right\}$, $\forall u \in T^{*2}M$. We shall denote the natural basis as

$$\partial_a = \frac{\partial}{\partial x^a}, \quad \dot{\partial}_a = \frac{\partial}{\partial y^a}, \quad \dot{\partial}^a = \frac{\partial}{\partial p_a}.$$

By means of (1.2), we can consider the following subdistributions of V :

$$V_1 : u \in T^{*2}M \longrightarrow V_1(u) \subset T_uT^{*2}M,$$

and

$$W_2 : u \in T^{*2}M \longrightarrow W_2(u) \subset T_uT^{*2}M,$$

locally generated by the vector fields $\left\{\dot{\partial}_a |_u, \ u \in T^{*2}M\right\}$ and $\left\{\dot{\partial}^a |_u, u \in T^{*2}M\right\}$ respectively. Clearly,we have

$$V(u) = V_1(u) \oplus W_2(u), \quad \forall u \in T^{*2}M. \tag{1.3}$$

Let us consider the following forms

$$\omega = p_a dx^a \text{ (Liouville 1-form), and} \quad \theta = d\omega = dp_a \wedge dx^a.$$

Theorem 1.1 1°. *The differential forms ω and θ are globally defined on the manifold $T^{*2}M$.*
2°. *The 2-form θ is closed and the rank of the form θ is $2n$.*
3°. *The form θ provides a is a presymplectic structure on $T^{*2}M$.*

We note that the following $\mathcal{F}(T^{*2}M)-$linear mapping

$$J : \mathcal{X}(T^{*2}M) \to \mathcal{X}(T^{*2}M),$$

defined by

$$J(\partial_a) = \dot{\partial}_a, \quad J(\dot{\partial}_a) = 0, \quad J(\dot{\partial}^a) = 0, \quad \forall u \in \widetilde{T^{*2}M},$$

has geometrical meaning. It is not difficult to prove the following result:

Theorem 1.2 1°. *J is a tensor field of type $(1, 1)$ on the manifold $T^{*2}M$.*
2°. *J is a tangent structure on $T^{*2}M$, i.e., $J_0J = 0$.*
3°. *J is an integrable structure.*
4°. $J_0J = J^2 = 0$.
5°. *Ker* $J = V_1 \oplus W_2$, $\mathrm{Im}\, J = V_1$.

With these object fields we can construct the geometry of the manifold $T^{*2}M$.

2 Nonlinear connections on $T^{*2}M$

We extend the classical definition of the nonlinear connection ([11]) to the total space of the dual bundle $(T^{*2}M, \pi^{*2}, M)$.

Definition 2.1 A nonlinear connection of the manifold $T^{*2}M$ is a regular distribution N on $T^{*2}M$, supplimentary to the vertical distribution V, i.e.,

$$T_uT^{*2}M = N(u) \oplus V(u), \forall u \in T^{*2}M. \tag{2.1}$$

Taking into account (1.3), it follows that the distribution N has the property

$$T_uT^{*2}M = N(u) \oplus V_1(u) \oplus W_2(u), \forall u \in T^{*2}M. \tag{2.2}$$

Therefore, the main geometrical objects on $T^{*2}M$ will be reported to the direct sum (2.2) of vector spaces.

We denote by

$$\left\{\frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^a}, \frac{\partial}{\partial p_a}\right\}, \quad (a = 1, ..., n), \tag{2.3}$$

a local basis adapted to N, V_1, W_2. Clearly, we have

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - N^b_{\ a}\frac{\partial}{\partial y^b} + N_{ab}\frac{\partial}{\partial p_b}. \tag{2.4}$$

The system of functions $(N^b_{\ a}(x, y, p), N_{ab}(x, y, p))$ form the *coefficients* of the nonlinear connection N.

With respect to the coordinate transformations (1.2), we have the rule of change:

$$\frac{\delta}{\delta x^a} = \frac{\partial \tilde{x}^b}{\partial x^a}\frac{\delta}{\delta \tilde{x}^b}, \quad \frac{\partial}{\partial y^a} = \frac{\partial \tilde{x}^b}{\partial x^a}\frac{\partial}{\partial \tilde{y}^b}, \quad \frac{\partial}{\partial p_a} = \frac{\partial x^a}{\partial \tilde{x}^b}\frac{\partial}{\partial \widetilde{p_b}} \tag{2.5}$$

Theorem 2.1 *With respect to (1.2), the coefficients* (N^a_b, N_{ab}) *of a nonlinear connection* N *on* $T^{*2}M$ *obey the rule*

$$\begin{aligned} \widetilde{N}^a_{\ c}\frac{\partial \tilde{x}^c}{\partial x^b} &= N^c_{\ b}\frac{\partial \tilde{x}^a}{\partial x^c} - \frac{\partial \tilde{y}^a}{\partial x^b}, \\ \widetilde{N}_{ab} &= \frac{\partial x^c}{\partial \tilde{x}^a}\frac{\partial x^d}{\partial \tilde{x}^b}N_{cd} + p_c\frac{\partial^2 x^c}{\partial \tilde{x}^a \partial \tilde{x}^b}. \end{aligned} \tag{2.6}$$

Conversely, if the system of functions (N^a_b, N_{ab}) *are given on the every domain of local chart of the manifold* $T^{*2}M$*, such that the equations (2.6 hold, then* (N^a_b, N_{ab}) *are the coefficients of a nonlinear connection on* $T^{*2}M$*.*

Assuming that the manifold M is paracompact it follows that the manifold $T^{*2}M$ is paracompact, too. Let $\gamma_{ab}(x)$, $x \in M$ be a Riemannian metric on M and $\gamma^a_{bc}(x)$ be its Christoffel symbols. Setting

$$f_b = \gamma^a_{bc}(x)p_a y^c.$$

Then, the system of functions

$$N^a_b = \dot{\partial}^a f_b, \quad N_{ab} = \dot{\partial}_b f_a, \tag{2.7}$$

are geometrical object fields on $T^{*2}M$, having the rules of transformations (2.6), with respect to the change of local coordinates (1.2). Hence we get the following

Theorem 2.2 *If the base manifold* M *is paracompact, then there exists a nonlinear connection on the manifold* $T^{*2}M$*.*

We shall further denote the basis (2.3) by:

$$\left\{\delta_a, \dot{\partial}_a, \dot{\partial}^a\right\}.$$

The dual basis of the adapted basis (2.3) is given by

$$\{dx^a, \delta y^a, \delta p_a\}, \tag{2.8}$$

where

$$\delta y^a = dy^a + N^a_b dx^b, \ \delta p_a = dp_a - N_{ba}dx^b.$$

With respect to (1.2), the covector fields (2.8) are transformed by the rules:

$$d\tilde{x}^a = \frac{\partial \tilde{x}^a}{\partial x^b}dx^b, \ \delta\tilde{y}^a = \frac{\partial \tilde{x}^a}{\partial x^b}\delta y^b, \ \delta\widetilde{p}_a = \frac{\partial x^b}{\partial \tilde{x}^a}\delta p_b.$$

3 Distinguished vector and covector fields. The Algebra of distinguished tensor fields.

Let N be a nonlinear connection on $T^{*2}M$. Let h, v_1, w_2 be the projectors defined by the distributions N, V_1, W_2 of the direct decomposition (2.2). We have

$$h + v_1 + w_2 = I,\ h^2 = h,\ v_1^2 = v_1,\ w_2^2 = w_2, \tag{3.1}$$
$$h \circ v_1 = v_1 \circ h = 0,\ h \circ w_2 = w_2 \circ h = 0,\ v_1 \circ w_2 = w_2 \circ v_1 = 0.$$

If $X \in \chi(\widetilde{T^{*2}M})$, then we denote

$$X^H = hX,\ X^{V_1} = v_1 X,\ X^{W_2} = w_2 X.$$

Therefore we have the unique decomposition

$$X = X^H + X^{V_1} + X^{W_2}. \tag{3.2}$$

Each of the components X^H, X^{V_1}, X^{W_2} are called *$d-$vector fields* on $\widetilde{T^{*2}M}$.

In the adapted basis (2.3) we get

$$X^H = X^{(0)a}\delta_a, \quad X^{V_1} = X^{(1)a}\dot{\partial}_a, \quad X^{W_2} = \underset{(2)}{X}{}_a \dot{\partial}^a.$$

By means of (2.5) we have

$$\widetilde{X}^{(0)a} = \frac{\partial \tilde{x}^a}{\partial x^b} X^{(0)b}, \quad \widetilde{X}^{(1)a} = \frac{\partial \tilde{x}^a}{\partial x^b} X^{(1)b}, \quad \underset{(2)}{\widetilde{X}}{}_a = \frac{\partial x^b}{\partial \tilde{x}^a} \underset{(2)}{X}{}_b,$$

i.e., the classical rules of the transformations of the local coordinates of vector and covector fields on M. Therefore, $X^{(0)a}, X^{(1)a}$ are called *$d-$vector fields* and $\underset{(2)}{X}{}_a$ is called a *$d-$covector field* on the manifold $T^{*2}M$.

A similar theory can be done for distinguished $1-$forms.

With respect to the direct decomposition (2.2) a $1-$form $\omega \in \chi^*(T^{*2}M)$ can be uniquely written in the form:

$$\omega = \omega^H + \omega^{V_1} + \omega^{W_2},$$

where

$$\omega^H = \omega \circ h, \quad \omega^{V_1} = \omega \circ v_1, \quad \omega^{W_2} = \omega \circ w_2.$$

In the adapted cobasis (2.8), we have

$$\omega = \underset{(0)}{\omega}{}_a dx^a + \underset{(1)}{\omega}{}_a \delta y^a + \omega^{(2)a}\delta p_a.$$

The quantities $\omega^H, \omega^{V_1}, \omega^{W_2}$ are called *$d-1-$forms*. The coefficients $\underset{(0)}{\omega}{}_a, \underset{(1)}{\omega}{}_a, \omega^{(2)a}$ are transformed by (1.2) as follows:

$$\underset{(0)}{\omega}{}_a = \frac{\partial \tilde{x}^b}{\partial x^a} \underset{(0)}{\widetilde{\omega}}{}_a, \quad \underset{(1)}{\omega}{}_a = \frac{\partial \tilde{x}^b}{\partial x^a} \underset{(1)}{\widetilde{\omega}}{}_b, \quad \widetilde{\omega}^{(2)a} = \frac{\partial \tilde{x}^a}{\partial x^b} \omega^{(2)b}.$$

Hence $\underset{(0)}{\omega}{}_a$ and $\underset{(1)}{\omega}{}_a$ are called *$d-$covector fields* and $\omega^{(2)a}$ is called a *$d-$vector field.*

Definition 3.1 A distinguished tensor (briefly, *d–tensor field*) on the manifold $T^{*2}M$ is a $d-$tensor field T of type (r,s) on $T^{*2}M$, with the property

$$T(\overset{1}{\omega},...,\overset{r}{\omega},\ \underset{1}{X},...,\underset{s}{X}) \quad = T(\overset{1}{\omega}{}^{H},...,\overset{r}{\omega}{}^{W_2},\ \underset{1}{X}{}^{H},...,\underset{s}{X}{}^{W_2}),$$
$$\forall\overset{1}{\omega},...,\overset{r}{\omega} \in \chi^*(T^{*2}M),\ \forall\underset{1}{X},...,\underset{s}{X} \in \chi(T^{*2}M).$$

For instance, every set of components X^H, X^{V_1}, X^{W_2} of a vector field X forms a $d-$tensor field of type $(1,0)$, and every set of components $\omega^H, \omega^{V_1}, \omega^{W_2}$ of a $1-$form ω is a $d-$tensor field of type $(0,1)$.

In the adapted basis $(\delta_a, \dot{\partial}_a, \dot{\partial}^a)$ and its dual basis $(dx^a, \delta y^a, \delta p_a)$, a $d-$tensor field T of type (r,s) can written in the form:

$$T = T^{\ a_1...a_r}_{\ b_1...b_s}(x,y,p)\delta_{a_1}\otimes...\otimes\dot{\partial}^{b_s}\otimes dx^{b_1}\otimes...\otimes\delta p_{a_r},$$

where

$$T^{\ a_1...a_r}_{\ b_1...b_s}(x,y,p) = T(dx^{b_1},...,\delta p_{a_r},\delta_{a_1},...,\dot{\partial}^{b_s}).$$

It follows that the set $\left\{1,\delta_a,\dot{\partial}_a,\dot{\partial}^a\right\}$ generates *the algebra* of the $d-$tensor fields over the ring of functions $\mathcal{F}(T^{*2}M)$.

With respect to the transformation of the coordinates on $T^{*2}M$, the local coefficients $T^{\ a_1...a_r}_{\ b_1...b_s}$ of T are transformed by the classical rule

$$\tilde{T}^{\ c_1...c_r}_{\ d_1...d_s} = \frac{\partial\tilde{x}^{c_1}}{\partial x^{a_1}}...\frac{\partial\tilde{x}^{c_r}}{\partial x^{a_r}}\frac{\partial x^{b_1}}{\partial\tilde{x}^{d_1}}...\frac{\partial x^{b_s}}{\partial\tilde{x}^{d_s}}T^{\ a_1...a_r}_{\ b_1...b_s}.$$

4 Lie brackets

In applications, the Lie brackets of the vector fields $(\delta_a, \dot{\partial}_a, \dot{\partial}^a)$ of the basis adapted to the direct decomposition (2.2), are important. By a direct calculus, we have:

Proposition 4.1 *The Lie brackets of the vector fields of the adapted basis are given by*

$$\begin{aligned}
[\delta_b,\delta_c] &= \underset{(01)}{R}{}^{a}{}_{bc}\dot{\partial}_a + \underset{(02)}{R}{}_{abc}\dot{\partial}^a,\\
\left[\delta_b,\dot{\partial}_c\right] &= \underset{(11)}{B}{}^{a}{}_{bc}\dot{\partial}_a + \underset{(12)}{B}{}_{abc}\dot{\partial}^a,\\
\left[\delta_b,\dot{\partial}^c\right] &= \underset{(21)}{B}{}^{a}{}_{b}{}^{c}\dot{\partial}_a + \underset{(22)}{B}{}_{ab}{}^{c}\dot{\partial}^a,\\
\left[\dot{\partial}_b,\dot{\partial}_c\right] &= 0,\ \left[\dot{\partial}_b,\dot{\partial}^c\right] = 0,\ \left[\dot{\partial}^b,\dot{\partial}^c\right] = 0,
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
&\underset{(01)}{R}{}^{a}{}_{bc} = \delta_c N^a_b - \delta_b N^a_c, && \underset{(02)}{R}{}_{abc} = \delta_b N_{ca} - \delta_c N_{ba},\\
&\underset{(11)}{B}{}^{a}{}_{bc} = \dot{\partial}_c N^a_b, && \underset{(12)}{B}{}_{abc} = -\dot{\partial}_c N_{ba},\\
&\underset{(21)}{B}{}^{a}{}_{b}{}^{c} = \dot{\partial}^c N^a_b, && \underset{(22)}{B}{}_{ab}{}^{c} = -\dot{\partial}^c N_{ba}.
\end{aligned} \tag{4.2}$$

Let us consider the followings coefficients from (4.1):

$$\underset{(11)}{B}{}^a{}_{bc} = \dot{\partial}_c N^a_b,\ -\underset{(22)}{B}{}_{ab}{}^c = \dot{\partial}^c N_{ba}\ \big(= -\underset{(22)}{B}{}^c{}_{ab}\big).$$

By means of (2.6) it follows

Proposition 4.2 *The coefficients* $\underset{(11)}{B}{}^a{}_{cb} = \underset{(11)}{U}{}^a{}_{bc}$, $-\underset{(22)}{B}{}^a{}_{bc} = \underset{(22)}{U}{}^a{}_{bc}$ *have the same rule of transformation with respect to the local change of coordinates (1.2) on* $T^{*2}M$*. This is*

$$\underset{(\beta\beta)}{\widetilde{U}}{}^a{}_{df}\frac{\partial x^d}{\partial x^b}\frac{\partial x^f}{\partial x^c} = \frac{\partial \tilde{x}^a}{\partial x^d}\underset{(\beta\beta)}{U}{}^d{}_{bc} - \frac{\partial^2 \tilde{x}^a}{\partial x^b \partial x^c}, \quad (\beta = 1, 2). \tag{4.3}$$

We will see that these coefficients are the horizontal coefficients of an $N-$linear connection on $T^{*2}M$. By straightforward direct computation, we obtain

Proposition 4.3 *The coefficients* $\underset{(01)}{R}{}^a{}_{bc}$, $\underset{(02)}{R}{}_{abc}$ *and*

$$\underset{(21)}{B}{}^a{}_b{}^c = \dot{\partial}^c N^a_b,\ \underset{(12)}{B}{}_{abc} = -\dot{\partial}_c N_{ba},$$

are $d-$*tensor fields on* $T^{*2}M$*, of type* $(1,2),(0,3),(2,1)$ *and, respectively,* $(0,3)$*, i.e.,*

$$\underset{(01)}{\widetilde{R}}{}^d{}_{cf} = \frac{\partial \tilde{x}^d}{\partial x^a}\frac{\partial x^b}{\partial \tilde{x}^c}\frac{\partial x^c}{\partial \tilde{x}^f}\underset{(01)}{R}{}^a{}_{bc},\ \ etc.$$

We will see that (4.4) can be the vertical coefficients of $N-$linear connection on $T^{*2}M$.
Also, we have

Proposition 4.4 *For the nonliner connection* $N(N^a{}_b, N_{ab})$ given by (2.7):

$$N^a{}_b = \gamma^a_{bc}(x)y^b, \quad N_{ab} = \gamma^c_{ab}(x)p_c, \tag{4.4}$$

the coefficients (4.2) of Lie brackets have the following expressions:

$$\begin{array}{ll} \underset{(01)}{R}{}^a{}_{bc} = r_b{}^a{}_{cd}(x)y^d, & \underset{(02)}{R}{}_{abc} = r_a{}^d{}_{bc}(x)p_d, \\ \underset{(11)}{B}{}^a{}_{bc} = \gamma^a_{bc}(x), & \underset{(12)}{B}{}_{abc} = 0, \\ \underset{(21)}{B}{}^a{}_b{}^c = 0, & \underset{(22)}{B}{}_{ab}{}^c = -\gamma^c_{ab}(x). \end{array} \tag{4.5}$$

5 The almost contact structure $\mathbb{F}$.

The nonlinear connection N being fixed, we have the direct decomposition (2.1), (2.2) and the corresponding adapted basis (2.3).

Let us consider the $\mathcal{F}(T^{*2}M)-$linear mapping:

$$\mathbb{F} : \chi(T^{*2}M) \longrightarrow \chi(T^{*2}M),$$

determined by

$$\mathbb{F}(\delta_a) = -\dot{\partial}_a, \quad \mathbb{F}(\dot{\partial}_a) = \delta_a, \quad \mathbb{F}(\dot{\partial}^a) = 0. \tag{5.1}$$

Then, we obtain

Theorem 5.1 *The mapping* $\mathbb{F}$ *has the following properties:*
1°. *It is globally defined on* $\widetilde{T^{*2}M}$.
2°. $\mathbb{F}$ *is a tensor field of type* $(1,1)$.
3°. $Ker\ \mathbb{F} = W_2$, $\operatorname{Im}\mathbb{F} = N \oplus V_1$.
4°. $rank\ \mathbb{F} = 2n$.
5°. $\mathbb{F}^3 + \mathbb{F} = 0$.

Proof. For 1° − 5° see [7, p. 259].

We say that $\mathbb{F}$ is a *natural almost contact structure* determined by the nonlinear connection N.

6 The Riemann structures on $\widetilde{T^{*2}M}$.

Let us consider a Riemannian structure $\mathbb{G}$ on the manifold $\widetilde{T^{*2}M}$.

In the natural basis, $\mathbb{G}$ is given locally by

$$\mathbb{G} = \underset{(00)}{\bar{g}}{}_{ab}dx^a \otimes dx^b + \underset{(01)}{\bar{g}}{}_{ab}dx^a \otimes dy^b + \underset{(02)}{\bar{g}}{}_a{}^b dx^a \otimes dp_b + ... + \underset{(22)}{\bar{g}}{}^{ab}dp_a \otimes dp_b,$$

where the matrix $\| \underset{(\alpha\beta)}{\bar{g}} \|$ is positively defined.

Let $\{\delta_a\}$, $(a = 1, ..., n)$, be the basis adapted to N:

$$\delta_a = \partial_a - N^b{}_a\dot{\partial}_b + N_{ab}\dot{\partial}^b.$$

and let $\{dx^a, \delta y^a, \delta p_a\}$ be the cobasis adapted to N

$$\delta y^a = dy^a + N^a{}_b dx^b, \ \delta p_a = dp_a - N_{ba}dx^b.$$

Then, after a direct calculation, the Riemann structure $\mathbb{G}$ can be written in the adapted cobasis, in the form

$$\mathbb{G} = \underset{(00)}{g}{}_{ab}dx^a \otimes dx^b + \underset{(01)}{g}{}_{ab}dx^a \otimes \delta y^b + \underset{(02)}{g}{}_a{}^b dx^a \otimes \delta p_b + ... + \underset{(22)}{g}{}^{ab}dp_a \otimes \delta p_b, \tag{6.1}$$

where $\underset{(00)}{g}{}_{ab}, \underset{(01)}{g}{}_{ab}, \underset{(02)}{g}{}_a{}^b, etc.$, are expressed by $\underset{(00)}{\bar{g}}{}_{ab}, \underset{(01)}{\bar{g}}{}_{ab}, \underset{(02)}{\bar{g}}{}_a{}^b, etc.$ and with the coefficients $N^a{}_b$ and N_{ab} of N given by (4.4).

Let $\mathbb{F}$ be the natural contact structure determined by the nonlinear connection N given by (4.4).

The following problem arises: when is the pair $(\mathbb{G}, \mathbb{F})$ a Riemannian almost contact structure?

For this, it is obviously necessary to have:

$$\mathbb{G}(\mathbb{F}X, Y) = -\mathbb{G}(X, \mathbb{F}Y), \quad \forall X, Y \in \chi(\widetilde{T^{*2}M}).$$

Consequently, we get

Theorem 6.1 *The pair* $(\mathbb{G}, \mathbb{F})$ *is a Riemannian almost contact structure if and only if in the adapted basis determined by* N *and* V *the tensor* $\mathbb{G}$ *has the form*

$$\mathbb{G} = g_{ab}dx^a \otimes dx^b + g_{ab}\delta y^a \otimes \delta y^b + h^{ab}\delta p_a \otimes \delta p_b. \tag{6.2}$$

Corollary 6.1 *With respect to the Riemannian structure (2.3), the distributions* N, V_1, W_2 *are orthogonal in pairs respectively.*

7 $N-$linear connections on $T^{*2}M$

A linear connection on $T^{*2}M$ is an mapping

$$D:\chi(T^{*2}M)\times\chi(T^{*2}M)\to\chi(T^{*2}M),\quad (X,Y)\longmapsto D_XY,$$

with the properties:

1. $D_{X_1+X_2}Y=D_{X_1}Y+D_{X_2}Y,$

$D_{fX}Y=fD_XY,\quad \forall f\in\mathcal{F}(T^{*2}M),\quad \forall X,X_1,X_2,Y\in\chi(T^{*2}M).$

2. $D_X(Y_1+Y_2)=D_XY_1+D_XY_2,\quad \forall X,Y_1,Y_2\in\chi(T^{*2}M).$

3. $D_X(fY)=(Xf)Y+fD_XY,\quad \forall X,Y\in\chi(T^{*2}M),\quad \forall f\in\mathcal{F}(T^{*2}M).$

We consider $X,Y\in\chi(T^{*2}M)$. With respect to the decomposition of type (3.2), we have

$$D_XY=\sum_{\alpha=0}^{2}(D_{X^H}Y^{V_\alpha}+D_{X^{V_1}}Y^{V_\alpha}+D_{X^{W_2}}Y^{V_\alpha}),$$

where $V_0=H$ and $V_2=W_2$.

The components $D_{X^H}Y^{V_\alpha}$, $D_{X^{V_1}}Y^{V_\alpha}$, $D_{X^{W_2}}Y^{V_\alpha}$, ($V_0=H$, $V_2=W_2$), are (not necessarily distinguished) vector fields.

The linear connection D on $T^{*2}M$ is uniquely determined by its 27 sets of coefficients, written in the adapted basis. To work with these 27 sets of coefficients is not imposible, but is laborious. We shall further use $N-$linear connections whose coefficients are much easier to determine and operate with.

Let N be a nonlinear connection on $T^{*2}M$.

Definition 7.1 A linear connection D on $T^{*2}M$ is called an *$N-$linear connection* if it preserves by parallelism the horizontal and the vertical distributions N,V_1 and W_2 on $T^{*2}M$.

By the general theory of connections on manifolds, the horizontal and vertical distributions are preserved by parallelilsm if for any $X\in\chi(T^{*2}M)$, D_X carries the horizontal vector fields to horizontal vector fields and the vertical vector fields to vertical vector fields. Thus D_XY^H is always an horizontal vector field, and $D_XY^{V_\beta}$ are vertical ones, ($\beta=1,2;\ V_2=W_2$).

Obviously, the local description of an $N-$linear connection $D\Gamma(N)$ on $T^{*2}M$ is given by *nine* unique sets of adapted coefficients:

$$D\Gamma(N):=(\underset{(00)}{H}{}^a{}_{bc},\underset{(10)}{H}{}^a{}_{bc},\underset{(20)}{H}{}_a{}^b{}_c,\underset{(01)}{C}{}^a{}_{bc},\underset{(11)}{C}{}^a{}_{bc},\underset{(21)}{C}{}_a{}^b{}_c,\underset{(02)}{C}{}^a{}_b{}^c,\underset{(12)}{C}{}^a{}_b{}^c,\underset{(22)}{C}{}_a{}^{bc}),$$

We have

Theorem 7.1 1°. *An $N-$ linear connection D on $T^{*2}M$ can be uniquely represented in the adapted basis $(\delta_a,\dot{\partial}_a,\dot{\partial}^a)$ in the form*

$$\begin{cases} D_{\delta_c}\delta_b=\underset{(00)}{H}{}^a{}_{bc}\delta_a, & D_{\delta_c}\dot{\partial}_b=\underset{(10)}{H}{}^a{}_{bc}\dot{\partial}_a, & D_{\delta_c}\dot{\partial}^b=-\underset{(00)}{H}{}_a{}^b{}_c\dot{\partial}^a,\\ D_{\dot{\partial}_c}\delta_b=\underset{(01)}{C}{}^a{}_{bc}\delta_a, & D_{\dot{\partial}_c}\dot{\partial}_b=\underset{(11)}{C}{}^a{}_{bc}\dot{\partial}_a, & D_{\dot{\partial}_c}\dot{\partial}^b=-\underset{(21)}{C}{}_a{}^b{}_c\dot{\partial}^a,\\ D_{\dot{\partial}^c}\delta_b=\underset{(02)}{C}{}^a{}_b{}^c\delta_a, & D_{\dot{\partial}^c}\dot{\partial}_b=\underset{(12)}{C}{}^a{}_b{}^c\dot{\partial}_a, & D_{\dot{\partial}^c}\dot{\partial}^b=-\underset{(22)}{C}{}_a{}^{bc}\dot{\partial}^a. \end{cases} \tag{7.1}$$

2°. *With respect to the coordinate transformation (1.2), the coefficients* $\underset{(\alpha 0)}{H}{}^{a}{}_{bc}$, $(\alpha = 0, 1, 2;\ \underset{(20)}{H}{}^{a}{}_{bc} := \underset{(20)}{H}{}_{b}{}^{a}{}_{c})$ *obey the rule of transformation:*

$$\underset{(\alpha 0)}{\widetilde{H}}{}^{a}{}_{de}\frac{\partial \tilde{x}^d}{\partial x^b}\frac{\partial \tilde{x}^e}{\partial x^c} = \frac{\partial \tilde{x}^a}{\partial x^e}\underset{(\alpha 0)}{H}{}^{e}{}_{bc} - \frac{\partial^2 \tilde{x}^a}{\partial x^b \partial x^c}.$$

3°. *The system of functions* $\underset{(\alpha 1)}{C}{}^{a}{}_{bc}, \underset{(\alpha 2)}{C}{}^{a}{}_{b}{}^{c}$, $(\alpha = 0, 1, 2;\ \underset{(21)}{C}{}^{a}{}_{bc} := \underset{(21)}{C}{}_{b}{}^{a}{}_{c};\ \underset{(22)}{C}{}^{a}{}_{b}{}^{c} := \underset{(22)}{C}{}_{b}{}^{ac}$ *are d−tensor fields of type* $(1, 2)$ *and* $(2, 1)$*, respectively.*

We have the following theorem of existence of $N-$linear connections on $T^{*2}M$.

Theorem 7.2 *If the manifold M is paracompact and N is a nonlinear connection on $T^{*2}M$ with coefficients N^a_b, N_{ab}, then there exists an $N-$linear connection on $T^{*2}M$.*

Proof. Since M is paracompact, then there exists a linear connection on M of local coefficients, say $\Gamma^a_{bc}(x)$. Let $N^a_b(x, y, p)$ and $N_{ab}(x, y, p)$ be the local coefficients of the nonlinear connection N. We set $\underset{(00)}{H}{}^{a}{}_{bc} = \Gamma^a_{bc}(x)$, $\underset{(10)}{H}{}^{a}{}_{bc} = \dot{\partial}_b N^a_c$, $\underset{(20)}{H}{}^{a}{}_{bc} = \dot{\partial}^a N_{bc}$. Thus, taking into account the previous results, we obtain three sets of functions which transform, with respect to (1.2), by (7.1). It results that $D\Gamma(N)$ given by

$$D\Gamma(N) = (\Gamma^a_{bc}(x), \underset{(11)}{B}{}^{a}{}_{cb}, -\underset{(22)}{B}{}^{a}{}_{bc}, 0, 0, 0, 0, 0, 0),$$

defines an $N-$linear connection on $T^{*2}M$.

In applications, we use the $N-$linear connection of the form

$$B\Gamma(N) = (\underset{(00)}{L}{}^{a}{}_{bc}, \underset{(11)}{B}{}^{a}{}_{cb}, -\underset{(22)}{B}{}^{a}{}_{bc}, 0, \underset{(11)}{C}{}^{a}{}_{bc}, 0, 0, 0, \underset{(22)}{C}{}_{b}{}^{ac})$$

called *$N-$linear connection of Berwald type on $T^{*2}M$.*

8 The $h_\alpha-, v_{1\alpha}-$ and $w_{2\alpha}-$covariant derivatives in the local adapted basis, $(\alpha = 0, 1, 2)$

The $N-$linear connection $D\Gamma(N)$ induces a linear connection on the $d-$tensors set of the 2−cotangent bundle $(T^{*2}M, \pi^{*2}, M)$, in a natural way. Thus, starting with a $d-$vector field X and a $d-$tensor field T, locally expressed by

$$X = X^{(0)a}\delta_a + X^{(1)a}\dot{\partial}_a + \underset{(2)}{X}{}_a\dot{\partial}^a,$$

$$T = T^{\ a_1 \ldots a_r}_{\ b_1 \ldots b_s}(x, y, p)\delta_{a_1} \otimes \ldots \otimes \dot{\partial}^{b_s} \otimes dx^{b_1} \otimes \ldots \otimes \delta p_{a_r},$$

we can define the covariant derivative

$$\begin{aligned} D_X T = & \{X^{(0)d}T^{\ a_1 \ldots a_r}_{\ b_1 \ldots b_s \overset{\alpha}{|} d} + \\ & + X^{(1)d}T^{\ a_1 \ldots a_r}_{\ b_1 \ldots b_s}|_{\alpha d} + \underset{(2)}{X}{}_d T^{\ a_1 \ldots a_r}_{\ b_1 \ldots b_s}|^{\alpha d}\Big\}\delta_{a_1} \otimes \ldots \otimes \delta p_{a_r}, \end{aligned}$$

where

$$T^{\ a_1 \ldots a_r}_{\ b_1 \ldots b_s \overset{\alpha}{|} d} = \delta_d T^{\ a_1 \ldots a_r}_{\ b_1 \ldots b_s} + \underset{(\alpha 0)}{H}{}^{a_1}{}_{cd}T^{\ c a_2 \ldots a_r}_{\ b_1 \ldots b_s} + \ldots + \underset{(\alpha 0)}{H}{}^{a_r}{}_{cd}T^{\ a_1 \ldots c}_{\ b_1 \ldots b_s} -$$

$$- \underset{(\alpha 0)}{H}{}^{c}{}_{b_1 d} T^{a_1 \ldots a_r}_{c b_2 \ldots b_s} - \ldots - \underset{(\alpha 0)}{H}{}^{c}{}_{b_s d} T^{a_1 \ldots a_r}_{b_1 \ldots c},$$

$$T^{a_1 \ldots a_r}_{b_1 \ldots b_s} |_{\alpha d} = \dot{\partial}_d T^{a_1 \ldots a_r}_{b_1 \ldots b_s} + \underset{(\alpha 1)}{C}{}^{a_1}{}_{cd} T^{c a_2 \ldots a_r}_{b_1 \ldots b_s} + \ldots + \underset{(\alpha 1)}{C}{}^{a_r}{}_{cd} T^{a_1 \ldots c}_{b_1 \ldots b_s} -$$
$$- \underset{(\alpha 1)}{C}{}^{c}{}_{b_1 d} T^{a_1 \ldots a_r}_{c b_2 \ldots b_s} - \ldots - \underset{(\alpha 1)}{C}{}^{c}{}_{b_s d} T^{a_1 \ldots a_r}_{b_1 \ldots c},$$

$$T^{a_1 \ldots a_r}_{b_1 \ldots b_s} |^{\alpha d} = \dot{\partial}^d T^{a_1 \ldots a_r}_{b_1 \ldots b_r} + \underset{(\alpha 2)}{C}{}_{c}{}^{a_1 d} T^{c a_2 \ldots a_r}_{b_1 \ldots b_s} + \ldots + \underset{(\alpha 2)}{C}{}_{c}{}^{a_r d} T^{a_1 \ldots c}_{b_1 \ldots b_s} -$$
$$- \underset{(\alpha 2)}{C}{}_{b_1}{}^{cd} T^{a_1 \ldots a_r}_{c b_2 \ldots b_s} - \ldots - \underset{(\alpha 2)}{C}{}_{b_s}{}^{cd} T^{a_1 \ldots a_r}_{b_1 \ldots c}.$$

Definition 8.1 The local derivative operators "$_{\shortmid \alpha d}$", "$|_{\alpha d}$" *and* "$|^{\alpha d}$" are called the $\mathbf{h}_\alpha-, \mathbf{v}_{\mathbf{1}\alpha}-$ ***and*** $\mathbf{w}_{\mathbf{2}\alpha}-$covariant derivatives of $D\Gamma(N), (\alpha = 0, 1, 2)$.

Remark 8.1 (i) In the particular case when T is a function $f(x, y, p)$ on $T^{*2}M$, the preceding covariant derivatives reduce to

$$f_{\shortmid \alpha d} = \delta_d f = \partial_d f - N^c{}_d \dot{\partial}_c,$$
$$f |_{\alpha d} = \dot{\partial}_d f, \quad f |^{\alpha d} = \dot{\partial}^d f, \quad \forall f \in \mathcal{F}(T^{*2}M).$$

(ii) Considering the $d-$tensor $T = Y$ as a $d-$tensor on $T^{*2}M$, locally expressed by

$$Y = Y^{(0)a} \delta_a + Y^{(1)a} \dot{\partial}_a + \underset{(2)}{Y}{}_a \dot{\partial}^a,$$

the following expressions of local covariant derivatives of $D\Gamma(N)$ hold good:

$$Y^{(0)a}{}_{\shortmid \alpha c} = \delta_c Y^{(0)a} + Y^{(0)b} \underset{(\alpha 0)}{H}{}^{a}{}_{bc}, Y^{(1)a}{}_{\shortmid \alpha c} = \delta_c Y^{(1)a} + Y^{(1)b} \underset{(\alpha 0)}{H}{}^{a}{}_{bc},$$
$$\underset{(2)}{Y}{}_{b \shortmid \alpha c} = \delta_c \underset{(2)}{Y}{}_b - \underset{(2)}{Y}{}_a \underset{(\alpha 0)}{H}{}^{a}{}_{bc},$$
$$Y^{(0)a} |_{\alpha c} = \dot{\partial}_c Y^{(0)a} + Y^{(0)b} \underset{(\alpha 1)}{C}{}^{a}{}_{bc}, Y^{(1)a} |_{\alpha c} = \dot{\partial}_c Y^{(1)a} + Y^{(1)b} \underset{(\alpha 1)}{C}{}^{a}{}_{bc},$$
$$\underset{(2)}{Y}{}_b |_{\alpha c} = \dot{\partial}_c \underset{(2)}{Y}{}_b - \underset{(2)}{Y}{}_a \underset{(\alpha 1)}{C}{}^{a}{}_{bc},$$
$$Y^{(0)a} |^{\alpha b} = \dot{\partial}^b Y^{(0)a} + Y^{(0)c} \underset{(\alpha 2)}{C}{}_{c}{}^{ab}, Y^{(1)a} |^{\alpha b} = \dot{\partial}^b Y^{(1)a} + Y^{(1)c} \underset{(\alpha 2)}{C}{}_{c}{}^{ab},$$
$$\underset{(2)}{Y}{}_b |^{\alpha b} = \dot{\partial}^b \underset{(2)}{Y}{}_c - \underset{(2)}{Y}{}_a \underset{(\alpha 2)}{C}{}_{c}{}^{ab}.$$

Proposition 8.1 *The quantities* $T^{a_1 \ldots a_r}_{b_1 \ldots b_s \shortmid \alpha d}$, $T^{a_1 \ldots a_r}_{b_1 \ldots b_s} |_{\alpha d}$, $T^{a_1 \ldots a_r}_{b_1 \ldots b_s} |^{\alpha d}$, $(\alpha = 0, 1, 2)$ *are* $d-$*tensor fields on* $T^{*2}M$*. The first six are of type* $(r, s + 1)$*, the last three are of type* $(r + 1, s)$*.*

9 Ricci identities. Local expressions of $d-$tensors of torsion and curvature

Let $D\Gamma(N)$ be an $N-$linear connection with the coefficients

$$D\Gamma(N) = (\underset{(00)}{H}{}^{a}{}_{bc}, \underset{(10)}{H}{}^{a}{}_{bc}, \underset{(20)}{H}{}^{a}{}_{bc}, \underset{(01)}{C}{}^{a}{}_{bc}, \underset{(11)}{C}{}^{a}{}_{bc}, \underset{(21)}{C}{}^{a}{}_{bc}, \underset{(02)}{C}{}_{a}{}^{bc}, \underset{(12)}{C}{}_{a}{}^{bc}, \underset{(22)}{C}{}_{a}{}^{bc}), \tag{9.1}$$

By a straightforward calculation we obtain:

Theorem 9.1 *For any $N-$linear connection D and any $d-$vector field $X \in \chi(T^{*2}M)$, the following Ricci formulae hold:*

$$
\begin{aligned}
&X^a{}_{|\alpha b|\alpha c} - X^a{}_{|\alpha c|\alpha b} = X^f \underset{(\alpha 00)}{R}{}_f{}^a{}_{bc} - \overset{0}{\underset{(\alpha 0)}{T}}{}^f{}_{bc} X^a{}_{|\alpha f} - \underset{(01)}{R}{}^f{}_{bc} X^a|_{\alpha f} - \underset{(02)}{R}{}_{fbc} X^a|^{\alpha f},\\
&X^a{}_{|\alpha b}|_{\alpha c} - X^a|_{\alpha c|\alpha b} = X^f \underset{(\alpha 01)}{R}{}_f{}^a{}_{bc} - \underset{(\alpha 1)}{C}{}^f{}_{bc} X^a{}_{|\alpha f} - \underset{(\alpha 1)}{P}{}^f{}_{bc} X^a|_{\alpha f} - \underset{(12)}{B}{}_{fbc} X^a|^{\alpha f},\\
&X^a{}_{|\alpha b}|^{\alpha c} - X^a|^{\alpha c}{}_{|\alpha b} = X^f \underset{(\alpha 02)}{R}{}_f{}^a{}_b{}^c - \underset{(\alpha 2)}{C}{}_b{}^{fc} X^a{}_{|\alpha f} - \underset{(21)}{B}{}^f{}_b{}^c X^a|_{\alpha f} - \underset{(\alpha 2)}{P}{}_{fb}{}^c X^a|^{\alpha f},\\
&X^a|_{\alpha b}|_{\alpha c} - X^a|_{\alpha c}|_{\alpha b} = X^f \underset{(\alpha 11)}{R}{}_f{}^a{}_{bc} - \underset{(\alpha 1)}{S}{}^f{}_{bc} X^a|_{\alpha f},\\
&X^a|_{\alpha b}|^{\alpha c} - X^a|^{\alpha c}|_{\alpha b} = X^f \underset{(\alpha 12)}{R}{}_f{}^a{}_b{}^c - \underset{(\alpha 2)}{C}{}_b{}^{fc} X^a|_{\alpha f} - \underset{(\alpha 1)}{C}{}^c{}_{fb} X^a|^{\alpha f},\\
&X^a|^{\alpha b}|^{\alpha c} - X^a|^{\alpha c}|^{\alpha b} = X^f \underset{(\alpha 22)}{R}{}_f{}^{abc} - \underset{(\alpha 2)}{S}{}_f{}^{bc} X^a|^{\alpha f}, \quad (\alpha = 0, 1, 2).
\end{aligned}
$$

where all the terms in $\underset{(01)}{R}{}^a{}_{bc}$, $\underset{(02)}{R}{}_{abc}$, $\underset{(12)}{B}{}_{abc}$, $\underset{(21)}{B}{}^a{}_b{}^c$ *are known from the Lie brackets (4.1), and the coefficients* $D\Gamma(N)$ *are given in (9.1).*

We further denote

$$
\overset{0}{\underset{(\alpha 0)}{T}}{}^a{}_{bc} = \underset{(\alpha 0)}{H}{}^a{}_{bc} - \underset{(\alpha 0)}{H}{}^a{}_{cb},\ \underset{(\alpha 1)}{P}{}^a{}_{bc} = \underset{(11)}{B}{}^a{}_{bc} - \underset{(\alpha 0)}{H}{}^a{}_{cb},\ \underset{(\alpha 2)}{P}{}_{ab}{}^c = \underset{(22)}{B}{}_{ab}{}^c + \underset{(\alpha 0)}{H}{}^c{}_{ab},
$$

$$
\underset{(\alpha 1)}{S}{}^a{}_{bc} = \underset{(\alpha 1)}{C}{}^a{}_{bc} - \underset{(\alpha 1)}{C}{}^a{}_{cb},\ \underset{(\alpha 2)}{S}{}_a{}^{bc} = -(\underset{(\alpha 2)}{S}{}_a{}^{bc} - \underset{(\alpha 2)}{C}{}_a{}^{cb}),
$$

and $\overset{0}{\underset{(00)}{T}}{}^a{}_{bc}$, $\overset{1}{\underset{(00)}{T}}{}^a{}_{bc}$, $\overset{2}{\underset{(00)}{T}}{}^a{}_{bc}$, $\overset{0}{\underset{(01)}{P}}{}^a{}_{bc}$, $\overset{1}{\underset{(01)}{P}}{}^a{}_{bc}$, $\overset{2}{\underset{(01)}{P}}{}^a{}_{bc}$, $\overset{0}{\underset{(02)}{P}}{}^a{}_{bc}$, $\overset{1}{\underset{(02)}{P}}{}^a{}_{bc}$, $\overset{2}{\underset{(02)}{P}}{}^a{}_{bc}$, $\overset{1}{\underset{(11)}{S}}{}^a{}_{bc}$, $\overset{1}{\underset{(12)}{Q}}{}^a{}_{bc}$, $\overset{2}{\underset{(12)}{Q}}{}_{ab}{}^c$, $\overset{2}{\underset{(22)}{S}}{}_a{}^{bc}$ are called *$d-$tensors of torsion* of D. These are given by:

$$
\begin{cases}
\overset{0}{\underset{(00)}{T}}{}^a{}_{bc} = \underset{(00)}{H}{}^a{}_{bc} - \underset{(00)}{H}{}^a{}_{cb},\ \overset{1}{\underset{(00)}{T}}{}^a{}_{bc} = \underset{(01)}{R}{}^a{}_{bc},\ \overset{2}{\underset{(00)}{T}}{}_{abc} = \underset{(02)}{R}{}_{abc},\\
\overset{0}{\underset{(01)}{P}}{}^a{}_{bc} = \underset{(01)}{C}{}^a{}_{bc},\ \overset{1}{\underset{(01)}{P}}{}^a{}_{bc} = \underset{(11)}{B}{}^a{}_{bc} - \underset{(10)}{H}{}^a{}_{cb},\ \overset{2}{\underset{(01)}{P}}{}_{abc} = \underset{(12)}{B}{}_{abc},\\
\overset{0}{\underset{(02)}{P}}{}^a{}_b{}^c = \underset{(02)}{C}{}^a{}_b{}^c,\ \overset{1}{\underset{(02)}{P}}{}^a{}_b{}^c = \underset{(21)}{B}{}^a{}_b{}^c,\ \overset{2}{\underset{(02)}{P}}{}_{ab}{}^c = \underset{(22)}{B}{}_{ab}{}^c + \underset{(20)}{H}{}^c{}_{ab},
\end{cases}
$$

$$
\begin{cases}
\overset{1}{\underset{(11)}{S}}{}^a{}_{bc} = \underset{(11)}{C}{}^a{}_{bc} - \underset{(11)}{C}{}^a{}_{cb}, \quad \overset{2}{\underset{(22)}{S}}{}_a{}^{bc} = -(\underset{(22)}{C}{}_a{}^{bc} - \underset{(22)}{C}{}_a{}^{cb})\\
\overset{1}{\underset{(12)}{Q}}{}^a{}_b{}^c = \underset{(12)}{C}{}^a{}_b{}^c =: \underset{(12)}{C}{}^{ac}{}_b,\ \overset{2}{\underset{(12)}{Q}}{}_{ab}{}^c = \underset{(21)}{C}{}_a{}^c{}_b =: \underset{(21)}{C}{}^c{}_{ab}.
\end{cases}
$$

We remark that $\underset{(11)}{P}{}^a{}_{bc} = \overset{1}{\underset{(01)}{P}}{}^a{}_{bc}$, $\underset{(22)}{P}{}_{ab}{}^c = \overset{2}{\underset{(02)}{P}}{}_{ab}{}^c$, *etc.* Also, $\underset{(\alpha 00)}{R}$, ..., are called *$d-$tensors of curvature* of D, and they are given by:

$$
\begin{cases}
\underset{(\alpha 00)}{R}{}_b{}^a{}_{cd} = \delta_d \underset{(\alpha 0)}{H}{}^a{}_{bc} - \delta_c \underset{(\alpha 0)}{H}{}^a{}_{bd} + \underset{(\alpha 0)}{H}{}^f{}_{bc} \underset{(\alpha 0)}{H}{}^a{}_{fd} - \underset{(\alpha 0)}{H}{}^f{}_{bd} \underset{(\alpha 0)}{H}{}^a{}_{fc} +\\
\qquad\qquad + \underset{(\alpha 1)}{C^a}{}_{bf} \underset{(01)}{R}{}^f{}_{cd} + \underset{(\alpha 2)}{C}{}_b{}^{af} \underset{(02)}{R}{}_{fcd},\\
\underset{(\alpha 01)}{R}{}_b{}^a{}_{cd} = \dot{\partial}_d \underset{(\alpha 0)}{H}{}^a{}_{bc} - \underset{(\alpha 1)}{C}{}^a{}_{bd|\alpha c} + \underset{(\alpha 1)}{C}{}^a{}_{bf} \underset{(\alpha 1)}{P}{}^f{}_{bc} + \underset{(\alpha 2)}{C}{}_b{}^{af} \underset{(12)}{B}{}_{fcd},\\
\underset{(\alpha 02)}{R}{}_b{}^a{}_c{}^d = \dot{\partial}^d \underset{(\alpha 0)}{H}{}^a{}_{bc} - \underset{(\alpha 2)}{C}{}_b{}^{ad}{}_{|\alpha c} + \underset{(\alpha 1)}{C}{}^a{}_{bf} \underset{(21)}{B}{}^f{}_c{}^d + \underset{(\alpha 2)}{C}{}_b{}^{af} \underset{(\alpha 2)}{P}{}_{fc}{}^d, \quad (\alpha = 0, 1, 2),
\end{cases}
$$

$$\begin{cases} \underset{(\alpha 11)}{R}{}_b{}^a{}_{cd} = \dot{\partial}_d \underset{(\alpha 1)}{C}{}^a{}_{bc} - \dot{\partial}_c \underset{(\alpha 1)}{C}{}^a{}_{bd} + \underset{(\alpha 1)}{C}{}^f{}_{bc} \underset{(\alpha 1)}{C}{}^a{}_{fd} - \underset{(\alpha 1)}{C}{}^f{}_{bd} \underset{(\alpha 1)}{C}{}^a{}_{fc}, \\ \underset{(\alpha 12)}{R}{}_b{}^a{}_c{}^d = \dot{\partial}^d \underset{(\alpha 1)}{C}{}^a{}_{bc} - \dot{\partial}_c \underset{(\alpha 2)}{C}{}_b{}^{ad} + \underset{(\alpha 1)}{C}{}^f{}_{bc} \underset{(\alpha 2)}{C}{}_f{}^{ad} - \underset{(\alpha 2)}{C}{}_b{}^{fd} \underset{(\alpha 1)}{C}{}^a{}_{fc}, \\ \underset{(\alpha 22)}{R}{}_b{}^{acd} = \dot{\partial}^d \underset{(\alpha 2)}{C}{}_b{}^{ac} - \dot{\partial}^c \underset{(\alpha 2)}{C}{}_b{}^{ad} + \underset{(\alpha 2)}{C}{}_b{}^{fc} \underset{(\alpha 2)}{C}{}_f{}^{ad} - \underset{(\alpha 2)}{C}{}_b{}^{fd} \underset{(\alpha 2)}{C}{}_f{}^{ac}, \quad (\alpha = 0, 1, 2). \end{cases}$$

10 Metric structures on the manifold $T^{*2}M$. Metric $N-$linear connections

Definition 10.1 A *metric structure* on the manifold $T^{*2}M$ is a symmetric covariant tensor field $\mathbb{G}$ of type (0,2) which is non-degenerate at each point $u \in T^{*2}M$ and of constant signature on $T^{*2}M$. If $\mathbb{G}$ is positive definite we say that it defines a Riemann structure on $T^{*2}M$.

Let us consider a metric structure $\mathbb{G}$ on $T^{*2}M$ for which the distributions N, V_1, W_2 are more general then (6.2), namely we have the decomposition:

$$\mathbb{G}(X,Y) = \mathbb{G}(X^H, Y^H) + \mathbb{G}(X^{V_1}, Y^{V_1}) + \mathbb{G}(X^{W_2}, Y^{W_2}), \quad \forall X, Y \in T^{*2}M. \tag{10.1}$$

In other words, $\mathbb{G}$ decomposes as a sum of three $d-$tensor fields,

(0) $\mathbb{G}^H$ of type $(0,2)$ defined by $\mathbb{G}^H(X,Y) = \mathbb{G}(X^H, Y^H)$,
(1) $\mathbb{G}^{V_1}$ of type $(0,2)$ defined by $\mathbb{G}^{V_1}(X,Y) = \mathbb{G}(X^{V_1}, Y^{V_1})$,
(2) $\mathbb{G}^{W_2}$ of type $(0,2)$ defined by $\mathbb{G}^{W_2}(X,Y) = \mathbb{G}(X^{W_2}, Y^{W_2})$.

Locally, these $d-$tensor fields can be written as

$$\mathbb{G}^H = \underset{(0)}{g}{}_{ab} dx^a \otimes dx^b, \quad \mathbb{G}^{V_1} = \underset{(1)}{g}{}_{ab} \delta y^a \otimes \delta y^b, \quad \mathbb{G}^{W_2} = \underset{(2)}{g}{}^{ab} \delta p_a \otimes \delta p_b,$$

where $\underset{(0)}{g}{}_{ab} = \mathbb{G}(\delta_a, \delta_b)$, $\underset{(1)}{g}{}_{ab} = \mathbb{G}(\dot{\partial}_a, \dot{\partial}_b)$, $\underset{(2)}{g}{}^{ab} = \mathbb{G}(\dot{\partial}^a, \dot{\partial}^b)$, and

$$rank \parallel \underset{(\alpha)}{g}{}_{ab} \parallel = n, \quad (\alpha = 0,1,2), \quad \parallel \underset{(2)}{g}{}_{ab} \parallel = \parallel \underset{(2)}{g}{}^{ab} \parallel^{-1}.$$

Thus the decomposition (10.1) looks locally as

$$\mathbb{G} = \underset{(0)}{g}{}_{ab} dx^a \otimes dx^b + \underset{(1)}{g}{}_{ab} \delta y^a \otimes \delta y^b + \underset{(2)}{g}{}^{ab} \delta p_a \otimes \delta p_b. \tag{10.2}$$

Definition 10.2 An $N-$linear connection D on $T^{*2}M$ endowed with a metric structure $\mathbb{G}$ is said to be a *metric $N-$linear connection* if $D_X \mathbb{G} = 0$ for every $X \in T^{*2}M$.

Let $\mathbb{G}$ be a metric structure on $T^{*2}M$ given by (10.2). We have

Proposition 10.2 *An $N-$linear connection on $T^{*2}M$ is a metric $N-$linear connection with respect to $\mathbb{G}$ given by (10.2) if and only if*

$$\underset{(\alpha)}{g}{}_{ab|\alpha c} = 0, \quad \underset{(\alpha)}{g}{}_{ab}|_{\alpha c} = 0, \quad \underset{(\alpha)}{g}{}^{ab}|^{\alpha c} = 0, \tag{10.3}$$

where $\parallel \underset{(\alpha)}{g}{}^{ab} \parallel = \parallel \underset{(\alpha)}{g}{}_{ab} \parallel^{-1}$, $(\alpha = 0,1,2)$.

Remark 10.1 The conditions (10.3) are equivalent with the conditions

$$\underset{(\alpha)}{g}{}^{ab}{}_{|\alpha c}=0,\quad \underset{(\alpha)}{g}{}^{ab}\,|_{\alpha c}=0,\quad \underset{(\alpha)}{g}{}_{ab}\,|^{\alpha c}=0,\quad (\alpha=0,1,2).$$

We shall now discuss the existence of metric $N-$linear connection on $T^{*2}M$. By straightforward calculation we get

Theorem 10.1 *If the manifold $T^{*}M$ is endowed with the metric structure $\mathbb{G}$ given by (10.2), then there exists on $T^{*2}M$ a metric $N-$linear connection, depending only on $\mathbb{G}$, whose $h(hh)-$, $v_1(v_1v_1)-$and $w_2(w_2w_2)-$tensors of torsion, $\underset{(00)}{\overset{0}{T}}{}^{a}{}_{bc}$, $\underset{(11)}{\overset{1}{S}}{}^{a}{}_{bc}$, $\underset{(22)}{\overset{2}{S}}{}_{a}{}^{bc}$ vanish. Its local coefficients defined by*

$$D\Gamma(N):=(\underset{(00)}{H}{}^{a}{}_{bc},\underset{(10)}{H}{}^{a}{}_{bc},\underset{(20)}{H}{}_{a}{}^{b}{}_{c},\underset{(01)}{C}{}^{a}{}_{bc},\underset{(11)}{C}{}^{a}{}_{bc},\underset{(21)}{C}{}_{a}{}^{b}{}_{c},\underset{(02)}{C}{}^{a}{}_{b}{}^{c},\underset{(12)}{C}{}^{a}{}_{b}{}^{c},\underset{(22)}{C}{}_{a}{}^{bc}),$$

are as follows

$$\begin{aligned}
&\underset{(00)}{\overset{c}{H}}{}^{a}{}_{bc}=\frac{1}{2}\underset{(0)}{g}{}^{ad}(\delta_c\underset{(0)}{g}{}_{bd}+\delta_b\underset{(0)}{g}{}_{dc}-\delta_d\underset{(0)}{g}{}_{bc}),\\
&\underset{(10)}{\overset{c}{H}}{}^{a}{}_{bc}=\underset{(11)}{B}{}^{a}{}_{cb}+\frac{1}{2}\underset{(1)}{g}{}^{ad}(\delta_c\underset{(1)}{g}{}_{bd}-\underset{(11)}{B}{}^{f}{}_{cb}\underset{(1)}{g}{}_{fd}-\underset{(11)}{B}{}^{f}{}_{cd}\underset{(1)}{g}{}_{bf}),\\
&\underset{(20)}{\overset{c}{H}}{}^{a}{}_{bc}=-\underset{(22)}{B}{}^{a}{}_{bc}+\frac{1}{2}\underset{(2)}{g}{}^{ad}(\delta_c\underset{(2)}{g}{}_{bd}+\underset{(22)}{B}{}^{f}{}_{bc}\underset{(2)}{g}{}_{fd}+\underset{(22)}{B}{}^{f}{}_{dc}\underset{(2)}{g}{}_{bf}),\\
&\underset{(01)}{\overset{c}{C}}{}^{a}{}_{bc}=\frac{1}{2}\underset{(0)}{g}{}^{ad}\dot{\partial}_c\underset{(0)}{g}{}_{bd},\qquad \underset{(02)}{\overset{c}{C}}{}_{a}{}^{bc}=-\frac{1}{2}\underset{(0)}{g}{}_{ad}\dot{\partial}_c\underset{(0)}{g}{}^{bd},\\
&\underset{(11)}{\overset{c}{C}}{}^{a}{}_{bc}=\frac{1}{2}\underset{(1)}{g}{}^{ad}(\dot{\partial}_c\underset{(1)}{g}{}_{bd}+\dot{\partial}_b\underset{(1)}{g}{}_{dc}-\dot{\partial}_d\underset{(1)}{g}{}_{bc}),\\
&\underset{(21)}{\overset{c}{C}}{}^{a}{}_{bc}=\frac{1}{2}\underset{(2)}{g}{}^{ad}\dot{\partial}_c\underset{(2)}{g}{}_{bd},\qquad \underset{(12)}{\overset{c}{C}}{}_{a}{}^{bc}=-\frac{1}{2}\underset{(1)}{g}{}_{ad}\dot{\partial}^c\underset{(1)}{g}{}^{bd},\\
&\underset{(22)}{\overset{c}{C}}{}_{a}{}^{bc}=-\frac{1}{2}\underset{(2)}{g}{}_{ad}(\dot{\partial}^c\underset{(2)}{g}{}^{bd}+\dot{\partial}^b\underset{(2)}{g}{}^{dc}-\dot{\partial}^d\underset{(2)}{g}{}^{bc}).
\end{aligned}\tag{10.4}$$

Definition 10.3 The metric $N-$linear connection given by (10.4) will be called the *canonical $N-$linear connection associated with* $\mathbb{G}$.

11 Berwald-Moor metrics on the manifold $T^{*2}M$

We further specialize the obtained results to the case when the base manifold is a Space-Time. Then $\dim M=4$, $\dim T^{*}M=8$ and $\dim T^{*2}M=12$. The nonlinear connection $N=(N^{a}{}_{b},N_{ab})$ given by (4.4), has the coefficients of the Lie brackets of the adapted basis satisfying the relations (4.5). We consider the Riemannian metric on $T^{*2}M$:

$$\mathbb{G}=g_{ab}(x)dx^{a}\otimes dx^{b}+g_{ab}(x)\delta y^{a}\otimes\delta y^{b}+h^{ab}(y)\delta p_{a}\otimes\delta p_{b}$$

where g_{ab} is a Riemannian metric on M and h^{ab} is the dual of the Berwald-Moor type metric

$$h_{ab}=\frac{1}{2}\frac{\partial^{2}F^{2}}{\partial y^{a}\partial y^{b}},\quad a,b=\overline{1,4},\tag{11.1}$$

where $F(y)=\sqrt[4]{|y^{1}y^{2}y^{3}y^{4}|}$. Then the structure $\mathbb{F}$ given in (5.1) satisfies the relation

$$\mathbb{G}(\mathbb{F}X,Y)=-\mathbb{G}(X,\mathbb{F}Y).\tag{11.2}$$

As well, the following results regarding the canonic $d-$linear connection hold true:

Theorem 11.1 1° *The canonic metrical linear d−connection* $D\overset{c}{\Gamma}(N)$ *has the components*

$$\begin{cases} \underset{(00)}{\overset{c}{H}}{}^a{}_{bc} = \gamma^a_{bc}(x), \ \underset{(10)}{\overset{c}{H}}{}^a{}_{bc} = \gamma^a_{bc}(x), \\ \underset{(20)}{\overset{c}{H}}{}^a{}_{bc} = \frac{1}{2}\left\{\gamma^a_{bc}(x) - \gamma^f_{cd}(x)h^{am}\left[y^d(\dot{\partial}_f h_{bm}) + \delta^d_m h_{bf}\right]\right\}, \end{cases}$$

and

$$\underset{(01)}{\overset{c}{C}}{}^a{}_{bc} = 0, \ \underset{(11)}{\overset{c}{C}}{}^a{}_{bc} = 0, \ \underset{(21)}{\overset{c}{C}}{}^a{}_{bc} = \frac{1}{2}h^{ad}\dot{\partial}_c h_{bd}, \ \underset{(02)}{\overset{c}{C}}{}_a{}^{bc} = 0, \ \underset{(12)}{\overset{c}{C}}{}_a{}^{bc} = 0, \ \underset{(22)}{\overset{c}{C}}{}_a{}^{bc} = 0.$$

2°. *The d−tensors of torsion are given by*

$$\begin{array}{l} \underset{(00)}{\overset{0}{T}}{}^a{}_{bc} = 0, \ \underset{(01)}{R}{}^a{}_{bc} = r_b{}^a{}_{cd}(x)y^d, \ \underset{(02)}{R}{}_{abc} = r_a{}^d{}_{bc}(x)p_d, \\ \underset{(01)}{\overset{0}{P}}{}^a{}_{bc} = 0, \ \underset{(01)}{\overset{1}{P}}{}^a{}_{bc} = 0, \ \underset{(01)}{\overset{2}{P}}{}_{abc} = 0, \\ \underset{(02)}{\overset{0}{P}}{}^a{}_b{}^c = 0, \ \underset{(02)}{\overset{1}{P}}{}^a{}_b{}^c = 0, \ \underset{(02)}{\overset{2}{P}}{}_{bc}{}^a = -\gamma^a_{bc}(x) + \underset{(20)}{\overset{c}{H}}{}^a{}_{bc}, \end{array}$$

and $\underset{(11)}{\overset{1}{S}}{}^a{}_{bc} = 0, \ \underset{(22)}{\overset{2}{S}}{}_a{}^{bc} = 0 \ \underset{(12)}{\overset{1}{Q}}{}^a{}_b{}^c = 0, \ \underset{(12)}{\overset{2}{Q}}{}_{bc}{}^a = \frac{1}{2}h^{ad}\dot{\partial}_c h_{bd}.$

3° *The d−tensors of curvature are given in the adapted basis by*

$$\begin{cases} \underset{(000)}{R}{}_b{}^a{}_{cd} = r_b{}^a{}_{cd}(x), \ \underset{(100)}{R}{}_b{}^a{}_{cd} = r_b{}^a{}_{cd}(x) \\ \underset{(200)}{R}{}_b{}^a{}_{cd} = \bar{\delta}_d \underset{(20)}{\overset{c}{H}}{}^a{}_{bc} - \bar{\delta}_c \underset{(20)}{\overset{c}{H}}{}^a{}_{bd} + \underset{(20)}{\overset{c}{H}}{}^f{}_{bc} \underset{(20)}{\overset{c}{H}}{}^a{}_{fd} - \underset{(20)}{\overset{c}{H}}{}^f{}_{bd} \underset{(20)}{\overset{c}{H}}{}^a{}_{fc} + \\ \qquad + \frac{1}{2}h^{am}(\dot{\partial}_f h_{bm})r_c{}^f{}_{dm}y^m, \end{cases}$$

where $\bar{\delta}_d = \partial_d - N^m{}_d\dot{\partial}_m$ *and*

$$\begin{cases} \underset{(001)}{R}{}_b{}^a{}_{cd} = 0, \ \underset{(101)}{R}{}_b{}^a{}_{cd} = 0 \\ \underset{(201)}{R}{}_b{}^a{}_{cd} = \dot{\partial}_d \underset{(20)}{\overset{c}{H}}{}^a{}_{bc} - \underset{(21)}{\overset{c}{C}}{}^a{}_{bd|2c} + \underset{(21)}{C}{}^a{}_{bf}(\gamma^f_{cd}(x) - \underset{(20)}{\overset{c}{H}}{}^f{}_{dc}), \end{cases}$$

$$\underset{(002)}{R}{}_b{}^a{}_c{}^d = 0, \ \underset{(102)}{R}{}_b{}^a{}_c{}^d = 0, \ \underset{(202)}{R}{}_b{}^a{}_c{}^d = 0,$$

$$\begin{cases} \underset{(011)}{R}{}_b{}^a{}_{cd} = 0, \ \underset{(111)}{R}{}_b{}^a{}_{cd} = 0, \\ \underset{(211)}{R}{}_b{}^a{}_{cd} = \dot{\partial}_d \underset{(21)}{\overset{c}{C}}{}^a{}_{bc} - \dot{\partial}_c \underset{(21)}{\overset{c}{C}}{}^a{}_{bc} + \underset{(21)}{\overset{c}{C}}{}^f{}_{bc} \underset{(21)}{\overset{c}{C}}{}^a{}_{fd} - \underset{(21)}{C}{}^f{}_{bc} \underset{(21)}{C}{}^a{}_{fc} = c_b{}^a{}_{cd}(y), \end{cases}$$

$$\underset{(012)}{R}{}_b{}^a{}_c{}^d = 0, \ \underset{(112)}{R}{}_b{}^a{}_c{}^d = 0, \ \underset{(212)}{R}{}_b{}^a{}_c{}^d = 0,$$

$$\underset{(022)}{R}{}_b{}^{acd} = 0, \ \underset{(122)}{R}{}_b{}^{acd} = 0, \ \underset{(222)}{R}{}_b{}^{acd} = 0$$

If we endow the space T^*M with the metric

$$\mathbb{G} = g_{ab}(x)dx^a \otimes dx^b + g_{ab}(x)\delta y^a \otimes \delta y^b + h^{ab}(p)\delta p_a \otimes \delta p_b,$$

where g_{ab} is a Riemannian metric on M and h^{ab} is the Berwald-Moor type metric

$$h^{ab} = \frac{1}{2}\frac{\partial^2 F^2}{\partial p_a \partial p_b}, \quad a, b = \overline{1,4}, \tag{11.3}$$

where $F(y) = \sqrt[4]{|p_1 p_2 p_3 p_4|}$, then the structure $\mathbb{F}$ given in (5.1) satisfies the relation (11.2). As well, we can state the following

Theorem 11.2 *1° The canonic metrical d−connection* $D\overset{c}{\Gamma}(N)$ *has the components:*

$$\begin{cases} \underset{(00)}{\overset{c}{H}}{}^a{}_{bc} = \gamma^a_{bc}(x), \quad \underset{(10)}{\overset{c}{H}}{}^a{}_{bc} = \gamma^a_{bc}(x) \\ \underset{(20)}{\overset{c}{H}}{}^a{}_{bc} = \gamma^a_{bc}(x) + \frac{1}{2} h^{ad}(N_{cf}\dot\partial^f h_{bd} - \gamma^f_{bc} h_{fd} - \gamma^f_{dc} h_{bf}), \end{cases}$$

$$\underset{(01)}{\overset{c}{C}}{}^a{}_{bc} = 0, \quad \underset{(11)}{\overset{c}{C}}{}^a{}_{bc} = 0, \quad \underset{(21)}{\overset{c}{C}}{}^a{}_{bc} = 0, \quad \underset{(02)}{\overset{c}{C}}{}_a{}^{bc} = 0, \quad \underset{(12)}{\overset{c}{C}}{}_a{}^{bc} = 0,$$

$$\underset{(22)}{C}{}_a{}^{bc} = -\frac{1}{2} h_{ad}(\dot\partial^c h^{bd} + \dot\partial^b h^{dc} - \dot\partial^d h^{bc}) = \Gamma^{bc}_a(p).$$

2° *The following sets of components of the d−tensors of torsion are nontrivial:*

$$\underset{(01)}{R}{}^a{}_{bc} = r_b{}^a{}_{cd}(x) y^d, \quad \underset{(02)}{R}{}_{abc} = r_a{}^d{}_{bc}(x) p_d, \quad \underset{(02)}{\overset{2}{P}}{}_{ab}{}^c = -\gamma^c_{ab} + \underset{(20)}{\overset{c}{H}}{}^c{}_{ab}.$$

3° *The following sets of components of the d−tensors of curvature are nontrivial:*

$$\underset{(000)}{R}{}_b{}^a{}_{cd} = r_b{}^a{}_{cd}(x), \quad \underset{(100)}{R}{}_b{}^a{}_{cd} = r_b{}^a{}_{cd}(x)$$

$$\underset{(200)}{R}{}_b{}^a{}_{cd} = \tilde\delta_d \underset{(20)}{\overset{c}{H}}{}^a{}_{bc} - \tilde\delta_c \underset{(20)}{\overset{c}{H}}{}^a{}_{bd} + \underset{(20)}{\overset{c}{H}}{}^f{}_{bc} \underset{(20)}{\overset{c}{H}}{}^a{}_{fd} - \underset{(20)}{H}{}^f{}_{bd} \underset{(20)}{H}{}^a{}_{fc} + \underset{(22)}{C}{}_b{}^{af} \underset{(02)}{R}{}_{fcd},$$

where $\tilde\delta_d = \partial_d - N_{df}\dot\partial^f$ *and* $\underset{(222)}{R}{}_b{}^{acd} = s_b{}^{acd}(p)$,

$$\underset{(202)}{R}{}_b{}^a{}_c{}^d = \dot\partial^d \underset{(20)}{H}{}^a{}_{bc} - \underset{(22)}{C}{}_b{}^{ad}{}_{|2c} + \underset{(22)}{C}{}_b{}^{af}(-\gamma^d_{fc} + \underset{(20)}{H}{}^d{}_{fc}).$$

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