PROPERTIES OF SPACES ASSOCIATED WITH COMMUTATIVE–ASSOCIATIVE H_3 AND H_4 ALGEBRAS

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In the first part of this work a real axis of the space associated with the H_3 algebra and the lines parallel to this axis are interpreted as the world lines of resting particles; surface of simultaneity is used for introduction of a distance between the real axis and a line parallel thereto. The coordinate system similar to a polar one can be introduced on this surface such that this allows us to reveal its simplest invariant transformations. In the second part of this paper the Lorentz transformations in form of special kind of rotations in the space associated with H_4 algebra are presented.

Introduction

The H_3 and H_4 algebras belong to the commutative–associative algebras of the H_n type which are of the simplest structure. These algebras are characterized by some preferred basis. The multiplication of numbers is realized in terms of this basis in a componentwise manner similarly to the addition in arbitrary algebras. On the other side, in H_n type algebras, which can be called hyperbolic, H_3 and H_4 algebras directly follow after the algebras of real (H_1) and double (H_2) numbers, which possessed important properties for their physical applications [6, 11]. We set forth an assumption of "inheriting" these properties by 3- and 4- dimensional algebras under consideration. As a motivation of this assumption we recall the relation between Berwald–Moor's metrics and H_4 algebra in Finsler generalization of the relativity theory [1]. From the point of view of possible applications, hyperbolic H_4 algebra is the most promising one because the $n = 4$ dimensional spaces have the topological preference [7]. However, H_3 algebra possesses one evident advantage. It is possible to use the computer visualization animation for figures, surfaces, and lines in the three dimensional metrical space associated with this algebra. Although it is not worth overestimating the analytical capacities of such applications, it gets a special visuality to geometric properties of this space. Therefore a sufficiently general approach to physical treatment of the hyperbolic space properties, offered in the first part of this paper, is represented for a space accounted with H_3 algebra. Its properties give the cube of norm as

$$|A|^3 = |a^1 a^2 a^3|,$$

where a^i are components of the vector in the preferred basis, combined from three numbers e_i, where $i = 1, 2, 3$, with properties $(e_i)^2 = e_i$, $e_i \cdot e_j = 0$ when $i \neq j$. Real numbers on a line can be shared in two classes: they are positive numbers, placed on the right side from zero, and the negative ones, placed on the left side from zero. Two isotropic lines in the double numbers algebra divide the pseudo-euclidian plane into 2^2 quadrants. Similarly to this the associated space is divided into 2^3 octants, and for all numbers appropriated to one octant points it is typical that the same sign combination of components is taken with respect to the preferred basis. The boundaries of the octants are three isotropic planes with equations $a^i = 0$, where $i = 1, 2, 3$. It will be noted also that since a hyperbolic algebras are algebras with a unity, defined by an expression

$$1 = e_1 + e_2 + \cdots + e_n,$$

two octants of the treated space can be preferably be selected. They are the octants, containing 1 and -1; they are characterized by numbers with all positive or all negative components, respectively.

Using considered algebras requires an availability of euclidian or pseudo-euclidian properties. In the order of algebras: the Dirac algebra [2], quaternions [3], biquaternions [5] – the existence of such properties provides a classical appearance of the norm of the number. However, there is a slight amount of such algebras, but amongst commutative-associative algebras only the double number algebra belongs to such class, in which a square of the norm of the numbers is given by

$$|A|^2 = |(a^1)^2 - (a^2)^2|$$

(see [4]). Chronogeometry method [8], [12] gives an other opportunity to establishing properties which are similar with the properties of euclidian or pseudo-euclidian spaces, in the spaces associated with the considered algebras; the first part of this paper is devoted to application of this method to H_3. Some more opportunity to establishing the sought properties appears on application of symmetric polyform associated with the algebra [9], which, for example, has the following form for H_3 algebra:

$$(A, B, C) = \frac{1}{3!}(a^1 b^2 c^3 + \cdots + a^3 b^2 c^1).$$

The second part of this paper is connected with such opportunity applied to H_4 algebra, where the form having appearance as pseudo-euclidian metric is determined by a polylinear form of four vectors.

1. A simultaneity surface in the commutative–associative algebras (as examplified by H_3)

1.1. *Axiomatics*

We shall treat the following statements, playing the role of axioms, as a principle to interpret physically the properties of the considered algebras class.

1. It is possible to connect an algebra number with some spatial-temporal event.

2. The real axis of the space, which direction is given by means of the unity of the algebras, is treated as a temporal axis, while the norm of the number is interpreted as an observer's time interval whose world line coincides with the vector corresponded to this number.

3. The increase of a relative velocity of particle or signal results in increasing an inclination of tangent line to the particles world line in the given point to the observer world line, and resting material points have world lines which are parallel to the observer line.

4. Light signals, which have a maximal velocity, are connected with isotropic hypersurfaces of the algebra; and it is supposed that the velocity of the light signals does not depend on their propagation direction. According to these statements two selected octants with 1 and -1, which are referred to above, are the analogs of the cone of the

future and the past Minkowski space in the space associated with H_3 algebra, respectively. Contrary to the Minkowski space in the considered space a domain outside these cones also possesses isotropic directions, because consists of six side cones. In this paper we restrict our attention to the most common particular case, when the observer world line coincides with the real axis.

1.2. *Exponential form of the H_3 algebra number representation with respect to the basis* $(1, j, k)$

Any number in the selected basis is represented as:

$$A = a^1 \cdot e_1 + a^2 \cdot e_2 + a^3 \cdot e_3.$$

For an exponential function in terms of this basis the following formula takes place:

$$\exp(a^1 \cdot e_1 + a^2 \cdot e_2 + a^3 \cdot e_3) = \exp(a^1) \cdot e_1 + \exp(a^2) \cdot e_2 + \exp(a^3) \cdot e_3. \tag{1}$$

Since in the considered algebra we get $|A|^3 = |a^1 a^2 a^3|$, any number with $a^i > 0$ is represented as

$$A = |A| \cdot \exp(b^1 e_1 + b^2 e_2 + b^3 e_3)$$

with a restriction

$$b_1 + b_2 + b_3 = 0, \tag{2}$$

which implies the identity:

$$|\exp(b^1 e_1 + b^2 e_2 + b^3 e_3)| = 1.$$

The other basis of the algebra is composed from vectors:

$$\begin{cases} 1 = e_1 + e_2 + e_3 \\ j = \sin\varphi_0 \cdot e_1 + \sin(\varphi_0 + 2\pi/3) \cdot e_2 + \sin(\varphi_0 + 4\pi/3) \cdot e_3 \\ k = \cos\varphi_0 \cdot e_1 + \cos(\varphi_0 + 2\pi/3) \cdot e_2 + \cos(\varphi_0 + 4\pi/3) \cdot e_3 \end{cases} \tag{3}$$

The vectors appearing in this basis are mutually orthogonal (in the usual euclidian sense), while an arbitrary parameter φ_0 can be treated in a certain sense as the angle of a simultaneous rotation of a pair of vectors j, k around the real axis. If t, x, y – are coordinates of the number in a new basis, then according to the transformation rules of coordinates of the number we have a system in the other basis:

$$\begin{cases} a^1 = t + \sin\varphi_0 \cdot x + \cos\varphi_0 \cdot y \\ a^2 = t + \sin(\varphi_0 + 2\pi/3) \cdot x + \cos(\varphi_0 + 2\pi/3) \cdot y \\ a^3 = t + \sin(\varphi_0 + 4\pi/3) \cdot x + \cos(\varphi_0 + 4\pi/3) \cdot y \end{cases} \tag{4}$$

from which it follows that $t = (a^1 + a^2 + a^3)/3$. Therefore by (2) the number representable in a exponential form in the basis (1,j,k) is given by

$$A = |A| \cdot e^{\alpha \cdot j + \beta \cdot k}.$$

If we modify this exponential representation, introducing an definition $\rho = \sqrt{\alpha^2 + \beta^2}$, we obtain

$$A = |A| \cdot e^{\rho(\cos\varphi \cdot j + \sin\varphi \cdot k)}. \tag{5}$$

Thus, in agreement with (5), the number at this representation is given by three parameters: the norm of the number $|A|$, the "radial coordinate" ρ, and the "angle coordinate" φ. Making use of (1) and (3), formula (5) takes simple and elegant form in components:

$$\begin{cases} a^1 = |A| \cdot \exp(\rho \sin[\varphi_0 + \varphi]) \\ a^2 = |A| \cdot \exp(\rho \sin[\varphi_0 + 2\pi/3 + \varphi]) \\ a^3 = |A| \cdot \exp(\rho \sin[\varphi_0 + 4\pi/3 + \varphi]) \end{cases}$$

1.3. *Method of setting the distance between the real axis and the parallel line.*

For determination of the distance between the world lines of resting particles, one of which lying on the real axis, we use the chronogeometry method. Consider the exchange of signals with the constant velocity $\nu \leq c$; for simplicity we shall arrange point-events of signal transmission and the reception of the reverse signal on the real axis symmetrically with respect to zero time moment. Because of an equality of lengths of straight and reverse signals velocity $|B - A_1| = |A_2 - B|$, so we have:

$$(a^1 + T)(a^2 + T)(a^3 + T) = (T - a^1)(T - a^2)(T - a^3),$$

where $a^i + T > 0$, $T - a^i > 0$, which after expanding takes form:

$$(a^1 + a^2 + a^3) \cdot T^2 + a^1 a^2 a^3 = 0. \tag{6}$$

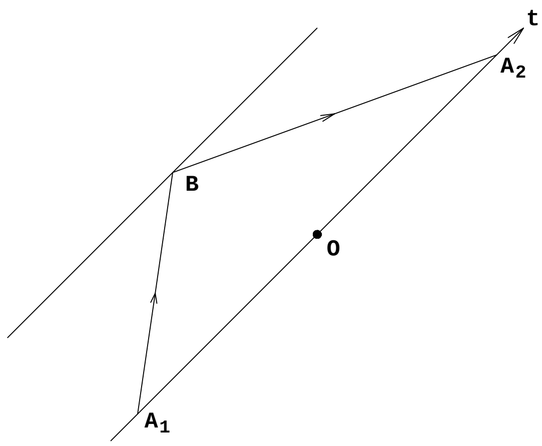

Figure 1: The measuring of a distance between the world lines by pre-light signals exchange.

The multitude of points-events satisfied to equation (6) form a surface of a simultaneity: it is for the observer on the real axis, being in the point with T coordinate, all these events are taking place in the same zero moment of time. Point $A = (0, 0, 0)$ belongs to the simultaneity surface, and the tangent plane to this surface in the origin has an equation:

$$a^1 + a^2 + a^3 = 0. \tag{7}$$

Substitution of (4) into (6) allows to obtain the equation of the simultaneity surface in form of the dependence of the time of the signal passing (on a clock of resting observer)

T from introduced coordinates $\{t, x, y\}$ of point of the simultaneity surface:

$$T^2 = \frac{1}{12}(x^2+y^2) - \frac{1}{3}\left\{t^2 + \frac{1}{t}\left[\frac{3}{4}xy(y\cdot\sin 3\varphi_0 - x\cdot\cos 3\varphi_0)\right.\right.$$
$$\left.\left. + x^3 \sin\varphi_0 \sin(\varphi_0+2\pi/3)\sin(\varphi_0+4\pi/3) + y^3\cos\varphi_0\cos(\varphi_0+2\pi/3)\cos(\varphi_0+4\pi/3)\right]\right\}.$$

According to this equation (and similar equations for other algebras, in particularly, H_4 algebra) the first items on the right side have an euclidian form, and then they dominate on other remaining items, square of travel time of signal depends linearly on square of the euclidian distance in the world lines space, which can be useful for the next physical interpretations.

1.4. *The system of curvilinear coordinates of the simultaneity surface and the transformations mapping it to itself*

Keeping in mind an important of an invariant transformations in modern physics, we shall briefly consider the topic of finding the transformations of the simultaneity surface, mapping it to itself. We introduce two-dimension coordinate system $\{\rho, \varphi\}$ on this surface, somewhat analogous to polar coordinate system on two-dimension plane to get:

$$\begin{cases} a^1 = (T-\rho)\cdot e^{R(\rho,\varphi)\sin(\varphi_0+\varphi)} - T, \\ a^2 = (T-\rho)\cdot e^{R(\rho,\varphi)\sin(\varphi_0+2\pi/3+\varphi)} - T, \\ a^3 = (T-\rho)\cdot e^{R(\rho,\varphi)\sin(\varphi_0+4\pi/3+\varphi)} - T, \end{cases} \tag{8}$$

where the function $R = R(\rho,\varphi)$ taken from transcendent equation is obtaining by using the coordinates (8) into (6):

$$\bar{Z}^3 - \bar{Z}^2\left[e^{-R\sin(\varphi_0+\varphi)} + e^{-R\sin(\varphi_0+2\pi/3+\varphi)} + e^{-R\sin(\varphi_0+4\pi/3+\varphi)}\right]$$
$$+2\bar{Z}\left[e^{R\sin(\varphi_0+\varphi)} + e^{R\sin(\varphi_0+2\pi/3+\varphi)} + e^{R\sin(\varphi_0+4\pi/3+\varphi)}\right] - 4 = 0,$$

where $\bar{Z} = (T-\rho)/T$.

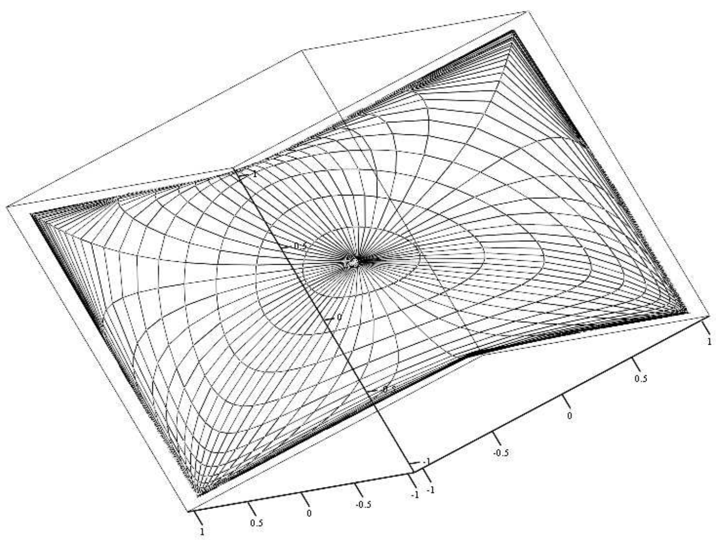

Figure 2: Curvilinear coordinates system ρ, ϕ on simultaneity surface.

In the vicinity of zero at $a^1, a^2, a^3 \ll 1,\ R \ll 1,\ \rho \ll 1$, the equations (8) are got simplified:

$$\begin{cases} a^1 \cong R \cdot T \cdot \sin(\varphi_0 + \varphi), \\ a^2 \cong R \cdot T \cdot \sin(\varphi_0 + 2\pi/3 + \varphi), \\ a^3 \cong R \cdot T \cdot \sin(\varphi_0 + 4\pi/3 + \varphi), \end{cases}$$

so that

$$a^1 + a^2 + a^3 \cong 0 \text{ and } (a^1)^2 + (a^2)^2 + (a^3)^2 \cong (R \cdot T)^2. \tag{9}$$

Thus, according to (9), the coordinate system (8) is distinguished: *in the vicinity of zero the parameter R is proportional to euclidian distance from a point, located on the simultaneity surface, to the center of this surface, in which $R = 0$.*

Then independent transformations of the simultaneity surface we seek are "rotations" by angle $\Delta\varphi(\varphi \to \varphi + \Delta\varphi)$ and "a similarity transformations" with a coefficient $K(\rho \to K \cdot \rho)$.

2. The representation a Lorentz transformations by rotations in the space, associated with H_4 algebra.

Following [10], we define the inner product of two arbitrary (with positive values of components) vectors A and B in the space under consideration by a symmetric four-form of H_4 space as:

$$(A, B) := \frac{(A, A, B, B)}{|A| \cdot |B|}.$$

The inner product of two vectors satisfying to properties of positiveness, homogeneity, and normality:
1. $(A, B) > 0$;
2. $(kA, B) = (A, kB) = k(A, B)$;
3. $(A, A) = |A|^2$.

The inner product of units vectors $a = A/|A|$ and $b = B/|B|$ may be regarded as an angle characteristic, setting a relation between two directions defined by these vectors – it is expressed via quotient components of these vectors ($d = b/a$):

$$(a, b) = (d_1 d_2 + d_1 d_3 + \dots d_3 d_4)/6. \tag{10}$$

Consider a basis in the space associated with H_4 algebra, consisting of these vectors:

$$\begin{cases} 1 = e_1 + e_2 + e_3 + e_4, \\ j' = 3e_1 - e_2 - e_3 - e_4, \\ k' = \sqrt{2}(2e_2 - e_3 - e_4), \\ l' = \sqrt{6}(e_3 - e_4). \end{cases}$$

We denote coordinates of relation of two considered vectors in a new basis via t_d, x_d, y_d, z_d and expressing (10) via these components, we obtain:

$$(a, b) = t_d^2 - x_d^2 - y_d^2 - z_d^2.$$

We shall denote the nonlinear transformation of 4-space, associated with H_4 algebra, which remains all vectors in the direction setting by vector A in rest, and retains the introduced inner product, as a *rotation* of vector B round a vector A. Thus, in addition to the other representations of Lorentz group [13] the representation by rotations round arbitrary time-like axis in the space, associated with H_4 algebra, can be used.

Results and conclusions

The method of determination of the distances between the world lines introduced for the space associated with a commutative-associative H_3 algebra (or H_4) allows to distinguish "a euclidian part".

A new geometric interpretation of the Lorentz transformations as rotations in the space connected with algebra H_4 is obtained. Arbitrary setting of a rotation axis is possible; all said above gives a hope on the application of such new interpretation in relativity physics.

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