FINSLEROID–SPACE SUPPLEMENTED BY ANGLE AND SCALAR PRODUCT

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The science of the past century has achieved great success on the basis of the geometrical quadratic concepts that were followed as logical and mathematical primaries. More profound ideas will imply using a more capacious class of geometries, for example the Finsler one which inscribes structures because the Finslerian indicatrices are no more isotropic in all directions. In the present work an attempt is made to resolve the respective difficulties of Finsler generalization by choosing the particular Finsleroid–type metric that implies one preferred direction, admitting the total axial symmetry around it. In this case, interesting constructive methods of introducing the concept of the angle and scalar product outside the frame of the Euclidean Geometry can conveniently be opened up.

"The Euclidean traditions are too strong to be rejected, and probably few generations of mathematicians are necessary to work off its influence." (Busemann [2], p. 8.)

Introduction

The quadratic method is the most convenient one to introduce the vector length. According to the method the length is defined by means of the square root of the quadratic form. For more than 20 centuries the Euclidean geometry and Euclidean rotations based on it have been served in theoretical constructions and predictions of results of experiments. The non-quadratic methods are developed in the Finsler geometry (see [1 – 6]).

Unfortunately, we must admit that much attention has not been paid in literature to studying the corresponding opportunities. By tradition the mathematical and theoretical physical concepts and equations are based on the method of introducing the vector length by the help of square root. And numerous interesting and deeply critical analysis (see, e.g., [7, 8]) of the geometrical structure of the space–time and methods of its generalization and comprehension usually go without even mentioning the existence of ideas and methods of the Finsler Geometry. In spite of high level of adequacy and accurate coincidence, it is still not clear how it is possible to express this degree of accuracy in numbers, for the Euclidean rotations do not possess a small parameter to evaluate.

In comparison with the common Euclidean metrics the Finslerian one introdudces the structure in metric geometry. While the unit surface of the Euclidean Geometry is a sphere that is isotropic in all directions, the introduction of geometrically preferred directions leads to generalizing the sphere and finally to generalizing the Euclidean Geometry. The corresponding, not isotropic, surface of the ends of the unit vectors (when issued from a fixed point) generates the Finsler metrics. Respective geometries can reflect the physical cases where the corresponding directional anisotropy is present. The Berwald–Moor metrics is totally anisotropic, for it supposes geometrically–emphasized directions whose number equals the number of manifold dimensions (accordingly, 3 in the three–dimensional case and 4 in the four–dimensional case). The Finsleroid–Geometry introduces only one preferred direction, supposing the total axial symmetry around it.

Actually, the task of generalizing the Euclidean metric function to the Finsler case seems to be too general and rather unclear to give a definite answer. But if we treat the problem from point of view of invariance and the possibility of introduction of the angle and scalar product, then we can endeavor to find constructive ways of defining the classes of Finsler spaces. As a result, there may emerge the methods of abandoning the Euclidean geometry.

Of course, no matter how motivated our desire to leave the borders of "quadratic conceptions" is, it is impossible to "overcome the square root completely". The hierarchy of geometries takes its root in generalizing. It is clear that methods and ideas of the Euclidean geometry are present and work in the Riemann geometry. Many authors of works on the Finsler Geometry used "the associative Riemann Geometry", introduced "the Riemann connection" or "the Finsler–Riemann connection", introduced "associative relative Riemann geometry along the vector fields", constructed "osculating Riemann space" , and "the Riemann development of the Finsler space along the curve", etc. The mathematicians applied steadily the associative Euclidean geometry in the Minkowski spaces.

Any theory that abandons the concepts dictated by the quadratic form has the shape of a pyramid: going down to the basis of the "unique super-geometry" the researcher must enter the area of "the associative Finsler Geometry", where in its turn appear different Riemann images, and then numerous Euclidean pictures.

The above facts are directly related to the Quadra-number geometry (developed recently in the work by Pavlov [9, 10]). In fact, it appears from examining the commutative hyper-numbers and relates the standard to them. By interpreting the component of the hyper-numbers as the component of the vector this metrics can be related to the type of "Berwald–Moor's Finsler metric function". Basing ourselves on the last case we can (and must) develop the theory of geometric correlations, including the introduction of the geodesic angle, perpendicularity,... – that do not coincide with analogous geometrical juxtaposition of the Riemann or Euclidean Geometry. Particularly, we cannot reject the latter one because we use graphic presentations and pictures, at least we have to simulate and construct them in the Euclidean space!

At the same time, this does not mean that the Finsler geometrical properties are prescribed uniquely to the Quadra–space. In fact, according to its own capacities, the poly–form theory makes it possible to introduce the corresponding angles and perpendicularity; in particular, such a generalization of the theory of "higher degree of metrics" was made in the works [9, 10].

Obviously, the Minkowski geometry has more invariants than the Euclidean one, and the Finsler aprroach – more than the Riemann one. In such context we should indicate that the Quadra–spaces have much more invariant objects, than the Finsler or Minkowski ones, and can offer a theory which is richer in geometrical concepts. In particular, this can be seen in the fact that the Euclidean geometry can be easily associated with the Quadra–space in many ways.

Philosophy and logic of associated problems. We can hardly overestimate the importance of Euclidean approaches and the fact that the Euclidean Quadratic geometry has already built up and keeps on building up the way of thinking and analysis of many scientists and researchers. For example, the Riemann geometry since its definition is based on the quadratic form (sometimes it used to be called "the geometry created by the quadratic form"), the theory of bundle spaces also applies the quadratic method (but it is more multifarious than the Riemann geometry), the Lagrangians in theories of physical fields are usually quadratic with respect to derivatives, the energy and impulse of the relativistic particle are connected by a quadratic form, etc. The Special and General Theory of Relativity are also based on the quadratic forms, but now possess pluses $\langle + \rangle$ as well as minuse $\langle - \rangle$ in the signature; the Lobachevski geometry is also related to the type.

Nowadays there are many books on geometry, where quite often different "models" of generalized geometries are presented and studied. In contrast to this the Geometry, and not "a model of geometry", is presented in Euclid's work.

Why the Euclidean geometry has lived through 2 millenniums? The reason is that the square root of a quadratic form is used to define length and vectors. We can come across this method everywhere: in practice, in mathematical and physical theories, and in experiments; it is also used nowadays. Logically it is the simplest way. But "the simplest" is not always "the most precise".

"The axiomaticians" during the last century have been analyzing the structure of the Euclidean geometry (remind Hilbert's famous work The Foundations of Geometry), and not the ways of constructive generalizing of the "quadraticity" of the Euclidean metrics.

It is quite easy to question any statement that declares about "high experimental accuracy" of the quadratic method of establishing the length. Has anyone and with what accuracy checked the Pythagorean theorem? Such check is hardly possible without the researcher using more general methods for comparison (profound research of the topic is out of the aim of the work, the readers may try to carry out their own analysis)

In fact, the Euclidivity of the geometry or its models is preserved till preserves the quadraticity of the definition of length. But we need something more than just courage to make the corresponding decisive step. This is a difficult task: we must find a good way to change the quadratic method of defining the length by a more general one and recast the equations of mathematical physics on the basis of the method in order to abandon the "Euclidivity". And this is a good task for the scientists of the new millennium. The conservative way of thinking as an obstacle in the way of geometrical progress can be effective only during a very short period of time.

The Length is the fundamental concept either in theoretical or applied science. We can compare it only with the concept of Number in its fundamentality. The development and application of the concept of the length have lead to creation of Geometry, and the concept of number – to Algebra.

Using the theory of the so–called Minkowski Space (it is also called Minkowski geometry) we can formulate quite a general and modern attitude. In the modern accurate mathematical language the Minkowski Space is often defined as the Finite–Dimensional Banach Space.

In the Minkowski spaces the length is introduced by the general definition that enables it to be defined by functions of a rather wide range of classes with minimum conditions on smoothness. The fibered manifold, where the fiberes are Minkowski spaces, are called the Finsler spaces.

During the last century many scientists have been studying the Minkowski geometry and Finsler geometry. More than 2000 works and a number of monographs have been published, but we should be very cautious while speaking about the achieved success. It is inevitable that we come across a large number of tensors in the Finsler geometry (that do not have non-trivial prototypes on the Riemann geometry), and it is not obvious that such numerical growth predetermines qualitative leap. By the latter case the Finsler geometry have spurned many mathematicians as it seems to be extremely difficult to study because of the great number of tensors (in comparison to this the Riemann geometry is quite economical: there is a metrical tensor, one set of coefficients of association and one curvature tensor).

But we should not be too pessimistic about the inconvenience of formalism. Especially nowadays, that there are few people that will be surprised by the multy–componentry of the objects neither in mathematics nor in theoretical physics. It is likely that the problems lie in another level, and to be more precise in the lack of definite key links. Here we can recall Busemann's remark, that the progress in motion should consist "not of the generalizing of the Riemann geometry, but of its results as well".

Should the development of the concept of "the Length" be in connection with the development of the concept of "the Number"?

If we turn to prehistory of the Euclidean geometry, to Pythagoras's activity, then we will learn about his tragedy when he learnt that the diagonal of the square is rationally not commensurable with the length of its sides. So, for Pythagoras it was a real catastrophe that the number did not correspond to the length. This "surprise" gave an impulse to development of the concept of the Number, and to be more precise, to creation of the theory of the irrational number. The developed correspondence between the Length and the Number made the basis of the Euclidean geometry and moreover of its axiomatics (for example suggested by Hilbert). In this regard the axiomatic of the Euclidean geometry developed by Hilbert was the culmination of the identity of the concept of arithmetical number and quadratic length, many geometric key concepts have been derived from arithmetical numerical properties.

The following move from the Euclidean geometry to the Riemann one does not add any new ideas to the dichotomy. The Riemann geometry is just "a fibration of the Euclidean geometries", so that in every level there works the Euclidean geometry and the common definition of length is used.

The Minkowski geometry abandons the definition of the length, but (though, as is well known, Minkowski started thinking about geometry while studying the theory of numbers) develops the problem without any connection with the concept of number. Pythagoras's tragedy does not matter any more! We can say the same about the modern Finsler spaces, that are just fibrations of the levels in the Minkowski space.

Such excursus into history enables us to show enough courage to state the following: *we should build the Finsler geometry in close connection with the development of the concept of the Number.*

We can hope that this idea will be principle for the successive development of the Finsler geometry in the present century. We should know on what level does the Finsler generalizing of the Length is needed to generalize the concept of the Number. The answer is not clear, though the reversed way of thinking is obvious: the non-Euclidean, not quadratic Finsler metric function should be the measure of the generalized number.

There emerges a very important question: where does poly-numbers are crucial in the Finsler geometry, so that you can do nothing without them? Pythagoras's tragedy is clear: the rational numbers are not enough to measure the length of the unit square diagonal. The origin of the transcendental numbers is also clear: the unit circumference diameter cannot be measured by the algebraic irrational number.

The anisotropy is presumed in generalizing the Euclidean geometry. In this connection the indicatrix becomes the key concept: it is the surface of ends of the unit vectors that issue at a fixed point. In the Euclidean geometry the sphere is the indicatrix. It symbolizes the isotropy of the space, equality of its properties in all directions. As the uniformity condition appears in the definition of the Minkowski space and the Finsler space, the indicatrix proves anisotrophy of any vectors (not necessarily the unit ones). The move from The Euclidean geometry to the Minkowski one symbolizes refusal from the total isotropy of the space and after the move the corresponding indicatrix cannot be a second-order surface any more.

From the point of view of anisotropy, The Berwald-Moor metrics is characterized by the presence of preferred directions, that in their number equal the number of dimensions of the space.

In the present work the necessary basic definitions and results of calculations of the associated values for the Finsleroid-geometry ($\mathcal{E}_g^{PD}$–geometry), that admit only one preferred direction are presented. Our previous investigation [5,6] showed that the study is promising. In fact, $\mathcal{E}_g^{PD}$-approach is applied to the development of new study of new field in the metric differential geometry and can be effective in Finsler and Minkowski geometries. The observation that the one-vector Finsler metric function associated with $\mathcal{E}_g^{PD}$-space quite naturally admits the promising two-vector generalizing , in this way generating the angle and the scalar product, is the key point of the article.

Attempts to introduce the angle and into the Finsler and Minkowski spaces always striked against the ambiguity:

"Therefore no particular angular measure can be entirely natural in Minkowski geometry. This is evidenced by the innumerable attempts to define such a measure, none of which found general acceptance". (Busemann [2], p. 279.)

"Unfortunately, there exists a number of distinct invariants in a Minkowskian space all of which reduce to the same classical euclidean invariant if the Minkowskian space degenerates into a euclidean space. Consequently, distinct definitions of the trigonometric functions and of angles have appeared in the literature concerning Minkowskian and Finsler spaces". (Rund [3], p. 26)

The fact that the attempts have never been unambiguous seems to be due to a lack of the proper tools. For the opinion was taken for granted that the angle ought to be defined or constructed in terms of the basic Finslerian metric tensor (and whence ought to be explicated from the initial Finslerian metric function). Let us doubt the opinion from the very beginning. Instead, we would like to raise alternatively the principle that the angle is a concomitant of the geodesics (and not of the metric function proper). The angle is determined by two vectors (instead of one vector in case of the length) and actually implies using a due extension of the Finslerian metric function to a two-vector metric function (to a scalar product). Below, the principle is applying to the Finsleroid space in a systematic way. The essence of the generalizing can be visualized in deformation of the Euclidean sphere (which is the indicatrix of the Euclidean space).

We devote the section 1 to geodesic equations. Remarkably, the equations admit a simple and clear solution. Then we can find the angle between two vectors. Usually it is expected that the angle measure should be additive (for the angles with the same vertex). The angle differs from the Euclidean angle in the quasi–Euclidean space only by the constant factor and consequently is additive. The cosine rule is held true when changing the Euclidean angle by the found angle. We get the corresponding scalar product.

Formally, the method of introducing the vector length with the help of the square root of a quadratic form lies in the basis of Euclidean conception. In the present work we use the concrete axial–symmetric generalizing of such method, basing ourselves on constructive ideas of the Finsler geometry. We introduce the corresponding Finsler metric function and in detail describe its basic properties and consequences. The generalizing is characterized by one non–dimentional parameter, that is denoted below as g.

Then the section 2 introduces designations, definitions and basic concepts of the space $\mathcal{E}_g^{PD}$. On this fact the supposition that the space includes one emphasized direction, that we will often call the Z axis, is based. The abbreviations FMF and FMT will be used to denote the Finsler metric function and Finsler metric tensor accordingly. The characteristic parameter g can take the value between -2 and 2; if $g=0$ the space $\mathcal{E}_g^{PD}$ is driven to the common Euclidean space. After preliminary introduction of characteristic quadratic form B, that differs from the Euclidean sum of squares by the presence of the mixed term (see (2.22)), we define FMF K of the space $\mathcal{E}_g^{PD}$ with the help of the formula (2.30) – (2.33). The characteristic feature of the formula is the presence of the function "arctan". Then we calculate the tensorial values of the space. There is a phenomenon that simplifies the construction: the associated Cartan tensor that turns out to have a simple algebraic structure (2.66) – (2.67). In particular this unique phenomenon leads to the conclusion that the indicatrix of the space $\mathcal{E}_g^{PD}$ is a space of constant positive curvature. The curvature value depends on the parameter g according to the rule (2.73).

The section 3 introduces the concept of quasi–Euclidean reflection of the $\mathcal{E}_g^{PD}$–space. The concept turns out to be quite promising because the quasi–Euclidean space is simple in many aspects, so that the corresponding transformation simplifies different calculations. It is not flat, but is conformally flat. The section 4 gives idea about some interesting properties of the quasi–Euclidean metric tensor. Figures that illustrate the Finsleroids with different values of the parameter g are placed in the Appendix.

1. Derivation of geodesics and angle in associated quasi–euclidean space

For the space under study, the geodesics should be obtained as solutions to the equation

$$\frac{d^2R^p}{ds^2}+C_q{}^p{}_r(g;R)\frac{\partial R^q}{\partial ds}\frac{\partial R^r}{\partial ds}=0 \tag{1}$$

which coefficients $C_p{}^q{}_r$ are given by the list placed at the end of Sec. 2. To avoid complications of calculations involved, it proves convenient to transfer the consideration in the quasi–euclidean approach (see Secs. 3 and 4). Accordingly, we put

$$\sqrt{g_{pq}(g;R)dR^pdR^q}=\sqrt{n_{pq}(g;t)dt^pdt^q} \tag{2}$$

and

$$R^p(s)=\mu^p(g;t^r(s)) \tag{3}$$

together with

$$\frac{dR^p(s)}{ds}=\mu^p_q(g;t^r(s))\frac{dt^q(s)}{ds}, \tag{4}$$

where $\mu^p(g;t^r)$ and $\mu^p_q(g;t^r)$ are the coefficients given, respectively by Eqs. (3.14) and (3.38) – (3.40). Let a curve C: $t^p=t^p(s)$ be given in the quasi-euclidean space, with *the arc-length parameter* s along the curve being defined by the help of the differential

$$ds=\sqrt{n_{pq}(g;t)dt^pdt^q}, \tag{5}$$

where $n_{pq}(g;t)$ is the associated quasi-euclidean metric tensor given by Eq. (3.49) in Part II. Respectively, *the tangent vectors*

$$u^p=\frac{dt^p}{ds} \tag{6}$$

to the curve are unit, in the sense that

$$n_{pq}(g;t)u^pu^q=1. \tag{7}$$

Since $L_p = \partial S/\partial t^p$, we have

$$L_p u^p = \frac{dS}{ds}. \tag{8}$$

Here, $S^2(t) = n_{pq}(g;t)t^p t^q = r_{pq}t^p t^q$ (see Eq. (3.46)). Using Eq. (4.16) leads through well-known arguments to the following equation of geodesics in the quasi–euclidean space:

$$\frac{d^2\mathbf{t}}{ds^2} = \frac{1}{4}G^2\frac{\mathbf{t}}{S^2}H_{pq}u^p u^q, \tag{9}$$

where $H_{pq} = h^2(n_{pq} - L_p L_q)$ (see Eq. (4.4)) and $\mathbf{t} = \{t^p\}$. We obtain

$$\frac{d^2\mathbf{t}}{ds^2} = \frac{1}{4}g^2\frac{\mathbf{t}}{S^2}\left(1 - (\frac{dS}{ds})^2\right) = \frac{1}{4}g^2(a^2 - b^2)\frac{\mathbf{t}}{S^4} \tag{10}$$

and

$$\frac{d^2\mathbf{t}}{ds^2} = \frac{1}{4}g^2(a^2 - b^2)\frac{\mathbf{t}}{S^4} \tag{11}$$

with

$$S^2(s) = a^2 + 2bs + s^2, \tag{12}$$

where a and b are two constants of integration.

If we put

$$S(\Delta s) = \sqrt{a^2 + 2b\Delta s + (\Delta s)^2} \tag{13}$$

and

$$\mathbf{t}_1 = \mathbf{t}(0), \qquad \mathbf{t}_2 = \mathbf{t}(\Delta s), \tag{14}$$

then we get

$$a = \sqrt{(\mathbf{t}_1\mathbf{t}_1)} \tag{15}$$

and

$$S(\Delta s) = \sqrt{(\mathbf{t}_2\mathbf{t}_2)} \tag{16}$$

together with

$$(\mathbf{t}_1\mathbf{t}_2) = aS(\Delta s)\cos\Big[h\arctan\frac{\sqrt{a^2 - b^2}\,\Delta s}{a^2 + b\Delta s}\Big]. \tag{17}$$

Here, $\mathbf{t}_1$ and $\mathbf{t}_2$ are two vectors with the fixed origin O; they point to the beginning of the geodesic and to the end of the geodesic, respectively. The notation parenthesis couple (..) is used for the euclidean scalar product, so that $(\mathbf{t}_1\mathbf{t}_1) = r_{pq}t_1^p t_1^q$, $(\mathbf{t}_1\mathbf{t}_2) = r_{pq}t_1^p t_2^q$, and r_{pq} is a euclidean metric tensor; $r_{pq} = \delta_{pq}$ in case of orthogonal basis; δ stands for the Kronecker symbol. From (1.15)-(1.17) it directly follows that

$$\frac{\sqrt{a^2 - b^2}\,\Delta s}{a^2 + b\Delta s} = \tan\Big[\frac{1}{h}\arccos\frac{(\mathbf{t}_1\mathbf{t}_2)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\Big]. \tag{18}$$

The equality (1.18) suggests the idea to introduce

DEFINITION. *The $\mathcal{E}_g^{PD}$-associated angle* is given by

$$\alpha \stackrel{\text{def}}{=} \frac{1}{h}\arccos\frac{(\mathbf{t}_1\mathbf{t}_2)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}, \tag{19}$$

so that

$$\alpha = \frac{1}{h}\alpha_{euclidean}. \tag{20}$$

Such an angle is obviously *additive*:

$$\alpha(\mathbf{t}_1, \mathbf{t}_3) = \alpha(\mathbf{t}_1, \mathbf{t}_2) + \alpha(\mathbf{t}_2, \mathbf{t}_3). \tag{21}$$

Also,

$$\alpha(\mathbf{t}, \mathbf{t}) = 0. \tag{22}$$

With the angle (1.19), we ought to propose

DEFINITION. Given two vectors $\mathbf{t}_1$ and $\mathbf{t}_2$, we say that the vectors are $\mathcal{E}_g^{PD}$-*perpendicular,* if

$$\cos\left(\alpha(\mathbf{t}_1, \mathbf{t}_2)\right) = 0. \tag{23}$$

Since the vanishing (1.23) implies

$$\alpha_{quasi-euclidean}(\mathbf{t}_1, \mathbf{t}_2) = \frac{\pi}{2}, \tag{24}$$

in view of 1.20) we ought to conclude that

$$\alpha_{euclidean}(\mathbf{t}_1, \mathbf{t}_2) = \frac{\pi}{2}h \le \frac{\pi}{2}. \tag{25}$$

Therefore, vectors perpendicular in the quasi-euclidean sense proper look like acute vectors as observed from associated euclidean standpoint.

With the equality

$$(\sqrt{a^2 - b^2}\,\Delta s)^2 + (a^2 + b\Delta s)^2 \equiv a^2 S^2(\Delta s), \tag{26}$$

we also establish the relations

$$\sqrt{a^2 - b^2}\,\Delta s = aS(\Delta s)\sin\alpha \tag{27}$$

and

$$a^2 + b\Delta s = aS(\Delta s)\cos\alpha. \tag{28}$$

They entail the equality

$$\frac{b}{\sqrt{a^2 - b^2}} = \frac{S(\Delta s)\cos\alpha - a}{S(\Delta s)\sin\alpha} \tag{29}$$

from which the quantity b can be explicated.

Thus *each member of the involved set* $\{a, b, \Delta s, S(\Delta s)\}$ *can be explicitly expressed through the input vectors* $\mathbf{t}_1$ *and* $\mathbf{t}_2$. For many cases it is worth rewriting the equality (1.24) as

$$S^2(\Delta s) = (\Delta s)^2 - a^2 + 2(a^2 + b\Delta s). \tag{30}$$

Thus we have arrived at the following substantive items:

The $\mathcal{E}_g^{PD}$-Case Cosine Theorem

$$(\Delta s)^2 = S^2(\Delta s) + a^2 - 2aS(\Delta s)\cos\alpha\,; \tag{31}$$

The $\mathcal{E}_g^{PD}$-Case Two-Point Length

$$(\Delta s)^2 = (\mathbf{t}_1\mathbf{t}_1) + (\mathbf{t}_2\mathbf{t}_2) - 2\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\cos\alpha\,; \tag{32}$$

The $\mathcal{E}_g^{PD}$-Case Scalar Product

$$< \mathbf{t}_1, \mathbf{t}_2 >= \sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\cos\alpha\,; \tag{33}$$

The $\mathcal{E}_g^{PD}$-Case Perpendicularity

$$< \mathbf{t}_1, \mathbf{t}_2 >= \sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}. \tag{34}$$

The identification

$$|\mathbf{t}_2 \ominus \mathbf{t}_1|^2 = (\Delta s)^2 \tag{35}$$

yields another lucid representation

$$|\mathbf{t}_2 \ominus \mathbf{t}_1|^2 = (\mathbf{t}_1\mathbf{t}_1) + (\mathbf{t}_2\mathbf{t}_2) - 2\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\cos\alpha\,. \tag{36}$$

The consideration can be completed by

THEOREM. *A general solution to the geodesic equation* (1.11) *can explicitly be found as follows:*

$$\mathbf{t}(s) =$$

$$= \frac{S(s)}{a}\,\frac{\sin\left[h\arctan\dfrac{\sqrt{a^2-b^2}\,(\Delta s - s)}{a^2+b\Delta s+(b+\Delta s)s}\right]}{\sin\left[h\arctan\dfrac{\sqrt{a^2-b^2}\,\Delta s}{a^2+b\Delta s}\right]}\,\mathbf{t}_1 + \frac{S(s)}{S(\Delta s)}\,\frac{\sin\left[h\arctan\dfrac{\sqrt{a^2-b^2}\,s}{a^2+bs}\right]}{\sin\left[h\arctan\dfrac{\sqrt{a^2-b^2}\,\Delta s}{a^2+b\Delta s}\right]}\,\mathbf{t}_2. \tag{37}$$

The euclidean limit proper is

$$\mathbf{t}(s)\Big|_{g=0} = \frac{(\Delta s - s)\mathbf{t}_1 + s\mathbf{t}_2}{\Delta s} = \mathbf{t}_1 + (\mathbf{t}_2 - \mathbf{t}_1)\frac{s}{\Delta s},$$

so that the geodesics become straight. From (1.35) the equality

$$(\mathbf{t}(s)\mathbf{t}(s)) = S^2(s) \tag{38}$$

follows, in agreement with (1.12). Since the general solution (1.35) is such that the right-hand side is spanned by two fixed vectors, $\mathbf{t}_1$ and $\mathbf{t}_2$, we are entitled concluding that *the geodesics under study are plane curves.*

2. Finsleroid-space $\mathcal{E}_g^{PD}$ of positive-definite type

Suppose we are given an N–dimensional vector space V_N. Denote by R the vectors constituting the space, so that $R \in V_N$. Any given vector R assigns a particular direction in V_N. Let us fix a member $R_{(N)} \in V_N$, introduce the straightline e_N oriented along the vector $R_{(N)}$, and use this e_N to serve as a R^N–coordinate axis in V_N. In this way we get the topological product

$$V_N = V_{N-1} \times e_N \tag{1}$$

together with the separation

$$R = \{\mathbf{R}, R^N\}, \qquad R^N \in e_N \quad \text{and} \quad \mathbf{R} \in V_{N-1}. \tag{2}$$

For convenience, we shall frequently use the notation

$$R^N = Z \tag{3}$$

and

$$R = \{\mathbf{R}, Z\}. \tag{4}$$

Also, we introduce a euclidean metric

$$q = q(\mathbf{R}) \tag{5}$$

over the $(N-1)$–dimensional vector space V_{N-1}.

With respect to an admissible coordinate basis $\{e_a\}$ in V_{N-1}, we obtain the coordinate representations

$$\mathbf{R} = \{R^a\} = \{R^1, \dots, R^{N-1}\} \tag{6}$$

and

$$R = \{R^p\} = \{R^a, R^N\} \equiv \{R^a, Z\}, \tag{7}$$

together with

$$q(\mathbf{R}) = \sqrt{r_{ab}R^aR^b}, \tag{8}$$

where r_{ab} are the components of a symmetric positive–definite tensor defined over V_{N-1}. The indices $(a, b, \dots)$ and $(p, q, \dots)$ will be specified over the ranges $(1, \dots, N-1)$ and $(1, \dots, N)$, respectively; vector indices are up, co–vector indices are down; repeated up–down indices are automatically summed; the notation δ^a_b will stand for the Kronecker symbol. The variables

$$w^a = R^a/Z, \quad w_a = r_{ab}w^b, \quad w = q/Z, \tag{9}$$

where

$$w \in (-\infty, \infty), \tag{10}$$

are convenient whenever $Z \neq 0$. Sometimes we shall mention the associated metric tensor

$$r_{pq} = \{r_{NN} = 1,\ r_{Na} = 0,\ r_{ab}\} \tag{11}$$

meaningful over the whole vector space V_N.

Given a parameter g subject to the inequality

$$-2 < g < 2, \tag{12}$$

we introduce the convenient notation

$$h = \sqrt{1 - \frac{1}{4}g^2}, \tag{13}$$

$$G = g/h, \tag{14}$$

$$g_+ = \frac{1}{2}g + h, \qquad g_- = \frac{1}{2}g - h, \tag{15}$$

$$g^+ = -\frac{1}{2}g + h, \qquad g^- = -\frac{1}{2}g - h, \tag{16}$$

so that

$$g_+ + g_- = g, \qquad g_+ - g_- = 2h, \tag{17}$$

$$g^+ + g^- = -g, \qquad g^+ - g^- = 2h, \tag{18}$$

$$(g_+)^2 + (g_-)^2 = 2, \tag{19}$$

$$(g^+)^2 + (g^-)^2 = 2, \tag{20}$$

and

$$g_+ \stackrel{g\to -g}{\Longleftrightarrow} -g_-, \qquad g^+ \stackrel{g\to -g}{\Longleftrightarrow} -g^-. \tag{21}$$

The characteristic quadratic form

$$B(g;R) = Z^2 + gqZ + q^2 \equiv \frac{1}{2}\Bigl[(Z + g_+q)^2 + (Z + g_-q)^2\Bigr] > 0 \tag{22}$$

is of the negative discriminant, namely

$$D_{\{B\}} = -4h^2 < 0, \tag{23}$$

because of Eqs. (2.12) and (2.13). Whenever $Z \neq 0$, it is also convenient to use the quadratic form

$$Q(g;w) \stackrel{\text{def}}{=} B/(Z)^2, \tag{24}$$

obtaining

$$Q(g;w) = 1 + gw + w^2 > 0, \tag{25}$$

together with the function

$$E(g;w) \stackrel{\text{def}}{=} 1 + \frac{1}{2}gw. \tag{26}$$

The identity

$$E^2 + h^2w^2 = Q \tag{27}$$

can readily be verified. In the limit $g \to 0$, the definition (2.22) degenerates to the quadratic form of the input metric tensor (2.11):

$$B\big|_{g=0} = r_{pq}R^pR^q. \tag{28}$$

Also

$$Q\big|_{g=0} = 1 + w^2. \tag{29}$$

In terms of this notation, we propose the FMF

$$K(g;R) = \sqrt{B(g;R)}\,J(g;R), \tag{30}$$

where

$$J(g;R) = \mathrm{e}^{\frac{1}{2}G\Phi(g;R)}, \tag{31}$$

$$\Phi(g;R) = \frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Bigl(\frac{q}{hZ} + \frac{G}{2}\Bigr), \qquad \text{if} \quad Z \geq 0, \tag{32}$$

$$\Phi(g;R) = -\frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Bigl(\frac{q}{hZ} + \frac{G}{2}\Bigr), \qquad \text{if} \quad Z \leq 0, \tag{33}$$

or in other convenient forms,

$$\Phi(g;R) = \frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Bigl(\frac{L(g;R)}{hZ}\Bigr), \qquad \text{if} \quad Z \ge 0, \tag{34}$$

$$\Phi(g;R) = -\frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Bigl(\frac{L(g;R)}{hZ}\Bigr), \qquad \text{if} \quad Z \le 0, \tag{35}$$

where

$$L(g;R) = q + \frac{g}{2}Z, \tag{36}$$

and

$$\Phi(g;R) = \frac{\pi}{2} - \arctan\frac{hq}{A(g;R)}, \qquad \text{if} \quad Z \ge 0, \tag{37}$$

$$\Phi(g;R) = -\frac{\pi}{2} - \arctan\frac{hq}{A(g;R)}, \qquad \text{if} \quad Z \le 0, \tag{38}$$

where

$$A(g;R) = Z + \frac{1}{2}gq. \tag{39}$$

This FMF has been normalized to show the handy properties

$$-\frac{\pi}{2} \le \Phi \le \frac{\pi}{2}, \tag{40}$$

$$\Phi = \frac{\pi}{2}, \quad \text{if} \quad q = 0 \quad \text{and} \quad Z > 0; \qquad \Phi = -\frac{\pi}{2}, \quad \text{if} \quad q = 0 \quad \text{and} \quad Z < 0. \tag{41}$$

We also have

$$\cot\Phi = \frac{hq}{A}, \qquad \Phi|_{Z=0} = \arctan\frac{G}{2}. \tag{42}$$

It is often convenient to use the indicator of sign ϵ_Z for the argument Z:

$$\epsilon_Z = 1, \quad \text{if} \quad Z > 0; \qquad \epsilon_Z = -1, \quad \text{if} \quad Z < 0; \tag{43}$$

Under these conditions, we call the considered space *the* $\mathcal{E}_g^{PD}$*–space*:

$$\mathcal{E}_g^{PD} = \{V_N = V_{N-1} \times e_N;\; R \in V_N;\; K(g;R);\; g\}. \tag{44}$$

The right–hand part of the definition (2.30) can be considered to be a function $\breve{K}$ of the arguments $\{g; q, Z\}$, such that

$$\breve{K}(g;q,Z) = K(g;R). \tag{45}$$

We observe that

$$\breve{K}(g;q,-Z) \ne \breve{K}(g;q,Z), \qquad \textit{unless} \quad g = 0. \tag{46}$$

Instead, the function $\breve{K}$ shows the property of *gZ–parity*

$$\breve{K}(-g;q,-Z) = \breve{K}(g;q,Z). \tag{47}$$

The $(N-1)$–space reflection invariance holds true

$$K(g;R) \overset{R^a \leftrightarrow -R^a}{\Leftrightarrow} K(g;R). \tag{48}$$

It is frequently convenient to rewrite the representation (2.30) in the form

$$K(g;R) = |Z|V(g;w), \tag{49}$$

whenever $Z \neq 0$, with *the generating metric function*

$$V(g;w) = \sqrt{Q(g;w)}\, j(g;w). \tag{50}$$

We have

$$j(g;w) = J(g;1,w).$$

Using (2.25) and (2.31)–(2.35), we obtain

$$V' = wV/Q, \qquad V'' = V/Q^2, \tag{51}$$

$$(V^2/Q)' = -gV^2/Q^2, \qquad (V^2/Q^2)' = -2(g+w)V^2/Q^3, \tag{52}$$

$$j' = -\frac{1}{2}gj/Q, \tag{53}$$

and also

$$\frac{1}{2}(V^2)' = wV^2/Q, \qquad \frac{1}{2}(V^2)'' = (Q-gw)V^2/Q^2, \tag{54}$$

$$\frac{1}{4}(V^2)''' = -gV^2/Q^3, \tag{55}$$

together with

$$\Phi' = -h/Q, \tag{56}$$

where the prime ($'$) denotes the differentiation with respect to w.

Also,

$$(A(g;R))^2 + h^2q^2 = B(g;R) \tag{57a}$$

and

$$(L(g;R))^2 + h^2Z^2 = B(g;R). \tag{57b}$$

Sometimes it is convenient to use the function

$$E(g;w) \stackrel{\text{def}}{=} 1 + \frac{1}{2}gw. \tag{58}$$

The simple results for these derivatives reduce the task of computing the components of the associated FMT to an easy exercise, indeed:

$$R_p \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial K^2(g;R)}{\partial R^p}:$$

$$R_a = r_{ab}R^b\frac{K^2}{B}, \qquad R_N = (Z+gq)\frac{K^2}{B}; \tag{59}$$

$$g_{pq}(g;R) \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial^2 K^2(g;R)}{\partial R^p \partial R^q} = \frac{\partial R_p(g;R)}{\partial R^q}:$$

$$g_{NN}(g;R) = [(Z+gq)^2 + q^2]\frac{K^2}{B^2}, \qquad g_{Na}(g;R) = gqr_{ab}R^b\frac{K^2}{B^2}, \tag{60}$$

$$g_{ab}(g;R) = \frac{K^2}{B}r_{ab} - g\frac{r_{ad}R^d r_{be}R^e Z}{q}\frac{K^2}{B^2}. \tag{61}$$

The reciprocal tensor components are

$$g^{NN}(g;R) = (Z^2+q^2)\frac{1}{K^2}, \qquad g^{Na}(g;R) = -gqR^a\frac{1}{K^2}, \tag{62}$$

$$g^{ab}(g;R) = \frac{B}{K^2} r^{ab} + g(Z+gq)\frac{R^a R^b}{q}\frac{1}{K^2}. \tag{63}$$

The determinant of the FMT given by Eqs. (2.59) – (2.60) can readily be found in the form

$$\det(g_{pq}(g;R)) = [J(g;R)]^{2N} \det(r_{ab}) \tag{64}$$

which shows, on noting (2.31) – (2.33), that

$$\det(g_{pq}) > 0 \quad \textit{over all the definition range} \quad V_N \setminus 0. \tag{65}$$

The associated angular metric tensor

$$h_{pq} \stackrel{\text{def}}{=} g_{pq} - R_p R_q \frac{1}{K^2}$$

proves to be given by the components

$$h_{NN}(g;R) = q^2 \frac{K^2}{B^2}, \qquad h_{Na}(g;R) = -Z r_{ab} R^b \frac{K^2}{B^2},$$

$$h_{ab}(g;R) = \frac{K^2}{B} r_{ab} - (gZ+q)\frac{r_{ad}R^d r_{be}R^e}{q}\frac{K^2}{B^2},$$

which entails

$$\det(h_{ab}) = \det(g_{pq})\frac{1}{V^2}.$$

The use of the components of the Cartan tensor (given explicitly in the end of the present section) leads, after rather tedious straightforward calculations, to the following simple and remarkable result.

PROPOSITION 1. *The Cartan tensor associated with the FMF* (2.30) *is of the following special algebraic form:*

$$C_{pqr} = \frac{1}{N}\left(h_{pq}C_r + h_{pr}C_q + h_{qr}C_p - \frac{1}{C_s C^s} C_p C_q C_r\right) \tag{66}$$

with

$$C_t C^t = \frac{N^2}{4K^2} g^2. \tag{67}$$

By the help of (2.65), elucidating the structure of the curvature tensor

$$S_{pqrs} \stackrel{\text{def}}{=} (C_{tqr} C_p{}^t{}_s - C_{tqs} C_p{}^t{}_r) \tag{68}$$

results in the simple representation

$$S_{pqrs} = -\frac{C_t C^t}{N^2}(h_{pr}h_{qs} - h_{ps}h_{qr}). \tag{69}$$

Inserting here (2.66), we are led to

PROPOSITION 2. *The curvature tensor of the space* $\mathcal{E}_g^{PD}$ *is of the special type*

$$S_{pqrs} = S^*(h_{pr}h_{qs} - h_{ps}h_{qr})/K^2 \tag{70}$$

with

$$S^* = -\frac{1}{4}g^2. \qquad (71)$$

DEFINITION. FMF (2.30) introduces an $(N-1)$–dimensional indicatrix hypersurface according to the equation

$$K(g;R) = 1. \qquad (72)$$

We call this particular hypersurface *the Finsleroid*, to be denoted as $\mathcal{F}_g^{PD}$.

Recalling the known formula $\mathcal{R} = 1 + S^*$ for the indicatrix curvature (see [4]), from (2.71) we conclude that

$$\mathcal{R}_{Finsleroid} = h^2 = 1 - \frac{1}{4}g^2, \qquad 0 < \mathcal{R}_{Finsleroid} \le 1. \qquad (73)$$

Geometrically, the fact that the quantity (2.70) is independent of vectors R means that the indicatrix curvature is constant. Therefore, we have arrived at

PROPOSITION 3. *The Finsleroid* $\mathcal{F}_g^{PD}$ *is a constant-curvature space with the positive curvature value* (2.73).

Also, on comparing between the result (2.73) and Eqs. (2.22)–(2.23), we obtain

PROPOSITION 4. *The Finsleroid curvature relates to the discriminant* (2.23) *of the input characteristic quadratic form* (2.22) *simply as*

$$\mathcal{R}_{Finsleroid} = -\frac{1}{4}D_{\{B\}}. \qquad (74)$$

Last, we write down the explicit components of the relevant Cartan tensor

$$C_{pqr} \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial g_{pq}}{\partial R^r}:$$

$$R^N C_{NNN} = gw^3V^2Q^{-3}, \qquad R^N C_{aNN} = -gww_aV^2Q^{-3},$$

$$R^N C_{abN} = \frac{1}{2}gwV^2Q^{-2}r_{ab} + \frac{1}{2}g(1-gw-w^2)w_aw_bw^{-1}V^2Q^{-3},$$

$$R^N C_{abc} = -\frac{1}{2}gV^2Q^{-2}w^{-1}(r_{ab}w_c + r_{ac}w_b + r_{bc}w_a) + gw_aw_bw_cw^{-3}\left(\frac{1}{2}Q + gw + w^2\right)V^2Q^{-3};$$

and

$$R^N C_N{}^N{}_N = gw^3/Q^2, \qquad R^N C_a{}^N{}_N = -gww_a/Q^2,$$

$$R^N C_N{}^a{}_N = -gw(1+gw)w^a/Q^2,$$

$$R^N C_a{}^N{}_b = \frac{1}{2}gwr_{ab}/Q + \frac{1}{2}g(1-gw-w^2)w_aw_b/wQ^2,$$

$$R^N C_N{}^a{}_b = \frac{1}{2}gw\delta^a_b/Q + \frac{1}{2}g(1+gw-w^2)w^aw_b/wQ^2,$$

$$R^N C_a{}^b{}_c = -\frac{1}{2}g\left(\delta^b_a w_c + \delta^b_c w_a + (1+gw)r_{ac}w^b\right)/wQ + \frac{1}{2}g(gwQ + Q + 2w^2)w_aw^bw_c/w^3Q^2.$$

The components have been calculated by the help of the formulae (2.50) – (2.53).
The use of the contractions

$$R^N C_a{}^b{}_c r^{ac} = -g\frac{w^b}{w}\frac{1+gw}{Q}\left(\frac{N-2}{2} + \frac{1}{Q}\right)$$

and

$$R^N C_a{}^b{}_c w^a w^c = -g\frac{w}{Q^2}(1+gw)w^b$$

is handy in many calculations.

Also,

$$R^N C_N = \frac{N}{2}gwQ^{-1}, \qquad R^N C_a = -\frac{N}{2}g(w_a/w)Q^{-1},$$

$$R^N C^N = \frac{N}{2}gw/V^2, \qquad R^N C^a = -\frac{N}{2}gw^a(1+gw)/wV^2,$$

$$C^N = \frac{N}{2}gwR^N K^{-2}, \qquad C^a = -\frac{N}{2}gw^a(1+gw)w^{-1}R^N K^{-2},$$

$$C_p C^p = \frac{N^2}{4K^2}g^2.$$

3. Quasi–euclidean map of Finsleroid

It is possible to indicate the diffeomorphism

$$\mathcal{F}_g^{PD} \stackrel{i_g}{\Longrightarrow} \mathcal{S}^{PD} \tag{1}$$

of the Finsleroid $\mathcal{F}_g^{PD} \subset V_N$ to the unit sphere $\mathcal{S}^{PD} \subset V_N$:

$$\mathcal{S}^{PD} = \{R \in \mathcal{S}^{PD}: \ S(R) = 1\}, \tag{2}$$

where

$$S(R) = \sqrt{r_{pq}R^pR^q} \equiv \sqrt{(R^N)^2 + r_{ab}R^aR^b} \tag{3}$$

is the input euclidean metric function (see (2.11)).

The diffeomorphism (3.1) can always be extended to get the diffeomorphic map

$$V_N \stackrel{\sigma_g}{\Longrightarrow} V_N \tag{4}$$

of the whole vector space V_N by means of the homogeneity:

$$\sigma_g \cdot (bR) = b\sigma_g \cdot R, \quad b > 0. \tag{5}$$

To this end it is sufficient to take merely

$$\sigma_g \cdot R = ||R|| i_g \cdot \Bigl(\frac{R}{||R||}\Bigr), \tag{6}$$

where

$$||R|| = K(g;R). \tag{7}$$

Eqs. (3.1)–(3.7) entail

$$K(g;R) = S(\sigma_g \cdot R). \tag{8}$$

The identity (2.57) suggests to take the map

$$\bar{R} = \sigma_g \cdot R \tag{9}$$

by means of the components

$$\bar{R}^p = \sigma^p(g;R) \tag{10}$$

with

$$\sigma^a = R^a h J(g;R), \qquad \sigma^N = A(g;R)J(g;R), \tag{11}$$

where $J(g;R)$ and $A(g;R)$ are the functions (2.31) and (2.39). Indeed, inserting (3.11) in (3.3) and taking into account Eqs. (2.30) and (2.57), we get the identity

$$S(\bar{R}) = K(g;R) \tag{12}$$

which is tantamount to the implied relation (3.8).

PROPOSITION 5. The map given explicitly by Eqs. (3.9)–(3.11) assigns *the diffeomorphism between the Finsleroid and the unit sphere* according to Eqs. (3.1)–(3.8).

Therefore, we may also call the operation (3.1) *the quasi–euclidean map of Finsleroid.*

The inverse

$$R = \mu_g \cdot \bar{R}, \qquad \mu_g = (\sigma_g)^{-1}, \tag{13}$$

of the transformation (3.9) – (3.11) can be presented by the components

$$R^p = \mu^p(g;\bar{R}) \tag{14}$$

with

$$\mu^a = \bar{R}^a / hk(g;\bar{R}), \qquad \mu^N = I(g;\bar{R})/k(g;\bar{R}), \tag{15}$$

where

$$k(g;\bar{R}) \stackrel{\text{def}}{=} J(g;\mu(g;\bar{R})) \tag{16}$$

and

$$I(g;\bar{R}) = \bar{R}^N - \frac{1}{2} G \sqrt{r_{ab}\bar{R}^a\bar{R}^b}. \tag{17}$$

The identity

$$\mu^p(g;\sigma(g;R)) \equiv R^p \tag{18}$$

can readily be verified. Notice that

$$\frac{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}}{\bar{R}^N} = \frac{hq}{A(g;R)}, \qquad w^a = \frac{R^a}{R^N} = \frac{\bar{R}^a}{hI(g;\bar{R})}, \tag{19}$$

and

$$\sqrt{B}/Z = S/I, \qquad \sqrt{Q} = S/I. \tag{20}$$

The σ_g–image

$$\phi(g;\bar{R}) \stackrel{\text{def}}{=} \Phi(g;R)\big|_{R=\mu(g;\bar{R})} \tag{21}$$

of the function Φ described by Eqs. (2.31) – (2.42) is of a clear meaning of angle:

$$\phi(g;\bar{R}) = \arccos \frac{\bar{R}^N}{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}} = \begin{cases} \dfrac{\pi}{2} - \arctan \dfrac{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}}{\bar{R}^N}, & \text{if} \quad \bar{R}^N \ge 0; \\ -\dfrac{\pi}{2} - \arctan \dfrac{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}}{\bar{R}^N}, & \text{if} \quad \bar{R}^N \le 0; \end{cases} \tag{22}$$

which ranges over

$$-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}. \tag{23}$$

We have

$$\phi = \frac{\pi}{2}, \quad \text{if} \quad \bar{R}^a = 0 \quad \text{and} \quad \bar{R}^N > 0; \qquad \phi = -\frac{\pi}{2}, \quad \text{if} \quad \bar{R}^a = 0 \quad \text{and} \quad \bar{R}^N < 0, \quad (24)$$

and also

$$\phi\big|_{\bar{R}^N=0} = 0. \quad (25)$$

Comparing Eqs. (3.16) and (2.31) shows that

$$k = \mathrm{e}^{\frac{1}{2}G\phi}. \quad (26)$$

The right–hand parts in (3.11) are homogeneous functions of degree 1:

$$\sigma^p(g; bR) = b\sigma^p(g; R), \quad b > 0. \quad (27)$$

Therefore, the identity

$$\sigma^p_s(g; R)R^s = \bar{R}^p \quad (28)$$

should be valid for the derivatives

$$\sigma^q_p(g; R) \overset{\text{def}}{=} \frac{\partial\sigma^q(g; R)}{\partial R^p}. \quad (29)$$

The simple representations

$$\sigma^N_N(g; R) = \left(B + \frac{1}{2}gqA\right)\frac{J}{B}, \quad (30)$$

$$\sigma^N_a(g; R) = -\frac{g(ZA - B)}{2q}\frac{Jr_{ab}R^b}{B}, \quad (31)$$

$$\sigma^a_N(g; R) = \frac{1}{2}gq\frac{JR^a h}{B}, \quad (32)$$

$$\sigma^a_b(g; R) = \left(B\delta^a_b - \frac{gr_{bc}R^c R^a Z}{2q}\right)\frac{Jh}{B}, \quad (33)$$

and also the determinant value

$$\det(\sigma^q_p) = h^{N-1}J^N \quad (34)$$

are obtained. The relations

$$\sigma^a_b R^b = JhR^a(AZ+q^2)/B, \qquad r^{cd}\sigma^a_c\sigma^b_d = J^2h^2\left[r^{ab} - g(R^aR^bZ/qB) + \frac{1}{4}g^2(R^aR^bZ^2/B^2)\right]$$

are handy in many calculations involving the coefficients $\{\sigma^q_p\}$.

Henceforth, to simplify notation, we shall use the substitution

$$t^p = \bar{R}^p. \quad (35)$$

Again, we can note the homogeneity

$$\mu^p(g; bt) = b\mu^p(g; t), \quad b > 0, \quad (36)$$

for the functions (3.15), which entails the identity

$$\mu^p_s(g; t)t^s = R^p \quad (37)$$

for the derivatives

$$\mu^p_q(g;t) \stackrel{\text{def}}{=} \frac{\partial \mu^p(g;t)}{\partial t^q}. \tag{38}$$

We find

$$\mu^N_N = 1/k(g;t) - \frac{1}{2}g\frac{m(t)I(g;t)}{k(g;t)(S(t))^2}, \qquad \mu^N_a = \frac{1}{2}g\frac{r_{ac}t^cI^*(g;t)}{k(g;t)(S(t))^2}, \tag{39}$$

$$\mu^a_N = -\frac{1}{2}g\frac{m(t)\,t^a}{hk(g;t)(S(t))^2}, \qquad \mu^a_b = \frac{1}{hk(g;t)}\delta^a_b + \frac{1}{2}g\frac{t^Nt^ar_{bc}t^c}{m(t)\,hk(g;t)(S(t))^2}, \tag{40}$$

where

$$m(t) = \sqrt{r_{ab}t^at^b}, \tag{41}$$

$$I^*(g;t) = hm(t) - \frac{1}{2}gt^N, \tag{42}$$

and

$$S(t) = \sqrt{r_{rs}t^rt^s} \equiv \sqrt{(t^N)^2 + r_{ab}t^at^b}. \tag{43}$$

The relations

$$\frac{\partial(1/k(g;t))}{\partial t^N} = -\frac{1}{2}g\frac{m(t)}{hk(g;t)(S(t))^2}, \qquad \frac{\partial(1/k(g;t))}{\partial t^a} = \frac{1}{2}g\frac{t^Nr_{ab}t^b}{m(t)hk(g;t)(S(t))^2}$$

are obtained.

Also

$$R_p\mu^p_q = t_q, \qquad t_p\sigma^p_q = R_q. \tag{44}$$

The unit vectors

$$L^p \stackrel{\text{def}}{=} \frac{t^p}{S(t)}, \qquad L_p \stackrel{\text{def}}{=} r_{pq}L^q \tag{45}$$

fulfil the relations

$$L^q = l^p\sigma^q_p, \qquad l^p = \mu^p_qL^q, \qquad l_p = \sigma^q_pL_q, \qquad L_p = \mu^q_pl_q, \tag{46}$$

where $l^p = R^p/K(g;R)$ and $l_p = g_{pq}(g;R)l^q$ are the initial Finslerian unit vectors.

Now we use the explicit formulae (2.61) – (2.62) and (3.29) – (3.32) to find the transform

$$n^{rs}(g;t) \stackrel{\text{def}}{=} \sigma^r_p\sigma^s_qg^{pq} \tag{47}$$

of the FMT g_{pq} under the $\mathcal{F}^{PD}_g$–induced map (3.9) – (3.11), which results in

PROPOSITION 6. *One obtains the simple representation*

$$n^{rs} = h^2r^{rs} + \frac{1}{4}g^2L^rL^s. \tag{48}$$

The covariant version reads

$$n_{rs} = \frac{1}{h^2}r_{rs} - \frac{1}{4}G^2L_rL_s. \tag{49}$$

The determinant of this tensor is a constant:

$$\det(n_{rs}) = h^{2(1-N)}\det(r_{ab}). \tag{50}$$

Notice that

$$L^pL_p = 1, \quad n_{pq}L^q = L_p, \quad n^{pq}L_q = L^p, \quad n_{pq}L^pL^q = 1, \quad n_{pq}t^pt^q = (S(t))^2.$$

Eq. (5.47) obviously entails

$$g_{pq} = n_{rs}(g;t)\sigma_p^r\sigma_q^s. \tag{51}$$

4. Quasi-euclidean metric tensor

Let us introduce

DEFINITION. The metric tensor (3.48) – (3.49) is called *quasi-euclidean.*

DEFINITION. *The quasi-euclidean space*

$$\mathcal{Q}_N = \{V_N; n_{pq}(g;t); g\} \tag{1}$$

is an extension of the euclidean space $\{V_N; r_{pq}\}$ to the case $g \neq 0$.

The transformation (3.47) can be inverted to read

$$g_{pq} = \sigma_p^r\sigma_q^s n_{rs}. \tag{2}$$

For the angular metric tensor (see the formula going below Eq. (2.64)), from (3.46) and (4.2) we infer

$$h_{pq} = \sigma_p^r\sigma_q^s H_{rs}\frac{1}{h^2}, \tag{3}$$

where

$$H_{rs} \stackrel{\text{def}}{=} r_{rs} - L_r L_s \tag{4}$$

is the tensor showing the orthogonality property

$$L^r H_{rs} = 0. \tag{5}$$

One can readily find that

$$H_{rs} = h^2(n_{rs} - L_r L_s).$$

PROPOSITION 7. The quasi-euclidean metric tensor (3.48) – (3.49) *is conformal* to the euclidean metric tensor.

Indeed, if we consider the map

$$\bar{R}^p \to \tilde{R}: \qquad \tilde{R}^p = f(g;\bar{R})\bar{R}^p/h \tag{6}$$

with

$$f(g;\bar{R}) = a\left(g;\ \frac{1}{2}S^2(\bar{R})\right) \tag{7}$$

and use the coefficients

$$k_q^p \stackrel{\text{def}}{=} \frac{\partial\tilde{R}^p}{\partial\bar{R}^q} = (f\delta_q^p + a'\bar{R}^p\bar{R}_q)/h \tag{8}$$

to define the tensor

$$c^{pq}(g;\tilde{R}) \stackrel{\text{def}}{=} k_r^p k_s^q n^{rs}(g;\bar{R}), \tag{9}$$

we find that

$$c^{pq} = f^2 r^{pq} \tag{10}$$

whenever

$$f=\left[\frac{1}{2}S^2(\bar{R})\right]^{\gamma/2},\tag{11}$$

where

$$\gamma=h-1\equiv\sqrt{1-\frac{g^2}{4}}-1\tag{12}$$

is the parameter. The proof of Proposition 7 is complete.

Let us now use the obtained quasi-euclidean metric tensor $n_{pq}(g;t)$ to construct the associated *quasi-euclidean Christoffel symbols* $N_p{}^r{}_q(g;t)$. We find consecutively:

$$n_{pq,r}\stackrel{\text{def}}{=}\frac{\partial n_{pq}}{\partial t^r}=-\frac{1}{4}G^2(H_{pr}L_q+H_{qr}L_p)/S,\tag{13}$$

and

$$N_p{}^r{}_q=n^{rs}N_{psq},\qquad N_{prq}=\frac{1}{2}(n_{pr,q}+n_{qr,p}-n_{pq,r}),\tag{14}$$

together with

$$N_{prq}(g;t)=-\frac{1}{4}G^2H_{pq}L_r/S,\tag{15}$$

which eventually yields

$$N_p{}^r{}_q(g;t)=-\frac{1}{4}G^2L^rH_{pq}/S.\tag{16}$$

Comparing the representation (4.16) with the identity (4.5) shows that

$$t^pN_p{}^r{}_q=0,\qquad N_p{}^s{}_s=0,\qquad N_t{}^s{}_rN_p{}^t{}_q=0.\tag{17}$$

Also,

$$\frac{\partial N_p{}^r{}_q}{\partial t^s}-\frac{\partial N_p{}^r{}_s}{\partial t^q}=-\frac{1}{4}G^2(H_{pq}H_s{}^r-H_{ps}H_q{}^r)/S^2.\tag{18}$$

Using the identities (4.17)-(4.18) in *the quasi-euclidean curvature tensor*:

$$R_p{}^r{}_{qs}(g;t)\stackrel{\text{def}}{=}\frac{\partial N_p{}^r{}_q}{\partial t^s}-\frac{\partial N_p{}^r{}_s}{\partial t^q}+N_p{}^w{}_qN_w{}^r{}_s-N_p{}^w{}_sN_w{}^r{}_q,\tag{19}$$

we arrive at the simple result:

$$R_{prqs}(g;t)=-\frac{1}{4}G^2(H_{pq}H_{rs}-H_{ps}H_{qr})/S^2.\tag{20}$$

This infers the identities

$$L^pR_{pqrs}=L^qR_{pqrs}=L^rR_{pqrs}=L^sR_{pqrs}=0.\tag{21}$$

Note. Because of the transformation rules (3.12) and (3.47), the representation (4.20) is tantamount to Eqs. (2.69)–(2.70). Therefore *we have got another rigorous proof of Proposition* 3, *and of Eq.* (2.71), *concerning the Finsleroid curvature.*

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Appendix

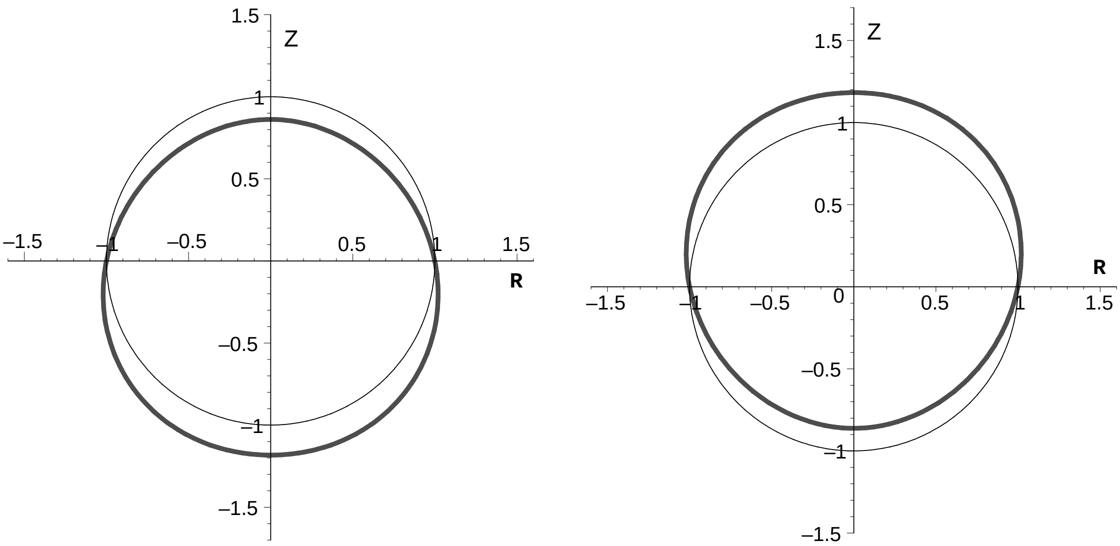

Figure 1: $g = 0.2$ and $g = -0.2$

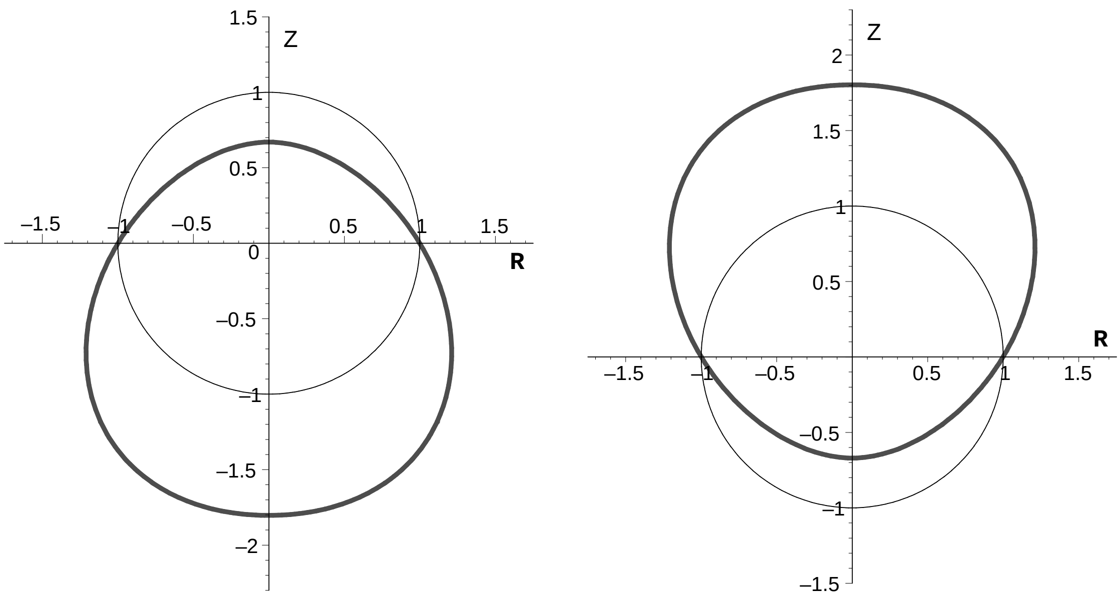

Figure 2: $g = 0.6$ and $g = -0.6$

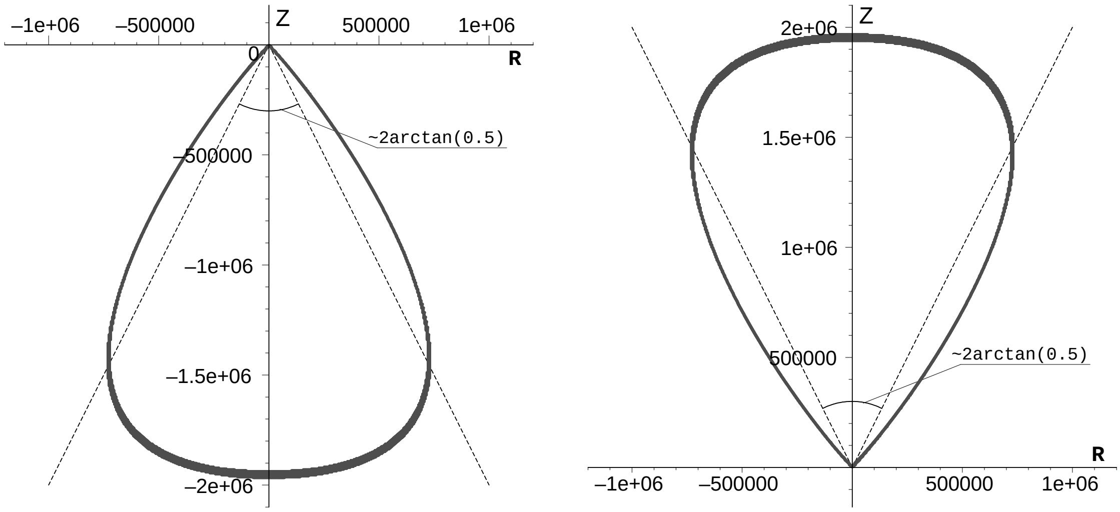

Figure 3: $g = 1.96$ and $g = -1.96$

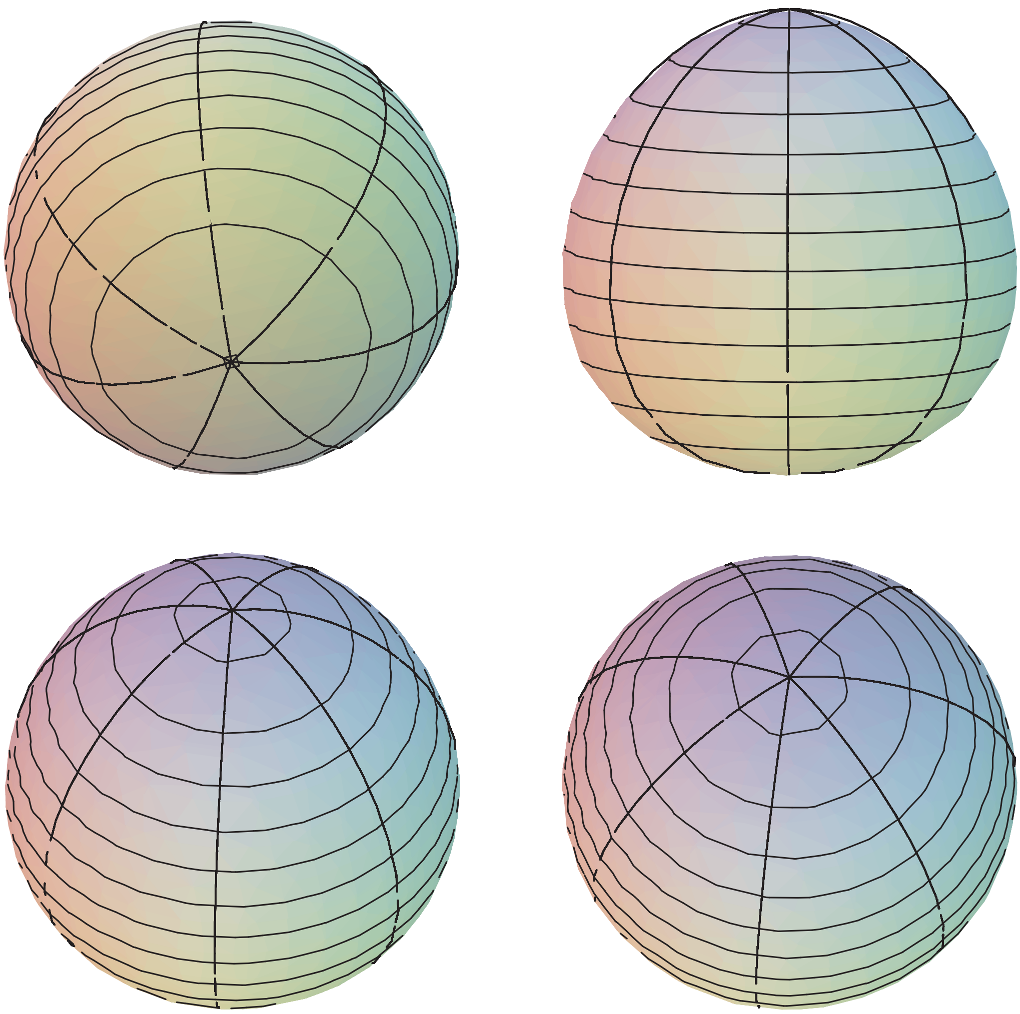

Figure 4: 3D-images of Finsleroid; $g = 0.6$