THREE-NUMBERS WHICH CUBE OF NORM IS NONDEGENERATE THREE-FORM

G. I. Garas'ko

Electrotechnical institute of Russia,
gri9z@mail.ru

Arbitrary three-form can be put in a canonical form. The requirement of existence of two-parametric Abelian Lie group to play the role of group of symmetry for three-form admits selecting the three-forms that correspond to three-numbers and finding all the three-numbers which cube of norm is a non-degenerate three-form with respect to a special coordinate system. There are exactly two (up to isomorphism) such sets of hypercomplex numbers, namely the sets: C_3, H_3. They can be regarded as generalizations of complex and binary (hyperbolic) bi-numbers to the case of three-numbers.

1. Introduction

The real number is a stoneconcept for both the mathematics and physics. The associative-commutative n-dimensional hypercomplex numbers over the field of real numbers, – which we shall call the n-numbers for short, – comprises an attractive extension of this concept. The complex numbers are well adopted in solving problems of mathematical and theoretical physics and present actually a particular case of such hypercomplex numbers, bi-numbers. Regrettably, the n-numbers at $n > 2$ have not been studied in great detail. It can be hoped that, possessing such simplified properties as associativity and commutativity and showing sufficient complexity in some cases, the associative-commutative hypercomplex numbers shall find their non-trivial application. At $n > 2$ the very classification and choice of the n-numbers for mathematical studies with the aim of farther application in physics is a non-trivial problem. The formulation of additional conditions to specify a narrow (but significant) class in all the set of n-numbers seems to be a convenient way to attack the problem. The stipulating of a special basis in term of which the coordinates of the n-numbers be similar (for example, the norm would be independent of permutation of coordinate labels, the more strength condition insists of fulfilling the requirement that n-th degree of norm of the n-number be non-degenerate with respect to such coordinates) can play the role of such a condition. For the sake of brevity, in the present work the n-form of the coordinates of the n-dimensional linear space is meant to be the highsymmetric poly-linear form of n-th degree, all the arguments of which being equal to a fixed vector. *Highsymmetry of form* means existence of such a basis that the relevant representation of the symmetric form of n-vector arguments does not change under permutation of of coordinates. *The non-degeneracy of form* means the impossibility to express the form as an integer degree of a form of lower degree. Below we shall often omit the term "non-degenerate" , implying merely an n-form.

The present work is devoted to studying the three-numbers, that is, the associative-commutative hypercomplex numbers of the form

$$X = x_1 + x_2 \cdot e_2 + x_3 \cdot e_3, \tag{1}$$

where e_2, e_3 are symbolic elements, and x_1, x_2, x_3 are real numbers applying as the coordinates with respect to the basis $e_1 \equiv 1, e_2, e_3$. If a number X admits the exponential representation

$$X = \rho \cdot \exp(\alpha \cdot e_2 + \beta \cdot e_3), \tag{2}$$

where $\rho > 0, \alpha, \beta$ – real numbers, then the quantity ρ can naturally be called the modulus of the three-number X. Let us search for only the three-numbers that with respect to a special basis (the latter is not necessary the basis $e_1 \equiv 1, e_2, e_3$) the cube of the norm $\rho(x_1, x_2, x_3)$ is a non-degenerate three-form of coordinates, that is,

$$\rho^3 = \Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3), \tag{3}$$

where the three-form of the general type

$$\Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3) = \omega_1 \Omega_1(x_1, x_2, x_3) + \omega_2 \Omega_2(x_1, x_2, x_3) + \omega_3 \Omega_3(x_1, x_2, x_3) \tag{4}$$

is an arbitrary linear combination with real numbers $\omega_i \quad (i = 1, 2, 3)$ at the basis three-forms:

$$\Omega_1(x_1, x_2, x_3) \equiv x_1^3 + x_2^3 + x_3^3, \tag{5}$$

$$\Omega_2(x_1, x_2, x_3) \equiv x_1 x_2^2 + x_1 x_3^2 + x_1^2 x_2 + x_2 x_3^2 + x_1^2 x_3 + x_2^2 x_3, \tag{6}$$

$$\Omega_3(x_1, x_2, x_3) \equiv x_1 x_2 x_3. \tag{7}$$

It will be noted that in the three-dimensional space the symmetric cubic form of three vector arguments, assuming the linearity in each argument, contains not three but ten arbitrary real parameters; that is, the form is a more general notion then the high-symmetric three-form and hence leads to the form which is more general then (4). The requirement of non-degeneracy of three-form reads

$$\Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3) \neq \Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3) \equiv \omega \cdot (x_1 + x_2 + x_3)^3. \tag{8}$$

In the sequel we shall assume the non-degenerate type, unless otherwise stated explicitly.

The multiplication of the number X by a unimodular number X_1 yields the number

$$Y = X_1 \cdot X, \tag{9}$$

which modulus is equal to the modulus of the number X, so that for such three-numbers we have

$$\Omega(y_1, y_2, y_3; \omega_1', \omega_2', \omega_3') = \Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3). \tag{10}$$

Thus in order that the cube of norm be three-form, the set of unimodular numbers of this hypercomplex system must form a two-parametrical continuous Abelian Lie group (the symmetry group which retains the form of the three-form) consisted of linear transformations (9) of the coordinate space of considered three-forms.

Let us assume that for definite values of parameters of three-form (4) we find the symmetry group which is two-parametric Abelian group of continuous linear transformations with generators E_2, E_3 given by real quadratic matrices 3×3. Then, as is known, the linear transformations themselves can be defined through generators of matrix $\hat{A}$ according to the formula

$$\hat{A} = \exp(\alpha \cdot \hat{E}_2 + \beta \cdot \hat{E}_3), \tag{11}$$

where α, β are real parameters. Let in this way the multiplication rules

$$\hat{E}_i \cdot \hat{E}_j = p_{ij}^k \cdot \hat{E}_k \tag{12}$$

obey for generators, where $i, j, k = 1, 2, 3$; $\hat{E}_1$ stands for the unit matrix (the generator of general scale transformation), p_{ij}^k is some real number; summation over repeated indices is assumed. Then $\hat{E}_1\hat{E}_2, \hat{E}_3$ can be regarded as a representation of the basis elements $e_1 \equiv 1, e_2, e_3$ of some set of three-numbers, whence the representation of the set of such numbers in the coordinate linear three-dimensional space x_1, x_2, x_3 in the form of linear quadratic matrices 3×3. It is obvious that the multiplication law for the basis elements $e_1 \equiv 1, e_2, e_3$ will be of the same form (12) with the same characteristic numbers p_{ij}^k

$$e_i \cdot e_j = p_{ij}^k \cdot e_k. \tag{13}$$

Now we can write the numbers representable in the exponential form (2). The coordinate linear space x_1, x_2, x_3 is not obliged to be introduced in the same basis, that is in accordance with the formula (1). Therefore, in general case there appears the following relation for numbers representable in exponential form:

$$x_1 \cdot e_1' + x_2 \cdot e_2' + x_3 \cdot e_3' = \rho \cdot \exp(\alpha \cdot e_2 + \beta \cdot e_3), \tag{14}$$

where e_1', e_2', e_3' is a basis differed in general case from $e_1 \equiv 1, e_2, e_3$, and such that e_1' may differ from real unity. Using three coordinate relations (14) and finding two real parameter α, β, we get the expression for the cube of norm through the coordinates x_1, x_2, x_3:

$$\rho^3 = f(x_1, x_2, x_3). \tag{15}$$

If an initial three-form enters the right-hand part of this formula, then the relevant three-numbers are found.

2. Transformation of three-form to a canonical type

Apart of general scale transformation, there exists but one linear coordinate transformation connected continuously with the identity by means of which an arbitrary three-form goes over again in a three-form. Let us write the transformation in the matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{3q} \begin{pmatrix} p+2 & p-1 & p-1 \\ p-1 & p+2 & p-1 \\ p-1 & p-1 & p+2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \tag{16}$$

where q is an arbitrary positive real number, and

$$p \equiv q^3. \tag{17}$$

With respect to new variables, the three-form $\Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3)$ transformed by (16) takes on the form

$$\Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3) = \Omega(y_1, y_2, y_3; \omega_1', \omega_2', \omega_3'), \tag{18}$$

where

$$\left.\begin{aligned} \omega_1' &\equiv u \cdot (w_1 p^3 + 3w_2 p + 2w_3), \\ \omega_2' &\equiv 3u(w_1 p^3 - w_3), \\ \omega_3' &\equiv 3u(2w_1 p^3 - 3w_2 p + 4w_3), \end{aligned}\right\} \tag{19}$$

$$u \equiv \frac{1}{27p}, \tag{20}$$

$$\left.\begin{aligned} w_1 &\equiv 3\omega_1 + 6\omega_2 + \omega_3, \\ w_2 &\equiv 6\omega_1 - \omega_3, \\ w_3 &\equiv 3\omega_1 - 3\omega_2 + \omega_3. \end{aligned}\right\} \tag{21}$$

Certainly, the classification of three-forms (by transforming to canonical type) can be performed in various ways. Let us start with stipulating that the three-forms connected by the linear non-degenerate coordinate transformation that does not affect the values of the three-form itself, are equivalent, — in the sense that they differ by only the choice of basis in three-dimensional linear space of x_1, x_2, x_3, that is by the choice of basis (symbolic) element in the space of three-numbers. When transforming three-form to a canonical form, we shall consider not all linear non-degenerate transformations but only the possible triple, namely, the transformation (16); the discrete transformation (changing simultaneously the sign for all three coordinates); general scale transformation (multiplying simultaneously all three coordinates by a fixed real positive number). The basis forms (5) – (7), because of their preferable type, are certainly regarded as canonical.

So, let us consider three-form of the general type (4) and go over by the help of the linear transformation (16) to new coordinates. Since the relationship between the quantities w_i and the parameters of the three-form ω_i is one-to-one, we shall try to diminish the number of parameters of three-form with respect to new coordinates, considering various variants and using the quantities w_i and the formulas (19).

1). If

$$\operatorname{sign}(w_1) = \operatorname{sign}(w_2) \neq 0, \tag{22}$$

then by the help of the coordinate transformation (16) with the parameter value

$$p = \sqrt[3]{\frac{w_3}{w_1}} \tag{23}$$

the initial three-form can be reduced to the form $\Omega(y_1, y_2, y_3; \omega'_1, 0, \omega'_3)$.

2). If

$$\operatorname{sign}(w_1) = -\operatorname{sign}(w_3) \neq 0, \tag{24}$$

then the two transformations (16) can always be found such that ω'_1 can be nullified by using one of them, whereas ω'_3 can be nullified by using another member, and in both cases the parameter ω'_2 gets strongly not equal to zero at any value w_2. Thus as a result, one is to choose either the form $\Omega(y_1, y_2, y_3; 0, \omega'_2, \omega'_3)$ or thereto equivalent three-form $\Omega(y_1, y_2, y_3; \omega'_1, \omega'_2, 0)$. In order to exclude ambiguity, we shall always choose the first version, that is the three-form $\Omega(y_1, y_2, y_3; 0, \omega'_2, \omega'_3)$. On so doing, the parameter p in the transformation (16) is a real positive root of the cubic equation

$$w_1 p^3 + 3w_2 p + 2w_3 = 0. \tag{25}$$

There remains to consider the case when the quantities vanish either separately or totally.

3). If

$$w_1 = 0, \quad \operatorname{sign}(w_2) = -\operatorname{sign}(w_3) \neq 0, \tag{26}$$

then by the help of the transformation (16) the three-form can be reduced to the canonical form $\Omega(y_1, y_2, y_3; 0, \omega'_2, \omega'_3)$, with $\omega'_2 \neq 0$ and $\omega'_3 \neq 0$, as well as

$$p = -\frac{2w_3}{3w_2}. \tag{27}$$

4). If

$$w_1 = 0, \quad \operatorname{sign}(w_2) = \operatorname{sign}(w_3) \neq 0, \tag{28}$$

then by the help of the transformation (16) the three-form can be reduced to the canonical form $\Omega(y_1, y_2, y_3; \omega'_1, \omega'_2, 0)$, with ω'_1 and $\omega'_2 \neq 0$, and

$$p = \frac{4w_3}{3w_2}. \tag{29}$$

5). If

$$w_1 = 0, \quad w_2 = 0, \quad w_3 \neq 0, \tag{30}$$

then

$$\omega'_1 = 2uw_3, \quad \omega'_2 = -3uw_3, \quad \omega'_3 = 12uw_3. \tag{31}$$

In this case the three-form can be presented by $\Omega(x_1, x_2, x_3; \omega_1, -\frac{3}{2}\omega_1, 6\omega_1)$, so that the coordinate transformation (16) leads to the representation $\Omega(y_1, y_2, y_3; \omega'_1, -\frac{3}{2}\omega'_1, 6\omega'_1)$ with $\omega'_1 \neq 0$, that is, the transformation (16) degenerates to a general scale transformation.

6). If

$$\operatorname{sign}(w_1) = -\operatorname{sign}(w_2) \neq 0, \quad w_3 = 0, \tag{32}$$

then the transformation (16) with the parameter

$$p = \sqrt{-\frac{3w_2}{w_1}} \tag{33}$$

transfers the initial three-form into the three-form $\Omega(y_1, y_2, y_3; 0, \omega'_2, \omega'_3)$, with $\omega'_2 \neq 0$ and $\omega'_3 \neq 0$.

7). If

$$\operatorname{sign}(w_1) = \operatorname{sign}(w_2) \neq 0, \quad w_3 = 0, \tag{34}$$

then under the linear transformation (16) with

$$p = \sqrt{\frac{3w_2}{2w_1}} \tag{35}$$

the three-form is reduced to $\Omega(y_1, y_2, y_3; \omega'_1, \omega'_2, 0)$, with $\omega'_1 \neq 0 and \omega'_2 \neq 0$.

8). If

$$w_1 \neq 0, \quad w_2 = 0, \quad w_3 = 0, \tag{36}$$

then

$$\omega'_1 = uw_1p^3, \quad \omega'_2 = 3uw_1p^3, \quad \omega'_3 = 6uw_1p^3, \tag{37}$$

and hence under the transformation (16) the three-form $\Omega(x_1, x_2, x_3; \omega_1, 3\omega_1, 6\omega_1)$ becomes $\Omega(y_1, y_2, y_3; \omega'_1, 3\omega'_1, 6\omega'_1)$, where $\operatorname{sign}(\omega'_1) = \operatorname{sign}(\omega_1)$. Thus, the transformation (16) in such a case is reduced to multiplication of the initial three-form by a real positive number, that is, to a general scale transformation. We exclude such case in constructing three-numbers, for the case is degenerate (see (8)).

9). Lastly, we are to consider the variant

$$w_1 = 0, \quad w_2 \neq 0, \quad w_3 = 0, \tag{38}$$

in which

$$\omega'_1 = 3 \cdot u \cdot w_2 \cdot p = \frac{w_2}{9} = \omega_1, \quad \omega'_2 = \omega_2 = 0, \quad \omega'_3 = -9 \cdot u \cdot w_2 \cdot p = -\frac{w_2}{3} = -3\omega_1, \tag{39}$$

that is, applying (16) with arbitrary p transforms the three-form $\Omega(x_1, x_2, x_3; \omega_1, 0, -3\omega_1)$ to $\Omega(y_1, y_2, y_3; \omega_1', 0, -3\omega_1')$. Thus, in the given case the transformation (16) does not influence parameters of three-form, so that we can conclude that this transformation is a symmetry transformation for three-form $\Omega(x_1, x_2, x_3; \omega_1, 0, -3\omega_1)$.

Subsequent simplifying the three-form can be performed by multiplying it by an arbitrary real number deviating from zero. Such an operation is reduced to the following two ones: the changing of sign for all the coordinates and the general scale transformation. As a result, one of the coefficients $\omega_i' \neq 0$ of the three-form can be put to be unity, that is, the normalization can be performed. The proposed scheme 1) – 9) together with the normalization does not contradict to selecting three basis forms to play the role of canonical coordinates and introducing the notion of non-degeneracy, for the given algorithm goes over the basis forms (5) – (7) to the same basis forms, and any degenerate form to a degenerate one.

Thus, we have arrived at the following conclusion. Studying three-form of the general type $\Omega(x_1, x_2, x_3; \omega_1, \omega_2, \omega_3)$ reduces to studying the 8 canonical three-forms:

$$\Omega(x_1, x_2, x_3; 1, 0, 0) \equiv \Omega_1(x_1, x_2, x_3); \tag{40}$$

$$\Omega(x_1, x_2, x_3; 0, 1, 0) \equiv \Omega_2(x_1, x_2, x_3); \tag{41}$$

$$\Omega(x_1, x_2, x_3; 0, 0, 1) \equiv \Omega_3(x_1, x_2, x_3); \tag{42}$$

$$\Omega(x_1, x_2, x_3; 1, -\frac{3}{2}, 6); \tag{43}$$

$$\Omega(x_1, x_2, x_3; 1, 3, 6) \equiv (x_1 + x_2 + x_3)^3, \qquad \text{(degenerate)}; \tag{44}$$

$$\Omega(x_1, x_2, x_3; 1, \omega, 0), \qquad \omega \in [-\frac{1}{2}; 0) \cup (0; 1;] \tag{45}$$

$$\Omega(x_1, x_2, x_3; 1, 0, \omega), \qquad \omega \neq 0; \tag{46}$$

$$\Omega(x_1, x_2, x_3; 0, 1, \omega), \qquad \omega \neq 0. \tag{47}$$

The condition on the parameter ω (45) for the sixth canonical three-form is necessary in order that the uncertainty be avoided that does exists under consideration of the variant 2) of values of parameters of the general-type three-form. The condition $\omega \neq 0$ for the sixth, seventh, and eighth canonical three-forms is necessary to exclude the basis three-forms that have been ascribed to a canonical type.

3. Three-forms which may relate to three-numbers

Instead of searching directly the linear transformations leaving the three-forms 1 (40) – 8 (47) unchanged, we shall try to find the linear transformations which are infinitely near to identical ones. This problem is reduced to finding relevant generators.

1. There does not exist any continuous two-parametric Abelian Lie group which leave the form of the first canonical three-from (40) unchanged.

2. There does not exist any continuous two-parametric Abelian Lie group which leave the form of the second canonical three-from (41) unchanged.

3. The third canonical three-form (42) has a two-parametric non-Abelian group Lie to act as a symmetry group with the generators

$$\hat{a}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \hat{a}_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{48}$$

4. The fourth canonical three-form (43) has a three-parametric non-Abelian group Lie to act as a symmetry group with the generators

$$\hat{a}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{a}_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{a}_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{49}$$

It is necessary to verify that whether a two-parametric Abelian sub-group exists in this group.

5. The fifth canonical three-form (44) is non-degenerate and, therefore, is excluded from searching the three-numbers corresponding thereto.

6. The sixth canonical three-form does involve any two-parametric Lie group (for any admissible-type parameter (45)), although at $\omega = 1$ this three-from has one-parametric group of symmetry. Therefore the sixth canonical three-from cannot relates to three-numbers.

7. $\omega = -3$ 7th is the only parameter value at which the three-form has a two-parametric Abelian Lie group to serve as a symmetry group with the generators

$$\hat{a}_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \hat{a}_7 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{50}$$

It will be noted that the transformation (16) with the generator presented by the sum of the generators (50) enters this symmetry group, so that the three-form $\Omega(x_1, x_2, x_3; 1, 0, -3)$ should related to special cases.

8. The eight canonical three-form (47) at $\omega = 3$ has a one-parametric symmetry group which cannot relate to three-numbers, and at $\omega = 2$ has a two-parametric Abelian symmetry group with the generators

$$\hat{a}_8 = \begin{pmatrix} -\frac{2}{3} & -1 & -1 \\ 0 & \frac{1}{3} & 1 \\ 0 & 1 & \frac{1}{3} \end{pmatrix}, \qquad \hat{a}_9 = \begin{pmatrix} \frac{1}{3} & 1 & 0 \\ 1 & \frac{1}{3} & 0 \\ -1 & -1 & -\frac{2}{3} \end{pmatrix}. \tag{51}$$

Thus among the canonical three-forms we are able to find the four non-degenerate types that may relate to three-numbers. Retaining the numeration of canonical three-forms, let us write down these four forms indicating the related generators of symmetry group:

1. $- - - - - - - - -$;
2. $- - - - - - - - -$;
3. $\Omega(x_1, x_2, x_3; 0, 0, 1) \equiv \Omega_3(x_1, x_2, x_3), \quad \{\hat{a}_1, \hat{a}_2\};$ (52)
4. $\Omega(x_1, x_2, x_3; 1, -\frac{3}{2}, 6) \quad \{\hat{a}_3, \hat{a}_4, \hat{a}_5\};$ (53)
5. $- - - - - - - - -$;
6. $- - - - - - - - -$;
7. $\Omega(x_1, x_2, x_3; 1, 0, -3), \quad \{\hat{a}_{12}, \hat{a}_{13}\};$ (54)
8. $\Omega(x_1, x_2, x_3; 0, 1, 2), \quad \{\hat{a}_{14}, \hat{a}_{15}\};$ (55)

4. Three-forms $\Omega_3(x_1, x_2, x_3), \Omega(x_1, x_2, x_3; 0, 1, 2)$ and three-numbers

Let us consider the three-form $\Omega_3(x_1, x_2, x_3)$ which, as have been clarified above, possesses a two-parametric continuous Lie group — namely the symmetry group with the generators $\hat{a}_1, \hat{a}_2$ (48). Juxtaposing to the unit matrix and generators $\hat{a}_1, \hat{a}_2$ the basis elements $e_1 \equiv 1, e_2, e_3$ of sought system of three-numbers, we get for them the following multiplication table:

$\times$	1	e_2	e_3
1	1	e_2	e_3
e_2	e_2	$\frac{1}{3}(2 - 2e_2 + e_3)$	$\frac{1}{3}(1 - e_2 - e_3)$
e_3	e_3	$\frac{1}{3}(1 - e_2 - e_3)$	$\frac{1}{3}(2 + e_2 - 2e_3)$

Table 1.

So the three-numbers which can relate to the three-form $\Omega_3(x_1, x_2, x_3)$ have been found. It remains to verify whether a system of linear coordinates exists with respect to which the cube of the found three-numbers is a three-form $\Omega_3(x_1, x_2, x_3)$.

From the form of generators (48) it is obvious that the obtained system of three-numbers is isomorphic to the algebra of the diagonal matrices 3×3; therefore, we denote such numbers as H_3 and introduce a linear coordinates x_1, x_2, x_3 in terms of the basis

$$\psi_1 = \frac{1}{3}(1 - e_2 - e_3), \quad \psi_2 = \frac{1}{3}(1 + 2e_2 - e_3), \quad \psi_3 = \frac{1}{3}(1 - e_2 + 2e_3) \tag{56}$$

with the multiplication table

$\times$	ψ_1	ψ_2	ψ_3
ψ_1	ψ_1	0	0
ψ_2	0	ψ_2	0
ψ_3	0	0	ψ_3

Table 2.

Whence,

$$x_1\psi_1 + x_2\psi_2 + x_3\psi_3 = \rho \cdot \exp(\alpha \cdot e_2 + \beta \cdot e_3) \tag{57}$$

or

$$x_1\psi_1 + x_2\psi_2 + x_3\psi_3 = \rho \cdot \exp[(-\alpha - \beta) \cdot \psi_1 + \exp(\alpha) \cdot \psi_2 + \exp(\beta) \cdot \psi_3] \tag{58}$$

Thus the exponential representation of the numbers H_3 is possible, if $x_i > 0$ for the coordinates. If the angles α, β are excluded from three relations, then we obtain the expression for the cube of norm

$$\rho^3 = x_1 \cdot x_2 \cdot x_3 \tag{59}$$

This is not a unique possibility of symmetric introducing linear coordinates. For the numbers H_3 there exists the basis involving two hyperbolic unities

$$1 = \psi_1 + \psi_2 + \psi_3, \quad j = -\psi_1 - \psi_2 + \psi_3, \quad k = -\psi_1 + \psi_2 - \psi_3 \tag{60}$$

$\times$	1	j	k
1	1	j	k
j	j	1	$-1 + j + k$
k	k	$-1 + j + k$	1

Table 3.

If linear coordinates are introduced with respect to this basis then the cube of norm of the numbers H_3 reads

$$\rho^3 = \Omega(x, x, x; 1, -1, 2) \tag{61}$$

A noncanonical form enters the right-hand part of the formula (61). By the help of the transformation (16) at $p = 4$, wich changing signs simultaneously for all the coordinates and applying the general scale transformation, the three-form $\Omega(x_1, x_2, x_3; 1, -1, 2)$ can be sent into the eighth canonical three-form $\Omega(x, x, x; 0, 1, 2)$. The linear coordinates x_i for the numbers H_3 can be introduce alternatively as

$$(x_2 + x_3)\psi_1 + (x_1 + x_3)\psi_2 + (x_1 + x_2)\psi_3 = \rho \cdot \exp(\alpha \cdot e_2 + \beta \cdot e_3), \tag{62}$$

in which case

$$\rho^3 = \Omega(x_1, x_2, x_3; 0, 1, 2) \tag{63}$$

is again the eighth canonical form (55).

On so doing, the three-forms $\Omega(x_1, x_2, x_3; 0, 0, 1) \equiv \Omega_3(x_1, x_2, x_3)$, $\Omega(x_1, x_2, x_3; 1, -1, 2)$, $\Omega(x_1, x_2, x_3; 0, 1, 2)$ relate to one and same three-numbers H_3 which isomorphic to the algebra of quadratic diagonal matrices 3×3. Although the three-forms $\Omega(x_1, x_2, x_3; 0, 0, 1) \equiv \Omega_3(x_1, x_2, x_3)$, $\Omega(x_1, x_2, x_3; 0, 1, 2)$ cannot be obtained one from another by applying continuous linear transformation (16) in conjunction with scale-general transformation and probably also changing the sign of all the three coordinates, the forms are nevertheless connected by discrete linear transformation of corrdinates:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{64}$$

5. Three-form $\Omega(x_1, x_2, x_3; 1, -\frac{2}{3}, 6)$

Let us consider the generators $\hat{a}_3, \hat{a}_4, \hat{a}_5$ of linear transformations which leave the three-form $\Omega(x_1, x_2, x_3; 1, -\frac{3}{2}, 6)$ unchanged. These generators does not commute with one another. To single out two commuting generators, let us comprise the following linear combinations for these operators:

$$\hat{E}_0 = \hat{a}_3 + \hat{a}_4 + \hat{a}_5, \qquad \hat{E}_2 = -\hat{a}_3 + \hat{a}_4, \qquad \hat{E}_3 = -\hat{a}_3 + \hat{a}_5. \tag{65}$$

For them the following multiplication table is operative:

$\times$	$\hat{E}_0$	$\hat{E}_2$	$\hat{E}_3$
$\hat{E}_0$	$3\hat{E}_0$	$3\hat{E}_2$	$3\hat{E}_3$
$\hat{E}_2$	0	0	0
$\hat{E}_3$	0	0	0

Table 4.

Thus,$\hat{E}_2, \hat{E}_3$ or arbitrary two linear-independent combination thereof can be taken to serve as a pair of commuting generators. Using Table 4, , it can readily be shown that apart of $\hat{E}_2, \hat{E}_3$ and their linear combinations, no linear combinations of three operators $\hat{E}_0, \hat{E}_2, \hat{E}_3$, that is, the operators $\hat{a}_3, \hat{a}_4, \hat{a}_5$, may exist which commute with one another.

Let us relate to $\hat{E}_2, \hat{E}_3$ the symbolic elements e_2, e_3 of the hypercomplex number. Then for the basis elements $e_1 \equiv 1, e_2, e_3$ we obtain the Kely table

$\times$	1	e_2	e_3
1	1	e_2	e_3
e_2	e_2	0	0
e_3	e_3	0	0

Table 6.

Three-numbers with such a multiplication table of symbolic elements may naturally be called dual and denoted by D_3. For such three-numbers,

$$\rho \cdot \exp(\alpha \cdot e_2 + \beta \cdot e_3) = \rho \cdot (1 + \alpha \cdot e_2 + \beta \cdot e_3). \tag{66}$$

Up to the nummeration order, the unique possibility to introduce linear coordinates x_i in a symmetric fashion is

$$X = x_1 + x_2 \cdot (1 + e_2) + x_3 \cdot (1 + e_3), \tag{67}$$

so that the three-form

$$\rho^3 = (x_1 + x_2 + x_3)^3 \equiv \Omega(x_1, x_2, x_3; 1, 3, 6) \tag{68}$$

is non-degenerate.

Thus no three-number which cube of norm is equal to this three-form $\Omega(x_1, x_2, x_3; 1, -\frac{3}{2}, 6)$ can be found.

6. Three-form $\Omega(x_1, x_2, x_3; 1, 0, -3)$

The generators $\hat{a}_6, \hat{a}_7$ of the group symmetry under which actions the form $\Omega(x_1, x_2, x_3; 1, 0, -3)$ leaves unchanged possess the following multiplication rules:

$$\hat{a}_6 \cdot \hat{a}_6 = \hat{a}_7, \qquad \hat{a}_7 \cdot \hat{a}_7 = \hat{a}_6, \qquad \hat{a}_6 \cdot \hat{a}_7 = \hat{a}_7 \cdot \hat{a}_6 = 1. \tag{69}$$

Juxtaposing with them the symbolic elements e_2, e_3 of the system of three-numbers, we obtain the following Kely table:

$\times$	1	e_2	e_3
1	1	e_2	e_3
e_2	e_2	e_3	1
e_3	e_3	1	e_2

Table 7.

The hypercomplex associative-commutative three-dimensional numbers with the multiplication law for basis elements that is indicated by Table 7 will be denoted as C_3. Using this Kely table, we get the formula

$$\begin{aligned} \exp(\alpha \cdot e_2 + \beta \cdot e_3) = \frac{1}{3}\, e^{\alpha+\beta}\{1 + 2e^{-\frac{3}{2}(\alpha+\beta)} \cdot \cos[\frac{\sqrt{3}}{2}(\alpha - \beta)]\} + \\ + \frac{1}{3} e^{\alpha+\beta}\{1 - 2e^{-\frac{3}{2}(\alpha+\beta)} \cdot \cos[\frac{\sqrt{3}}{2}(\alpha - \beta) + \frac{\pi}{3}]\} \cdot e_2 + \\ \frac{1}{3} e^{\alpha+\beta}\{1 - 2e^{-\frac{3}{2}(\alpha+\beta)} \cdot \cos[\frac{\sqrt{3}}{2}(\alpha - \beta) - \frac{\pi}{3}]\} \cdot e_3. \end{aligned} \tag{70}$$

Let us introduce a coordinate system x_1, x_2, x_3 with respect to the same basis as follows:

$$x_1 + x_2 \cdot e_2 + x_3 \cdot e_3 = \exp(\alpha \cdot e_2 + \beta \cdot e_3). \tag{71}$$

Using the formula (70) and three coordinate relations (71), we get two relations

$$x_1 + x_2 + x_3 = \rho \cdot e^{(\alpha+\beta)}, \qquad x_1^2 + x_2^2 + x_3^2 = \frac{1}{3}\rho^2 \cdot e^{2(\alpha+\beta)}\{1 + 2 \cdot e^{-3(\alpha+\beta)}\}, \tag{72}$$

which are no more involving any difference of parameters $(\alpha - \beta)$. Expressing the sum of parameters $(\alpha + \beta)$ from (70), we get two relations

$$\rho^3 = \frac{3}{2} \cdot (x_1 + x_2 + x_3\} \cdot (x_1^2 + x_2^2 + x_3^2\} - \frac{1}{2} \cdot (x_1 + x_2 + x_3\}^3 \equiv \Omega(x_1, x_2, x_3; 1, 0, -3\}. \tag{73}$$

Thus we observe that for the three-numbers C_3 the cube of modulus is a three-form $\Omega(x_1, x_2, x_3; 1, 0, -3)$.

Although for the numbers C_3, by using symbolic element and unity, one can comprise the linear combination

$$j = \frac{1}{3}[1 - 2(e_2 + e_3)], \qquad j^2 = 1, \tag{74}$$

which is a hyperbolic unity ($j^2 = 1$), that is the numbers C_3 really present a generalization of hyperbolic (binary) numbers, it proves impossible to form a linear combination which would be the elliptic unity (with $i^2 = -1$); in a sense, the three-numbers C_3 present a generalization also for complex numbers for which the symbolic unity is a solution of the algebraic equation $x^2 = -1$. For the numbers C_3 the basis elements $1, e_2, e_3$ are roots for the cubic equation $x^3 = 1$, or with modified sign $-1, -e_2, -e_3$ they are roots for the equation $x^3 = -1$. Thus, from one side, in terms of complex numbers the equation $x^3 = 1$ has three roots

$$1, \quad -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \tag{75}$$

from which an imaginary unity can be singled out as their linear combination; from another side, the formulas (70) involve trigonometric functions, so that (in just this sense) the numbers C_3 may be regarded as a generalization of not only binary (hyperbolic) but also complex numbers for the three-dimensional case.

6. Conclusion

Up to isomorphism, two systems of hypercomplex three-dimensional numbers C_3 and H_3 are the only systems that can be selected from all the set of systems of associative-commutative hypercomplex numbers by setting forth the requirement of existence of a basis which respect to which the cube of norm of three-number (if it exists) is a non-degenerate three-form. The numbers C_3 can be juxtaposed by canonical three-form $\Omega(x_1, x_2, x_3; 1, 0, -3)$ (see Section 6 of present work), whereas the three-numbers H_3 by canonical three-forms $\Omega_3(x_1, x_2, x_3), \Omega(x_1, x_2, x_3; 0, 1, 2)$ ((see Section 4 of present work).

It is hoped that the result obtained permits entailing that also for the n-numbers with $n > 3$ the requirement of existence of a basis in term of which the n-degree of norm (provided the latter be exist) of n-number is equal to the n-form of coordinates, would select a narrow class of the hyperbolic numbers to play the role of generalization of complex and hyperbolic numbers (bi-numbers). Probably it is the hyperbolic numbers of such a type that primary find applications in mathematics and physics, being applied to the problems which involve in a sense the symmetry with respect to permutation of coordinates or some transformation "mixing" coordinates and simultaneously retaining their legitimacy.