QUATERNIONS: ALGEBRA, GEOMETRY AND PHYSICAL THEORIES

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A review of modern study of algebraic, geometric and differential properties of quaternionic (Q) numbers with their applications. Traditional and "tensor" formulation of Q-units with their possible representations are discussed and groups of Q-units transformations leaving Q-multiplication rule form-invariant are determined. A series of mathematical and physical applications is offered, among them use of Q-triads as a moveable frame, analysis of Q-spaces families, Q-formulation of Newtonian mechanics in arbitrary rotating frames, and realization of a Q-Relativity model comprising all effects of Special Relativity and admitting description of kinematics of non-inertial motion. A list of "Quaternionic Coincidences" is presented revealing surprising interconnection between basic relations of some physical theories and Q-numbers mathematics.

Introduction

The discovery of quaternionic (Q) numbers dated by 1843 is usually attributed to Hamilton [1, 2], but in the previous century Euler and Gauss made a contribution to mathematics of Q-type objects; moreover Rodriguez offered multiplication rule for elements of similar algebra [3-5]. Active opposition of Gibbs and Heaviside to Hamilton's disciples gave a start to the modern vector algebra, and later to vector analysis, and quaternions practically ceased to be a tool of mathematical physics, despite of exclusive nature of their algebra confirmed by Frobenius theorem. At the beginning of 20 century last bastion of Q-numbers amateurs, "Association for the Promotion of the Study of Quaternions", was ruined. The only reminiscence of once famous hypercomplex numbers was the set of Pauli matrices. Later on quaternions appeared incidentally as a mathematical mean for alternative description of already known physical models [6, 7] or due to surprising simplicity and beauty they were used to solve rigid body cinematic problems [8]. An interest to quaternionic numbers essentially increased in last two decades when a new generation of theoreticians started feeling in quaternions deep potential yet undiscovered (e. g. [9 – 11]).

This work is an attempt to give more systematic overview of contemporary state of Q-number mathematics, its applications to physical theories and possible perspectives in this area. In the context some quite specific even surprising physical models, but worth to pay attention to, are shortly discussed.

The review arranged as follows. In section 1 general relations of the quaternionic algebra are briefly described in the traditional hamiltonian formulation as well as in tensor-like format. Section 2 is devoted to description of structure of three "imaginary" quaternionic units. In section 3 the elements of differential Q-geometry are given with examples of their mathematical application. Section 4 comprises Q-formulation of Newtonian mechanics in the rotating frames of reference. Quaternionic Relativity Theory with a number of cinematic relativistic effects is found in section 5. Section 6 contains the list of "Great Quaternion Coincidences" and final discussion.

1. Algebra of quaternions

Traditional approach

According to Hamilton, a quaternion is a mathematical object of the form

$$Q \equiv a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},$$

where a, b, c, d are real numbers, a is a coefficient at real unit "1", and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ – three imaginary quaternion units. The multiplication rule for these units given by Hamilton and often used in literature is

$$\begin{gathered} 1\mathbf{i} = \mathbf{i}1 \equiv \mathbf{i}, \qquad 1\mathbf{j} = \mathbf{j}1 \equiv \mathbf{j}, \qquad 1\mathbf{k} = \mathbf{k}1 \equiv \mathbf{k}, \\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \qquad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j} \end{gathered}$$

These very cumbersome equations mean, that Q-multiplication loses a commutativity.

$$Q_1 Q_2 \neq Q_2 Q_1,$$

so that a notion of the right and the left multiplication appears, but it remains associative.

$$(Q_1 Q_2) Q_3 = Q_1 (Q_2 Q_3).$$

Two rather different algebraic parts are separated naturally in a quaternion, these once could be denoted as scalar:

$$scal\ Q = a,$$

and vector

$$vect\ Q = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}.$$

Addition (subtraction) of quaternions is performed by components, scalar and vector parts are added (subtracted) separately. With respect to addition the Q-algebra is commutative and associative.

Further step is quaternion conjugation introduced similarly to that of the complex numbers

$$\bar{Q} \equiv scal\ Q - vect\ Q = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k},$$

modulus of a Q-number is defined as

$$|Q| \equiv \sqrt{Q\bar{Q}} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

This permit to formulate a quaternionic division being as multiplication "right" and "left"

$$Q_L = \frac{Q_1 \bar{Q}_2}{|Q_2|^2}, \qquad Q_R = \frac{\bar{Q}_2 Q_1}{|Q_2|^2}.$$

Definition of Q-modulus enhances the famous four squares identity

$$|Q_1 Q_2|^2 = |Q_1|^2 |Q_2|^2 .$$

Due to the properties mentioned above the Q-numbers form the algebra, which belongs to the elite group of four the so-called exclusive – "very good" – algebras: of real, complex, quaternionic numbers and the octonions (Frobenious and Horwits theorems of 1878-1898 [12]).

Special attention should be paid to Q-units representations. In terms of Hamilton real unit is simply 1 while three imaginary units similarly to complex numbers algebra are denoted as **i, j, k**. Later a simple 2×2 matrices representation of these units was revealed

$$\mathbf{i} = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{j} = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \mathbf{k} = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This representation of course is not unique. Here is a simple example. If in the above expressions imaginary unit i of complex numbers is represented as 2×2 with real elements

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then three vector Q-units turn out to be represented by real 4×4 matrices. The procedure of the matrix rank duplication can obviously be continued further.

"Tensor" form and representations

If each Q-unit is endowed with its proper number (as components of a tensor)

$$(\mathbf{i}, \mathbf{j}, \mathbf{k}) \rightarrow (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \mathbf{q}, \qquad k, j, k, l, m, n, \ldots = 1, 2, 3,$$

then quaternionic multiplication rule acquires compact form

$$1\mathbf{q}_k = \mathbf{q}_k 1 = \mathbf{q}_k, \qquad \mathbf{q}_j \mathbf{q}_k = -\delta_{jk} + \varepsilon_{jkn} \mathbf{q}_n,$$

where δ_{kn} and ε_{knj} – respectively, 3-dimension (3D) symbols Kronecker and Levi-Chivita.

It is easy to show that a number of the Q-units representations even only by 2×2 matrices is infinite. Indeed for any 2×2 matrices with properties

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \qquad B = \begin{pmatrix} d & e \\ f & -d \end{pmatrix}, \qquad TrA = TrB = 0,$$

the first two Q-units can be constructed as follows

$$\mathbf{q}_1 = \frac{A}{\sqrt{\det A}}, \qquad \mathbf{q}_2 = \frac{B}{\sqrt{\det B}},$$

while the third one is

$$\mathbf{q}_3 \equiv \mathbf{q}_1 \mathbf{q}_2 = \frac{AB}{\sqrt{\det A \det B}} \qquad \text{provided that } \ Tr(AB) = 0.$$

The scalar unit is always invariant:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Transformations of Q-units and invariancy of the multiplication rule

a. Spinor-type transformations

If U is an operator changing at once all the units, and there is an inverse operator $U^{-1}: \quad UU^{-1} = E$, then transformations

$$\mathbf{q}_{k'} \equiv U\mathbf{q}_k U^{-1} \quad \text{and} \quad 1' \equiv U1U^{-1} = E1 = 1$$

retain the multiplication rule

$$1\mathbf{q}_k = \mathbf{q}_k 1 = \mathbf{q}_k, \quad \mathbf{q}_j\mathbf{q}_k = -\delta_{jk} + \varepsilon_{jkn}\mathbf{q}_n$$

form-invariant

$$\mathbf{q}_{k'}\mathbf{q}_{n'} = U\mathbf{q}_k U^{-1}U\mathbf{q}_n U = U\delta_{kn}U^{-1} + \varepsilon_{knj}U\mathbf{q}_j U^{-1} = \delta_{kn} + \varepsilon_{knj}\mathbf{q}_{j'}.$$

Such operator can be represented for example by 2×2 matrix

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det U = 1,$$

or unimodular quaternion,

$$U = \frac{a+d}{2} + \sqrt{1 - \left(\frac{a+d}{2}\right)^2}\,\mathbf{q},$$

where

$$\mathbf{q} \equiv \left(\sqrt{1 - \left(\frac{a+d}{2}\right)^2}\right)^{-1} \begin{pmatrix} \frac{a-d}{2} & b \\ c & -\frac{a-d}{2} \end{pmatrix}.$$

In general this transformation contains 3 independent complex parameter (or 6 real ones), then $U \in SL(2, C)$. In special case of only three real parameters, then $U \in SU(2)$.

b. Vector type transformations

Vector Q-units can be transformed by 3×3 matrix $O_{k'n}$

$$\mathbf{q}_{k'} = O_{k'n}\mathbf{q}_n.$$

The requirement of Q-multiplication form-invariance forces the transformation matrix to be orthogonal and unimodular

$$O_{k'n}O_{j'n} = \delta_{kn} \Rightarrow O^{-1}_{nk'} = O_{k'n}, \quad \det O = 1.$$

This transformation in general has 6 independent real parameters, then $O \in SO(3, C)$. In the special case of three parameters $O \in SO(3, R)$. Below a variant of representation of the transformation matrix O is given with x, y, z being arbitrary real or complex functions

$$O = \begin{pmatrix} \sqrt{1 - x^2 - z^2} & -\frac{x\sqrt{1-y^2-z^2} + yz\sqrt{1-x^2-z^2}}{1-z^2} & \frac{xy - z\sqrt{1-x^2-z^2}\sqrt{1-y^2-z^2}}{1-z^2} \\ x & \frac{\sqrt{1-x^2-z^2}\sqrt{1-y^2-z^2} - xyz}{1-z^2} & \frac{-y\sqrt{1-x^2-z^2} - xz\sqrt{1-y^2-z^2}}{1-z^2} \\ z & y & \sqrt{1 - y^2 - z^2} \end{pmatrix}.$$

This matrix can be represented as a product of three irreducible multipliers

$$O=\begin{pmatrix}\sqrt{\frac{1-x^2-z^2}{1-z^2}} & -\frac{x}{\sqrt{1-z^2}} & 0\\ \frac{x}{\sqrt{1-z^2}} & \sqrt{\frac{1-x^2-z^2}{1-z^2}} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{1-z^2} & 0 & -z\\ 0 & 1 & 0\\ z & 0 & \sqrt{1-z^2}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \sqrt{\frac{1-y^2-z^2}{1-z^2}} & -\frac{y}{\sqrt{1-z^2}}\\ 0 & \frac{y}{\sqrt{1-z^2}} & \sqrt{\frac{1-y^2-z^2}{1-z^2}}\end{pmatrix}.$$

after substitutions $z\equiv\sin\mathrm{B},\ x\equiv-\sin\mathrm{A}\cos\mathrm{B},\ y\equiv-\sin\Gamma\cos\mathrm{B}$, where A, B, Γ – are complex "angles", it takes the form

$$O=\begin{pmatrix}\cos\mathrm{A} & \sin\mathrm{A} & 0\\ -\sin\mathrm{A} & \cos\mathrm{A} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\mathrm{B} & 0 & -\sin\mathrm{B}\\ 0 & 1 & 0\\ \sin\mathrm{B} & 0 & \cos\mathrm{B}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\Gamma & \sin\Gamma\\ 0 & -\sin\Gamma & \cos\Gamma\end{pmatrix}=O_3^{\mathrm{A}}O_2^{\mathrm{B}}O_1^{\Gamma}.$$

If the angles are real: $\mathrm{A}=\alpha,\ \mathrm{B}=\beta,\ \Gamma=\gamma$, then this transformation is an ordinary vector rotation consisting of three simple rotations around numbered orthogonal axes: $O\Rightarrow R, R=R_3^{\alpha}R_2^{\beta}R_1^{\gamma}$. Correlation between related "spinor" and "vector" transformations is easily determined:

$$O_{k'n}=-\frac{1}{2}Tr(U\mathbf{q}_kU^{-1}\mathbf{q}_n),\qquad U=\frac{1-O_{k'n}\mathbf{q}_k\mathbf{q}_n}{2\sqrt{1+O_{mm'}}}.$$

Q-geometry in three dimensional space

Hamilton was the first to note that triad of Q-units behaves as three strictly tied unit vectors (with length i) initiating Cartesian coordinate system, somewhat exotic because of its "imaginarity". Due to the fact the Q-triad in 3D-space $(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)$ will be called 'quaternionic basis' (Q-basis). Now Q-units transformations have apparent geometrical sense of various rotations of the Q-basis. An example: a simple rotation by real angle α around axis # 3

$$\mathbf{q}'=R_3^{\alpha}\mathbf{q}.$$

Notion of Q-basis helps to introduce 3D quaternionic vectors (Q-vectors), defined as

$$\mathbf{a}=a_k\mathbf{q}_k,$$

here all its components a_k are real. The most important property of Q-vector – is its invariancy with respect to vector transformations from the group SO(3,R)

$$\mathbf{a}'=a_{k'}\mathbf{q}_{k'}=a_{k'}R_{k'j}\mathbf{q}_j=a_j\mathbf{q}_j=\mathbf{a}.$$

The projection of Q-vector onto arbitrary coordinate axis (represented by any different Q-unit) can be found in two ways. First, if at least one set of projections of Q-vector and rotation matrices $R_{nk'}$ are known then projections of this vector on rotated axis are immediately found

$$a_{k'}=a_nR_{nk'}.$$

The second approach is related to existence of internal structure of the Q-units; a brief analysis of it is given in the next section.

2. Structure of quaternionic "imaginary" units

Eigenfunctions of Q-units [13]

Each vector Q-unit can be thought of as operator, so eigenfunctions and eigenvalues problem can be formulated for it

$$\mathbf{q}\psi = \lambda\psi, \qquad \varphi\mathbf{q} = \mu\varphi.$$

The solution of this problem are the eigenvalues ("imaginary length" of Q-unit with division by parity)

$$\lambda = \mu = \pm i,$$

and two sets of eigenfunctions(one for each parity), possible given by columns $\psi^{\pm}$ and rows $\varphi^{\pm}$, being the functions of components $\mathbf{q}$.

Here is an example explicit form of eigenfunction: for the Q-unit represented by matrix

$$\mathbf{q} = -\frac{i}{T}\begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

where $T \equiv a^2 + bc \neq 0, b \neq 0, c \neq 0$, its eigenfunctions are defined as

$$\varphi^{\pm} = x\begin{pmatrix} 1 & \pm\frac{b}{T\pm a} \end{pmatrix}, \qquad \psi^{\pm} = y\begin{pmatrix} 1 \\ \mp\frac{c}{T\pm a} \end{pmatrix},$$

where x, y are arbitrary complex factors.

The freedom of components, arising in the calculations is reduced by convenient normalization condition

$$\varphi^{\pm}\psi^{\pm} = 1,$$

while the eigenfunctions orthogonality (by parity) is an inherited property

$$\varphi^{\mp}\psi^{\pm} = 0.$$

One can construct tensor products of eigenfunctions and obtain 2×2 matrices

$$C^{\pm} \equiv \psi^{\pm}\varphi^{\pm},$$

possessing a properties reciprocal with respect to the ones of vector $\mathbf{q}$:

$$\det C = 0, \quad Tr\, C = 1,$$

whereas

$$\det \mathbf{q} = 1, \quad Tr\, \mathbf{q} = 0.$$

Matrix C is idempotent

$$C^n = C,$$

and can be expressed throw their own unit Q-vector

$$C^{\pm} = \frac{1 \pm i\mathbf{q}}{2}.$$

When inversed the latter expression gives information about internal structure of Q-unit

$$\mathbf{q} = \pm i(2C^{\pm} - 1) = \pm i(2\psi^{\pm}\varphi^{\pm} - 1),$$

which turns out to consist of a combination of its eigenfunctions and scalar units.

Since each Q-unit has its own eigenfunctions the Q-triad as a whole possesses unique set of eigenfunctions $\{\varphi^{\pm}_{(k)}, \psi^{\pm}_{(k)}\}$. There is an interesting algebraic observation concerning this set. Three Q-units are interrelated by obviously nonlinear combination – multiplication e. g.

$$\mathbf{q}_3 = \mathbf{q}_1\mathbf{q}_2,$$

but it is easy to show that corresponding eigenfunctions depend on each other linearly:

$$\varphi^{\pm}_{(3)} = \sqrt{\mp i}\varphi^{\pm}_{(1)} \pm \sqrt{i}\varphi^{\pm}_{(2)}, \quad \psi^{\pm}_{(3)} = \sqrt{\pm i}\psi^{\pm}_{(1)} \pm \sqrt{-i}\psi^{\pm}_{(2)}.$$

Q-eigenfunctions help to represent a spinor-type transformation of Q-units retaining Q-multiplication invariant in the familiar form

$$\psi^{\pm}_{(k')} = U\psi^{\pm}_{(k)}, \quad \varphi^{\pm}_{(k')} = \varphi^{\pm}_{(k)}U^{-1},$$

so that the eigenfunctions can be regarded as a set of specific spinor functions, allowing in subject in general to $SL(2C)$ transformations. Yet another mathematical observation should be noted: from pairs of eigenfunctions, belonging to different Q-units of one triad and having one parity, one can construct 24 scalar invariants $SL(2C)$ group; these invariants are real or complex numbers, e. g.:

$$\sigma^{+}_{12} \equiv \varphi^{+}_{(1)}\psi^{+}_{(2)} = \sqrt{-\frac{i}{2}} = \frac{1-i}{2}.$$

Quaternionic eigenfunctions as projectors

Eigenfunctions act on their own Q-basis as following

$$\varphi^{\pm}_{(1)}\mathbf{q}_1\psi^{\pm}_{(1)} = \pm i, \quad \varphi^{\pm}_{(1)}\mathbf{q}_2\psi^{\pm}_{(1)} = 0, \quad \varphi^{\pm}_{(1)}\mathbf{q}_3\psi^{\pm}_{(1)} = 0,$$

or in general

$$\varphi^{\pm}_{(k)}\mathbf{q}_n\psi^{\pm}_{(k)} = \pm i\delta_{kn} \qquad \text{(no summation by k).}$$

It looks like that eigenfunctions select a projection of the unit Q-vector, generating them. This idea is confirmed by an example of an action of eigenfunctions of one Q-basis onto the vectors of the rotated Q-basis

$$\varphi^{\pm}_{(k)}\mathbf{q}_{n'}\psi^{\pm}_{(k)} = \varphi^{\pm}_{(k)}R_{n'm}\mathbf{q}_m\psi^{\pm}_{(k)} = \pm iR_{n'k} = \pm i\cos\angle(\mathbf{q}_{n'}, \mathbf{q}_k) \quad \text{(no summation by k),}$$

the result of the action is 'nearly' projection of Q-basis $\mathbf{q}'$ on $\mathbf{q}$. It is convenient to denote precise projection as

$$\langle \mathbf{q}_{\mathbf{n}'} \rangle_k \equiv \mp i\varphi^{\pm}_{(k)}\mathbf{q}_{n'}\psi^{\pm}_{(k)} = \cos\angle(\mathbf{q}_{n'}, \mathbf{q}_k) \qquad \text{(no summation by k).}$$

It is now easy to formulate rule of calculation of projection of a Q-vector a onto arbitrary direction, defined by vector $\mathbf{q}_j$ (e. g. with help of eigenfunctions of positive parity)

$$\langle \mathbf{a} \rangle^{+}_{j} \equiv -ia_{k'}\varphi^{+}_{(j)}\mathbf{q}_{k'}\psi^{+}_{(j)} = a_{k'}R_{k'j} = a_j \qquad \text{(no summation by j).}$$

Thus quaternionic eigenfunctions with their own interesting properties are more fundamental mathematical objects then Q-units and too can serve as useful tool for practical purposes such as computing projections of Q-vectors.

4. Differential Q-geometry

Quaternionic connection

If vectors of Q-basis are smooth functions of parameters $\mathbf{q}_k(\Phi_\xi)$ (index ξ enumerates parameters), then

$$d\mathbf{q}_k(\Phi) = \omega_{\xi\ kj}\mathbf{q}_j d\Phi_\xi,$$

where an object $\omega_{\xi\ kj}$ is called quaternionic connection. Q-connection is antisymmetric in vector indices

$$\omega_{\xi\ kj} + \omega_{\xi\ jk} = 0,$$

and has the following number of independent components

$$N = Gp(p-1)/2,$$

where G is an number of parameters and p = 3 – is a number of space dimensions. If G = 6 [a case of group SO(3,C)], then N = 18; if G = 3 [a case of group SO(3,R)], then N = 9. Q-connection can be calculated at least in three ways:

using vectors of Q-basis $\qquad \omega_{\xi\ kn} = \left\langle \dfrac{\partial \mathbf{q}_k}{\partial \Phi_\xi} \right\rangle_n^+,$

using matrices U from the group SL(2C) (general case) and special representation of constant Q-units $\mathbf{q}_{\tilde{k}} = -i\sigma_k$, where σ_k – Pauli matrices

$$\omega_{\xi\ kn} = \left\langle U^{-1}\frac{\partial U}{\partial \Phi_\xi}\mathbf{q}_{\tilde{k}} - \mathbf{q}_{\tilde{k}} U \frac{\partial U^{-1}}{\partial \Phi_\xi} \right\rangle_n^+,$$

and, finally, using matrices O from $SO(3, C)$ (in a general case)

$$\omega_{\xi\ kn} = \frac{\partial O_{k\tilde{j}}}{\partial \Phi_\xi} O_{n\tilde{j}}.$$

All the formulae of course provide same result.

From the point of view of vector transformations a Q-connection is not a tensor. If $\mathbf{q}_k = O_{kp'}\mathbf{q}_{p'}$, then transformed components of connection are expressed throw original ones with addition of inhomogeneous term

$$\omega_{\xi\ kj} = O_{kp'}O_{jn'}\omega_{\xi\ p'n'} + O_{jp'}\frac{\partial O_{kp'}}{\partial \Phi_\xi}.$$

In 3D space Q-connectivity has clear geometrical and physical treatment as moveable Q-basis with behavior of Cartan 3-frame. Parameters of its ordinary rotations can depend on spatial coordinates $\Phi_\xi = \Phi_\xi(x_k)$, then $\partial_n \mathbf{q}_k = \Omega_{nkj}\mathbf{q}_j$, then components of slightly modified Q-connection

$$\Omega_{nkj} \equiv \omega_{\xi\ kj}\partial_n \Phi_\xi$$

have a sense of Ricci rotation coefficients. Parameters can also depend on the length of line of motion of the Q-basis or on the observer's time. Then $\Phi_\xi = \Phi_\xi(t), \partial_t \mathbf{q}_k = \Omega_{kj}\mathbf{q}_j$, and components of Q-connection

$$\Omega_{kj} \equiv \omega_{\xi\ kj}\partial_t \Phi_\xi$$

became generalized angular velocities of rotations of the frame.

The typical examples of Q-frames and Q-connection are

a) Frene frame. For the smooth curve $x_{\tilde{k}}(s)$ defined in constant basis the Frene frame is represented by the triad $\mathbf{q}_k$, obeying the equations

$$\frac{d}{ds}\mathbf{q}_1 = R_I(s)\mathbf{q}_2,\ \frac{d}{ds}\mathbf{q}_2 = -R_I(s)\mathbf{q}_1 + R_{II}(s)\mathbf{q}_3,\ \frac{d}{ds}\mathbf{q}_3 = -R_{II}(s)\mathbf{q}_2,$$

where the first and the second curvatures are

$$R_I = \Omega_{12}, R_{II} = \Omega_{23}.$$

b) Twisted straight line. For a given straight line $x_{\tilde{1}} = u,\ x_{\tilde{2}} = x_{\tilde{3}} = 0$, one can construct a Q-basis associated with it so that one vector is tangent to the line. In this case Q-connection is not zero and represented the only component describing torsion (or rather twist) of the line about itself.

$$\mathbf{q}_1 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{q}_2 = -i\begin{pmatrix} 0 & -ie^{-i\gamma(u)} \\ ie^{i\gamma(u)} & 0 \end{pmatrix}, \qquad \Omega_{23} = \frac{d\gamma}{du},$$

here $\gamma(u)$ is the angle, which is an arbitrary but smooth function of the line length.

Quaternionic spaces

Tangent Q-space [15]. It is known that on every N-dimensional differentiable manifold U_N with coordinates $\{y^A\}$ one can construct a tangent space T_N with coordinates $\{X^{(A)}\}$ so that $dX^{(A)} = g_B^{(A)} dy^B$, where $g_B^{(A)}$ – Lame coefficients. By an extra rotation one can construct a tangent Q-space $T(U, \mathbf{q})$, with coordinates $\{x_k\}$, k = 1,2,3, which associated with Q-frame vectors.

$$dx_k = h_{k(A)} dX^{(A)} = h_{k(A)} g_B^{(A)} dy^B,$$

where $h_{k(A)}$ are in general non-square matrices normalized by projectors of the basic space onto 3D one or vice versa.

Proper quaternionic space itself $\mathbf{U}_3$ is defined as 3D-space, locally identical to own tangent space $T(\mathbf{U}_3, \mathbf{q})$. The Q-space has the following basic features. Its Q-metric represented by vector part of the Q-multiplication rule $\mathbf{q}_j\mathbf{q}_k = -\delta_{jk} + \varepsilon_{jkn}\mathbf{q}_n$ is non-symmetric, its antisymmetric part is Q-operator (matrix), so that every point $\mathbf{U}_3$ has internal quaternionic structure. Q-connection $\mathbf{U}_3$ can be: (i) proper (metric) $\Omega_{nkj} \equiv \omega_{\xi kj}\partial_n\Phi_\xi$, for variable Q-basis it is always non zero, and (ii) affine (non-metric), independent from Q-basis. Q-torsion does not vanish in both cases, whereas Q-curvature $r_{knab} = \partial_a\Omega_{bkn} - \partial_b\Omega_{akn} + \Omega_{ajn}\Omega_{bjk} - \Omega_{bjk}\Omega_{ajn}$ for the metric Q-connection identically equals zero, but can be present in the space of affine Q-connection.

Once Q-space is introduced, there appears a new field of investigation of differential manifolds and spaces. Thus in the preliminary classification of Q-spaces based on presence and nature of curvature, torsion and non-metricity at least 10 different families can be distinguish [15]. In addition Q-spaces can be a nontrivial background for classical and quantum theories and problems.

4. Newton mechanics in Q-basis

Dynamics equations in rotating frame [16]

The Q-basis endowed with clock becomes a classical (non-relativistic) reference system. For an inertial observer the dynamic equations of classical mechanics can be written in constant Q-basis

$$m\frac{d^2}{dt^2}x_{\tilde{k}}\mathbf{q}_{\tilde{k}} = F_{\tilde{k}}\mathbf{q}_{\tilde{k}}.$$

$SO(3,R)$-invariance of two Q-vectors, the radius-vector $\mathbf{r} \equiv x_k\mathbf{q}_k$ and force $\mathbf{F} \equiv F_k\mathbf{q}_k$ allow to represent these equations in Q-vector form

$$m\frac{d^2}{dt^2}(x_k\mathbf{q}_k) = F_k\mathbf{q}_k, \qquad \text{or} \quad m\ddot{\mathbf{r}} = \mathbf{F}$$

In explicit form these equations possess enough complicated structure

$$m(\frac{d^2}{dt^2}x_n + 2\frac{d}{dt}x_k\Omega_{kn} + x_k\frac{d}{dt}\Omega_{kn} + x_k\Omega_{kj}\Omega_{jn}) = F_n$$

which nevertheless can be simplified and interpreted from physical points of view. Due to antisymmetry of the connection (generalized angular velocity)

$$\Omega_j \equiv \Omega_{kn}\frac{1}{2}\varepsilon_{knj}, \qquad \Omega_{kn} = \Omega_j\varepsilon_{knj},$$

the dynamic equations can be rewritten in vector components

$$m(a_n + 2v_k\Omega_j\varepsilon_{knj} + x_k\frac{d}{dt}\Omega_j\varepsilon_{knj} + x_k\Omega_j\Omega_m\varepsilon_{jkp}\varepsilon_{mpn}) = F_n$$

or by conventional vector notation

$$m(\vec{a} + 2\vec{\Omega}\times\vec{v} + \dot{\vec{\Omega}}\times\vec{r} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})) = \vec{F}.$$

Among left hand side terms one easily recognizes 4 classical accelerations: linear, Coriolis, angular and centripetal. However this traditional interpretation is good only for simple rotation; in the case of combination of many Q-frame rotations number of components of generalized accelerations highly increases, and the equations become much more complicated. However it is worth noting that derivation of these equations for the most complicated rotations with the help of Q-basis and Q-connection is extremely simple.

Samples of Q-formulation of problems of classical mechanics

'Chasing' Q-basis – is a frame with one of its vectors, say $\mathbf{q}_1$ is always directed to observed particle. Dynamic equations for this case are written in explicit form in following manner

$$\ddot{r} - r(\Omega_2^2 + \Omega_3^2) = F_1/m,$$
$$2\dot{r}\Omega_3 + r\dot{\Omega}_3 + r\Omega_2\Omega_1 = F_2/m,$$
$$2\dot{r}\Omega_2 + r\dot{\Omega}_2 + r\Omega_1\Omega_3 = -F_3/m.$$

Components of Q-connection are defined as functions of angles of two rotations, the first (an angle α) – around vector $\mathbf{q}_3$, the second (an angle β) – around $\mathbf{q}_2$

$$\Omega_1 = \dot{\alpha}\sin\beta, \qquad \Omega_2 = -\dot{\beta}, \qquad \Omega_3 = \dot{\alpha}\cos\beta.$$

The chasing Q-basis approach is convenient to solve a number of mechanical problems related to rotations, some times very complicated, of observed objects and systems of reference. Here is an illustration.

Rotating oscillator. One seeks for motion law $r(t)$ of a harmonic oscillator (mass m, spring elasticity k) which has a freedom of motion along rigid smooth rod rotating in the plane around one of its ends (here one end of the spring is fixed) with angular velocity ω; the equilibrium point is located at the distance l from the rotation center, there is no gravity. Radial and tangent dynamic equations in the chasing Q-basis (F is unknown rod reaction force)

$$\ddot{r} - r\omega^2 = -\frac{k}{m}(r - l), \qquad 2\dot{r}\omega = \frac{1}{m}F,$$

admit the following family of solutions:

$$\text{(i)} \qquad r(t) = r_0 + v_0 t + at^2$$

mass moves away from the center of rotation with quadratic (or linear) law,

$$\text{(ii)} \qquad r(t) = const + Ae^{iwt} + Be^{-iwt}, \qquad w \equiv \sqrt{k/m - \omega^2}$$

here are three different situations depending on a relation of the quantities under the square root:

– r = const,

– harmonic oscillators,

– exponential motion away from the center of rotation.

It is interesting that the variants of rotating classical oscillator behavior with $l = 0$ are precisely similar to behavior of four known cosmological models of Einstein-DeSitter-Friedman considered in the General Relativity.

5. Construction of Quaternionic Relativity

Hyperbolic rotations and biquaternions [17]

It was noted above, that SO(3,C)-transformations of Q-units admit pure imaginary parameters. In this case rotations become hyperbolical (H – from hyperbolic); e. g. simple H-rotation $\mathbf{q}' = H_3^\psi \mathbf{q}$ is performed by matrix of the form

$$H_3^\psi = \begin{pmatrix} \cosh\psi & -i\sin\psi & 0 \\ i\sin\psi & \cosh\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and 2×2 -matrices of Q-units representation are no longer hermitian:

$$\mathbf{q}_{1'} = -i \begin{pmatrix} 0 & e^{\psi} \\ e^{-\psi} & 0 \end{pmatrix}.$$

This is the time to recall the notion of so called biquaternionic vectors (BQ). BQ-vector is defined as Q-vector with complex components $\mathbf{u} = (a_k + ib_k)\mathbf{q}_k$. Obviously for vectors of this type the norm (or modulus) in general can not be defined; but among all BQ-vectors there is a subset of "good" elements with well definable norm by $\mathbf{u}^2 = b^2 - a^2$. These vectors appear to be form-invariant with respect to transformations of subgroup

$SO(2,1) \subset SO(3,C)$, and in particular, with respect to simple H-rotations $\mathbf{q}' = H\mathbf{qu} = u_k\mathbf{q}_k = u_{k'}\mathbf{q}_{k'}$, but only when reciprocally imaginary components $a_k b_k = 0$ are orthogonal to each other.

Quaternionic Relativity

The made above observation allows to suggest a space-time BQ-vector "interval"

$$d\mathbf{z} = (dx_k + idt_k)\mathbf{q}_k,$$

with specific properties:

(i) Temporal interval is defined by imaginary vector,

(ii) space-time of the model appears to have six-dimensional (6D),

(iii) vector of the displacement of the particle and vector of corresponding time change must always be normal to each other $dx_k dt_k = 0$.

In this case BQ-vector-interval is invariant under group $SO(2,1) \subset SO(3,C)$, as well as of course its square (which differs from the square of norm only by sign) $d\mathbf{z}^2 = dt^2 - dr^2$, the latter has precisely the same form as a space-time interval of Special Relativity of Einstein. This 6D-model was initially named the Quaternionic Relativity. Temporal and spatial variables symmetrically enter the expression of BQ-vector-interval, and the Q-triad related to it describes relativistic system of reference $\Sigma \equiv (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$. Transition from one reference system to another is performed with the help of 'rotational equations' of the type $\Sigma' = O\Sigma$ with matrix O from the group $SO(2,1)$ is a product of matrices of real and hyperbolic rotations. So the theory could also be named (may be more correctly) 'Rotational Relativity'. The meaning of a simple H-rotation is immediately revealed from the first line of equation $\Sigma' = H_3^{\psi}\Sigma$ in the explicit form

$$i\mathbf{q}_{1'} = i\cosh\psi(\mathbf{q}_1 + \tanh\psi\mathbf{q}_2).$$

If like in Special Relativity $\cosh\psi = dt/dt'$, then

$$idt'\mathbf{q}_{1'} = idt(\mathbf{q}_1 + V\mathbf{q}_2),$$

which describes motion of reference system Σ' relative to Σ with velocity V along direction $\mathbf{q}_2$. It is easy to show that $SO(2,1)$-rotations of Q-reference system enhance Lorenz coordinate transformations and therefore all cinematic effects of Special Relativity.

It should be noted here that parameters of real and hyperbolic rotations can be variable for instance dependent on observer's time. This hints to expect of the discussed theory a possibility to describe non-inertial motions. Analysis of the rotational equations confirms the expectation. Well-known relativistic model of reference system constantly accelerated with respect to the inertial one (hyperbolic motion), frequently found in literature and normally regarded with use of assumption beyond frames of Special Relativity, in quaternionic theory is solved naturally and fast not only from the inertial observer viewpoint, but from position of accelerated frame too [18].

The kinematic problem of other non-inertial motion – relativistic circular motion – can be completely and precisely resolved by means of the rotation equation $\Sigma' = H_2^{\psi(t)}R_1^{\alpha(t)}\Sigma$, where Σ' is reference system rotating along the circle around the immobile frame Σ. This problem also can be solved both from the point of view of inertial observer, in this case the result has the form

$$t = \int dt' \cosh\psi(t'), \qquad \alpha(t) = \frac{1}{R}\int dt' \tanh\psi(t'),$$

$$a_{\tan}(t) = \frac{1}{\cosh^2\psi}\frac{d\psi}{dt}, \qquad a_{norm}(t) = R\left(\frac{d\alpha(t)}{dt}\right)^2,$$

and from the point of view of the observer in the reference system arbitrary moving along circular orbit.

The solution of the problem of "classical" Thomas precession in the framework of Special Relativity also needs additional assumptions, while in the quaternionic theory has a single line form – the first row of the matrix of rotation equation $\Sigma'' = R_1^{-\alpha(t)} H_2^{\psi} R_1^{\alpha(t)} \Sigma$, in this case of course correct value of precession frequency is obtained

$$\omega_T = (1 - \cosh\psi) \approx -\frac{1}{2}\omega V^2.$$

Moreover, the quaternionic theory of relativity appears to be able to describe Thomas precession for the vectors moving along trajectories of general type. The basic rotational equation in this case naturally generalized: $\Sigma'' = R^{-\theta(t)} H^{\psi(t)} R^{\theta(t)} \Sigma$, here $\theta(t)$ – an angle of instant rotation. Requirement that an axis of hyperbolic rotation be normal to the plane formed by the radius-vector of observed frame and its velocity vector, is also significant. In this case formula of variable frequency of general Thomas precession has the form

$$\Omega_T = \frac{d}{dt}(\theta - \theta').$$

An example of such Thomas precession is an apparent displacement of mercurial perihelion, for which calculations give a value $\Delta\varepsilon = 2,7''/100$ years.

Universal character of motion of the bodies (including non-inertial motions) in the Quaternionic Relativity suggests seeking for new cinematic relativistic effects. One is found in Solar System planets' satellites motion. Relative velocity of the Earth and other planets changes with time and sometimes achieves significant value comparable somehow to value of the fundamental velocity. This can lead to discrepancy between calculated and observed from the Earth cinematic magnitudes characterizing cyclic processes on this planet or near it. In particular there must be a deviation of the planetary satellite position. Such an angular difference is surprisingly found to be linearly dependent upon the time of observation

$$\Delta\varphi \approx \frac{\omega V_E V_P}{c^2} t,$$

here ω is an angular velocity of satellite motion around the planet, V – are linear velocities of the Earth and the planet around the sun. The magnitude of the effect is the following for the closest to the Jupiter and "the fasters" Jupiter satellite $\Delta\varphi \cong 12'$ for 100 terrestrial years; for the Mars satellite (Phobos) $\Delta\varphi \cong 20'$ for 100 terrestrial years [19]. Both values are big enough for the effect to be noticed in prolonged and precise observations.

One can say that space-time model and kinematics of the Quaternionic Relativity are nowadays studied in enough details and can be used as an effective mathematical tool for calculation of many relativistic effects. But respective relativistic dynamic has not been yet formulated, there are no quaternionic field theory; Q-gravitation, electromagnetism, weak and strong interactions are still remote projects. However, there is a hope that it is only beginning of a long way, and the theory will "mature". This hope is supported by observation of number of remarkable "Quaternionic Coincidences" forming a discrete mosaic of physical and mathematical facts; probably one day it will turn into a logically consistent picture providing new instruments and extending our insight of physical laws.

6. Remarkable "quaternionic coincidences"

There are, at least, five such coincidences (all of them given below), noted by different authors in various time.

1. *The Maxwell equations as an conditions of the analyticity of functions of quaternionic variable.*

In 1937 year Fueter [20] noted, that Cauchy-Riemann $\partial f/\partial z^* = 0$ equations defining the differentiability of complex variable function and modeling physically a flat motion of liquid without sources and whirls, have the following quaternionic analogue

$$\left(i\frac{\partial}{\partial t} - \mathbf{q}_{\tilde{k}}\frac{\partial}{\partial x_{\tilde{k}}}\right)\mathbf{H} = 0, \qquad \mathbf{H} = (B_{\tilde{n}} + iE_{\tilde{n}})\mathbf{q}_{\tilde{n}}.$$

Surprising fact is that the equations of classic Maxwell electrodynamics in vacuum prove to be corresponding physical model

$$div\vec{E} = 0, \quad div\vec{B} = 0, \quad rot\vec{E} - \frac{\partial\vec{B}}{dt} = 0, \quad rot\vec{B} + \frac{\partial\vec{E}}{dt} = 0.$$

2. *Classical mechanics in the rotating reference systems.*

The compact form of Newton equations in quaternion frame is described above in section 4. Finally it should be stressed that the form of dynamics equations naturally arising and externally primitive

$$m\ddot{\mathbf{r}} = \mathbf{F}$$

hides all possible combinations of rotations of reference systems or observed bodies. Using differential quaternionic objects helps to easily obtain explicit form of the equations whose elements have obvious physical meaning.

3. *The quaternionic theory of relativity.*

1:1 isomorphism of the Lorenz group of Special Relativity and the group of invariance of quaternionic multiplication $SO(3, C)$ leads to non-standard theory of relativity with symmetric six-dimensional space-time. This theory significantly differs from Einstein Special Relativity in origin, model, possibilities and mathematical tools, but predicts absolutely similar cinematic effects. Invariance of specific biquaternionic vector "interval" $d\mathbf{z} = (dx_k n + i\, dt_k)\mathbf{q}_k$ under subgroup $SO(2, 1)$ with in general variable parameters admits calculation of relativistic effects for non-inertial motion of reference systems.

4. *Pauli equations* [21].

Consider the quantum particle with electric charge e, mass m, and generalized momentum

$$P_k \equiv -i\hbar\frac{\partial}{\partial x_k} - \frac{e}{c}A_k$$

in the simplest quaternionic space (all the parameters are constant, connection, nonmetricity, torsion and curvature equal to zero). Hamiltonian of such particle in Q-metrics

$$H \equiv -\frac{1}{2m}P_k P_m \mathbf{q}_k \mathbf{q}_m$$

is the exact copy of Hamilton function of Pauli equation

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{2mc}\vec{B}\cdot\vec{\sigma},$$

and the spin term "automatically" acquires a coefficient equal to Bohr magneton.

5. *Young-Mills field strength.*

If one constructs a "potential" vector in an arbitrary quaternionic space from Q-connection components Ω_{amn} (indices a, b, c enumerate coordinates of basic Q-space, indices j, k, m, n enumerate vectors of tangent triad)

$$A_{ka} \equiv \frac{1}{2}\varepsilon_{kmn}\Omega_{amn},$$

and similarly construct a "field strength" vector

$$F_{kab} \equiv \frac{1}{2}\varepsilon_{kmn}r_{mnab},$$

from quaternionic curvature components

$$r_{knab} = \partial_a\Omega_{bkn} - \partial_b\Omega_{akn} + \Omega_{ajn}\Omega_{bjk} - \Omega_{bjk}\Omega_{ajn}$$

then these two geometrical objects are interconnected in the similar manner as the field strength and potential of the Young-Mills field

$$F_{kab} \equiv \partial_b A_{ka} - \partial_a A_{kb} + \varepsilon_{kmn}A_{ma}A_{nb}.$$

(formula) It should be stressed that for the Q-spaces with metric (not affine) connection curvature (field strength) identically vanish.

Discussion

Quaternionic numbers of course are first of all mathematical objects, so the problem of development of their algebra, analysis and geometry is self-consistent. But history of modern science states that once the geometry, in particular differential geometry, is discussed the presence of physics is unavoidable. There is a known point of view that Einstein who suggested General Relativity was a pioneer in geometrization of physics. But it is also known that quite earlier Maxwell formulated his electrodynamics in terms of quaternions convenient for description of 'etheric tensions' which were thought to represent field strength vectors. But since that the geometrical language has not been utilized for many decades.

The aspects of quaternionic mathematics given in this review once again draw attention to 'genetic relations' between physics and geometry: from description of frames rotations to quaternionic field structure phenomena in Pauli equations and Young-Mills theory.

Wide variety of possibilities provided by Q-approach and derived within it non-traditional physical models, like six-dimensional space-time or mentioned above coincidences may lead to opinion that quaternions are still a mathematical play, something like 'lego' elements, from which one can build many exotic constructions.

As a comment there are the following two observations.

1. Producing non-standard physical models Q-method nonetheless allows to successfully solve physical problems thus being a useful tool for practical purposes. A typical example: inherited exponential character of representation of simple rotations helps to simply formulate summation of different rotations, including, of course, imaginary rotations, describing relativistic boosts. Recall that in classical mechanics summation of ordinary rotations is quite a task.

2. All physical quaternionic theories are not heuristically invented, but appear naturally from fundamental mathematical lows, as though confirming Pythagorean idea on "world – number" dependence. Indeed, Q-algebra, the last associative algebra, describes well physical quantities, all of them up to our knowledge being associative with respect to multiplication: from observable kinematic and dynamic one, to tensors and spinors incorporated in the theories. All this gives a hope that further efforts in the research "quaternions – physical laws" relations will once grow into wide scientific programme. Yet another small, but persevering step in this direction has been recently made, when the author of this review succeeded to found an exact solution for relativistic oscillator problem in the framework Quaternionic Relativity. Details of the solution will be published elsewhere.

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